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Real analytic actions of SL (2, ℝ) on a surface

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Abstract. Real analytic actions of connected Lie groups locally isomorphic to SL (2, ℝ) on compact surfaces, possibly with boundary, are classified up to topological conjugacy and up to real analytic conjugacy. Finite dimensional universal unfoldings of the real analytic conjugacy relation are also constructed. These are local transversals to the conjugacy classes in the space of actions. Sometimes the unfolding is a variety but not a manifold, and thus the space of actions is not naturally modelled on a vector space. We find many rigid actions and some unexpected bifurcation.

0. *Introduction*

The general problem addressed here is to determine how a non-compact semisimple Lie group can act differentiably on a manifold. Of course, this is a very general problem, and we do not attempt a full attack. Rather, we consider a special case – that of the group SL (2, ℝ) acting on a surface – and see what happens.

One of the charming aspects of the general problem is that it might be solvable to some extent. If, instead of considering actions of a semisimple Lie group G – i.e. homomorphisms from G into some group of diffeomorphisms – one considers homomorphisms from G into a Lie group H, then the problem has essentially been solved. For one thing, a homomorphism from G and H is always rigid; composing it with small inner automorphisms of H provides all nearby homomorphisms. Also one can, in principle, list the conjugacy classes of such homomorphisms using the representation theory for the Lie algebra of G. Thus the true object of interest

$$\mathrm{Hom}\,(G, H)/H$$

is a discrete, describable space.

For a compact group G, R. Palais has shown [**5**] that the same rigidity carries over to general actions. That is, if M is a manifold, then

$$\mathrm{Act}\,(G, M)/\mathrm{Diff}\,(M)$$

is discrete in the appropriate topology. Our primary goal is to determine the extent to which actions of semisimple groups are rigid. As is well known [**7**, § IV. 2] and as will be seen here, some are not rigid. Still, it seems possible that the action of Diff (M) on Act (G, M) (by conjugation) admits finite-dimensional local cross-sections near many points. This is at least the case if M is a surface and G is SL (2, ℝ).

The approach here is naive. In the first section, the orbit types which may occur in an action of SL (2, ℝ) on a surface – i.e. the homogeneous spaces of SL (2, ℝ) of dimensions zero to two – are classified. In § 2, we determine normal forms for the generating vector fields near a stationary point or a one-dimensional orbit. This computation is the heart of the investigation, for knowledge of dynamics near orbits of less than maximal dimension and the ways in which these local dynamics may be perturbed lead to global conclusions. The normal forms near a stationary point are, according to [**1**] and [**2**], just the linear ones. Near a one-dimensional orbit, the list of normal forms is infinite but quite simple, as might be expected since $\mathfrak{sl}(2, \mathbb{R})$ is simple. In § 3 we succumb to the temptation to list the analytic and topological conjugacy classes of actions. In particular, we find that the only compact surfaces on which $\mathrm{SL}(2, \mathbb{R})$ can act non-trivially are the sphere, projective plane, torus, Klein bottle, disk, Möbius band, and cylinder. This is not too surprising, since $\mathrm{SL}(2, \mathbb{R})$ contains $\mathrm{SO}(2)$, but the same holds for the universal cover $\widetilde{\mathrm{SL}(2, \mathbb{R})}$, which contains no non-trivial compact subgroup. Finally, § 4 describes the finite-dimensional unfolding of a neighbourhood of a non-degenerate action. We find that a special few of these non-degenerate actions are *fragile* – i.e. they can be perturbed into actions with vastly different characteristics. The other actions exhibit some degree of stability. In particular, we find that actions having a *hyperbolic* or *elliptic* two-dimensional orbit are rigid and that actions having a stationary point are structurally stable in the sense of being topologically conjugate to all nearby actions but are not rigid.

This summary of results should have been prefaced by the disclaimer that they hold only in the space of real analytic actions. The initial reason for considering this case was that it facilitated the computations of § 2. In fact, a similar list of normal forms for C^r actions, $1 \leq r \leq \infty$, would be quite complicated. However, if one is willing to require that a certain eigenvalue (eventually called λ) associated with each one-dimensional orbit does not lie in a certain range (roughly, $\lambda \notin [-4r^{-1}, 0]$), then it seems that much of the analysis here carries over at least to C^r-Hölder actions.

In order to bring out phenomena determined by the algebraic, rather than topological, properties of a group, we actually consider actions by the universal covering group $\widetilde{\mathrm{SL}(2, \mathbb{R})}$. It is easy to go from such actions to actions by $\mathrm{SL}(2, \mathbb{R})$ itself, since the latter may be interpreted as those actions by $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ which send

$$\exp 2\pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to the identity. However, it is by no means clear that phenomena arising here should be typical of actions by general semisimple groups. For one thing, of course, surfaces are small. More important, perhaps, is the fact that the centre of $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ is infinite, whereas the centre of $\widetilde{\mathrm{SL}(n, \mathbb{R})}$ for $n > 2$, for example, is finite. These drawbacks are balanced in part by the role of $\mathfrak{sl}(2, \mathbb{R})$ as a fundamental constituent of semisimple Lie algebras.

1. $\widetilde{\mathrm{SL}(2, \mathbb{R})}$

Here we present the basic facts about $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ which will be needed later and classify its homogeneous spaces of dimensions one and two. For brevity, we denote

$\widetilde{\mathrm{SL}(2,\mathbb{R})}$ by G and its Lie algebra of left invariant vector fields by $\mathscr{g}$. This notation will remain in force throughout this paper.

The standard basis for $\mathscr{g}$ consists of the matrices

$$H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad X=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad\text{and}\quad Y=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

These satisfy the relations

$$[H,X]=2X,\qquad [H,Y]=-2Y,\quad\text{and}\quad [X,Y]=H.$$

It follows that the adjoint group

$$\mathrm{Ad}\,(G)\subseteq \mathrm{GL}\,(\mathscr{g})$$

is the component of the identity among all linear automorphisms of $\mathscr{g}$ which preserve the Casimir form

$$\theta(aH+bX+cY)=8(a^2+bc).$$

Portrayed in figure 1, this adjoint action is the key to the algebraic structure of G.

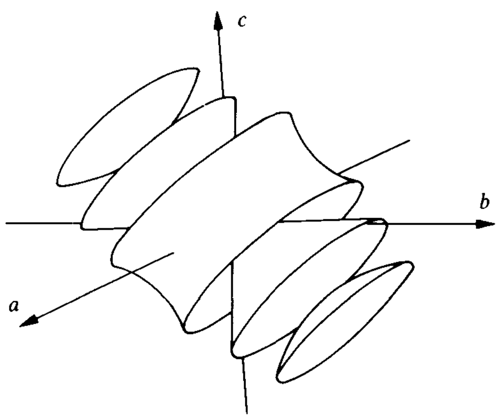

FIGURE 1. Orbits of the adjoint action.

As for topology, G is an infinite cyclic cover of SL (2, $\mathbb{R}$), homeomorphic to $\mathbb{R}^3$, with infinite centre generated by $\exp \pi(Y-X)$. Moreover, the one-parameter subgroup $\exp \mathbb{R}(Y-X)$ infinitely covers SO (2). These observations follow from the fact that every matrix in SL (2, $\mathbb{R}$) may be factored uniquely as

$$\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}u & v\\ 0 & u^{-1}\end{pmatrix}$$

with $u>0$.

An action $\alpha\in \mathrm{Hom}\,(G, \mathrm{Diff}^{\omega}(M))$ decomposes a surface M as a union of orbits, each of which can be viewed as a homogeneous space of G. The action therefore can be described by saying which homogeneous spaces are involved and how they fit together. In order to describe 'which', we classify the homogeneous spaces in theorem 1.2 below; the results of § 2 will allow us to say 'how'. The relevant terminology is that for $x \in M$, the closed subgroup

$$J(x)=\{g \in G: \alpha(g)x = x\}$$

is called the *isotropy group* of x, its Lie algebra $\mathscr{j}(x)$ is called the *isotropy algebra*, and $\alpha(G)x$ is called the *orbit through* x. The mapping

$$gJ(x) \mapsto \alpha(g)x$$

from the left coset space $G/J(x)$ into M is always a C^ω injective immersion, and it intertwines the natural action on $G/J(x)$ with α. Since the isotopy group of a typical point on the orbit through x is

$$J(\alpha(g)x) = gJ(x)g^{-1},$$

orbit types correspond to conjugacy classes of closed subgroups of G.

To classify the closed subgroups of G, we begin by classifying their Lie algebras. The proof, a computation, is omitted.

THEOREM 1.1. *Every two-dimensional subalgebra of* $\mathscr{g}$ *is conjugate under the adjoint group to* span $\{H, X\}$. *Every one-dimensional subalgebra is conjugate to one of*:

$$\operatorname{span}\{H\}, \qquad \operatorname{span}\{X\}, \quad \textit{or} \quad \operatorname{span}\{Y-X\},$$

and no two of these three are conjugate.

We call a one-dimensional subalgebra *hyperbolic*, *parabolic*, or *elliptic* according to its conjugacy to span $\{H\}$, span $\{X\}$, or span $\{Y-X\}$, respectively, and extend this terminology to one-dimensional subgroups and two-dimensional homogeneous spaces.

Let $\mathscr{j}$ be a subalgebra of $\mathscr{g}$,

$$N=\{g \in G: \operatorname{Ad}_g \mathscr{j} = \mathscr{j}\}$$

its normalizer, and J_0 the subgroup generated by $\exp(\mathscr{j})$. Using theorem 1.1, it is easy to see that J_0 is closed. There is a one-to-one correspondence between the closed subgroups of G having Lie algebra $\mathscr{j}$ and the discrete subgroups of N/J_0, namely that $J \subseteq G$ corresponds to

$$J/J_0 \subseteq N/J_0.$$

Also, for two such subgroups J_1 and J_2 to be conjugate in G, it is necessary and sufficient that J_1/J_0 and J_2/J_0 be conjugate in N/J_0. In the cases considered below, N turns out to be the semidirect product of J_0 with an abelian group K, so there is a bijection between the conjugacy classes of closed subgroups with Lie algebra $\mathscr{j}$ and the discrete subgroups of K. We omit the computations.

$\mathscr{j}$	K
span $\{H, X\}$	centre $(G) = \langle \exp \pi(Y-X)\rangle$
span $\{H\}$	$\langle \exp \frac{1}{2}\pi(Y-X)\rangle$
span $\{X\}$	$\exp \mathbb{R}H \times$ centre (G)
span $\{Y-X\}$	$\{1_G\}$

Here $\langle g\rangle$ denotes $\{g^n : n \in \mathbb{Z}\}$ for $g \in G$. In summary:

THEOREM 1.2. *The conjugacy classes of closed subgroups of G of dimensions one and two are parameterized by*:

two-dimensional	*subgroups of* $\mathbb{Z}$
hyperbolic	*subgroups of* $\mathbb{Z}$
parabolic	*discrete subgroups of* $\mathbb{R}\times\mathbb{Z}$
elliptic	*singleton.*

Note that a subgroup of G corresponds to a subgroup of SL $(2, \mathbb{R})$ precisely when it contains $\exp 2\pi(Y-X)$.

Soon it will be convenient to have explicit realizations of the homogeneous spaces just classified. Besides, the dynamics of the natural actions on them are interesting in their own right. Let $\mathfrak{j}$ be any one of the four subalgebras considered above and J_0 the corresponding subgroup. If J is any closed subgroup of G with Lie algebra $\mathfrak{j}$, then G/J may be viewed as the quotient of G/J_0 by the right action of $J\cap K$, where this right action is defined by

$$(gJ_0)k = gJ_0k = gkJ_0$$

for $k\in K$ and $g\in G$. Thus to describe the homogeneous spaces G/J, we will be content with an explicit realization of G/J_0 and the right action of K in each of the four cases. To obtain G/J_0, we place coordinates on the universal cover of the orbit through a point x of any convenient action with $\mathfrak{j}(x)=\mathfrak{j}$.

For the case $\mathfrak{j} = \operatorname{span}\{H, X\}$, consider the natural representation of G on $\mathbb{R}^2$. This induces a transitive action on $\mathbb{R}P^1$. Mapping the real line onto $\mathbb{R}P^1$ via

$$s \mapsto [\exp(s\sqrt{-1})]$$

and lifting the infinitesimal generators, we obtain vector fields

$$\bar{H} = -\sin 2s\frac{\partial}{\partial s}$$

$$\bar{X} = \tfrac{1}{2}(-1+\cos 2s)\frac{\partial}{\partial s}$$

$$\bar{Y} = \tfrac{1}{2}(1+\cos 2s)\frac{\partial}{\partial s}$$

on the real line. The right action of K is generated by the mapping

$$s\cdot \exp \pi(Y-X) = s+\pi.$$

Thus G/J is a circle except when $J=J_0$. For these one-dimensional spaces, we will call the index of $J\cap K$ in K the *length* of G/J. The situation is sketched in figure 2.

For the hyperbolic case, consider the action on the sphere $S(\mathfrak{g})$ which is induced by the adjoint representation (see figure 3). The orbit through $\mathbb{R}^+H\in S(\mathfrak{g})$ is a hyperbolic cylinder. In coordinates, its universal cover becomes $\mathbb{R}\times(-1, 1)$, and the

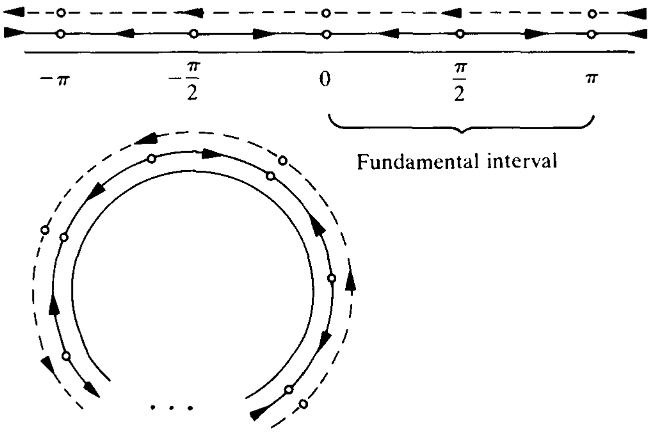

FIGURE 2. One-dimensional homogeneous spaces. —◂— = flow generated by $\bar{H}$, – – ◂ – – = flow generated by $\bar{X}$, —O— = zero of vector field.

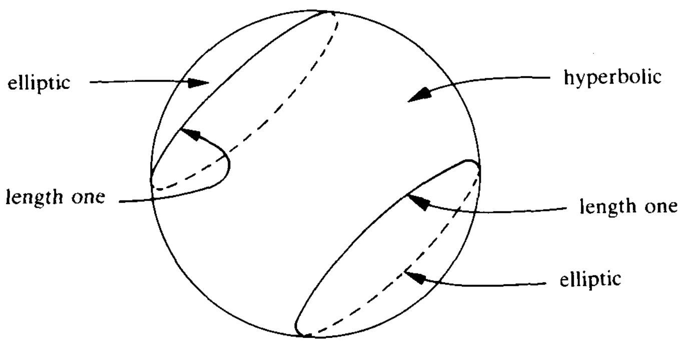

FIGURE 3. Orbits of the induced action on $S(\mathscr{g})$.

infinitesimal generators take the form

$$\bar{H} = -t \sin 2s \frac{\partial}{\partial s} + 2(1-t^2) \cos 2s \frac{\partial}{\partial t}$$

$$\bar{X} = \tfrac{1}{2}(-1 + t \cos 2s) \frac{\partial}{\partial s} + (1-t^2) \sin 2s \frac{\partial}{\partial t}$$

$$\bar{Y} = \tfrac{1}{2}(1 + t \cos 2s) \frac{\partial}{\partial s} + (1-t^2) \sin 2s \frac{\partial}{\partial t}.$$

This model is sketched in figure 4(a). A computation reveals that the right action of K is generated by

$$(s, t) \cdot \exp \tfrac{1}{2}\pi(Y - X) = (s + \tfrac{1}{2}\pi, -t).$$

Thus for J properly containing J_0, G/J is either a cylinder or a Möbius band, depending on the parity of the index of $J \cap K$ in K.

FIGURE 4. The simply connected two-dimensional homogeneous spaces.

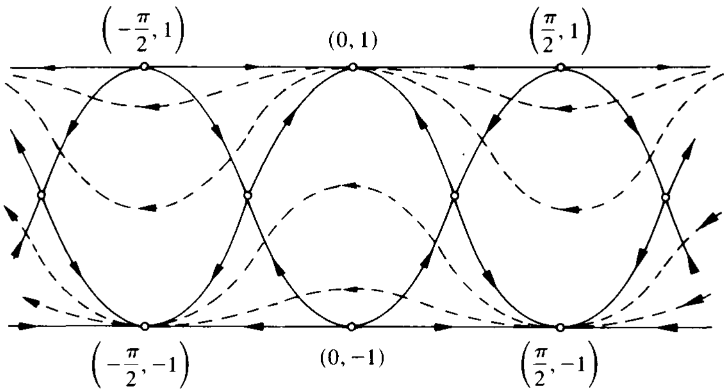

FIGURE 4(a). The hyperbolic strip.

The parabolic case $\mathfrak{j} = \text{span}\{X\}$ is complicated by the fact that K is not cyclic. Since the punctured plane $\mathbb{R}^2\backslash\{(0, 0)\}$ is a parabolic orbit of the natural representation, G/J_0 may be obtained by placing coordinates on its universal cover. This yields the strip $\mathbb{R} \times (-1, 1)$ with vector fields

$$\bar{H} = -\sin 2s \frac{\partial}{\partial s} + \tfrac{1}{2}(1-t^2) \cos 2s \frac{\partial}{\partial t}$$

$$\bar{X} = \tfrac{1}{2}(-1+\cos 2s) \frac{\partial}{\partial s} + \tfrac{1}{4}(1-t^2) \sin 2s \frac{\partial}{\partial t}$$

$$\bar{Y} = \tfrac{1}{2}(1+\cos 2s) \frac{\partial}{\partial s} + \tfrac{1}{4}(1-t^2) \sin 2s \frac{\partial}{\partial t}.$$

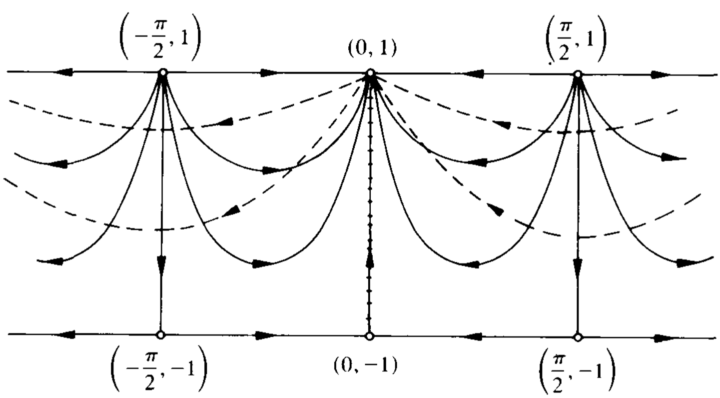

FIGURE 4(b). The parabolic strip. ++++ = curves along which $\bar{X}$ vanishes.

