WALAILAK JOURNAL
http://wjst.wu.ac.th Engineering and Physical Sciences

RBF Meshless Method of Lines for the Numerical Solution of Nonlinear Sine-Gordon Equation

Marjan UDDIN[1,*], Arshad HUSSAIN[2], Sirajul HAQ[2] and Amjad ALI[1]

[1]*Department of Basic Sciences, University of Engineering and Technology, Peshawar, Pakistan*
[2]*Faculty of Engineering Sciences, Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, Topi, Pakistan*

([*]Corresponding author's e-mail: marjankhan1@hotmail.com)

Abstract

In this paper, a meshless method of lines (MOL) is applied for the numerical solution of nonlinear sine-Gordon equations. This technique does not require a mesh in the problem domain, and only a set of scattered nodes provided by initial data is required for the solution of the problem using some radial basis functions (RBF). The scheme is tested for single solitons, collision of breathers and soliton doublets. The results obtained from the method are compared with the exact solutions and earlier work.

Keywords: MOL, RBF, sine-Gordon equation, single soliton, soliton doublets

Introduction

The study of solutions of nonlinear wave equations has gained much attention in the last decade. These wave equations have applications in many branches of applied mathematics and theoretical physics. Zabusky and Kruskal [1] observed that the KdV equation leads to solitary waves called solitons, localized entities which keep identity after interaction. John Scott Russell [2] noted solitary waves while observing the movement of a canal barge. The sine-Gordon equation also models a soliton wave, which occurs in several physical situations [3]. The exact solution of soliton type equations has been developed using Backlund transformation [4], the painleve method [5], the Lie group of methods [6], the tangent hyperbolic method [7] and the inverse scattering method [8,9]. The sine-Gordon equation has also been solved numerically by using the finite-difference method, finite element method, pseudo spectral method and Adomian decomposition method [10-18].

Recently, the theory of radial basis functions (RBFs) has enjoyed great success as a scattered data interpolating technique. A radial basis function, $\phi(x - x_j) = \phi(\|x - x_j\|)$ is a continuous spline which depends upon the separation distances of a subset of data centers, $X \subset \Re^n$, $\{x_j \in X, j = 1,2,\ldots,N\}$. Due to spherical symmetry about the centers x_j, the RBFs are called radial. The distances, $\|x - x_j\|$, are usually taken to be the Euclidean metric.

The method of lines (MOL) [19] is a general procedure for the solution of time dependent partial differential equations (PDEs). The basic idea of the MOL is to replace the spatial (boundary value) derivatives in the PDEs with algebraic approximations. Once this is done, the spatial derivatives are no longer stated explicitly in terms of the spatial independent variables. Thus, only the initial value variable, typically time in a physical problem, remains. In other words, a system of ODEs that approximate the original PDE is obtained. Now any integration algorithm can be applied for initial value ODEs to compute an approximate numerical solution to the PDE. Thus, one of the salient features of the MOL is the use of existing, and generally well established, numerical methods for ODEs. Recently, G. Fasshauer [20] developed a meshless method of lines by connecting RBF method with Pseudo-spectral method. Recently, this approach has been applied for various types of PDEs [21-24].

RBF meshless method of lines has achieved a very high accuracy in solving partial differential equations with very easy implementation [25-27]. This method has an edge over finite element, finite difference and finite volume methods, because it does not require a mesh in the problem domain. Derivatives of the RBFs are used to approximate spatial derivatives using a set of nodes scattered in the problem domain. However, the interpolation matrix obtained in this technique becomes highly ill-conditioned when the number of nodes increases [28]. Several techniques have been used to overcome this drawback [29,30].

In this work, MOL is applied with RBFs approximation method, using the multiquadric radial basis function (MQ) {$\left(r^2+c^2\right)^{1/2}$, where c is a shape parameter}, for the numerical solution of sine-Gordon equation given by;

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{\partial^2 u(x,t)}{\partial x^2}-\sin u(x,t), \qquad x\in[a,b]\times[0,T] \tag{1}$$

with the initial and boundary conditions;

$$u(x,0)=u_1(x),\ \ u_t(x,0)=u_2(x) \tag{2}$$

$$u(a,t)=g_1(t),\ \ u(b,t)=g_2(t). \tag{3}$$

The sine-Gordon equation is transformed into coupled equations, given by;

$$\frac{\partial u(x,t)}{\partial t}=v(x,t),\ \frac{\partial v(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}-\sin u(x,t),\ x\in[a,b]\times[0,T]. \tag{4}$$

The rest of the paper is organized as follows: in Section 2, the meshless method of lines using RBFs is discussed. Section 3 is devoted to the numerical tests of the method on the problems related to the sine-Gordon equation. In Section 4, the results are concluded.