See figure 4(b). We identify

$$K = \exp(\mathbb{R}H) \times \text{centre}(G)$$

with $\mathbb{R} \times \mathbb{Z}$. The right action is generated by a vertical vector field

$$\tfrac{1}{2}(1-t^2)\frac{\partial}{\partial t}$$

and a horizontal translation

$$(s, t) \mapsto (s+\pi, t).$$

For parabolic subgroups J properly containing J_0, we then have the following cases:

Case 1. $J \cap K$ is not cyclic. Then $J \cap K$ has generators $(a, 0)$ and (b, n) with $a \in \mathbb{R}^+$, $b \in \mathbb{R}$, and $n \in \mathbb{Z}^+$. Moreover, a and n are uniquely determined, and b is determined modulo a. Here G/J is a torus, and the family of tori is parameterized by

$$\mathbb{R}^+ \times S^1 \times \mathbb{Z}^+.$$

Case 2. $J \cap K$ is generated by an element of the form $(a, 0)$. This case provides a family of cylinders parameterized by $\mathbb{R}^+$, which in fact never occur in an action of G on a compact surface.

Case 3. $J \cap K$ is generated by an element of the form (a, n) with $n \in \mathbb{Z}^+$. This gives a family of cylinders parameterized by $\mathbb{R} \times \mathbb{Z}^+$.

Finally, suppose that $\mathfrak{j} = \operatorname{span}\{Y - X\}$. There is just one orbit type, an example of which is either cap on $S(\mathfrak{g})$ (figure 3). In coordinates, this cap becomes the disk $s^2+t^2<1$ with vector fields

$$\bar{H} = -2st\frac{\partial}{\partial s} + 2(1-t^2)\frac{\partial}{\partial t}$$

$$\bar{X} = (s^2+t-1)\frac{\partial}{\partial s} + s(t-1)\frac{\partial}{\partial t}$$

$$\bar{Y} = (s^2-t-1)\frac{\partial}{\partial s} + s(t+1)\frac{\partial}{\partial t}.$$

See figure 4(c).

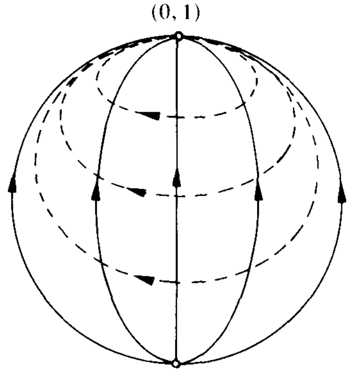

FIGURE 4(c). The elliptic disc.

Each model here has the property that the generating vector fields extend analytically to its boundary in $\mathbb{R}^2$. We will see in theorem 3.1 that the behaviour near the boundary is topologically quite natural; it occurs along the edge of any orbit of that type in an action on a surface.

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2. *Local structure*

Let M be a real analytic surface, possibly with boundary. A real analytic (C^ω) action of G on M is a homomorphism from G into $\mathrm{Diff}^\omega(M)$ for which the evaluation map from $G\times M$ into M is real analytic. The infinitesimal generator of the action α is denoted

$$\alpha_*: \mathfrak{g} \to \chi^\omega(M, \partial M).$$

That is,

$$\alpha_*(Z)x = \frac{d}{dt}\bigg|_{t=0} \alpha(\exp tZ)x$$

for $Z\in\mathfrak{g}$ and $x\in M$; the notation $\chi^\omega(M,\partial M)$ refers to C^ω vector fields on M which are tangent to the boundary. Since α is a homomorphism, the identities

$$\alpha_*(Z)\alpha(g)x = D\alpha(g)|_x\alpha_*(\mathrm{Ad}_{g^{-1}}Z)x$$

and

$$\alpha_*[Z, W] = -[\alpha_*(Z), \alpha_*(W)]$$

hold. In particular, the space $\mathrm{Act}^\omega(G, M)$ of all real analytic actions injects into the space of anti-homomorphisms from $\mathfrak{g}$ into $\chi^\omega(M,\partial M)$. If M is compact, this is a bijection.

In this section, we look for normal forms for α_* near a point $x\in M$ which is either stationary or lies on a one-dimensional orbit. To describe the possibilities in the latter case, we may assume that the isotropy algebra $\mathfrak{j}(x)$ is $\mathrm{span}\{H, X\}$. For if x is any point on a one-dimensional orbit, there is a group element g with

$$\mathrm{Ad}_g\mathfrak{j}(x) = \mathrm{span}\{H, X\},$$

and always

$$\mathfrak{j}(\alpha(g)x) = \mathrm{Ad}_g\mathfrak{j}(x).$$

If ϕ is a diffeomorphism from a neighbourhood of $\alpha(g)x$ onto a subset of $\mathbb{R}^2$ which transforms α_* into some normal form a_*, then $\phi\circ\alpha(g)$ transforms α_* near x into the normal form $a_*\circ\mathrm{Ad}_g$. That is,

$$D(\phi\circ\alpha(g))|_y\alpha_*(Z)y = a_*(\mathrm{Ad}_g Z)\phi\circ\alpha(g)y$$

for all $Z\in\mathfrak{g}$ and all y near x.

Let x be a stationary point of α. Evaluating derivatives at x, we obtain the isotropy representation of $\mathfrak{g}$ on T_xM:

$$Z\in\mathfrak{g}\mapsto D\alpha_*(Z)|_x\in\mathfrak{gl}(T_xM).$$

If this representation is not trivial, then it must be equivalent to the natural representation, since these are the only two-dimensional representations of $\mathfrak{g}$. Since the natural representation is irreducible, it can only occur when x is interior to M. The linearization theorem of Guillemin and Sternberg, **[1]** (see also **[2]**) thus transcribes as:

THEOREM 2.1 (Guillemin and Sternberg). *Let x be stationary for $\alpha\in\mathrm{Act}^\omega(G, M)$. Then either α_* vanishes near x, or else there exist local C^ω coordinates in which x*

becomes the origin of $\mathbb{R}^2$ *and* α_* *takes the normal form*

$$\bar{H} = s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t}$$

$$\bar{X} = t\frac{\partial}{\partial s}$$

$$\bar{Y} = s\frac{\partial}{\partial t}.$$

In the latter case, x is interior to M.

A wider variety of normal forms arises when the orbit through x is one-dimensional. Suppose that $\mathscr{J}(x) = \operatorname{span}\{H, X\}$. We first choose coordinates in which x becomes the origin of $\mathbb{R}^2$ and $\alpha_*(Y)$ becomes the constant vector field $\bar{Y} = \partial/\partial s$. We then perform a sequence of coordinate changes within $\mathbb{R}^2$ which leave the origin fixed and do not destroy the results of previous constructions.

The relation $[\bar{H}, \bar{Y}] = 2\bar{Y}$ forces

$$\bar{H} = (Q(t) - 2s)\frac{\partial}{\partial s} + R(t)\frac{\partial}{\partial t},$$

where Q and R vanish at zero. A change of coordinates of the kind

$$(s, t) \mapsto (s, f(t))$$

then transforms R into one of the forms

(i) $R(t) = at \qquad a \in \mathbb{R}$, or

(ii) $R(t) = \pm t^m + bt^{2m-1} \qquad m \geq 2, b \in \mathbb{R}$.

A second change of coordinates of the kind

$$(s, t) \mapsto (s + f(t), t)$$

transforms Q to zero except in a few cases – namely, the instances of (i) above with $a = -2m^{-1}$ for some positive integer m. The upshot is that we may obtain coordinates in which $\bar{Y} = \partial/\partial s$ and $\bar{H}$ is one of the following:

(1) $\bar{H} = -2s\frac{\partial}{\partial s} + at\frac{\partial}{\partial t}, \qquad a \in \mathbb{R}$;

(2) $\bar{H} = (\pm t^m - 2s)\frac{\partial}{\partial s} - 2m^{-1}t\frac{\partial}{\partial t}, \qquad m \in \mathbb{Z}^+$;

(3) $\bar{H} = -2s\frac{\partial}{\partial s} + (\pm t^m + bt^{2m-1})\frac{\partial}{\partial t}, \qquad m \geq 2, b \in \mathbb{R}$.

These cases are almost pairwise inequivalent under C^ω change of coordinates; the only exception is that the transformation

$$(s, t) \mapsto (s, -t)$$

interchanges some cases.

Since $[\bar{X}, \bar{Y}] = -\bar{H}$, we have

$$\bar{X} = (S(t) + sQ(t) - s^2)\frac{\partial}{\partial s} + (T(t) + sR(t))\frac{\partial}{\partial t},$$

where Q and R are as above and

$$S(0)=T(0)=0.$$

The relation $[\bar{H}, \bar{X}]=-2\bar{X}$ becomes ($'$ denoting d/dt):

(*) $Q^2-Q'T+4S+RS'=0$, and

(**) $QR+RT'-TR'+2T=0$.

In case (1), this means that

$$at\,S'(t)+4S(t)=0,$$

and

$$at\,T'(t)+(2-a)T(t)=0.$$

If $a=0$, then $S=T=0$. Otherwise, the general solution is

$$S(t)=\sigma t^{-4/a}, \qquad T(t)=\tau t^{1-(2/a)}.$$

If τ is not zero, then for T to be analytic with $T(0)=0$ it must be that $2/a$ is a negative integer, say $a=-2m^{-1}$, $m\in\mathbb{Z}^+$. The change of coordinates

$$(s, t)\mapsto(s-\tfrac{1}{2}\tau m t^m, t)$$

then leaves $\bar{Y}$ and $\bar{H}$ unchanged while putting $\bar{X}$ in the form

$$\bar{X}=((\sigma-\tfrac{1}{4}\tau^2m^2)t^{2m}-s^2)\frac{\partial}{\partial s}-2m^{-1}st\frac{\partial}{\partial t}.$$

In summary, we can always choose coordinates in case (1) so that $T=0$. In these coordinates, S vanishes except when $a=-4m^{-1}$, $m\in\mathbb{Z}^+$, in which case $S(t)=ct^m$. A linear change of the t-coordinate then yields $c=0$, $+1$, or -1.

In case (2), the equation (**) simplifies to

$$tT'(t)-(m+1)T(t)\pm t^{m+1}=0.$$

The general solution

$$T(t)=\mp t^{m+1}\ln t+ct^{m+1}$$

is not analytic near $t=0$. Thus (2) in fact never occurs. In case (3), (*) and (**) become

$$(\pm t^m+bt^{2m-1})S'(t)+4S(t)=0,$$

and

$$(\pm t^m+bt^{2m-1})T'(t)+(2\mp mt^{m-1}-(2m-1)bt^{2m-2})T(t)=0.$$

Since $m\geq 2$, the only solutions analytic near $t=0$ are $S=T=0$.

In summary, we have the following complete list of normal forms for α_* near a point x having isotropy algebra $\mathfrak{j}(x)=\text{span}\{H, X\}$:

Normal form (I, a), $a\in\mathbb{R}$.

$$\bar{H}=-2s\frac{\partial}{\partial s}+at\frac{\partial}{\partial t}$$

$$\bar{X}=-s^2\frac{\partial}{\partial s}+ast\frac{\partial}{\partial t}$$

$$\bar{Y}=\frac{\partial}{\partial s}.$$

Normal form $(II, m, \pm)$, $m\in\mathbb{Z}^+$.

$$\bar H=-2s\frac{\partial}{\partial s}-4m^{-1}t\frac{\partial}{\partial t}$$

$$\bar X=(\pm t^m-s^2)\frac{\partial}{\partial s}-4m^{-1}st\frac{\partial}{\partial t}$$

$$\bar Y=\frac{\partial}{\partial s}.$$

Normal form $(III, m, b, \pm)$, $m\geq 2$, $b\in\mathbb{R}$.

$$\bar H=-2s\frac{\partial}{\partial s}+(\pm t^m+bt^{2m-1})\frac{\partial}{\partial t}$$

$$\bar X=-s^2\frac{\partial}{\partial s}+s(\pm t^m+bt^{2m-1})\frac{\partial}{\partial t}$$

$$\bar Y=\frac{\partial}{\partial s}.$$

The restriction of each to the upper half plane provides a complete list of normal forms when x lies in the boundary of M. As before, the forms listed here are almost pairwise inequivalent, the only exceptions being that the transformation

$$(s,t)\mapsto(s,-t)$$

converts $(II, m, \pm)$ to $(II, m, \mp)$ when m is odd and $(III, m, b, \pm)$ to $(III, m, b, \mp)$ when m is even.

In all cases, the s-axis is invariant, and the restriction to the s-axis is always

$$\bar H=-2s\frac{\partial}{\partial s},\qquad \bar X=-s^2\frac{\partial}{\partial s},\qquad \bar Y=\frac{\partial}{\partial s}.$$

The degenerate normal form $(I, 0)$ is just the product of this representation with the trivial representation on the t-axis. In all the other normal forms, however, the s-axis is flanked by two dimensional (local) orbits as follows:

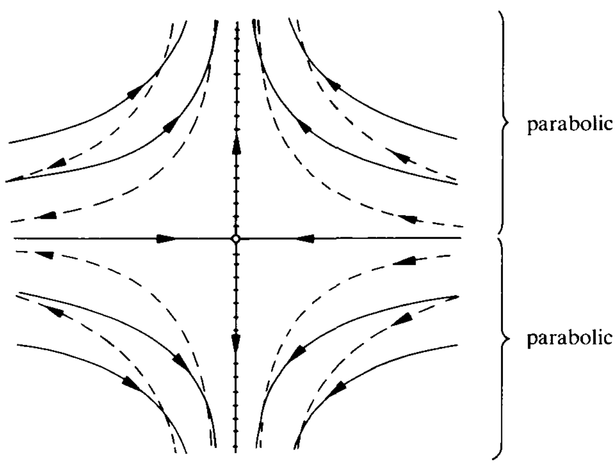
Normal form	Local orbit above s-axis	Local orbit below s-axis
(I, a), $a\neq 0$	parabolic	parabolic
$(II, m, +)$, m even	elliptic	elliptic
$(II, m, -)$ m even	hyperbolic	hyperbolic
$(II, m, +)$ m odd	elliptic	hyperbolic
$(II, m, -)$ m odd	hyperbolic	elliptic
(III)	parabolic	parabolic

The normal forms may also be grouped according to the qualitative behaviour of the restriction of $\bar H$ to the t-axis. Except for the case $(I, 0)$, each half of the t-axis is invariant under the local flow generated by $\bar H$, and the origin either attracts or repels in each half. This phenomenon has a more natural interpretation – namely, whether the origin attracts or fails to attract an entire neighbourhood in the corresponding half plane under this local flow.

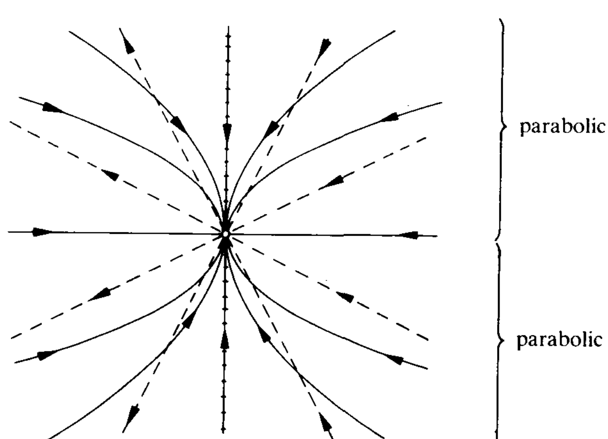
Normal form	Positive t-axis	Negative t-axis
(I, a), $a > 0$	repels	repels
(I, a), $a < 0$	attracts	attracts
(II)	attracts	attracts
$(III, m, b, +)$, m even	repels	attracts
$(III, m, b, -)$, m, even	attracts	repels
$(III, m, b, +)$, m odd	repels	repels
$(III, m, b, -)$, m odd	attracts	attracts

Examples are sketched in figure 5.

FIGURE 5. Normal forms. Symbolism as in Fig. 4.



$(I, 1)$



$(I, -1)$

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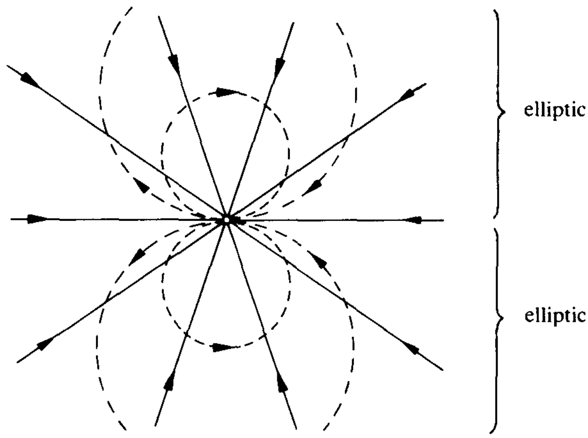
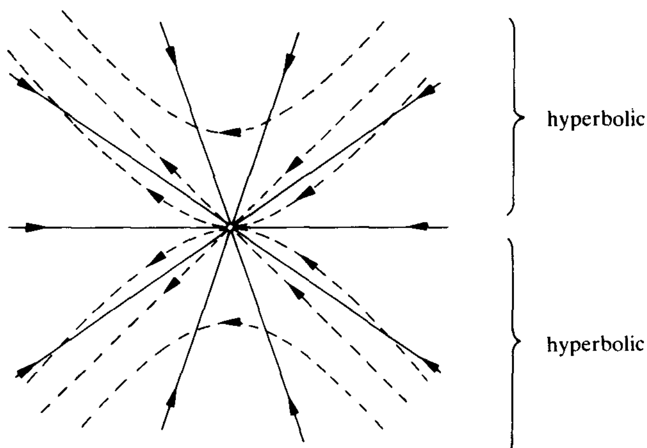
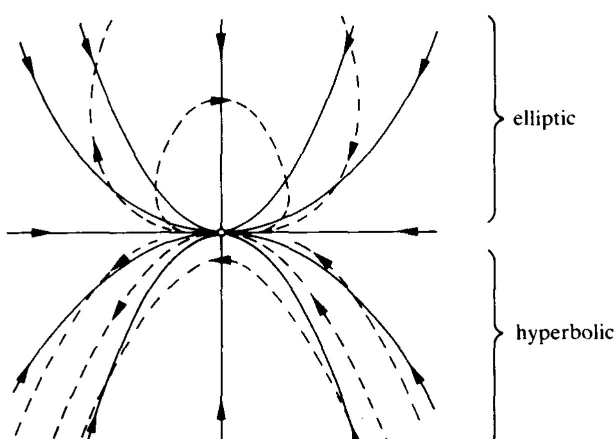


$(II, 2, +)$



$(II, 2, -)$



$(II, 1, +)$

Reflecting in s-axis yields $(II, 1, -)$.