RBF meshless method of lines for sine-Gordon equation

Consider the transformed form of the sine-Gordon equation as coupled equations;

$$u_t(x,t)=v(x,t),\ v_t(x,t)=u_{xx}(x,t)-\sin u(x,t), \tag{5}$$

with boundary conditions;

$$u(a,t)=f_1(t),\ \ u(b,t)=f_2(t),\ \ v(a,t)=g_1(t),\ \ v(b,t)=g_2(t),\ \ t>0, \tag{6}$$

and initial conditions;

$$u(x,0)=f(x),\ \ v(x,0)=g(x),\ \ a\le x\le b. \tag{7}$$

For a given set of N collocation points $\{x_i\}_{i=1}^{N}$ in the domain $[a,b]$, the RBF approximation for u and v of Eq. (5) are given by;

$$U(x)=\sum_{j=1}^{N}\lambda_{1j}\phi_j(r), \qquad V(x)=\sum_{j=1}^{N}\lambda_{2j}\phi_j(r), \tag{8}$$

where $\{\lambda_j\}_{j=1}^{N}$ are the unknown constants to be determined, $\phi_j(r)=\phi(\|x-x_j\|)$ can be any well known radial basis function and $r_j=\|x-x_j\|$ is the Euclidean norm between points x and x_j. Here the multiquadric $\phi(r)=\sqrt{r^2+c^2}$ radial basis function is used. Now, for each node $x_i, i=1,2,3,\ldots,N$ in the domain $[a,b]$, Eq. (8) can be written as;

$$\mathbf{U}=\mathbf{A}\boldsymbol{\lambda}_1, \qquad \mathbf{V}=\mathbf{A}\boldsymbol{\lambda}_2, \tag{9}$$

where

$$\mathbf{U}=[U(x_1),U(x_2),U(x_3),\ldots,U(x_N)]^T,\ \boldsymbol{\lambda}_i=[\lambda_{i1},\lambda_{i2},\lambda_{i3},\ldots,\lambda_{iN}]^T,\ i=1,2,$$

$$\mathbf{A}=\begin{bmatrix}\boldsymbol{\Phi}^T(x_1)\\ \boldsymbol{\Phi}^T(x_2)\\ \cdots\\ \boldsymbol{\Phi}^T(x_N)\end{bmatrix}=\begin{bmatrix}\phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_N(x_1)\\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_N(x_2)\\ \vdots & \vdots & \ddots & \vdots\\ \phi_1(x_N) & \phi_2(x_N) & \cdots & \phi_N(x_N)\end{bmatrix},$$

$$\boldsymbol{\Phi}^T(x_i)=[\phi_1(x_i),\phi_2(x_i),\phi_3(x_i),\ldots,\phi_N(x_i)],\quad i=1,2,3,\ldots,N.$$

Eq. (8) is written as;

$$U(x)=\boldsymbol{\Phi}^T(x)\mathbf{A}^{-1}\mathbf{U}=\mathbf{D(x)U},\ V(x)=\boldsymbol{\Phi}^T(x)\mathbf{A}^{-1}\mathbf{V}=\mathbf{D(x)V}, \tag{10}$$

where

$$\mathbf{D(x)}=\boldsymbol{\Phi}^T(x)\mathbf{A}^{-1}=[D_1(x),D_2(x),\ldots,D_N(x)].$$

Using the approximation $U_i(t)$ and $V_i(t)$ of the solution $u(x_i,t)$ and $v(x_i,t)$ given in Eq. (9), Eq. (5) at each node $x_i, i=1,2,3,\ldots,N$ can be written as;

$$\frac{dU_i}{dt}=V_i, \qquad \frac{dV_i}{dt}=D_{xx}(x_i)\mathbf{U}-\sin(U_i) \tag{11}$$

where

$$\mathbf{D_x(x_i)}=[D_{1x}(x_i),D_{2x}(x_i),\ldots,D_{Nx}(x_i)],\ D_{jx}(x_i)=\frac{\partial}{\partial x}D_j(x_i),\ j=1,2,\ldots,N,$$

$$\mathbf{D_{xx}(x_i)}=[D_{1xx}(x_i),D_{2xx}(x_i),\ldots,D_{Nxx}(x_i)],\ D_{jxx}(x_i)=\frac{\partial^2}{\partial x^2}D_j(x_i),\ j=1,2,\ldots,N.$$