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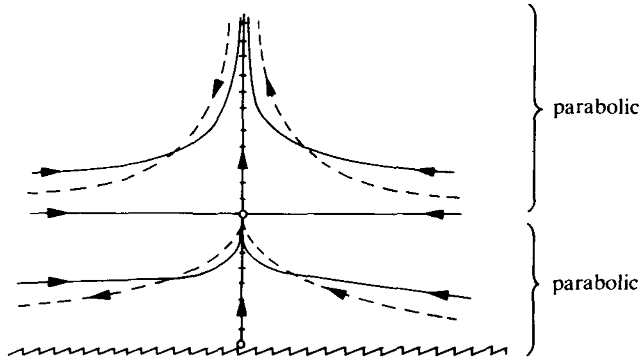
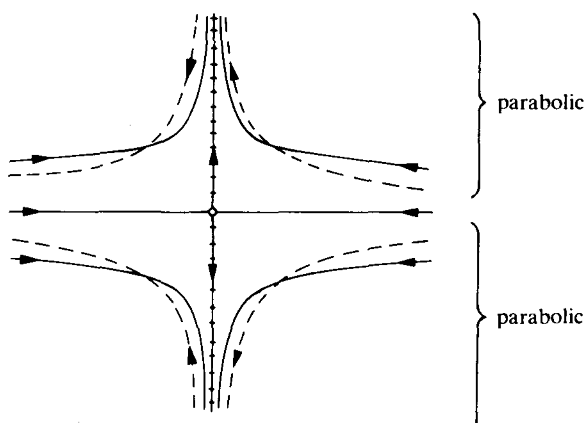
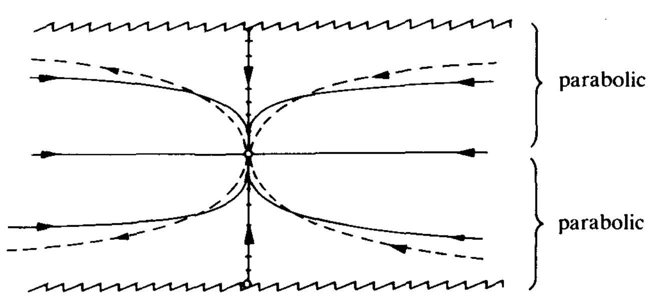


$(III, 2, 1, +)$



$(III, 3, 1, +)$



$(III, 3, 1, -)$

Imagine an orientation-preserving homeomorphism from one neighbourhood of the origin in $\mathbb{R}^2$ onto another which fixes the origin and intertwines the local actions generated by two of these normal forms. Such a homeomorphism must be the identity on the s-axis and therefore preserve the upper and lower half planes. It follows that the two normal forms must have the same gross properties specified in the two tables above. The following theorem says that, conversely, two normal forms which could possibly be topologically conjugate are so.

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THEOREM 2.2. *Consider the nine families of normal forms determined by the two tables above – the case $(I, 0)$, the four families from case (II), and the four families from cases (I) and (III). Two normal forms are conjugate by an orientation-preserving local homeomorphism which fixes the origin if and only if they belong to the same family.*

If we allow orientation-reversing conjugacy, the number of inequivalent normal forms is reduced to seven.

Proof. Consider two normal forms which lie in the same family; number them one and two. If they are of type (II), the homeomorphism

$$(s, t) \mapsto (s, \operatorname{sign}(t)|t|^{m_1/m_2})$$

conjugates one with the other.

In the parabolic case (i.e. each is of type (I) or (III)), it is simpler to give a general argument than to come up with such an explicit conjugacy. Let f be a local homeomorphism of the t-axis, defined near $t=0$, which conjugates the local flow generated by $\bar{H}_1$ with that generated by $\bar{H}_2$. Then f is a diffeomorphism away from zero. Set

$$\phi(s, t) = (s, 0) + f(0, t).$$

With ϕ_* denoting the forward push of vector fields, we have

$$\phi_* \bar{Y}_1 = \bar{Y}_2,$$

$$\phi_* \bar{H}_1 = \bar{H}_2 \qquad \text{along } \{(0, t)\colon t \neq 0\},$$

and

$$\phi_* \bar{X}_1 = \bar{X}_2 = 0 \qquad \text{along } \{(0, t)\colon t \neq 0\}.$$

Both $\phi_* \bar{H}_1$ and $\bar{H}_2$ are solutions to

$$\frac{\partial}{\partial s} \bar{Z} = -2 \frac{\partial}{\partial s},$$

and they agree along the t-axis, so they must be equal. Similarly, both $\phi_* \bar{X}_1$ and $\bar{X}_2$ are solutions to

$$\frac{\partial}{\partial s} \bar{Z} = \bar{H}_2$$

and vanish along the t-axis, so they must be equal. □

We have ignored the more difficult but important question of how the normalizing coordinates depend on α, at least if they are chosen carefully. This question will be taken up in § 4.

3. *Classification of actions on a surface*

The previous results allow us to classify, up to topological conjugacy and up to real analytic conjugacy, the real analytic actions of G on a surface. To keep the description simple, we consider only compact, connected surfaces, possibly with boundary.

Let M be such a surface. It follows from theorem 2.1 (and is well known) that the set of stationary points of an action of G on M is either discrete, hence finite, or is all of M. Of course, every manifold admits the trivial action.

Another possibility is that an action may have just one orbit – i.e. that M is a homogeneous space. As determined in § 1, the compact two-dimensional homogeneous spaces of G constitute a family of tori, all parabolic, parameterized by $\mathbb{R}^+ \times S^1 \times \mathbb{Z}^+$. Topological and C^ω conjugacy are equivalent here, for the differential structure on a homogeneous space is determined by the requirement that the natural action be real analytic.

A third possibility is that an action may have only one-dimensional orbits. Let α be such an action. By § 2, the set

$$P = \{x \in M: \mathcal{J}(x) = \text{span}\{H, X\}\}$$

is a neat ([**3**, p. 30]) one-dimensional submanifold which is transverse to the orbit foliation. Let x be a point in P, P_0 the component of P containing x, and

$$g_0 = \exp \pi(Y - X) \in G.$$

Since P has finitely many components and $\alpha(g_0)P = P$, there is a smallest positive integer n with

$$\alpha(g_0^n)P_0 = P_0.$$

Let $f: P_0 \to P_0$ denote the restriction of $\alpha(g_0^n)$, and let L denote the explicit realization of the homogeneous space $G/J_0(x)$ which is sketched in figure 2. Then α is C^ω equivalent to the quotient of $P_0 \times L$, which carries an action on the second factor, by the equivariant diffeomorphism

$$(p, s) \mapsto (f^{-1}p, s + n\pi).$$

Thus the C^ω conjugacy class of α is determined by the diffeomorphism class of P_0, the integer n, and the conjugacy class of f in $\text{Diff}^\omega(P_0)$. The topological conjugacy class of α is similarly determined by P_0, n, and the toplogical conjugacy class of f. Here M is a cylinder, Möbius band, torus, or Klein bottle, when f is in $\text{Diff}^+[0, 1]$, $\text{Diff}^-[0, 1]$, $\text{Diff}^+ S^1$, or $\text{Diff}^- S^1$, respectively.

The remaining possibilities, in which the action has at least one two-dimensional orbit and at least one orbit of lower dimension, cannot be disposed of so quickly. We first determine the toplogical conjugacy classes of such actions by analyzing the closure of an open orbit.

Let α be such an action, w a point with isotropy algebra span $\{H\}$, and W the orbit through w. There is a unique equivariant mapping ϕ from the hyperbolic strip $\mathbb{R} \times (-1, 1)$ onto W with

$$\phi(\pi/4, 0) = w.$$

Let x be a point of accumulation of $\phi(0, t)$ as t tends to one. Taking limits in

$$\alpha_*((1 + t_i)X + (1 - t_i)Y)\phi(0, t_i) = 0,$$

we have $\alpha_*(X)x = 0$. Clearly x cannot lie in a parabolic orbit, nor can x be stationary, since any stationary point is surrounded by a parabolic orbit. So x lies in a one-dimensional orbit, and $\mathcal{J}(x)$, since it contains X, must equal span $\{H, X\}$. The list of normal forms shows that any such point contained in the closure of a hyperbolic orbit is a sink of $\alpha_*(H)$. Thus

$$\lim_{t \to 1} \phi(0, t) = x,$$

and for any $s_0 \in \mathbb{R}$,

$$\lim_{(s,t)\to(s_0,1)} \phi(s,t) = \lim_{(s,t)\to(s_0,1)} \alpha(\exp s(Y-X))\phi(0,t)$$
$$= \alpha(\exp s_0(Y-X))x.$$

Therefore ϕ extends continuously and equivariantly to the upper boundary $\mathbb{R}\times\{1\}$. Similarly, ϕ extends continuously and equivariantly to the lower boundary $\mathbb{R}\times\{-1\}$.

The zero set of $\alpha_*(H)$ is discrete and therefore finite. With h denoting $\exp\frac{1}{2}\pi(Y-X)$, $\alpha(h)$ leaves this set invariant, so there is a smallest positive integer n with $\alpha(h^n)w = w$. Then

$$J(w) = \exp(\mathbb{R}H)\langle h^n\rangle,$$

and ϕ induces an equivariant mapping from the quotient of $\mathbb{R}\times[-1,1]$ by the right action of h^n which was discussed in § 1. Let

$$\bar{\phi}: Q \to M$$

denote this induced mapping. For n odd, Q is a closed Möbius band whose boundary has length n. For n even, Q is a closed cylinder, each boundary component having length $\frac{1}{2}n$. In either case, $\bar{\phi}$ maps the interior of Q diffeomorphically onto W, and since Q is compact, $\bar{\phi}(Q)$ equals the closure of W. For toplogical reasons, it is clear that $\bar{\phi}$ cannot map three distinct points on the boundary of Q to the same image. Thus the only manner in which $\bar{\phi}$ can collapse the boundary of Q is by an equivariant mapping of order two of a boundary component to itself or, in the case that n is even, an equivariant mapping between the boundary components.

In summary, we obtain a complete list of toplogical models for the closure of a hyperbolic orbit when G acts analytically on a compact surface. With $n \in \mathbb{Z}^+$ playing the same role as above, the pairwise (topologically) inequivalent models are easily seen to be: one Möbius band for each odd n, one cylinder for each even n, one torus for each even n and each integer k in the range $0 \le k < \frac{1}{4}n$, and, for each n divisible by four, one Möbius band and one Klein bottle.

The same techniques apply to analyzing the closure of a parabolic orbit, although the details differ. The key here is to focus on the zero set $P \subseteq M$ of $\alpha_*(X)$. Suppose that α has a parabolic orbit W. Then all two-dimensional orbits are parabolic, and P is a neat one-dimensional submanifold of M, invariant under the flow generated by $\alpha_*(H)$, which intersects any one-dimensional orbits of α transversely. If p is a stationary point of α, then p is a source for the restriction of the H-flow to P. If on the other hand p is a point of intersection of P with a one-dimensional orbit, then under the H-flow p may independently attract or repel each half of a neighbourhood within P, depending on the normal form of α_* near p.

Let $w \in W \cap P$. Recall that $J(w)$ is the semidirect product of $\exp(\mathbb{R}X)$ and a discrete subgroup of

$$K = \exp(\mathbb{R}H) \times \mathrm{centre}(G).$$

As in § 1, we identify K with $\mathbb{R}\times\mathbb{Z}$. Let ϕ be the equivariant mapping from the parabolic strip $\mathbb{R}\times(-1,1)$ onto W which sends the origin to w. The set S of points of intersection of P with zero- and one-dimensional orbits is finite and may be characterized as the zero set of $\alpha_*(H)$ on P. For each integer n, ϕ maps the segment $\{n\pi\}\times(-1, 1)$ to a connected, H-invariant, subset of $P\backslash S$ – i.e. a component of $P\backslash S$. Since there are only finitely many such components, the inevitable duplication implies that $J(w)\cap K$ contains some element (a, n) with $n\neq 0$. Were $J(w)\cap K$ a lattice, then W would be a torus, hence equal to M, which is not the case we are considering here. Thus $J(w)\cap K$ is cyclic, and we choose the generator (a, n) with $n\in\mathbb{Z}^+$. In particular, W is homeomorphic to an open cylinder, and ϕ maps the segment $\{0\}\times(-1, 1)$ homeomorphically onto a component of $P\backslash S$.

Since

$$\lim_{t\to 1}\phi(0, t)\equiv x\in S \quad\text{and}\quad \lim_{t\to -1}\phi(0, t)\equiv y\in S$$

exist, it follows as before that ϕ extends continuously and equivariantly to $\mathbb{R}\times[-1, 1]$. It is possible that y is stationary for α, but since x attracts at least a half neighbourhood in P under the H-flow, x cannot be stationary. If y is stationary, we have by theorem 2.1 that $(a, n)=(0, 2)$. Then, since $\alpha(\exp 2\pi(Y-X))$ fixes $\phi(0, t)$ for all $t\in(-1, 1)$, the orbit through x has length one or two. In this case, the closure of W is topologically conjugate either to the closed disk $s^2+t^2\leq 1$ with action generated by

$$\bar{H}=s(1-s^2+t^2)\frac{\partial}{\partial s}-t(1+s^2-t^2)\frac{\partial}{\partial t}$$

$$\bar{X}=t(1-s^2)\frac{\partial}{\partial s}-st^2\frac{\partial}{\partial t}$$

$$\bar{Y}=-s^2t\frac{\partial}{\partial s}+s(1-t^2)\frac{\partial}{\partial t}$$

or to the projective space obtained by gluing the boundary of this disk to itself antipodally. In the case that y is not stationary, let Q be the quotient of $\mathbb{R}\times[-1, 1]$ by the right action of $J(W)\cap K$. As before, ϕ induces an equivariant mapping $\bar{\phi}$ from Q onto the closure of W which maps the interior of Q diffeomorphically onto W and collapses the boundary of Q in at worst a two-to-one manner. Notice that Q is a closed cylinder, and each boundary component has length n.

Recall that the pair $(a, n)\in\mathbb{R}\times\mathbb{Z}^+$ is an invariant of the orbit W, being independent of the choice of $w\in P\cap W$. In summary, we have the following complete list of topological models for the closure of a parabolic orbit: one closed disk with $(a, n)=(0, 2)$, one projective space with $(a, n)=(0, 2)$, one cylinder for each pair (a, n), one torus for each pair (a, n) and each integer k in the range $0\leq k<n$, and, for each pair (a, n) with n even, two distinct Möbius bands and a Klein bottle.

There is just one type of elliptic homogeneous space, an example of which is the elliptic disk $s^2+t^2<1$ presented in § 1. If W is an elliptic orbit of some action and ϕ is the unique equivariant mapping from this elliptic disk onto W, then the same techniques as above show that ϕ extends continuously and equivariantly to topological embedding of the closed disk. Thus there is just one topological model for the closure of an elliptic orbit.

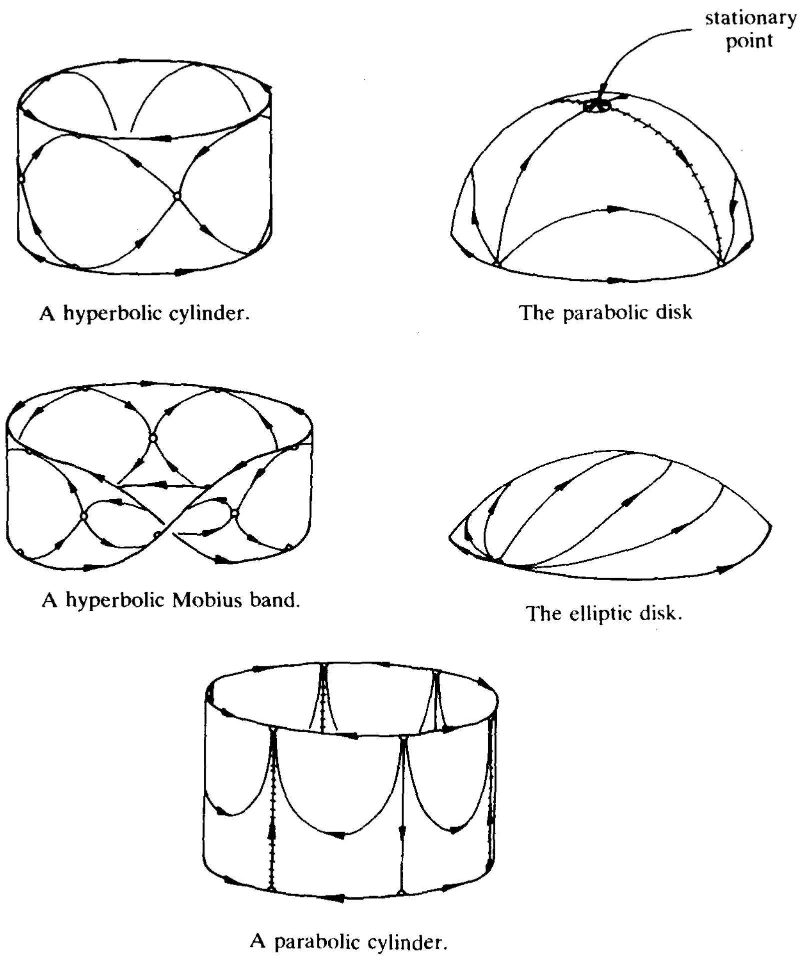

FIGURE 6. Homogeneous spaces with boundary. Note: all other 'topological building blocks' are obtained from these by equivariant gluing of boundary components to themselves or to each other.

Examples of some of these homogeneous spaces with boundary are sketched in figure 6. Notice that an analytic action on a compact surface has finitely many open orbits, since each point has a neighbourhood which intersects at most two open orbits. If the surface is connected, then we have already observed that either every open orbit is parabolic or none is. Finally, if there are parabolic orbits, the pair $(a, n)\in\mathbb{R}\times\mathbb{Z}^+$ associated to each parabolic orbit must be constant. This follows from analyticity if nothing else, since if

$$\exp(aH)\cdot\exp n\pi(Y-X)$$

acts as the identity on some component of $P\backslash S$, where P and S are as in the discussion of parabolic orbits above, then it must act as the identity on the corresponding component, say P_0, of P. Since every orbit intersects P_0, (a, n) is an integral multiple of the pair associated with each parabolic orbit. Choosing n to be minimal among these pairs establishes the claim. Combining these observations with the previous

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analysis, we have proved the following:

THEOREM 3.1. *Consider the family of real analytic actions of G on compact, connected surfaces which have at least one open orbit and at least one other orbit. For such an action, either all open orbits are parabolic or none is. If all are parabolic, then $J(x)$ is constant among those points x with $\mathscr{j}(x) = \text{span}\{X\}$. Subject to these two restrictions, a complete list of topological models for such actions is obtained by gluing finitely many of the 'homogeneous spaces with boundary' listed above by equivariant homeomorphisms of their boundaries.*

Some crude consequences are that G can act non-trivially on only those surfaces enumerated in the introduction, that an action can have no more than two elliptic orbits, and that a non-trivial action can have no more than two stationary points.

To obtain a complete list of topological models for actions by SL $(2, \mathbb{R})$, we use as building blocks only those homogeneous spaces with boundary upon which $\exp 2\pi(Y - X)$ acts as the identity. There are only finitely many of these building blocks. In particular, the pair (a, n) associated with each parabolic orbit must be $(0, 1)$ or $(0, 2)$.

Theorem 3.1 seems to suggest that each topological model admits a differential structure making the action real analytic. We will now prove this and enumerate the differential structures which work.