In a more compact form, Eq. (11) can be written as;

$$\frac{d\mathbf{U}}{dt}=\mathbf{V}, \qquad \frac{d\mathbf{V}}{dt}=\mathbf{D}_{xx}\mathbf{U}-\sin(\mathbf{U}), \tag{12}$$

where

$$\mathbf{D}_x=\left[D_{jx}(x_i)\right]_{i,j=1}^{N},\ \mathbf{D}_{xx}=\left[D_{jxx}(x_i)\right]_{i,j=1}^{N}.$$

Eq. (12) is written as;

$$\frac{d\mathbf{U}}{dt}=F_1(\mathbf{U},\mathbf{V}),\ \ \frac{d\mathbf{V}}{dt}=F_2(\mathbf{U},\mathbf{V}), \tag{13}$$

where

$$F_1(\mathbf{U},\mathbf{V})=\mathbf{V},\ \ F_2(\mathbf{U},\mathbf{V})=\mathbf{D}_{xx}\mathbf{U}-\sin(\mathbf{U}).$$

Now the system of ODEs (13) is solved by using Fourth-order Runge-Kutta scheme;

$$\begin{aligned}
&K_1^n=F_1(U^n,V^n),\ \ J_1^n=F_2(U^n,V^n),\\
&K_2^n=F_1(U^n+(\delta t/2)K_1^n,V^n+(\delta t/2)J_1^n),\ \ J_2^n=F_2(U^n+(\delta t/2)K_1^n,V^n+(\delta t/2)J_1^n),\\
&K_3^n=F_1(U^n+(\delta t/2)K_2^n,V^n+(\delta t/2)J_2^n),\ \ J_3^n=F_2(U^n+(\delta t/2)K_2^n,V^n+(\delta t/2)J_2^n),\\
&K_4^n=F_1(U^n+\delta tK_3^n,V^n+\delta tJ_3^n),\ \ J_4^n=F_2(U^n+\delta tK_3^n,V^n+\delta tJ_3^n),\\
&U^{n+1}=U^n+\frac{\delta t}{6}(K_1^n+2K_2^n+2K_3^n+K_4^n),\ \ V^{n+1}=V^n+\frac{\delta t}{6}(J_1^n+2J_2^n+2J_3^n+J_4^n).
\end{aligned}$$

Numerical examples

In this section, the proposed method for the numerical solution of sine-Gordon equations is applied. The accuracy of the meshless method of lines is tested in terms of L_2, L_∞ error norms and the conservation of energy of the sine-Gordon equation. These error norms and energy are defined as;

$$L_2=\|u-U\|_2=\left[\delta x\sum_{j=1}^{N}(u-U)^2\right]^{1/2}, \tag{14}$$

$$L_\infty=\|u-U\|_\infty=max_j\left|u-U\right|, \tag{15}$$

$$E=1/2\int_{-\infty}^{\infty}\left[(u_t)^2+(u_x)^2+2(1-\cos u)\right]dx, \tag{16}$$

where U and u denote the numerical and exact solution respectively. The test problems are given below.

Problem 1. Single soliton:

Sine-Gordon Eq. (1) is considered as a system of 2 equations;

$$u_t = v, \quad v_t = u_{xx} - \sin u. \tag{17}$$

The exact solutions of the Eq. (17) are given as;

$$u(x,t) = 4\tan^{-1}\left(\exp\left[\gamma(x-Ct)+\beta\right]\right), \; v(x,t) = \frac{-4\gamma C\exp\left[\gamma(x-Ct)+\beta\right]}{1+\exp\left[\gamma(x-Ct)+\beta\right]^2}, \tag{18}$$

where $\gamma = (1-C^2)^{-1/2}$. The problem is solved in the spatial interval $-2 \le x \le 58$, by the present method using MQ radial basis functions along with RK4 time integration scheme, at times $t = 0,9,36,108$, when time step size $\delta t = 0.001$, number of collocation points $N = 121$ and MQ shape parameter $c = 1$, $C = 0.5$, $\beta = 0$. The L_∞, L_2 error norms and the conserved quantity $E(t)$ are given in **Table 1**. In **Figure 1**, the numerical solutions, along with exact solutions at times $t = 0,9,36,108$, are shown. It is observed that the energy $E(t)$ is almost constant, which confirms the accuracy of the meshless method. The results are compared with the results in [10]. The results when MQ is used have a good agreement with the exact solution and is better than in earlier work [10]. It is observed that the solution converges when $0.2 < c < 4.4$.

Table 1 Error norms and energy constant for single soliton using MQ when $\delta t = 0.001$, $N = 121$, $c = 2$, $C = 0.5$, $\beta = 0$, $E(0) = 9.147397$ in $[-2,58]$ corresponding to Problem 1.

t	$L_\propto$ (RK4)	$\lvert E(t)-E(0)\rvert (RK4)$	L_∞ [10]	$\lvert E(t)-E(0)\rvert$[10]
9	6.296E-004	0.000159	4.518E-001	0.000321
36	8.109E-004	0.000162	2.293E+000	0.000297
108	2.568E-003	0.000199	5.120E+000	0.000280

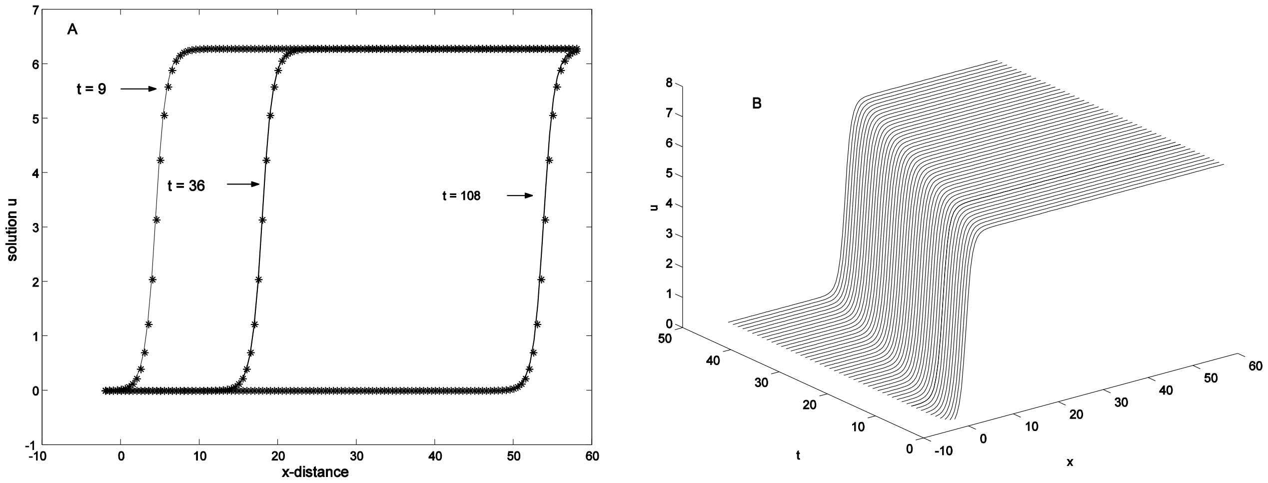


Figure 1 Single soliton: (A) the dot curves represent the exact solution, while the full curve shows the numerical solution at times t = 9,36,108 and (B) the numerical solution is shown over the time interval [1,48], when C = 0.5, β = 0, δt = 0.001, c = 2, N = 121 corresponding to Problem 1.

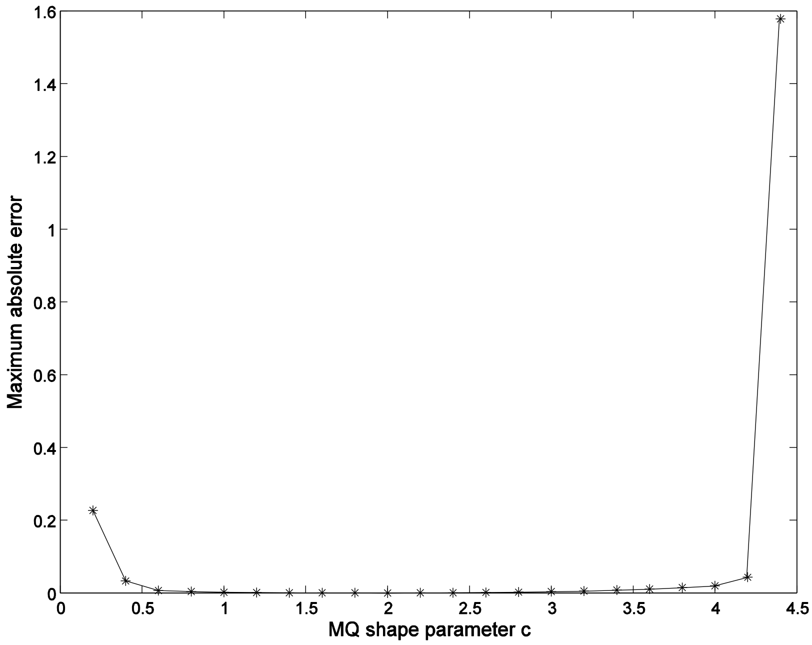


Figure 2 Single soliton: plots show maximum absolute error versus MQ shape parameter c, at time t = 1, when C = 0.5, β = 0, N = 121 corresponding to Problem 1.