Let M be a toplogical manifold. A *local action* of G near $x \in M$ is a continuous mapping

$$(g, y) \mapsto \alpha(g)y$$

from a neighbourhood of $(1_G, x)$ in $G \times M$ into M such that, if g and h are sufficiently near 1_G and y is sufficiently near x, then

$$\alpha(1_G)y = y$$

and

$$\alpha(gh)y = \alpha(g)\alpha(h)y.$$

An *action germ* at x is the germ of a local action. Two action germs at points which may lie in different manifolds will be said to be *topologically equivalent* if there is a local homeomorphism from a neighbourhood of one point onto a neighbourhood of the other which conjugates them. Given an action germ $[\alpha]$ at x, we will say that $[\alpha]$ *admits the germ e* of a C^ω differential structure near x if, in some neighbourhood of $(1_G, x)$, the mapping $(g, y) \mapsto \alpha(g)y$ is C^ω with respect to representatives of these germs. There is similarly a notion of C^ω equivalence between two given action germs, each with a given admissible structure germ.

Now let (M, α) be one of the toplogical models described in theorem 3.1. Let $\mathscr{E}$ denote the sheaf over M consisting of the C^ω structure germs admitted by the action germs $[\alpha]_x$, $x \in M$. The stalk of $\mathscr{E}$ over x will be denoted $\mathscr{E}_x$. For each $x \in M$, we have a complete set of C^ω normal forms for $[\alpha]_x$ – that is, a family $\mathbb{B}$ of action germs at the origin of $\mathbb{R}^2$ (or of the upper half plane if $x \in \partial M$) which are C^ω in the usual structure $\mathscr{S}$ and such that:

(1) $[\alpha]_x$ is topologically conjugate to each $[\beta] \in \mathbb{B}$, and

(2) for each element of $\mathscr{E}_x$, there is a unique $[\beta] \in \mathbb{B}$ to which $[\alpha]_x$ is C^ω conjugate.

Each topological conjugacy $[\phi]$ from $[\alpha]_x$ to $[\beta]\in\mathcal{B}$ determines an element $[\phi^{-1}]\cdot\mathcal{S}$ of $\mathcal{E}_x$, and

$$[\phi^{-1}]\cdot\mathcal{S}=[\psi^{-1}]\cdot\mathcal{S}$$

if and only if $[\phi\psi^{-1}]$ and $[\psi\phi^{-1}]$ are C^ω. Thus $\mathcal{E}_x$ is in one-to-one correspondence with the union over $\mathcal{B}$ of the coset spaces

$$\mathcal{G}_{[\beta]}/\mathcal{H}_{[\beta]},$$

where $\mathcal{G}_{[\beta]}$ is the group of toplogical self-conjugacies of $[\beta]$ and $\mathcal{H}_{[\beta]}$ is the subgroup consisting of those which are germs of C^ω diffeomorphisms.

The case-by-case analysis of $\mathcal{E}_x$ runs as follows. First, if x lies in an open orbit, it is well known that $\mathcal{E}_x$ is a singleton. It is also true that $\mathcal{E}_x$ is a singleton when x is stationary, for here theorem 2.1 gives exactly one normal form, and it is easily verified that in this case $\mathcal{G}$ consists of the germs of scalar multiplications. Hence $\mathcal{G}=\mathcal{H}$, proving the assertion. Now suppose that x lies on a one-dimensional orbit. Using the obvious bijection from $\mathcal{E}_x$ onto $\mathcal{E}_{\alpha(g)x}$ induced by $\alpha(g)$ for each $g\in G$, we may assume that $\jmath(x)=\operatorname{span}\{H,X\}$. If x is interior to M, we have (theorem 2.2) the following six possibilities for the family $\mathcal{B}$ of normal forms:

(1) $\{(II, m, +)\colon m \text{ even}\}$
(2) $\{(II, m, -)\colon m \text{ even}\}$
(3) $\{(II, m, +)\colon m \text{ odd}\}$
(4) $\{(I, a)\colon a>0\}\cup\{(III, m, b, +)\colon m \text{ odd}\}$
(5) $\{(I, a)\colon a<0\}\cup\{(III, m, b, -)\colon m \text{ odd}\}$
(6) $\{(III, m, b, +)\colon m \text{ even}\}$.

Recall that $(II, m, +)$ is C^ω equivalent to $(II, m, -)$ when m is odd, and $(III, m, b, +)$ is equivalent to $(III, m, b, -)$ when m is even. To obtain uniqueness in families (3) and (6), we have selected just one of the pair. The normal forms of (1), (2), (4), and (5) admit at least one orientation-reversing self-conjugacy which is C^ω – the mapping $(s, t)\mapsto(s, -t)$ – while those of (3) and (6) admit no orientation-reversing self-conjugacy. Thus in either case $\mathcal{G}/\mathcal{H}$ is equivalent to $\mathcal{G}^0/\mathcal{H}^0$, where the superscript denotes preservation of orientation. The following is easily verified:

LEMMA 3.2. *For each of the normal forms of* (1), (2), *and* (3), $\mathcal{G}^0$ *is trivial. For those of* (4), (5), *and* (6), $\mathcal{G}^0$ *consists of the germs of*

$$(s,t)\mapsto\begin{cases}(s,\Psi_a t) & \text{if } t\geq 0\\ (s,\Psi_b t) & \text{if } t\leq 0\end{cases}$$

as a and b vary independently over $\mathbb{R}$. *Here* Ψ *denotes the restriction of the H-flow to the t-axis. Such a germ is in* $\mathcal{H}^0$ *if and only if* $a=b$.

Thus $\mathcal{G}^0/\mathcal{H}^0$ is trivial in cases (1), (2), and (3) and is a line in cases (4), (5), and (6). Finally, if $x\in\partial M$, then $\mathcal{B}$ is the restriction of one of the following families to the upper half plane:

(1′) $\{(II, m, +)\colon m\in\mathbb{Z}^+\}$
(2′) $\{(II, m, -)\colon m\in\mathbb{Z}^+\}$
(3′) $\{(I, a)\colon a>0\}\cup\{(III, m, b, +)\colon m\geq 2\}$
(4′) $\{(I, a)\colon a<0\}\cup\{(III, m, b, -)\colon m\geq 2\}$.

It follows from lemma 3.2 that $\mathscr{G} = \mathscr{H}$ in all of these cases.

Before enumerating the sections of $\mathscr{E}$ – i.e. the differential structures on M in which α is real analytic – we need the following facts. These hold for general reasons having nothing to do with the particular group G or particular pairs (M, α) considered here or even with analyticity.

LEMMA 3.3. (a) *The natural action $\tilde{\alpha}$ on $\mathscr{E}$ induced by α is continuous.*

(b) *If P is a topologically embedded orbit of α, then $\mathscr{E}|_P$ is a covering space.*

Proof. (a) Let $x \in M$, $g \in G$, and $e \in \mathscr{E}_x$ be given. By definition, any neighbourhood of $\tilde{\alpha}(g)e$ contains the germs $[\mathscr{J}]_y$, $y \in U$, of some admissible C^ω structure $\mathscr{J}$ on a neighbourhood U of $\alpha(g)x$, where

$$[\mathscr{J}]_{\alpha(g)x} = \tilde{\alpha}(g)e.$$

Let $\mathscr{J}'$ denote the differential structure on

$$U' = \alpha(g)^{-1}U$$

induced by $\alpha(g)^{-1}$. Then $\mathscr{J}'$ is admissible, and

$$[\mathscr{J}']_x = \tilde{\alpha}(g)^{-1}[\mathscr{J}]_{\alpha(g)x} = e.$$

Now let V be a neighbourhood of the identity in G and $W \subseteq U$ a neighbourhood of $\alpha(g)x$ such that the mapping

$$(v, y) \mapsto \alpha(v)y$$

takes $V \times W$ into U and is C^ω with respect to $\mathscr{J}$. Then if $h \in V \cap V^{-1}$ and $z \in \alpha(g)^{-1}W$, we have

$$\tilde{\alpha}(hg)[\mathscr{J}']_z = \tilde{\alpha}(h)[\mathscr{J}]_{\alpha(g)z} = [\mathscr{J}]_{\alpha(hg)z}.$$

Since $\{[\mathscr{J}']_z \colon z \in \alpha(g)^{-1}W\}$ is a neighbourhood of e and $(V \cap V^{-1})g$ is a neighbourhood of g, this shows that $\tilde{\alpha}$ is continuous at (g, e).

(b) Let $x \in P$ and $e \in \mathscr{E}_x$ be given. Since the mapping

$$g \mapsto \tilde{\alpha}(g)e$$

takes $J(x)$ continuously into $\mathscr{E}_x$, the identity component J_0 of $J(x)$ must fix e. Thus the mapping

$$F\colon G/J_0 \to \mathscr{E}|_P$$
$$F(gJ_0) = \tilde{\alpha}(g)e$$

is well defined. On the other hand, F is open. To show this, it suffices to demonstrate that F is open at $J_0 \in G/J_0$. Let $\mathscr{J}$ be an admissible structure on a neighbourhood U of x such that $[\mathscr{J}]_x = e$, and let V be a neighbourhood of 1_G and $W \subseteq U$ a neighbourhood of x such that the mapping $(g, y) \mapsto \alpha(g)y$ takes $V \times W$ into U and is C^ω with respect to $\mathscr{J}$. Now let S be any neighbourhood of 1_G. If $g \in S \cap V \cap V^{-1}$, then

$$F(gJ_0) = \tilde{\alpha}(g)[\mathscr{J}]_x = [\mathscr{J}]_{\alpha(g)x}.$$

Since $\alpha(S \cap V \cap V^{-1})x$ is a neighbourhood of x in P, this shows that $F(SJ_0)$ includes the neighbourhood

$$\{[\mathscr{J}]_y \colon y \in \alpha(S \cap V \cap V^{-1})x\}$$

of e in $\mathscr{E}|_P$. Thus F is open. It follows that the image of F is open and is a covering space of P. Since $\mathscr{E}|_P$ is disjoint union of such sets, assertion (b) is proved. □

Now let (M, α) be one of the topological models, and choose one point x from each one-dimensional orbit P. By lemma 3.3, the possible sections of $\mathscr{E}$ are in one-to-one correspondence with the ways of choosing, for each P, a structure germ $e \in \mathscr{E}_x$ which is fixed by $\tilde{\alpha}(J(x))$. Write

$$J(x) = J_0 \times \langle \exp n\pi(Y-X)\rangle,$$

where J_0 is the component of the identity and n is the length of P. Since $\tilde{\alpha}(J_0)$ fixes $\mathscr{E}_x$ pointwise, the question is: which structure germs are fixed by

$$\tilde{\alpha}(\exp n\pi(Y-X)).$$

In other words, which admissible structure germs also make

$$[\alpha(\exp n\pi(Y-X))]_x$$

and its inverse real analytic?

Let us choose $x \in P$ with $\mathscr{J}(x) = \operatorname{span}\{H, X\}$, and let $\mathscr{B}$ denote the family of C^ω normal forms for $[\alpha]_x$. Let $e \in \mathscr{E}_x$, and choose a C^ω conjugacy $[\phi]$ from $[\alpha]_x$ to some $[\beta] \in \mathscr{B}$. Of course, $[\beta]$ is uniquely determined by e. Let u denote the germ of $\alpha(\exp n\pi(Y-X))$ at x. Since $\exp n\pi(Y-X)$ is central, u commutes with $[\alpha]_x$, so

$$[\phi]u[\phi]^{-1} \in \mathscr{G} = \mathscr{G}_{[\beta]}.$$

Except when P is locally flanked by parabolic orbits, this implies (lemma 3.2) that $[\phi]u[\phi]^{-1} \in \mathscr{H}$ – i.e. that $\tilde{\alpha}(\exp n\pi(Y-X))$ fixes e.

Now suppose that P is locally flanked by parabolic orbits. Let $(a, m) \in \mathbb{R} \times \mathbb{Z}^+$ be the pair which characterizes all parabolic orbits of α. That is,

$$J(y) = \exp(\mathbb{R}X)\langle \exp(aH) \cdot \exp m\pi(Y-X)\rangle$$

for all $y \in M$ with $\mathscr{J}(y) = \operatorname{span}\{X\}$ (see theorem 3.1). If u preserves orientation, then $m = n$; if u reverses orientation, then $m = 2n$. In the former case, $[\phi]u[\phi]^{-1}$ is a self-conjugacy of $[\beta]$ whose restriction to the t-axis is the time-$(-a)$ flow of H. By lemma 3.2, $[\phi]u[\phi]^{-1}$ is the germ of

$$(s, t) \mapsto (s, \Psi_{-a}t)$$

(notation as in lemma 3.2). In particular, it is in $\mathscr{H}$, so $\alpha(\exp n\pi(Y-X))$ fixes e. In the final case, where u reverses orientation, no such conclusion holds. We can only say that the square of $[\phi]u[\phi]^{-1}$ takes (s, t) to $(s, \Psi_{-a}t)$, so $[\phi]u[\phi]^{-1}$ is the germ, for some $b, c \in \mathbb{R}$ with $b + c = -a$, of

$$(s, t) \mapsto \begin{cases} (s, -\Psi_b t) & \text{if } t \geq 0 \\ (s, -\Psi_c t) & \text{if } t \leq 0. \end{cases}$$

For $d \in \mathbb{R}$, denote by f_d the mapping

$$(s, t) \mapsto \begin{cases} (s, \Psi_d t) & \text{if } t \geq 0 \\ (s, t) & \text{if } t \leq 0. \end{cases}$$

Then, as we have seen, $\mathbb{R}$ is in one-to-one correspondence with those elements of $\mathscr{E}_x$ which make $[\alpha]_x$ C^ω conjugate to $[\beta]$, and this correspondence is given explicitly by

$$d \mapsto [\phi^{-1} f_d] \cdot \mathscr{S},$$

where $\mathcal{S}$ is the standard C^ω structure. There is precisely one value of d, namely $d = \frac{1}{2}(c-b)$, such that

$$[f_d^{-1}\phi]u[\phi^{-1}f_d] \in \mathcal{H},$$

and this is the unique value of d such that u and u^{-1} are C^ω in the structure germ $[\phi^{-1}f_d]\cdot\mathcal{S} \in \mathcal{E}_x$.

In summary, we have proved the following:

THEOREM 3.4. *Let (M, α) be one of the topological models described in theorem* 3.1. *From each one-dimensional orbit P_i choose a point x_i with $\jmath(x_i) = \text{span}\{H, X\}$, and let $\mathbb{B}_i$ denote the family of C^ω normal forms for $[\alpha]_{x_i}$. ($\mathbb{B}_i$ is one of the $6+4$ families discussed earlier.) If P_i is two-sided in the sense of separating some neighbourhood of itself and is flanked by parabolic orbits, then for each $[\beta] \in \mathbb{B}_i$ there is a one-parameter family of germs along P_i of admissible differential structures (near P_i) in which $[\alpha]_{x_i}$ is C^ω equivalent to $[\beta]$. Otherwise, there is for each $[\beta] \in \mathbb{B}_i$ a unique such germ. Each choice of germs for the various orbits P_i determines a unique admissible differential structure on M, and all admissible structures on M are obtained in this way.*

If we were to take the entire collection of topological models and assign to each model the entire range of admissible C^ω structures, we would of course obtain every C^ω conjugacy class for actions of the kind considered here. However, there would be duplication. The question arises: when do different structures produce actions which are C^ω equivalent?

For a real analytic action α, the normal form $[\beta]$ corresponding to a point x with $\jmath(x) = \text{span}\{H, X\}$ depends only on the orbit P in which x lies. Let $[\beta](P)$ denote this normal form. Then, for a topological conjugacy ϕ from $\alpha_1 \in \text{Act}^\omega(G, M_1)$ to $\alpha_2 \in \text{Act}^\omega(G, M_2)$ to be a C^ω diffeomorphism, it is necessary that

$$[\beta](\phi P) = [\beta](P)$$

for all one-dimensional orbits P of α_1. By theorem 3.4, this necessary condition is sufficient except when there exists a two-sided orbit P flanked by parabolic orbits, and it fails to be sufficient in that case.

Suppose that α_1 has parabolic orbits, that M_1 is not a torus, and that

$$[\beta](\phi P) = [\beta](P)$$

for all one-dimensional orbits P. By theorem 3.1, we can order the open orbits $W_1, \ldots, W_p$ of α_1 so that

$$\text{clos}(W_i) \cap \text{clos}(W_{i+1})$$

is a one-dimensional orbit P_i. Note that $P_1, \ldots, P_{p-1}$ are disjoint and that there are two other orbits P_0 and P_p, each a circle or a point, with

$$\text{clos}(W_1) = P_0 \cup W_1 \cup P_1$$

and

$$\text{clos}(W_p) = P_{p-1} \cup W_p \cup P_p.$$

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For $i=1,\ldots,p$, choose a point $w_i\in W_i$ with $\mathscr{J}(w_i)=\operatorname{span}\{X\}$, and for $t\in\mathbb{R}$, $x\in M_1$, define

$$\Phi_i(t)x=\begin{cases}\alpha_i(g\exp(tH))w_i & \text{if } x=\alpha_i(g)w_i\\ x & \text{if } x\notin W_i.\end{cases}$$

Then $\Phi_i(t)$ is continuous, well defined, and commutes with α_i. Now, ϕ may not be a diffeomorphism, but by theorem 3.4, its restriction to clos (W_i) is a C^ω embedding for each i. By lemma 3.2, we can find $t_2\in\mathbb{R}$ such that

$$\phi\circ\Phi_2(t_2)$$

is an embedding of clos $(W_1\cup W_2)$, and then find $t_3\in\mathbb{R}$ such that

$$\phi\circ\Phi_2(t_2)\circ\Phi_3(t_3)$$

is an embedding of clos $(W_1\cup W_2\cup W_3)$, etc. Ultimately we obtain a diffeomorphism. Thus the existence of ϕ in this case implies that α_1 and α_2 are C^ω equivalent. Finally, suppose that everything is the same as above except that M_1 is a torus. We ask: are α_1 and α_2 necessarily C^ω equivalent? The answer is no. The set-up is the same as above, except that in this case $P_0=P_p$, a one-dimensional orbit. We can again find $t_2,\ldots,t_p\in\mathbb{R}$ such that

$$\phi\circ\Phi_2(t_2)\circ\cdots\circ\Phi_p(t_p),$$

when restricted to any half-neighbourhood of P_0, is an embedding, but the derivative may have a jump discontinuity along P_0. Now suppose that ψ were a C^ω conjugacy with

$$\psi(W_i)=\phi(W_i)\quad\text{for all } i.$$

Replacing ψ with

$$\psi\circ\alpha_1(\exp n\pi(Y-X))$$

for an appropriate integer n, we may assume that

$$\psi|_{P_i}=\phi|_{P_i}\quad\text{for all } i.$$

By lemma 3.2, in order to be smooth near $P_1\cup P_2\cup\cdots\cup P_{p-1}$,

$$\psi^{-1}\circ\phi\circ\Phi_2(t_2)\circ\cdots\circ\Phi_p(t_p)$$

must be of the form

$$\Phi_1(t)\circ\cdots\circ\Phi_p(t)$$

for some real number t. This forces it to be smooth near P_0, implying that

$$\phi\circ\Phi_2(t_2)\circ\cdots\circ\Phi_p(t_p)$$

is so. Thus if we failed originally, no such conjugacy ψ exists. Furthermore, we have seen that the differential structure of M_1 near P_0 may be admissibly changed in a one-parameter way without affecting $[\beta](P_0)$. Thus failure is to be expected. There might be (?) a C^ω conjugacy between α_1 and α_2 such that, say,

$$\psi(W_1)=\phi(W_2),\ldots,\psi(W_p)=\phi(W_1),$$

but this would be a fluke.