Problem 2. Soliton-antisoliton breather:

Sine-Gordon Eq. (1) is considered as a system of 2 equations;

$$u_t = v, \quad v_t = u_{xx} - \sin u. \tag{19}$$

The exact solutions of a breather (or bion), which is a bound kink-antikink pair, and the energy, are given by;

$$\begin{aligned} u(x,t) &= 4\tan^{-1}\left[C^{-1}\sin(\bar{\gamma}Ct)\operatorname{sech}(\bar{\gamma}x)\right], \\ v(x,t) &= \frac{4\bar{\gamma}\cos(\bar{\gamma}Ct)\operatorname{sech}(\bar{\gamma}x)}{1+\left[C^{-1}\sin(\bar{\gamma}Ct)\operatorname{sech}(\bar{\gamma}x)\right]^2}, \end{aligned} \tag{20}$$

where $\bar{\gamma} = (1+C^2)^{-1/2}$, $E = 16\bar{\gamma}$. This problem is solved in the spatial interval $[-10,10]$, using RK4 scheme, when time step size $\delta t = 0.001$, number of collocation points $N = 201$, MQ shape parameter $c = 0.5$, and parameter $C = 0.5$, $\beta = 0$ are used. In **Figure 3(a) - 3(c)**, the numerical solutions, along with exact solutions at times $t = 1, 10, 30.$ are shown. **In Figure 3(d)**, the numerical solution over the time interval $[1,30]$ is shown. It is observed that the energy $E(t)$ is almost constant, showing the accuracy of the meshless method. The results are compared with the results in [10], which have a good agreement with the exact solution and the earlier work by [10], as shown in **Table 2**.

Table 2 Error norms and energy constant for single soliton when $\delta t = 0.0001$, $N = 201$, $c = 0.5$, $C = 0.5$, $\beta = 0$ in $[-10,10]$ corresponding to Problem 2.

t	L_∞ (RK4)	$\lvert E(t)-E(0)\rvert (RK4)$	L_∞ [10]	$\lvert E(t)-E(0)\rvert$[10]
1	2.485E-008	0.000001	0.988E-003	0.000369
10	3.145E-007	0.000004	0.1622E-002	0.003498
20	1.875E-006	0.000001	0.103E-002	0.007407

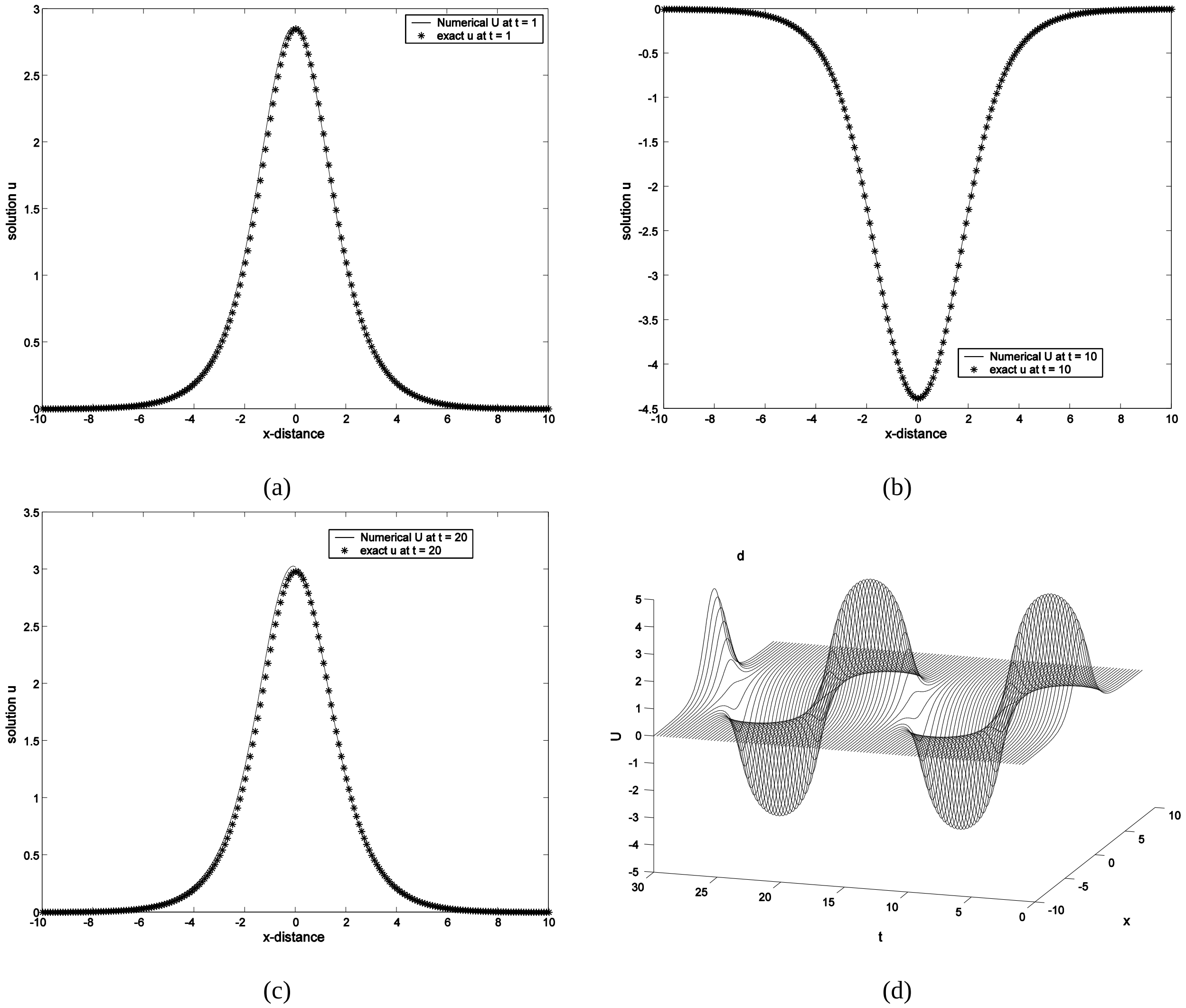


Figure 3 Soliton-antisoliton breather: (a), (b), (c) the dot curves represent the exact solution, while the full curve represents the numerical solution at times $t = 1, 10, 20$ and (d) the numerical solution is shown over the time interval [1,30] for $C = 0.5, \quad \beta = 0, \quad \delta t = 0.001, \quad N = 201,$ corresponding to Problem 2.

Problem 3. Soliton-soliton doublets:

Sine-Gordon Eq. (1) is considered as a system of 2 equations;

$$u_t = v, \quad v_t = u_{xx} - \sin u. \tag{21}$$

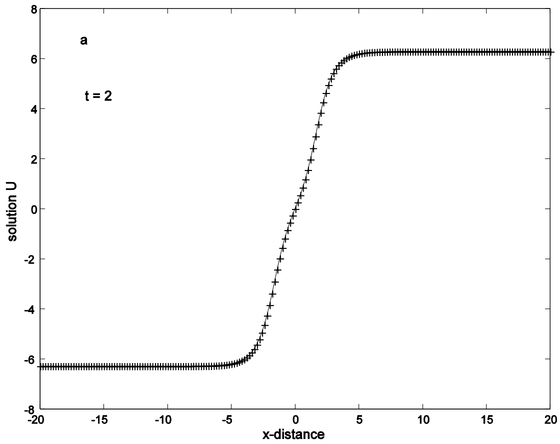
The exact solutions, and the energy, are given by;

$$u(x,t) = 4\tan^{-1}\left[C\text{sech}(\gamma Ct)\sinh(\gamma x)\right],$$
$$v(x,t) = \frac{-4C^2\gamma\text{sech}(\gamma Ct)\tanh(\gamma Ct)\sinh(\gamma x)}{1+\left[C\text{sech}(\gamma Ct)\sinh(\gamma x)\right]^2}, \quad (22)$$

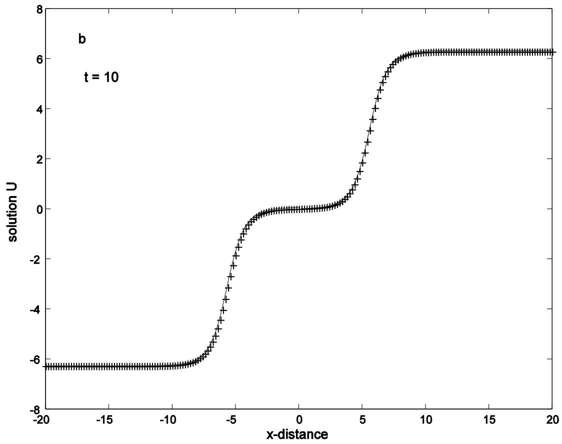
where $\gamma = (1-C^2)^{-1/2}$, $E = 16\gamma$. The problem is solved in [–20, 20]. The L_∞, L_2 error norms and the conserved quantity $E(t)$ using MQ, when time step size $\delta t = 0.001$, number of collocation points $N = 121$, MQ shape parameter $c = 1$, and parameter $C = 0.5$, $\beta = 0$ in the spatial interval $[-20, 20]$, are shown in **Table 3**. In **Figure 4 (A) - 4(C)**, the numerical solutions, along with exact solutions at times $t = 2,10,20.$, are shown. In **Figure 3(C)**, the numerical solution over the time interval $[1,30]$ is shown. It is clear that the energy $E(t)$ is almost constant, showing the accuracy of the meshless method of lines. The results are compared with the results in [10]. It is noted that the results of the present method are in good agreement with the exact solution and the earlier work by [10], as shown in **Table 3**.