In the language of theorem 3.4, these comments amount to the following:

THEOREM 3.5. *Let (M, α) be one of the topological models described in theorem* 3.1, *$P_1, \ldots, P_r$ its one-dimensional orbits, and $\mathcal{S}$, $\mathcal{J}$ admissible structures yielding the same values for $[\beta](P_1), \ldots, [\beta](P_r)$. If there is no two-sided P_i flanked by parabolic orbits, then*

$$\mathcal{S} = \mathcal{J}.$$

If α has parabolic orbits and M is not a torus, then $(M, \alpha, \mathcal{S})$ is C^ω equivalent to $(M, \alpha, \mathcal{J})$. If α has parabolic orbits and M is a torus, there is precisely a one-parameter family of such structures $\mathcal{J}$ such that the family $\{(M, \alpha, \mathcal{J})\}$ is pairwise inequivalent under C^ω conjugacy leaving each orbit invariant.

4. *Stability and unfolding in* $\mathrm{Act}^\omega\,(G, M)$

The group of diffeomorphisms of a surface M acts on the space of actions of G on M by conjugation. Specifically, the diffeomorphism f transforms the action α into the action α^f defined by

$$\alpha^f(g) = f\alpha(g)f^{-1}, \qquad g \in G.$$

In appropriate topologies to be discussed below, this action is continuous. The object of this section is to show that for many values of α there is a finite dimensional universal unfolding (with respect to the action of Diff (M)) of a neighbourhood of α. That is, there is a continuous mapping

$$\beta \mapsto f(\beta) \in \mathrm{Diff}\,(M)$$

defined on a neighbourhood of α such that the operation

$$\beta \mapsto \beta^{f(\beta)}$$

retracts this neighbourhood onto a subset V which is homeomorphic to a finite-dimensional algebraic variety. The elements of V are interpreted as models for perturbations of α.

This definition of universal unfolding is somewhat arbitrary in that it simply matches what we do here. Certainly a less restrictive definition could be useful (that given in [**6**], for example). However, in the presence of a finite-dimensional unfolding as defined above, it should in general be easy to determine whether an unfolding exists which satisfies additional conditions. For example, V might be required to be a local cross section to the action of Diff (M) in the sense that if v and v' are distinct elements of V, then there should be no diffeomorphism near the identity which transforms v into v'. Similarly, V might be required to be a local section in the stronger sense obtained by removing 'near the identity' from the previous sentence. One might also require that V be a manifold transverse to orbits in appropriate Banach structures. Although we will not explore any of these additional conditions, they turn out to be satisfied by most of the unfoldings we construct.

We continue to work in the C^ω category and to assume that M is compact and connected. Considered as a subset of $C^\omega(G \times M, M)$, $\mathrm{Act}^\omega\,(G, M)$ inherits a weak

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C^∞ topology ([**3**, p. 35]). This topology is metrizable, and $\alpha_1, \alpha_2, \ldots$ converge to α if and only if, for each $Z \in \mathscr{g}$, $\alpha_{1*}(Z)$, $\alpha_{2*}(Z), \ldots$ converge to $\alpha_*(Z)$ in the C^∞ sense.

Let P be a one-dimensional orbit of an action α. The normal form for α_* near a point $x \in P$ with $\mathscr{j}(x) = \operatorname{span}\{H, X\}$ depends only on P. It therefore makes sense to define $\lambda(P)$ as the eigenvalue other than -2 of the derivative of $\alpha_*(H)$ at such a point x. Thus:

$\lambda = a$ if the normal form is (I, a);

$\lambda = -4m^{-1}$ if the normal form is $(II, m, \pm)$;

and $\lambda = 0$ if the normal form is $(III, m, b, \pm)$.

We say that P is *infinitesimally isolated* if $\lambda(P) \neq 0$ and that α is non-degenerate if every one-dimensional orbit is infinitesimally isolated.

For $N \in \mathbb{Z}^+$, let

$$G_N = G/\langle \exp N\pi(Y - X)\rangle.$$

For example, $G_2 = \mathrm{SL}(2, \mathbb{R})$.

THEOREM 4.1. *The set of non-degenerate actions is open in* $\mathrm{Act}^\omega(G, M)$. *For every positive integer N, the set of non-degenerate actions is dense (and open) in* $\mathrm{Act}^\omega(G_N, M)$.

Proof. The trivial action is isolated even in the space of C^1 actions of G on M, [**8**]. If α is non-trivial and non-degenerate, the points at which $\alpha_*(H)$ vanishes are hyperbolic. Any C^1 perturbation of the H-flow therefore has this same number of fixed points, each hyperbolic. In particular, an action which is even C^1 close to α must be non-degenerate.

Let N be a positive integer, and let $\alpha \in \mathrm{Act}^\omega(G_N, M)$ be degenerate. The zero set Q of $\alpha_*(X)$ is a neat one-dimensional submanifold, and

$$\alpha(\exp \mathbb{R}(Y - X))Q = M.$$

Let S_0 denote the set of stationary points of α. On $M \backslash S_0$, define the multi-valued function θ by the statement that

$$x \in \alpha(\exp \theta(x)(Y - X))Q.$$

Notice that θ is real analytic on $M \backslash S_0$, defined up to multiples of π, and that

$$\alpha_*(Y - X)(\theta) \equiv 1.$$

Now let $f: Q \to Q$ denote the restriction of $\alpha(\exp \pi(Y - X))$. Since f has finite order, it is easy to see that f leaves invariant some C^ω vector field Z_Q on Q which vanishes on $\partial Q = Q \cap \partial M$ and in addition satisfies:

(1) if α has only one-dimensional orbits, then Z_Q is Morse-Smale;

(2) if α has at least one open orbit, then $Z_Q(x) = DZ_Q(x) = 0$ if $x \in S_0$, and $Z_Q(x) = 0$, $DZ_Q(x) \neq 0$ if x is a point of intersection of Q with a one-dimensional orbit.

Now, Z_Q extends uniquely to a vector field Z on M which commutes with $\alpha_*(Y - X)$; clearly it extends so to $M \backslash S_0$, and the assumptions on Z_Q near a stationary point easily show that the extension is analytic near stationary points. Let

$$H_\varepsilon = \alpha_*(H) + \varepsilon \cos(2\theta) Z$$
$$X_\varepsilon = \alpha_*(X) + \tfrac{1}{2}\varepsilon \sin(2\theta) Z$$
$$Y_\varepsilon = \alpha_*(Y) + \tfrac{1}{2}\varepsilon \sin(2\theta) Z.$$

These satisfy the bracket relations

$$[H_\varepsilon, X_\varepsilon] = -2X_\varepsilon, \quad [H_\varepsilon, Y_\varepsilon] = 2Y_\varepsilon, \quad [X_\varepsilon, Y_\varepsilon] = -H_\varepsilon.$$

We omit the computations, but they are based on the identities

$$\sin(2\theta)\alpha_*(H) + (-1-\cos(2\theta))\alpha_*(X) + (1-\cos(2\theta))\alpha_*(Y) = 0$$
$$\alpha_*(H)(\theta) = -\sin(2\theta)$$
$$\alpha_*(X)(\theta) = \tfrac{1}{2}(-1+\cos(2\theta))$$
$$\alpha_*(Y)(\theta) = \tfrac{1}{2}(1+\cos(2\theta))$$
$$Z(\theta) = 0$$
$$[\alpha_*(Y-X), Z] = 0.$$

When ε is small but not zero, the zeros of H_ε are hyperbolic, so the action generated by H_ε, X_ε, and Y_ε is non-degenerate. Since

$$Y_\varepsilon - X_\varepsilon = \alpha_*(Y-X),$$

this is indeed an action by G_N. □

Surprisingly, non-degenerate actions are not always dense in $\mathrm{Act}^\omega(G, M)$. For example, there is a diffeomorphism f of S^1 such that for any h near f in the C^2 topology:

$$\{k \in \mathrm{Diff}(S^1) \colon hk = kh\} = \{h^n \colon n \in \mathbb{Z}\},$$

(see [**4**]). Build a torus by identifying (p, s) with $(f^{-1}p, s+\pi)$ in $S^1 \times \mathbb{R}$. This torus carries the degenerate action generated by vector fields

$$\alpha_*(H) = -\sin(2s)\frac{\partial}{\partial s}$$

$$\alpha_*(X) = \tfrac{1}{2}(-1+\cos(2s))\frac{\partial}{\partial s}$$

$$\alpha_*(Y) = \tfrac{1}{2}(1+\cos(2s))\frac{\partial}{\partial s}.$$

The zero set Q_α of $\alpha_*(X)$ is a circle, and the first return mapping of $\alpha_*(Y-X)$ is equivalent to f by construction. If β is near α, there is a unique $\beta_*(H)$-invariant submanifold Q_β near Q_α, which by uniqueness must be invariant under $\beta(\exp \pi(Y-X))$. Since the restriction of $\beta(\exp \pi(Y-X))$ to Q_β is equivalent to a mapping near f, $\beta_*(H)$ must vanish along Q_β. Thus β is degenerate.

The main result of this section is the following.

THEOREM 4.2. *There is a finite-dimensional universal unfolding near each non-degenerate action in* $\mathrm{Act}^\omega(G, M)$ *or in* $\mathrm{Act}^\omega(G_N, M)$, $N \in \mathbb{Z}^+$.

The precise nature of the unfolding cannot easily be summarized; there are many cases. The trivial action and transitive actions will be discussed below. For actions having a hyperbolic or elliptic orbit, the variety V is a point, i.e. such actions are rigid. For a non-degenerate, non-transitive action α having a parabolic orbit, V is usually a vector space. There is one dimension for each one-dimensional orbit P of α, indicating the variation of $\lambda(P)$. If the unfolding is taking place in $\mathrm{Act}^\omega(G, M)$ and α has no stationary point, there is one additional dimension indicating a variation of the conjugacy class of subgroup which characterizes all parabolic orbits (cf. theorem 3.1). If M is a torus and the unfolding occurs in either $\mathrm{Act}^\omega(G, M)$ or $\mathrm{Act}^\omega(G_N, M)$, there is an additional degree of freedom as suggested by theorem 3.5. In one unusual and interesting case, however, V is not a vector space. In addition to the vector space of parameters described above, there is in this case a transverse line along which all elements but α have hyperbolic orbits.

Let us dispose of the special cases. Since the trivial action is isolated [**8**], there certainly is an unfolding in this case. Next, consider a transitive action α; M is then a (parabolic) torus. Clearly any action near α is also transitive. Choose $x \in M$ with $\mathscr{J}_\alpha(x) = \mathrm{span}\{X\}$, and write

$$J_\alpha(x) = \langle X, (a, 0), (b, n)\rangle \qquad \text{with } a \in \mathbb{R}^+, b \in \mathbb{R}, n \in \mathbb{Z}^+.$$

The notation means that $J_\alpha(x)$ is generated by

$$\exp(\mathbb{R}X) \cup \{\exp(aH)\} \cup \{\exp(bH) \cdot \exp n\pi(Y-X)\}.$$

If β is near α, then $\mathscr{J}_\beta(x)$ must be a parabolic subalgebra near $\mathrm{span}\{X\}$, so there is a unique value of θ near zero with

$$\mathscr{J}_\beta(x) = \mathrm{span}\{\mathrm{Ad}_{\exp(-\theta(Y-X))} X\}.$$

Set $x_\beta = \beta(\exp \theta(Y-X))x$; then $\mathscr{J}_\beta(x_\beta) = \mathrm{span}\{X\}$. It is easy to show that there are unique small constants δ and ε with

$$J_\beta(x_\beta) = \langle X, (a+\delta, 0), (b+\varepsilon, n)\rangle,$$

It is also easy to show that there is a continuous mapping

$$(\delta, \varepsilon) \mapsto \Gamma(\delta, \varepsilon) \in \mathrm{Act}^\omega(G, M)$$

with $\Gamma(0, 0) = \alpha$ and

$$J_{\Gamma(\delta,\varepsilon)}(x) = \langle X, (a+\delta, 0), (b+\varepsilon, n)\rangle.$$

Now for β near α, let $f(\beta)$ be the unique conjugacy between β and $\Gamma(\delta(\beta), \varepsilon(\beta))$ which sends x_β to x. This produces a two dimensional universal unfolding near α. If we consider only actions by G_N, then in the previous construction b is a multiple of anN^{-1}, and a perturbation β lies in $\mathrm{Act}^\omega(G_N, M)$ if and only if $\varepsilon(\beta)$ is the same multiple of $\delta(\beta)nN^{-1}$. Thus the unfolding in $\mathrm{Act}^\omega(G_N, M)$ is one-dimensional.

For the remaining cases, in which α is non-degenerate, non-trivial, and non-transitive, we employ a method which seems fairly general, although for the most part we fill in the details *ad hoc*. Let $\{x_1, \ldots, x_p\}$ consist of the stationary points of α together with one point having isotropy algebra $\mathrm{span}\{H, X\}$ from each one-dimensional orbit. Fix normalizing coordinates

$$\phi_i : U_i \to \mathbb{R}^2$$

about each point x_i, and if x_i is not stationary let n_i be the length of the orbit of α through x_i. The first step is to find what might be called an unfolding of the local action near x_i determined by α. We consider perturbations of the coordinate representations of $\alpha_*(H)$, $\alpha_*(X)$, and $\alpha_*(Y)$ which satisfy the same bracket relations. If x_i is not stationary, we also consider perturbations of

$$\phi_i \circ \alpha(\exp n_i\pi(Y-X)) \circ \phi_i^{-1}$$

which commute with the perturbed vector fields. We obtain a space $\mathscr{m}_i$ of models for such perturbations; $\mathscr{m}_i$ is a variety in the appropriate topology. Formally, $\mathscr{m}_i$ consists of triples of vector fields defined on some fixed disk or half disk $s^2+t^2<R_i$ together with an embedding from some fixed smaller disk or half disk $s^2+t^2<\rho_i$ into the larger one when x_i is not stationary. For β near α, we obtain coordinates

$$\phi_i^\beta : U_i' \to \mathbb{R}^2$$

about x_i with:

(i) $U_i' \subseteq U_i$ and ϕ_i^α is the restriction of ϕ_i;

(ii) ϕ_i^β varies continuously with β;

(iii) the image of ϕ_i^β contains the disk or half disk $s^2+t^2<R_i$;

(iv) ϕ_i^β transforms β_* and $\beta(\exp n_i\pi(Y-X))$ (when x_i is not stationary) into some $\mu_i(\beta) \in \mathscr{m}_i$.

This construction is the local unfolding.

For the second step, let $\mathscr{e}$ denote the collection of ends of open orbits of α. Parabolic orbits have two ends, elliptic orbits have one, and hyperbolic orbits have one or two. A covering

$$\{V_i\}_{1\le i\le p} \cup \{V_E\}_{E\in \mathscr{e}}$$

of M is defined as follows. If x_i is stationary, choose a small disk $V_i \subseteq U_i$ about x_i. If x_i is not stationary, let V_i be a regular tubular neighbourhood of the orbit of α through x_i small enough that

$$V_i \subseteq \beta(\exp[0, n_i\pi](Y-X))(\phi_i^\beta)^{-1}\{(s,t)\colon s^2+t^2<\rho_i\}$$

for all β near α. These sets V_i should be chosen small enough that their closures are disjoint. Now, $\mathscr{e}$ is represented by the components of $V_i\backslash$ (orbit of α through x_i) as i ranges from one to p. Let U be such a component representing $E \in \mathscr{e}$, and W the orbit of α containing U. Taking

$$\operatorname{clos}(U) \cup (W\backslash \operatorname{clos}(V_1 \cup \cdots \cup V_p))$$

and then removing a very small closed tubular neighbourhood of the orbit through x_i, we obtain an open set V_E. The following properties hold:

(1) $\{V_i\}_{1\le i\le p} \cup \{V_E\}_{E\in\mathscr{e}}$ is an open cover of M;

(2) there exists a unique index $i(E)$ with $V_E \cap V_{i(E)} \neq \varnothing$;

(3) $V_E \cap V_{i(E)}$ is connected;

(4) $\operatorname{clos}(V_E)$ is contained in an open orbit of α.

For each $E \in \mathscr{e}$ fix a choice of

$$y_E \in V_E \cap V_{i(E)} \cap U'_{i(E)},$$

and for β near α define

$$y_E^\beta = (\phi_{i(E)}^\beta)^{-1}\phi_{i(E)}^\alpha y_E.$$

We will in fact find it convenient to choose

$$y_E = (\phi^{\alpha}_{i(E)})^{-1}(0, t(E))$$

where $t(E)$ is moderately small in absolute value. Now let W be a two-ended open orbit of α, with $D, E \in \mathscr{e}$ being its ends. Fix $g = g(W) \in \mathbb{G}$ such that

$$\alpha(g) y_D = y_E.$$

If β is near α, then $\beta(g) y^{\beta}_D$ is near y^{β}_E, so

$$\phi^{\beta}_{i(E)}(\beta(g) y^{\beta}_D) = u(\beta, W)$$

with $u \in \mathbb{R}^2$ near $(0, t(E))$. If $\{W_1, \ldots, W_q\}$ are the two-ended open orbits of α, we write $u_j(\beta)$ for $u(\beta, W_j)$. Define

$$\Gamma: \text{neighbourhood of } \alpha \to \mathscr{m}_1 \times \cdots \times \mathscr{m}_p \times \underbrace{\mathbb{R}^2 \times \cdots \times \mathbb{R}^2}_{q \text{ times}}$$

$$\Gamma(\beta) = (\mu_1(\beta), \ldots, \mu_p(\beta), u_1(\beta), \ldots, u_q(\beta)).$$

This Γ is the primary tool for unfolding a neighbourhood of α.