Table 3 Error norms and energy constant for single soliton when $\delta t = 0.0001$, $N = 201$, $c = 0.5$, $C = 0.5$, $\beta = 0$ in $[-20, 20]$ corresponding to problem 3.

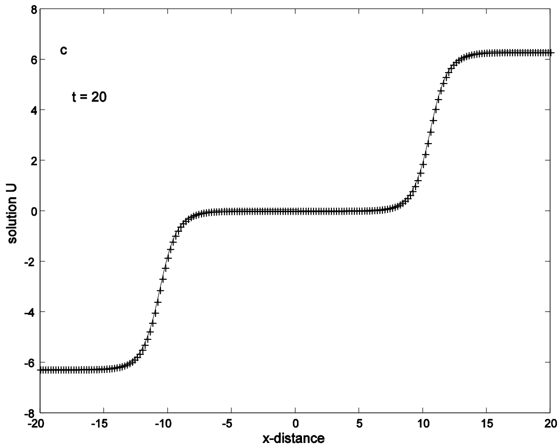
t	L_∞ (RK4)	$\lvert E(t)-E(0)\rvert (RK4)$	L_∞ [10]	$\lvert E(t)-E(0)\rvert$ [10]
2	4.006E-006	0.000003	0.127E-003	0.000344
10	2.152E-005	0.000005	0.191E-003	0
20	4.352E-005	0.000005	0.251E-003	0



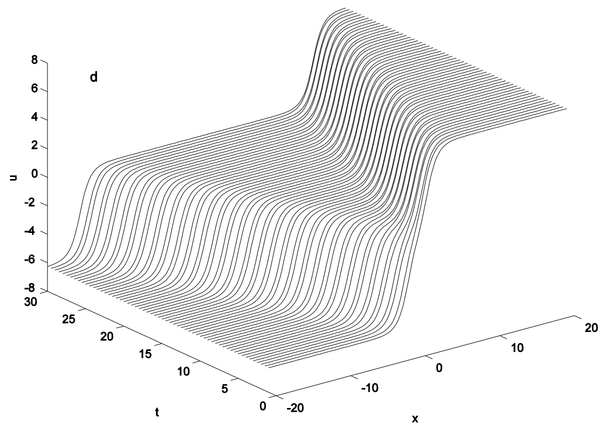
(a)



(b)



(c)



(d)

Figure 4 Soliton-soliton doublets: (a), (b), (c) the dot curves represent the exact solution, while the full curve represents the numerical solution at times $t = 1, 10, 20$ and (d) the numerical solution is shown over the time interval $[1, 30]$ when $C = 0.5,\ \beta = 0,\ \delta t = 0.001,\ N = 201,$ corresponding to Problem 3.

Concluding remarks

In this paper, a meshless method of lines using radial basis functions is applied for the numerical solution of sine-Gordon equations. The results of this method are in good agreement with the exact solution and with earlier work [10]. As a whole, the present method produces better results with an ease of implementation. The technique used in this paper provides an efficient alternative for the solution of time-dependent nonlinear partial differential equations. From application viewpoints, the implementation of this method is very simple and straightforward.

References

[1] NJ Zabusky and MD Kruskal. Interaction of solitons in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett*. 1965; **15**, 240-3.

[2] RJ Scott. Report of the committee of waves, Report of the 7th Meeting of the British Association for the Advancement of Science, Liverpool, 1838, p. 417-96.

[3] PJ Caudrey, JC Eilbeck and JD Gibbon. The sine-Gordon equation as a model classical field theory. *Nuovo. Cimento.* 1975; **25**, 497-511.

[4] HD Wahlquist and FB Estabrook. Bäcklund transformation for solutions of the Korteweg-de Vries equation. *Phys. Rev. Lett*. 1973; **31**, 1386-90.

[5] MJ Ablowitz, BM Herbst and C Schober. Numerical simulation of quasi-periodic solutions of the sine-Gordon equation. *Physica D* 1995; **87**, 37-47.

[6] PJ Olver. Applications of Lie groups to differential equations, Graduate Texts in Mathematics, Vol 107, Springer, Berlin, 1986.

[7] W Malfliet. Solitary wave solutions of nonlinear wave equations. *Am. J. Phys*. 1992; **60**, 650-4.

[8] MJ Ablowitz and PA Clarkson. Solitons, nonlinear evolution equations and inverse scattering. *London Math. Soc. Lecture Note Ser*. 1991; **149**, 1-532.

[9] CS Gardner, JM Greene, MD Kruskal and RM Miura. Method for solving the Korteweg-de Vries equation. *Phys. Rev. Lett*. 1967; **19**, 1095-7.

[10] AG Bratsos. A fourth order numerical scheme for the one-dimensional sine-Gordon equation. *Int. J. Comput. Math*. 2008; **85**, 1083-95.

[11] JC Eilbeck. *Numerical Studies of Solitons. In*: AR Bishop and T Sneider (eds.). Solitons and Condensed Matter Physics, Springer-Verlang, Berlin, 1978, p. 28-43.

[12] WA Strauss and L Vazquez. Numerical solution of a nonlinear Klein-Gordon equation. *J. Comput. Phys*. 1978; **28**, 271-78.

[13] BY Guo, PJ Pascual, MJ Rodriguez and L Vazquez. Numerical solution of the sine-Gordon equation. *Appl. Math. Comput*. 1986; **18**, 1-14.

[14] AQM Khaliq, B Abukhodair, Q Sheng and MS Ismail. A predictor-corrector scheme for the sine-Gordon equation. *Numer. Meth. D* 2000; **16**, 133-46.

[15] JI Ramos. The sine-Gordon equation in the finite line. *Appl. Math. Comput*. 2001; **124**, 45-93.

[16] AG Bratsos. The solution of two-dimensional sine-Gordon equation using the method of lines. *Appl. Math. Comput*. 2007; **206**, 251-77.

[17] M Dehghan and A Ghesmati. Numerical simulation of two-dimensional sine-Gordon solitons via a local weak meshless technique based on the radial point interpolation method (RPIM). *Comp. Phys. Comm*. 2010; **181**, 772-86.

[18] M Dehghan and A Shokri. A numerical method for solution of the two-dimensional sine-Gordon equation using the radial basis functions. *Math. Comp. Simul.* 2008; **79**, 700-15.

[19] WE Schiesser. *The Numerical Method of Lines: Integration of Partial Differential Equations*. Academic Press, San Diego, 1991.

[20] EG Fasshauer. *RBF Collocation Method and Pseudospectral Methods*. Illinois Institute of Technology, 2005.

[21] M Uddin. RBF-PS scheme for solving the equal width equation. *Appl. Math. Comput*. 2013; **222**, 619-31.

[22] M Uddin, S Haq, M Ishaq. RBF-Pseudospectral Method for the Numerical Solution of Good Boussinesq Equation. *Appl. Math. Sci*. 2012; **6**, 2403-10.

[23] S Haq, A Hussain and M Uddin. RBFs meshless method of lines for the numerical solution of time-dependent nonlinear coupled partial differential equation. *Appl. Math.* 2011; **2**, 414-23.

[24] S Haq and A Hussain. On the numerical solution of nonlinear Burgers-type equations using meshless method of lines. *Appl. Math. Comput*. 2012; **218**, 6280-90.

[25] M Uddin and S Haq. RBFs approximation method for time fractional partial differential equations. *Comm. Nonlinear Sci. Numer. Simulat*. 2011; **16**, 4208-14.

[26] M Uddin. On the selection of good value of shape parameter in solving time-dependent partial differential equations using RBF. *Appl. Math. Model*. 2014; **38**, 145-44.

[27] M Uddin and G Qassem. Radial basis functions approximation method for numerical solution of good Boussinesq equation. *Struc. Long*. 2012; **8**, 91-103.

[28] R Schaback. Error estimates and condition numbers for radial basis function interpolation. *Adv. Comput. Math*. 1995; **3**, 251-64.

[29] EJ Kansa and YC Hon. Circumventing the ill-conditioning problem with multi- quadric radial basis functions applications to elliptic partial differential equations. *Comput. Math. Appl*. 2000; **39**, 123-37.

[30] JC Li and YC Hon. Domain decomposition for radial basis meshless methods. *Numer. Meth. Differ. Eq*. 2004; **20**, 450-62.