It turns out that $J_\beta((\phi^{\beta}_i)^{-1}(s, t))$ may be determined from the normal form $\mu_i(\beta)$. In particular,

$$J_\beta(y^{\beta}_E) = J_\beta((\phi^{\beta}_{i(E)})^{-1}(0, t(E)))$$

may be determined. Such a phenomenon probably would not hold in a more general problem – we would only expect to determine the intersection of

$$J_\beta((\phi^{\beta}_i)^{-1}(s, t))$$

with large compact subsets of G – but we will not look a gift horse in the mouth. If D and E are the ends of W_j and $g = g(W_j)$, then the relation

$$J_\beta(\beta(g) y^{\beta}_D) = g \cdot J_\beta(y^{\beta}_D) \cdot g^{-1}$$

imposes a restriction on the possible values of Γ. We write this as the 'consistency condition'

$$(*) \qquad J_{\mu_{i(E)}(\beta)}(u_j(\beta)) = g \cdot J_{\mu_{i(D)}(\beta)}(0, t(D)) \cdot g^{-1}.$$

Thus there are q different consistency conditions. Let

$$\mathscr{U} \subseteq \mathscr{m}_1 \times \cdots \times \mathscr{m}_p \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$$

be the elements satisfying all these conditions. We will show:

THEOREM 4.3. *If β and γ are near α with $\Gamma(\beta) = \Gamma(\gamma)$, there is a canonical $F = F(\beta, \gamma) \in \operatorname{Diff}^{\omega}(M)$ with $\beta^F = \gamma$. The dependence of F upon (β, γ) is continuous, and $F(\beta, \beta) = \mathrm{id}_M$.*

THEOREM 4.4. *There is a continuous mapping B from a finite-dimensional algebraic variety into $\operatorname{Act}^{\omega}(G, M)$ such that the image contains α and $\Gamma \circ B$ is a homeomorphism onto a neighbourhood of $\Gamma(\alpha)$ in $\mathscr{U}$.*

These theorems achieve a universal unfolding of a neighbourhood of α. The mapping

$$\beta \mapsto \beta^{F(\beta, B(\Gamma \circ B)^{-1}\Gamma(\beta))}$$

retracts a neighbourhood of α onto the image of B.

The unfolding thus constructed is not the smallest possible and does not live up to the advance billing given after the statement of theorem 4.2. As a final step, we briefly describe an unfolding *of the image of B* which, composed with the original unfolding, completes the project.

We begin with step one. The following lemma is surely widely known and is therefore offered without proof.

LEMMA 4.5. *Let $\mathscr{X}$ be the family of C^ω vector fields on the real line which vanish only at the origin and have non-zero derivative there. If $Z \in \mathscr{X}$, there is a unique C^ω embedding $f:\mathbb{R}\to\mathbb{R}$ with $Df(0)=1$ which linearizes Z. Furthermore, f depends continuously on Z in the weak C^∞ topologies.*

Consider the normal form for α_* near a stationary point. Let $\bar{H}$, $\bar{X}$, and $\bar{Y}$ be C^ω vector fields which are close to $s(\partial/\partial s)-t(\partial/\partial t)$, $t(\partial/\partial s)$, and $s(\partial/\partial t)$, respectively, in some fixed neighbourhood of the origin and satisfy the appropriate bracket relations. Translating, we may assume that $\bar{H}$ vanishes at the origin. Let β denote the local action of G generated by these vector fields. Since

$$D\bar{H}|_{(0,0)} \approx \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we have

$$\|D\beta(\exp tH)|_{(0,0)}v\| \le \exp|3t/2|\,\|v\|$$

for all $v\in\mathbb{R}^2$ and $t\in\mathbb{R}$. On the other hand,

$$D\beta(\exp tH)|_{(0,0)}\bar{X}(0,0) = \exp(2t)\bar{X}(0,0)$$
$$D\beta(\exp tH)|_{(0,0)}\bar{Y}(0,0) = \exp(-2t)\bar{Y}(0,0)$$

by virtue of the bracket relations. Thus $\bar{X}$ and $\bar{Y}$ also vanish at the origin.

Let $h=\beta(\exp 2\pi(Y-X))$. Since $D(\bar{Y}-\bar{X})|_{(0,0)}$ is a commutator, it has trace zero; thus $Dh(0,0)$ has determinant one. Also, $Dh(0,0)$ commutes with matrices known to be near

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so it must be a scalar multiplication. Since it is near the identity, $Dh(0,0)$ must therefore be the identity. Now, the unstable manifold of the origin for $\bar{H}$ is C^ω (proved below using transversality) and invariant under h. Since h commutes with $\bar{H}$, it follows from lemma 4.5 that h fixes all points of this unstable manifold. Then, since the image of this unstable manifold under $\beta(\exp\mathbb{R}(Y-X))$ contains a neighbourhood of the origin, h must fix all points. We apply the familiar change of coordinates

$$(s,t)\mapsto \frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\beta(\exp\theta(Y-X))(s,t)\,d\theta.$$

In the new coordinates,

$$\bar{Y}-\bar{X}=-t\frac{\partial}{\partial s}+s\frac{\partial}{\partial t}.$$

The argument showing that $\bar{X}$ vanishes at the origin also shows that $\bar{X}$ vanishes along the entire unstable manifold. On the other hand, the points at which the $\partial/\partial s$-component of $\bar{X}$ vanishes constitute, by transversality, the graph $t=f(s)$ of a C^ω function f which depends continuously upon $\bar{X}$. Also, since the unstable manifold is invariant under scalar multiplication by -1 – i.e. under $\beta(\exp \pi(Y-X))$ – f is odd. Write

$$f(s)=s\cdot u(s^2)$$

and apply the change of coordinates

$$(s,t)\mapsto(s-t\cdot u(s^2+t^2),\, t+s\cdot u(s^2+t^2)).$$

This flattens the unstable manifold without affecting $\bar{Y}-\bar{X}$. Finally, let F be the unique transformation of the s-axis with $DF(0)=1$ which linearizes $\bar{H}$. Again, F must be odd. Writing

$$F(s)=s\cdot v(s^2),$$

apply the transformation

$$(s,t)\mapsto(s\cdot v(s^2+t^2),\, t\cdot v(s^2+t^2)).$$

This linearizes $\bar{H}$ along the s-axis without affecting $\bar{Y}-\bar{X}$. The system

$$\left[-t\frac{\partial}{\partial s}+s\frac{\partial}{\partial t}, \bar{H}\right]=-2(\bar{Y}+\bar{X})$$

$$\left[-t\frac{\partial}{\partial s}+s\frac{\partial}{\partial t}, \bar{Y}+\bar{X}\right]=2\bar{H}$$

with initial data

$$\bar{H}(s,0)=as\frac{\partial}{\partial s}, \qquad (\bar{Y}+\bar{X})(s,0)=s\frac{\partial}{\partial t}$$

has unique solutions. One can easily show that if $\bar{H}$ is analytic through the origin, a must equal one. Thus $\bar{H}$, $\bar{X}$, and $\bar{Y}$ constitute the natural linear representation of $\mathscr{g}$ in the final coordinate system.

This shows that perturbations of the linear normal form may be converted to this normal form by a C^ω change of coordinates which depends continuously on the perturbation. The model space $\mathscr{m}_i$ is therefore a point in this case. Although we have been loose about the domain of the change of coordinates, it is clear that if we consider perturbations defined on some disk $s^2+t^2<R$, and if $R'<R$ is given, then the domain of the change of coordinates will contain the disk $s^2+t^2<R'$ provided the perturbation is near enough the original on the larger disk. Similar considerations hold true for the local unfoldings exhibited below.

Let $(\bar{H}, \bar{X}, \bar{Y})$ be a perturbation of either the normal form (I, a), $a \neq 0$:

$$\bar{H}_0 = -2s\frac{\partial}{\partial s} + at\frac{\partial}{\partial t}$$

$$\bar{X}_0 = -s^2\frac{\partial}{\partial s} + ast\frac{\partial}{\partial t}$$

$$\bar{Y}_0 = \frac{\partial}{\partial s}$$

or $(II, m, \pm)$:

$$\bar{H}_0 = -2s\frac{\partial}{\partial s} - 4m^{-1}t\frac{\partial}{\partial t}$$

$$\bar{X}_0 = (\pm t^m - s^2)\frac{\partial}{\partial s} - 4m^{-1}st\frac{\partial}{\partial t}$$

$$\bar{Y}_0 = \frac{\partial}{\partial s}.$$

Translating, we again assume that the unique zero of $\bar{H}$ is the origin. The transformation

$$(s, t) \mapsto \beta(\exp sY)(0, t)$$

straightens Y into the constant vector field $\partial/\partial s$. We now have:

$$\bar{H} = (Q(t) - 2s)\frac{\partial}{\partial s} + R(t)\frac{\partial}{\partial t}$$

$$\bar{X} = (S(t) + sQ(t) - s^2)\frac{\partial}{\partial s} + (T(t) + sR(t))\frac{\partial}{\partial t}$$

where $Q(0) = R(0) = 0$. Applying lemma 4.5, we may assume that $R(t) = \lambda t$ with $\lambda \approx a$ in the first case and $\lambda \approx -4m^{-1}$ in the second. The function

$$u(t) = \begin{cases} (\lambda t)^{-1}(T(t) - T(0) - tDT(0)) & t \neq 0 \\ 0 & t = 0 \end{cases}$$

depends continuously on T. After the transformation $(s, t) \mapsto (s + u(t), t)$, we have

$$\bar{H} = (U(t) - 2s)\frac{\partial}{\partial s} + \lambda t\frac{\partial}{\partial t}$$

$$\bar{X} = (V(t) + sU(t) - s^2)\frac{\partial}{\partial s} + (\lambda st + p + qt)\frac{\partial}{\partial t}$$

$$\bar{Y} = \frac{\partial}{\partial s},$$

where $p = T(0)$ and $q = DT(0)$. The relation $[\bar{H}, \bar{X}] = -2\bar{X}$ then works out to ($'$ denoting d/dt):

$$U^2 + \lambda tV' + 4V - (p + qt)U' = 0$$

$$\lambda tU - \lambda p + 2p + 2qt = 0.$$

Since $U(0)=0$ and $\lambda \neq 0$, the latter equation forces $U(t)=0$ and $q=0$, and it forces $p=0$ except when $\lambda=2$. The former equation therefore has the solution

$$V(t)=\tau t^{-4/\lambda}.$$

Of course, V must also vanish if $\lambda \notin \{-4,-2,-4/3,\ldots\}$.

In summary, we have unfolded the perturbations of $(\bar{H}_0, \bar{X}_0, \bar{Y}_0)$, producing a space of models which turns out as follows:

Case 1. The normal form (I, a) with $a \notin \{0,2\} \cup \{-4,-2,-4/3,\ldots\}$ has the one-dimensional unfolding:

$$\bar{H}=-2s\frac{\partial}{\partial s}+(a+\varepsilon)t\frac{\partial}{\partial t}$$

$$\bar{X}=-s^2\frac{\partial}{\partial s}+(a+\varepsilon)st\frac{\partial}{\partial t}$$

$$\bar{Y}=\frac{\partial}{\partial s}.$$

Case 2. $(I, -4m^{-1})$, $m \in \mathbb{Z}^+$, has an unfolding whose models constitute a union of two interesecting lines. One line is described in case one, and the other is:

$$\bar{H}=-2s\frac{\partial}{\partial s}-4m^{-1}t\frac{\partial}{\partial t}$$

$$\bar{X}=(\varepsilon t^m-s^2)\frac{\partial}{\partial s}-4m^{-1}st\frac{\partial}{\partial t}$$

$$\bar{Y}=\frac{\partial}{\partial s}.$$

Notice that when $\varepsilon \neq 0$, this model is equivalent to $(II, m, \operatorname{sign}(\varepsilon))$. However, the change of coordinates converting it to $(II, m, \operatorname{sign}(\varepsilon))$ would be discontinuous at $\varepsilon=0$.

Case 3. $(I, 2)$ also unfolds to the union of two intersecting lines – that described in case one, and the line:

$$\bar{H}=-2s\frac{\partial}{\partial s}+2t\frac{\partial}{\partial t}$$

$$\bar{X}=-s^2\frac{\partial}{\partial s}+(2st+\varepsilon)\frac{\partial}{\partial t}$$

$$\bar{Y}=\frac{\partial}{\partial s}.$$

Notice that when $\varepsilon \neq 0$, the one-dimensional orbit has disappeared.

Case 4. $(II, m, \pm)$ is rigid. Here one further change of coordinates is needed. Fix $t_0 \neq 0$ such that $(0, t_0)$ lies in the common domain of the perturbations being considered. If a perturbation is near the normal form, then after the previous change of coordinates we have

$$\bar{X}(0, t_0) \approx \pm t_0^m \frac{\partial}{\partial s}.$$

However, if λ is near but not equal to $-4m^{-1}$, we have seen that $\bar{X}(0, t_0) = 0$. Thus in fact $\lambda = -4m^{-1}$ for all perturbations. Now apply the transformation

$$(s, t) \mapsto (s, |\tau|^{1/m} t),$$

where

$$\bar{X}(0, t_0) = \tau t_0^m \frac{\partial}{\partial s}.$$

This transformation depends continuously on $\bar{X}$ and converts $(\bar{H}, \bar{X}, \bar{Y})$ to the normal form $(II, m, \pm)$.

The analysis above also applies to normal forms for α_* near the boundary of M. The exceptional models $(\bar{H}, \bar{X}, \bar{Y})$ of case 3 do not arise in unfoldings along the boundary, but with this exception the boundary and interior unfoldings are identical.

As already mentioned, our goal is to find unfoldings not for perturbations of $(\bar{H}_0, \bar{X}_0, \bar{Y}_0)$ but for perturbations of $(\bar{H}_0, \bar{X}_0, \bar{Y}_0, h_0)$, where h_0 is the coordinate representation of

$$\alpha(\exp n_i \pi(Y - X)).$$

To perform this, we make no further change of coordinates but only analyze the possibilities for h near h_0 to commute with model vector fields $(\bar{H}, \bar{X}, \bar{Y})$. While we are at it, we also compute the isotropy group $J(0, t)$ of a point with coordinates $(0, t)$, $t \neq 0$. The isotropy group $J(s, t)$, $t \neq 0$, is then determined by the formula

$$J(s, t) = \exp(sY) \cdot J(0, t) \cdot \exp(-sY).$$

Since the analysis here is trivial and rather uninteresting, we omit it.

Case 1 *revisited.* Write

$$\mu_i(\alpha) = \begin{cases} (\bar{H}_0, \bar{X}_0, \bar{Y}_0) = (I, a) \\ h_0 : (s, t) \to (s, At). \end{cases}$$

The models of the unfolding are then given by

$$\begin{cases} (I, a + \varepsilon) \\ (s, t) \mapsto (s, (A + \delta)t), \end{cases}$$

and there is no restriction on the pair (ε, δ). Thus $\mathfrak{m}_i$ is a plane. In such a model,

$$J(0, t) = \begin{cases} \left\langle X, \exp(n_i \pi(Y - X)) \cdot \exp\left(-\dfrac{\ln(A+\delta)}{a+\varepsilon} H\right)\right\rangle & \text{if } A + \delta > 0 \\[2ex] \left\langle X, \exp(2n_i \pi(Y - X)) \cdot \exp\left(-\dfrac{\ln(A+\delta)^2}{a+\varepsilon} H\right)\right\rangle & \text{if } A + \delta < 0. \end{cases}$$

Case 2 *revisited.* Here

$$\mu_i(\alpha) = \begin{cases} (I, -4m^{-1}) \\ (s, t) \mapsto (s, At). \end{cases}$$

If either $A \notin \{-1, 1\}$ or if m is odd and $A = -1$, $\mathfrak{m}_i$ is just the plane of case 1. If m is odd and $A = 1$, or if m is even and $A \in \{-1, 1\}$, $\mathfrak{m}_i$ is the union of this plane with the line consisting of

$$\begin{cases} \text{the exceptional model in which } \tilde{X} = (\varepsilon t^m - s^2)\dfrac{\partial}{\partial s} - 4m^{-1}st\dfrac{\partial}{\partial t} \\ (s, t) \mapsto (s, At). \end{cases}$$

In such a model with $\varepsilon \neq 0$,

$$J(0, t) = \begin{cases} \langle X - \varepsilon t^m Y, \exp n_i\pi(Y - X)\rangle & \text{if } A = 1 \\ \langle X - \varepsilon t^m Y, \exp 2n_i\pi(Y - X)\rangle & \text{if } A = -1. \end{cases}$$

Case 3 *revisited.* Here

$$\mu_i(\alpha) = \begin{cases} (I, 2) \\ (s, t) \mapsto (s, At). \end{cases}$$

If $A \neq 1$ or if $x_i \in \partial M$, m_i is just the plane of case 1. If $A = 1$ and x_i is interior to M, m_i is the union of this plane with the line:

$$\begin{cases} \text{the exceptional model in which } \tilde{X} = -s^2\dfrac{\partial}{\partial s} + (2st + \varepsilon)\dfrac{\partial}{\partial t} \\ (s, t) \mapsto (s, t). \end{cases}$$

In this model with $\varepsilon \neq 0$,

$$J(0, t) = \langle \varepsilon H - 2tX, \exp n_i\pi(Y - X)\rangle.$$

Case 4 *revisited.* We have

$$\mu_i(\alpha) = \begin{cases} (II, m, \pm) \\ (s, t) \mapsto (s, At) \end{cases}$$

where $A = 1$ if m is odd, and $A \in \{-1, 1\}$ if m is even. Clearly m_i is a point. Also,

$$J(0, t) = J_\alpha(\phi_i^\alpha)^{-1}(0, t).$$

For completeness, we should also point out that in the normal coordinates near a stationary point we have

$$J(0, t) = \langle Y, \exp 2\pi(Y - X)\rangle$$

for $t \neq 0$, and

$$J(r\cos\theta, r\sin\theta) = g \cdot J(0, r) \cdot g^{-1},$$

where $g = \exp((\theta - \frac{1}{2}\pi)(Y - X))$. This completes step one of the programme for unfolding.

Proof of theorem 4.3. For $1 \leq i \leq p$, let D_i denote the disk of radius R_i about the origin of $\mathbb{R}^2$ or, if $x_i \in \partial M$, the half disk. For β, γ near α, define

$$F_i(\beta, \gamma) = (\phi_i^\gamma)^{-1} \circ \phi_i^\beta : (\phi_i^\beta)^{-1}D_i \xrightarrow{\cong} (\phi_i^\gamma)^{-1}D_i.$$

Suppose that $\mu_i(\beta) = \mu_i(\gamma)$. Then $F_i(\beta, \gamma)$ intertwines β_* with γ_*. If x_i is not stationary for α, we also have

$$F_i(\beta, \gamma) \circ \beta(\exp n_i\pi(Y - X)) = \gamma(\exp n_i\pi(Y - X)) \circ F_i(\beta, \gamma)$$

on $(\phi_i^\beta)^{-1}\{(s,t): s^2+t^2<\rho_i\}$. This allows $F_i(\beta,\gamma)$ to be extended to

$$\beta(\exp([0, n_i\pi](Y-X)))(\phi_i^\beta)^{-1}\{(s,t): s^2+t^2<\rho_i\}$$

by setting

$$F_i(\beta,\gamma)(\beta(\exp\theta(Y-X))y)=\gamma(\exp\theta(Y-X))F_i(\beta,\gamma)y$$

for $\theta\in[0, n_i\pi]$ and $y\in(\phi_i^\beta)^{-1}\{(s,t): s^2+t^2<\rho_i\}$. The extension is well defined and intertwines β_* with γ_*. By construction of V_i, the domain of this extension includes V_i.

Let E be an element of $\mathscr{e}$. By definition,

$$F_{i(E)}(\beta,\gamma)y_E^\beta=(\phi_{i(E)}^\gamma)^{-1}(0, t(E))=y_E^\gamma.$$

We have observed that the isotropy group of a point with given coordinates may be deduced from the model. Thus

$$J_\beta(y_E^\beta)=J_{\mu_{i(E)}(\beta)}(0, t(E))=J_\gamma(y_E^\gamma).$$

Finally, if β is near α then

$$V_E\subseteq\beta(G)y_E^\beta$$

by property (4) in the construction of V_E. It therefore makes sense to define

$$F_E(\beta,\gamma): V_E\to M$$

by the formula

$$F_E(\beta,\gamma)(\beta(g)y_E^\beta)=\gamma(g)y_E^\gamma,$$

where $g\in G$ is arbitrary so long as $\beta(g)y_E^\beta\in V_E$. Clearly $F_E(\beta,\gamma)$ intertwines β_* with γ_*.

We claim that $F_E(\beta,\gamma)$ and $F_{i(E)}(\beta,\gamma)$ agree on their common domain $V_E\cap V_{i(E)}$. Certainly the points of agreement constitute a closed subset containing y_E^β. On the other hand, this subset if open. If the two mappings agree at $y\in V_E\cap V_{i(E)}$ and if z is near enough to y, there exists $Z\in\mathscr{g}$ with

$$\beta(\exp(tZ))y\in V_{i(E)}$$

for all $t\in[0,1]$ and $\beta(\exp Z)y=z$. Then

$$\begin{aligned}F_E(\beta,\gamma)z&=F_E(\beta,\gamma)(\beta(\exp Z)y)\\&=\gamma(\exp Z)F_E(\beta,\gamma)y=\gamma(\exp Z)F_{i(E)}(\beta,\gamma)y\\&=F_{i(E)}(\beta,\gamma)(\beta(\exp Z)y)=F_{i(E)}(\beta,\gamma)z.\end{aligned}$$

Since $V_E\cap V_{i(E)}$ is connected, this proves the claim.

The question remains: for distinct $D, E\in\mathscr{e}$, do $F_D(\beta,\gamma)$ and $F_E(\beta,\gamma)$ agree on $V_D\cap V_E$? The only time this intersection is non-empty is when D and E are the ends of some W_j. Set $g=g(W_j)$. The yet unused assumption that

$$u_j(\beta)=u_j(\gamma)$$

means that there is some $h\in G$, which may be determined solely from

$$\mu_{i(E)}(\beta)=\mu_{i(E)}(\gamma)\quad\text{and}\quad u_j(\beta)=u_j(\gamma),$$

with

$$\beta(h)y_E^\beta=\beta(g)y_D^\beta$$
$$\gamma(h)y_E^\gamma=\gamma(g)y_D^\gamma.$$

At a point $z=\beta(g_1)y_D^\beta \in V_D \cap V_E$, then,

$$\begin{aligned} F_D(\beta,\gamma)z &= \gamma(g_1)y_D^\gamma = \gamma(g_1 g^{-1}h)y_E^\gamma \\ &= F_E(\beta,\gamma)(\beta(g_1g^{-1}h)y_E^\beta) = F_E(\beta,\gamma)z. \end{aligned}$$

□

The next step is to prove theorem 4.4. Although a general proof would be desirable, we offer a specialized proof in five cases – the case that α has a hyperbolic or elliptic orbit and four cases in which α has a parabolic orbit – and we define the desired mapping B by fairly explicit formulae.

The first case is easy. If α has a hyperbolic or elliptic orbit, then each space $\mathscr{m}_i$ consists of a single point $\bar{\mu}_i$. Let W_j be a two-ended open orbit of α with ends $D, E \in \mathscr{e}$, and let $g = g(W_j)$. In each model $\bar{\mu}_i$, the function $(s,t)\mapsto J(s,t)$ is locally one to one. Thus if β is near α, the consistency condition

$$J_{\bar{\mu}_{i(E)}}(u_j(\beta)) = g\cdot J_{\bar{\mu}_{i(D)}}(0,t(D))\cdot g^{-1}$$

has no solution $u_j(\beta)$ near $(0,t(E))$ other than $(0,t(E))$ itself. Therefore $\mathscr{U}$ is a point; α is rigid.

From now on, we assume that α is non-degenerate, non-trivial, and non-transitive and has at least one parabolic orbit. Let us determine exactly what $\mathscr{U}$ is by finding what the consistency condition (*) says about $\mu_{i(D)}$, $\mu_{i(E)}$, and u_j for a parabolic orbit W_j with ends D and E.

Suppose that $x_{i(D)}$ is stationary for α (in which case $x_{i(E)}$ is not stationary). Write

$$\mu_{i(E)}(\alpha) = \begin{cases} (I,\lambda) \\ (s,t)\mapsto(s,At). \end{cases}$$

Since $J_\alpha(y_E^\alpha)$ is conjugate to $J_\alpha(y_D^\alpha)=\langle Y, \exp 2\pi(Y-X)\rangle$, either $n_{i(E)}=2$ and $A=1$ or else $n_{i(E)}=1$ and $A=-1$. Now let

$$(\bar{\mu}_1,\ldots,\bar{\mu}_p,\bar{u}_1,\ldots,\bar{u}_q)$$

be a point near $\Gamma(\alpha)$ in $\mathscr{U}$. As $\mathscr{m}_{i(D)}$ is a point, the consistency condition (*) says that

$$\begin{aligned} J_{\bar{\mu}_{i(E)}}(\bar{u}_j) &= g\cdot\langle Y, \exp 2\pi(Y-X)\rangle\cdot g^{-1} \\ &= J_\alpha(y_E^\alpha) = \langle X, \exp 2\pi(Y-X)\rangle. \end{aligned}$$

No matter if there are exceptional models in $\mathscr{m}_{i(E)}$, this equation forces $\bar{u}_j$ to lie on the t-axis and forces

$$\bar{\mu}_{i(E)} = \begin{cases} (I,\lambda+\varepsilon) \\ (s,t)\mapsto(s,At). \end{cases}$$

The values of $\bar{u}_j$ and ε are independent.

Now consider the case that neither $x_{i(D)}$ nor $x_{i(E)}$ is stationary. Write

$$\mu_{i(D)}(\alpha) = \begin{cases} (I,\lambda_D) \\ (s,t)\mapsto(s,A_Dt) \end{cases}$$

$$\mu_{i(E)}(\alpha) = \begin{cases} (I,\lambda_E) \\ (s,t)\mapsto(s,A_Et). \end{cases}$$

One of λ_D and λ_E is positive, the other negative; assume that

$$\lambda_D < 0 < \lambda_E.$$

Since $\mathscr{J}_\alpha(y_D^\alpha)=\mathscr{J}_\alpha(y_E^\alpha)=\text{span}\{X\}$, the element $g=g(W_j)$ must be of the form

$$g=\exp(n\pi(Y-X))\cdot\exp(bH)\cdot\exp(cX).$$

Let $(\bar{\mu}_1,\ldots,\bar{\mu}_p,\bar{u}_1,\ldots,\bar{u}_q)$ be near $\Gamma(\alpha)$ in $\mathscr{U}$. Clearly the consistency condition implies that either both of $\bar{\mu}_{i(D)}$ and $\bar{\mu}_{i(E)}$ are exceptional or neither is. If neither is, let

$$\bar{\mu}_{i(D)}=\begin{cases}(I,\lambda_D+\varepsilon_D)\\(s,t)\mapsto(s,(A_D+\delta_D)t)\end{cases}$$
$$\bar{\mu}_{i(E)}=\begin{cases}(I,\lambda_E+\varepsilon_E)\\(s,t)\mapsto(s,(A_E+\delta_E)t).\end{cases}$$

The condition (*) forces $\bar{u}_j$ to lie on the t-axis and imposes precisely one relation among ε_D, δ_D, ε_E, and δ_E. This relation takes four different forms, depending on the one-sidedness or two-sidedness of the orbits of α through $x_{i(D)}$ and $x_{i(E)}$. When both are two-sided, for example, it is that

$$\frac{\ln(A_D+\delta_D)}{\lambda_D+\varepsilon_D}=\frac{\ln(A_E+\delta_E)}{\lambda_E+\varepsilon_E}.$$

In no case is the relation degenerate; it always has a three-parameter family of small solutions. Finally, suppose that $\bar{\mu}_{i(D)}$ and $\bar{\mu}_{i(E)}$ are exceptional. This only occurs when $\lambda_D=-4m^{-1}$, $m\in\mathbb{Z}^+$, either m is odd and $A_D=1$ or m is even and $A_D=\pm1$, $\lambda_E=2$, $A_E=1$, and $x_{i(E)}\in\text{int}(M)$. In this very special case, we can have

$$\bar{\mu}_{i(D)}=\begin{cases}\text{the exceptional model with } \bar{X}=(\varepsilon_D t^m-s^2)\dfrac{\partial}{\partial s}-4m^{-1}st\dfrac{\partial}{\partial t}\\(s,t)\mapsto(s,A_D t)\end{cases}$$
$$\bar{\mu}_{i(E)}=\begin{cases}\text{the exceptional model with } \bar{X}=-s^2\dfrac{\partial}{\partial s}+(2st+\varepsilon_E)\dfrac{\partial}{\partial t}\\(s,t)\mapsto(s,t).\end{cases}$$

Write $\bar{u}_j=(\bar{s},\bar{t})$. After some calculation, one finds that the consistency equation (*) works out to:

$$(**)\qquad\begin{aligned}\varepsilon_D&=-\exp(4b)(\bar{s})^2(1-\bar{s}c\exp(2b))^{-2}t(D)^{-m}\\\varepsilon_E&=-2\bar{s}\bar{t}(1-\bar{s}c\exp(2b))(1-2\bar{s}c\exp(2b))^{-1}.\end{aligned}$$

(Recall that $g=\exp(n\pi(Y-X))\cdot\exp(bH)\cdot\exp(cX)$.)

We are now in a position to describe $\mathscr{U}$ in the parabolic cases. We say that α is *fragile* if there are points near $\Gamma(\alpha)$ in $\mathscr{U}$ having what we have called 'exceptional' models among their first p coordinates. Assuming the validity of theorem 4.4, this is to say that a fragile action is one which may be perturbed into an action having hyperbolic or elliptic orbits. The description of $\mathscr{U}$ falls into four cases:

I. Either α has a stationary point or the unfolding is in $\text{Act}^\omega(G_N,M)$ and α is not fragile;

II. α has no stationary point and is not fragile, and the unfolding is in $\text{Act}^\omega(G,M)$;

III. α is fragile and the unfolding is in $\text{Act}^\omega(G_N,M)$;

IV. α is fragile and the unfolding is in $\text{Act}^\omega(G,M)$.

There is no ambiguity here – an action with stationary point is not fragile – since clearly all of the first p coordinates of an element of $\mathcal{U}$ are exceptional or none is. In the following discussion, we denote the orbit of α through x_i by P_i, and for each one-dimensional orbit P_i we let

$$\mu_i(\alpha)=\begin{cases}(I,\lambda_i)\\(s,t)\mapsto(s,A_it).\end{cases}$$

Consider case I. Here each A_i is ± 1. Let

$$\eta=(\bar{\mu}_1,\ldots,\bar{\mu}_p,\bar{u}_1,\ldots,\bar{u}_q)$$

be a point near $\Gamma(\alpha)$ in

$$m_1\times\cdots\times m_p\times\mathbb{R}^2\times\cdots\times\mathbb{R}^2.$$

Then:

I. $\eta\in\mathcal{U}$ if and only if each $\bar{u}_j$ lies on the t-axis and, for each one-dimensional orbit P_i of α, $\bar{\mu}_i$ has the form

$$\bar{\mu}_i=\begin{cases}(I,\lambda_i+\varepsilon_i)\\(s,t)\mapsto(s,A_it).\end{cases}$$

In particular, a neighbourhood of $\Gamma(\alpha)$ in $\mathcal{U}$ is homeomorphic to a vector space of dimension $q+r$, where r is the number of one-dimensional orbits of α. From a constructive point of view, this indicates a freedom to vary each λ_i and to slide V_D and V_E along each other equivariantly for each pair of ends D and E of an open orbit W_j. In case II, we have this same freedom and one additional degree of freedom in varying the common conjugacy class associated to all open orbits. More formally:

II. There are smooth functions f_i, $1\leq i\leq p$, defined near one such that $f_i(1)=1$, $f_1(\delta)\equiv\delta$, and $\eta\in\mathcal{U}$ if and only if each $\bar{u}_j$ lies on the t-axis and each $\bar{\mu}_i$ has the form

$$\bar{\mu}_i=\begin{cases}(I,\lambda_i+\varepsilon_i)\\(s,t)\mapsto(s,\operatorname{sign}(A_i)\cdot|A_if_i(\delta)|^{1+(\varepsilon_i/\lambda_i)}\cdot t).\end{cases}$$

The instance $\delta=1$ corresponds to the case that an action β with $\Gamma(\beta)=\eta$ is topologically conjugate to α. In fact, the functions f_i can be chosen uniquely so that two actions β and γ near α are topologically conjugate if and only if the same value of δ occurs in $\Gamma(\beta)$ as in $\Gamma(\gamma)$. In case II, then, a neighbourhood of $\Gamma(\alpha)$ in $\mathcal{U}$ is homeomorphic to a vector space of dimension $p+q+1$.

Before describing $\mathcal{U}$ in cases III and IV, we determine necessary and sufficient conditions for α to be fragile. We assume that α has no stationary point, since if it had one it would not be fragile. Index the orbits of α so that they are arranged according to the following diagram:

$$P_1\leftarrow W_1\rightarrow P_2\leftarrow\cdots\leftarrow W_{p-1}\rightarrow P_p(\leftarrow W_p).$$

An arrow from W_j to P_i means that $P_i\subseteq\operatorname{clos}(W_j)$. This diagram is meant to indicate any one of four cases – that $q=p-1$ and M is a cylinder with boundary $P_1\cup P_p$; that $q=p-1$ and M is a Möbius band with boundary P_1; that $q=p-1$ and M is a Klein bottle; or that $q = p$ and M is a torus. If α is to be fragile, it is clearly necessary that:

(a) for each boundary component P_i, $\lambda_i = -4m^{-1}$, $m \in \mathbb{Z}^+$, and $A_i = 1$;

(b) for each interior orbit P_i, either

(i) $\lambda_i = 2$ and $A_i = 1$, or

(ii) $\lambda_i = -4m^{-1}$, m an even positive integer, and $A_i = \pm 1$, or

(iii) $\lambda_i = -4m^{-1}$, m an odd positive integer, and $A_i = 1$.

In fact, if (iii) occurs then α is not fragile. If P_i is interior to M and $A_i = 1$, there are distinct ends $D, D' \in \mathscr{E}$ with $i(D) = i(D') = i$, and $t(D)$, $t(D')$ have opposite sign. If

$$\bar{\mu}_i = \begin{cases} \text{the exceptional model with } \bar{X} = (\varepsilon t^m - s^2)\dfrac{\partial}{\partial s} - 4m^{-1}\, st\dfrac{\partial}{\partial t} \\ (s, t) \mapsto (s, t) \end{cases}$$

is the ith coordinate of a point near $\Gamma(\alpha)$ in $\mathscr{U}$, then (**) shows that the sign of ε is opposite that of $t(D)^m$ as well as opposite that of $t(D')^m$. This is impossible if m is odd. Necessary conditions for fragility can therefore be rephrased:

(1) α has no stationary point;

(2) for some $n \in \mathbb{Z}^+$, $\alpha(\exp n\pi(Y - X)) = \mathrm{id}_M$;

(3) for each boundary component P_i, $\lambda_i \in \{-4, -2, -4/3, \ldots\}$;

(4) for each interior orbit P_i, either $\lambda_i = 2$ and P_i is two-sided or else $\lambda_i \in \{-2, -1, -2/3, \ldots\}$.

Condition (2) is equivalent to saying that $A_i = \pm 1$ for some (equivalently, every) index i. Surprisingly, these conditions are not sufficient for α to be fragile.

We now assume that α satisfies (1)–(4). We also assume that $\lambda_1 < 0$. This is automatic if M is a cylinder, Möbius band, or Klein bottle, and since the signs of $\lambda_1, \lambda_2, \ldots$ alternate we can obtain this when M is a torus by shifting the indices over. Define $m_1, m_3, m_5, \ldots \in \mathbb{Z}^+$ by the equation

$$\lambda_i = -4m_i^{-1} \qquad (\lambda_i = 2 \text{ for } i \text{ even}).$$

If P_i is a boundary component, m_i may be odd, but otherwise m_i is even. Define the ends D_j, E_j of W_j by the diagram:

$$P_1 \leftarrow D_1 E_1 \rightarrow P_2 \leftarrow E_2 D_2 \rightarrow \cdots.$$

In other words, $\lambda_{i(D_j)} < 0$ and $\lambda_{i(E_j)} = 2$ for all j. Finally, we make simplifying assumptions about the choices of $t(E)$, $E \in \mathscr{E}$, and of $g_j = g(W_j)$, $1 \le j \le q$. These will not detract from the generality of our proof but only simplify the formulae, since it is easy to show that different choices lead to isomorphic objects $\mathscr{U}$ and Γ. We assume that there is some positive number t_0 such that $|t(E)| = t_0$ for all $E \in \mathscr{E}$, and that each g_j is of the form

$$g_j = \exp(\nu_j \pi(Y - X)) \cdot \exp(b_j H).$$

Recall that g_j was necessarily of the form $\exp(n\pi(Y - X)) \cdot \exp(bH) \cdot \exp(cX)$ anyway, and we simply drop the last factor. We maintain the convention that $\alpha(g_j)$ maps $y^{\alpha}_{D_j}$ to $y^{\alpha}_{E_j}$ rather than the other way around.

Suppose that M is a torus, in which case $m_1, m_3, \ldots, m_{p-1}$ are even. Let

$$\eta = (\bar{\mu}_1, \ldots, \bar{\mu}_p, \bar{u}_1, \ldots, \bar{u}_p)$$

be an exceptional point near $\Gamma(\alpha)$ in $\mathscr{U}$. Define $\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{p-1}$ by

$$\bar{\mu}_i = \begin{cases} \text{the exceptional model with } \bar{X} = (\varepsilon_i t^{m_i} - s^2)\dfrac{\partial}{\partial s} - 4m_i^{-1}\, st\dfrac{\partial}{\partial t} \\ (s, t) \mapsto (s, t). \end{cases}$$

and $\varepsilon_2, \varepsilon_4, \ldots, \varepsilon_p$ by

$$\bar{\mu}_i = \begin{cases} \text{the exceptional model with } \bar{X} = -s^2\dfrac{\partial}{\partial s} + (2st + \varepsilon_i)\dfrac{\partial}{\partial t} \\ (s, t) \mapsto (s, t). \end{cases}$$

Set $\bar{u}_j = (\bar{s}_j, \bar{t}_j)$. By (**):

$$\varepsilon_1 = -\exp(4b_1)(\bar{s}_1)^2 t_0^{-m_1}$$
$$\varepsilon_2 = -2\bar{s}_1\bar{t}_1 = -2\bar{s}_2\bar{t}_2$$
$$\varepsilon_3 = -\exp(4b_2)(\bar{s}_2)^2 t_0^{-m_3}.$$

Then

$$\frac{\varepsilon_3}{\varepsilon_1} = \exp[4(b_2 - b_1)] t_0^{m_1 - m_3}(\bar{t}_1)^2(\bar{t}_2)^{-2}.$$

Similarly, we obtain formulae for the ratio of ε_5 to ε_3, etc., on out to the ratio of ε_1 to ε_{p-1}. The product of these ratios being one, we obtain:

(***) $\qquad 1 = \exp[4(-b_1 + b_2 - \cdots + b_p)](\bar{t}_1\bar{t}_3 \cdots \bar{t}_{p-1})^2(\bar{t}_2\bar{t}_4 \cdots \bar{t}_p)^{-2}.$

If α is fragile, then η can be chosen arbitrarily close to $\Gamma(\alpha)$, which places each $|\bar{t}_j|$ arbitrarily close to t_0. Thus there is another necessary condition for α to be fragile when M is a torus, namely:

(5) $-b_1 + b_2 - \cdots + b_p = 0.$

We will try later to clarify the meaning of this condition, but the reader may convince himself that it does not follow from (1)–(4).

If α satisfies (1)–(5), then it is indeed fragile. To describe the exceptional elements in $\mathscr{U}$ near $\Gamma(\alpha)$, choose real numbers $\bar{t}_j$ near $t(E_j)$ for $j = 1, \ldots, q$ and choose $\bar{s}_1$ near zero. If M is a torus, (***) requires that

$$|\bar{t}_1\bar{t}_3 \cdots \bar{t}_{p-1}| = |\bar{t}_2\bar{t}_4 \cdots \bar{t}_p|,$$

but other than this the choices may be made independently. The formulae (**) then determine a unique element of $\mathscr{U}$ exhibiting these parameters. If $\bar{s}_1 = 0$, this element is not exceptional. We conclude:

III. A neighbourhood of $\Gamma(\alpha)$ in $\mathscr{U}$ is homeomorphic to the union of a vector space of dimension $p+q$ (the non-exceptional elements) and a vector space of dimension p (consisting mostly of exceptional elements) which intersect along a vector subspace of dimension $p-1$.

IV. A neighbourhood of $\Gamma(\alpha)$ in $\mathscr{U}$ is homeomorphic to the union of a vector space of dimension $p+q+1$ and a vector space of dimension p which intersect along a subspace of dimension $p-1$.

This description is a bit flat, but in the interest of simplicity we leave a fuller description to the reader.

To define the mapping B sought in theorem 4.4, we introduce notation applicable to all cases I–IV. Let Q be the zero set of $\alpha_*(X)$. As in the proof of theorem 4.1, define the multi-valued function θ on $M \backslash \{$stationary points of $\alpha\}$ by the statement that $x \in \alpha(\exp \theta(x)(Y-X))Q$ whenever x is not stationary; θ is defined up to multiples of π. Recall that there exist $a \in \mathbb{R}$ and $n \in \mathbb{Z}^+$ such that

$$J_\alpha(x) = \langle X, \exp(n\pi(Y-X)) \cdot \exp(aH) \rangle$$

whenever $\mathscr{j}_\alpha(x) = \operatorname{span}\{X\}$. In cases I, III, and IV, $a = 0$. The vector field

$$C = \cos 2\theta\alpha_*(H) + \sin 2\theta\alpha_*(Y+X)$$

is C^ω even through stationary points of α and commutes with $\alpha_*(H)$, $\alpha_*(X)$, and $\alpha_*(Y)$. This is most easily seen by observing that C commutes with $\alpha_*(Y-X)$ everywhere and that C is tangent to Q and commutes with $\alpha_*(H)$ and $\alpha_*(X)$ along Q. In fact, C generates the component of the identity among the analytic self-conjugacies of α. Set

$$R = \alpha_*(Y-X) + a(n\pi)^{-1}C;$$

R satisfies the same bracket relations with $\alpha_*(H)$, $\alpha_*(X)$, and $\alpha_*(Y)$ as does $\alpha_*(Y-X)$. Since $R(\theta) = 1$, the time-π flow along R sends Q back into Q. Let $f: Q \to Q$ be the restriction of this mapping. For $x \in Q$,

$$f^n x = \alpha(\exp n\pi(Y-X))\alpha(\exp aH)x = x.$$

That is, f has period n. The only time that $f^m x = x$ with $0 < m < n$ is when $m = \frac{1}{2}n$ and either x is stationary or x lies on a one-sided interior one-dimensional orbit. Let $Q' = Q/f$. If M is a torus, then Q' is a circle. Otherwise Q' is a segment. We also denote by C the vector field on Q' induced by the f-invariant vector field

$$C|_Q = \alpha_*(H)|_Q.$$

For $i = 1, \dots, p$, let $z_i = [x_i] \in Q'$. These are the zeros of C, and $DC(z_i) = 1$ when x_i is stationary and $DC(z_i) = \lambda_i$ when x_i is not. We assume that these occur in order $z_1, \dots, z_p$ along Q'. The normal coordinates ϕ_i^α about x_i provide a coordinate system about z_i in which C is linear, and this extends uniquely to a mapping τ_i from the segment between z_{i-1} and z_{i+1} (the half-segment if $z_i \in \partial Q'$) onto the real line (the positive half-line if $z_i \in \partial Q'$) which linearizes C. The segment from z_i to z_{i+1} corresponds to an open orbit of α, which we index as W_i in keeping with conventions discussed above. Notice that $q = p - 1$ except when M is a torus, in which case $q = p$ and we interpret the indices modulo p. The change of coordinates $\tau_{i+1} \circ \tau_i^{-1}$ on this segment is of the form

$$t \mapsto c|t|^{\lambda_{i+1}/\lambda_i};$$

c may be determined from $g(W_i)$, $t(D)$, and $t(E)$, where D and E are the ends of W_i. We have

$$\alpha_*(H) = \cos 2\theta \cdot C - \sin 2\theta(R - a(n\pi)^{-1}C)$$

$$\alpha_*(Y+X) = \sin 2\theta \cdot C + \cos 2\theta(R - a(n\pi)^{-1}C)$$

$$\alpha_*(Y-X) = R - a(n\pi)^{-1}C.$$

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Let U be a vector field on Q'. There is a smooth extension of U to M which commutes with R provided the following holds:

(i) If x_i is stationary for α or lies on a one-sided interior orbit of α, then in the coordinates τ_i the even order derivatives of U vanish at the origin.

Denote this extension as U also. Then $U(\theta)=0$ everywhere. For the extension to be tangent to ∂M, we need:

(ii) If $x_i \in \partial M$, then $U(z_i)=0$.

Now let δ be a real number, and consider the vector fields

$$H' = \cos 2\theta(C+U) - \sin 2\theta(R+(\delta - a(n\pi)^{-1})(C+U))$$
$$(Y+X)' = \sin 2\theta(C+U) + \cos 2\theta(R+(\delta - a(n\pi)^{-1})(C+U))$$
$$(Y-X)' = R+(\delta - a(n\pi)^{-1})(C+U).$$

In cases I and III, we will always take $\delta=0$. In light of this and the fact that $a=0$ in case I (as well as in cases III and IV), smoothness of H', X', and Y' through stationary points will be guaranteed by the additional assumption:

(iii) If x_i is stationary for α, then $DU(z_i)=0$.

These vector fields satisfy the necessary bracket relations. Let $\beta=\beta_{\delta,U}$ denote the action they generate. Then Q is also the zero set of $\beta_*(X)$, and for $x \in Q$:

$$\beta(\exp n\pi(Y-X))x = \beta(\exp(n\pi\delta - a)H)x.$$

Thus

$$J_\beta(x) = \langle X, \exp(n\pi(Y-X)) \cdot \exp((a-n\pi\delta)H)\rangle$$

whenever $\mathscr{j}_\beta(x) = \operatorname{span}\{X\}$. Thus varying δ has the effect of varying the conjugacy class of isotropy groups associated to the open orbits, which explains why we take $\delta=0$ in cases I and III.

For each one-dimensional orbit P_i of α, let U_i be a vector field on Q' which satisfies (i)–(iii) and satisfies:

$$U_i(z_i)=0, \qquad DU_i(z_i)=1$$
$$U_i(z_j) = DU_i(z_j) = 0 \quad \text{for } j \neq i.$$

Let β_ε denote the action obtained by setting $\delta=0$ and $U=\varepsilon U_i$ above, where ε is any real number. If

$$\mu_i(\alpha) = \begin{cases} (I, \lambda_i) \\ (s,t) \mapsto (s, A_i t), \end{cases}$$

then

$$\mu_i(\beta_\varepsilon) = \begin{cases} (I, \lambda_i+\varepsilon) \\ (s,t) \mapsto (s, \operatorname{sign}(A_i)|A_i|^{1+(\varepsilon/\lambda_i)}t) \end{cases}$$

and $\mu_j(\beta_\varepsilon)=\mu_j(\alpha)$ for $j \neq i$. Now for each open orbit W_j, let U^j be a vector field on Q' which satisfies (i)–(iii), approximates C on a large compact subset of the interval from z_j to z_{j+1}, is extremely small off this interval, and satisfies:

$$U^j(z_i) = DU^j(z_i) = 0 \quad \text{for all } i.$$

Again, let β_ε denote the action corresponding to the choices $\delta=0$ and $U=\varepsilon U^j$. Then

$$\mu_i(\beta_\varepsilon) = \mu_i(\alpha) \quad \text{for } 1 \leq i \leq p,$$

and if $k \neq j$ then

$$\left\| \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} u_k(\beta_\varepsilon) \right\| \ll \left\| \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} u_j(\beta_\varepsilon) \right\|.$$

This statement presupposes something we have not proved – that our local unfoldings depend smoothly upon parameters. This is the case, but we leave the proof to the reader. Now let r be the number of one-dimensional orbits of α. It follows from the inverse function theorem and our analysis of $\mathcal{U}$ that in case I the mapping

$$(\varepsilon_1, \ldots, \varepsilon_r, \varepsilon^1, \ldots, \varepsilon^q) \mapsto U = \varepsilon_1 U_1 + \cdots + \varepsilon_r U_r + \varepsilon^1 U^1 + \cdots + \varepsilon^q U^q$$
$$\mapsto \beta = \beta_{0,U} \mapsto \Gamma(\beta)$$

takes a neighbourhood of the origin in $\mathbb{R}^{r+q}$ homeomorphically onto a neighbourhood of $\Gamma(\alpha)$ in $\mathcal{U}$. In case II $r = p$, and we have the $(p+q+1)$-dimensional unfolding

$$(\delta, \varepsilon_1, \ldots, \varepsilon^q) \mapsto (\delta, U = \varepsilon_1 U_1 + \cdots + \varepsilon^q U^q) \mapsto \beta = \beta_{\delta,U} \mapsto \Gamma(\beta).$$

These mappings also unfold the non-exceptional actions near α in cases III and IV.

If Q' is a segment – that is, if M is not a torus – then we can reduce the dimension of the unfolding by q units as follows. Let β, γ be actions near α with $\mu_i(\beta) = \mu_i(\gamma)$ for all i in the range $1 \leq i \leq p$. The composition $(\phi_1^\gamma)^{-1} \circ \phi_1^\beta$ provides the germ of a unique analytic conjugacy between β and γ, as we have already seen in theorem 3.5, and this conjugacy varies continuously with (β, γ). Let Φ be the vector space

$$\operatorname{span}\{U_1, \ldots, U_r, U^1, \ldots, U^q\}$$

in case I and

$$\mathbb{R} \times \operatorname{span}\{U_1, \ldots, U_p, U^1, \ldots, U^q\}$$

in case II, and

$$B: \Phi \to \mathrm{Act}^\omega (G, M)$$

the mapping just constructed. Let

$$\Phi_0 = \{x \in \Phi \colon u_j(Bx) = u_j(\alpha) \quad \text{for all } j, 1 \leq j \leq q\}.$$

Then Φ_0 is homeomorphic to a vector space of dimension r in case I or of dimension $p+1$ in case II. For each $x \in \Phi$, there is a unique $\pi(x) \in \Phi_0$ with

$$\mu_i(Bx) = \mu_i(B\pi(x)) \quad \text{for } 1 \leq i \leq p.$$

Furthermore, the local unfolding about x_1, say, determines a canonical analytic conjugacy from Bx to $B\pi(x)$ which varies continuously with $x \in \Phi$. We have thus unfolded the image of B to obtain the smaller unfolding described after the statement of theorem 4.2.

If Q' is a circle – i.e. M is a torus – then the same ideas apply, but as we have seen (theorem 3.5) the relation

$$\mu_i(\beta) = \mu_i(\gamma)$$

for all i does not imply that β and γ are analytically conjugate. Rather, there is an additional restriction. An ambitious reader may work out a formula for this restriction, but for our purposes it suffices to point out that if $\Gamma(\beta)$ is fixed then after arbitrary choices of $u_1(\gamma), \ldots, u_{q-1}(\gamma)$, there is a unique value of $u_q(\gamma)$ which guarantees that γ is analytically conjugate to β. In this case, then we can reduce the models of the unfolding to a set which is homeomorphic to a vector-space of dimension $r+1$ in case I and $p+2$ in case II.

These comments also apply to the unfolding of the non-exceptional elements near α when α is fragile. Again, let Φ be the parameter space for the unfolding of the non-exceptional elements,

$$B:\Phi \to \mathrm{Act}^\omega\,(G, M)$$

the mapping just constructed, $\Phi_0 \subseteq \Phi$ the reduced parameter space, and $\pi:\Phi \to \Phi_0$ the projection. For $x \in \Phi$, let $f_x \in \mathrm{Diff}^\omega\,(M)$ be the analytic conjugacy from $B\pi(x)$ to Bx with germ determined by the local unfolding near x_1. Suppose we can construct a curve of exceptional actions in $\mathrm{Act}^\omega\,(G, M)$ such that $\beta_0 = \alpha$, the evaluation map $\mathbb{R} \times G \times M \to M$ is C^∞, and

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Gamma(\beta_\varepsilon) \neq 0.$$

Then for $x \in \pi^{-1}\{0\}$, set

$$B(\varepsilon, x) = (\beta_\varepsilon)^{(f_x)}.$$

It follows from the implicit function theorem (again using smooth dependence of the local unfoldings upon parameters) that $\Gamma \circ B$ sends a neighbourhood of the origin in

$$(\{0\} \times \Phi) \cup (\mathbb{R} \times \pi^{-1}\{0\})$$

homeomorphically onto a neighbourhood of $\Gamma(\alpha)$ in $\mathscr{U}$. Furthermore, we may obviously reduce the parameter space to

$$(\{0\} \times \Phi_0) \cup (\mathbb{R} \times \{0\}),$$

thus obtaining the smaller unfolding described after theorem 4.2.

It remains to construct such a line of exceptional actions when α is fragile. We dispense with some of the earlier notation, since now $a = 0$,

$$R = \alpha_*(Y - X),$$

and α has no stationary point. However, we maintain the conventions introduced in the discussion of fragile actions above. Let U and V be vector fields on Q' and u and v real-valued functions. These admit smooth extensions to M, denoted by the same letters, which satisfy

$$[\alpha_*(Y-X), U] = [\alpha_*(Y-X), V] = 0$$
$$\alpha_*(Y-X)u = \alpha_*(Y-X)v = 0$$

provided:

(i) If x_i lies on a one-sided interior orbit of α, then in the coordinates τ_i the even order derivatives of U and V and the odd order derivatives of u and v vanish at the origin.

Again, $U(\theta) = V(\theta) = 0$ everywhere, and we assume:

(ii) If $x_i \in \partial M$, then $U(z_i) = V(z_i) = 0$.

Consider vector fields

$$\begin{aligned} H' &= \cos 2\theta(U + u \cdot \alpha_*(Y-X)) - \sin 2\theta(V + v \cdot \alpha_*(Y-X)) \\ (Y+X)' &= \sin 2\theta(U + u \cdot \alpha_*(Y-X)) + \cos 2\theta(V + v \cdot \alpha_*(Y-X)) \\ (Y-X)' &= \alpha_*(Y-X). \end{aligned}$$

These satisfy the desired bracket relations provided:

(iii) $[U, V] + 2uU + 2vV = 0$

(iv) $U(v) - V(u) + 2(u^2 + v^2) = 2.$

Notice that the action α corresponds to the case

$$U = C, \quad V = 0, \quad u = 0, \quad v = 1.$$

We would like to find solutions

$$(U_\varepsilon(z), V_\varepsilon(z), u_\varepsilon(z), v_\varepsilon(z))$$

to (i)–(iv) such that the resulting action is exceptional for $\varepsilon \neq 0$ and such that the dependence on ε and z is C^ω. However, we only succeed in producing solutions which are C^ω in z and C^∞ in both variables.

Let $c_2, c_4, \ldots, c_{2[p/2]}$ be non-zero real numbers, and define a function ϕ on $Q' \backslash \{z_1, \ldots, z_p\}$ by:

$$\phi(z) = c_i \tau_i(z)^{-1} \quad \text{if } z \in \text{domain } \tau_i, \quad i \text{ even}.$$

Notice that $C(\phi) + 2\phi = 0$, since in the coordinates τ_i with i even C takes the form

$$2t\frac{\partial}{\partial t}.$$

Now let i be odd, $1 \leq i \leq p$, and consider the expression for ϕ in terms of τ_i. If z_{i+1} makes sense (that is, except when $i = p$ and Q' is a segment), then by our conventions

$$\alpha(\exp \nu_i \pi(Y-X))\alpha(\exp(b_i H)) y^\alpha_{D_i} = y^\alpha_{E_i}.$$

Therefore with ψ denoting the flow on Q' generated by C:

$$\begin{aligned} \tau_i[y^\alpha_{E_i}] &= \tau_i(\psi_{b_i}[y^\alpha_{D_i}]) \\ &= \exp(-4b_i/m_i)\tau_i[y^\alpha_{D_i}] = \exp(-4b_i/m_i) t(D_i). \end{aligned}$$

Since $\tau_{i+1}[y^\alpha_{E_i}] = t(E_i)$, this shows that on the segment from z_i to z_{i+1} we have

$$\tau_{i+1}(z) = t(E_i) t(D_i)^{\frac{1}{2}m_i} \exp(-2b_i) \tau_i(z)^{-\frac{1}{2}m_i}$$

and therefore

$$\phi(z) = c_{i+1} t(E_i)^{-1} t(D_i)^{-\frac{1}{2}m_i} \exp(2b_i) \tau_i(z)^{\frac{1}{2}m_i}.$$

Similarly if z_{i-1} makes sense (that is, except when $i = 1$ and Q' is a segment), then on the segment from z_{i-1} to z_i

$$\phi(z) = c_{i-1} t(E_{i-1})^{-1} t(D_{i-1})^{-\frac{1}{2}m_i} \exp(2b_{i-1}) \tau_i(z)^{\frac{1}{2}m_i}.$$

We claim that c_2, c_4, etc., can be chosen so that ϕ^2 is analytic through z_1, z_3, etc. All that is necessary here is that whenever i is odd with z_i interior to Q', then:

$$\left|\frac{c_{i+1}}{c_{i-1}}\right| = \left| t(E_{i-1})^{-1} t(D_{i-1})^{-\frac{1}{2}m_i} t(E_i) t(D_i)^{\frac{1}{2}m_i} \right| \exp[2(b_{i-1}-b_i)]$$
$$= \exp[2(b_{i-1}-b_i)].$$

Clearly this can be achieved when Q' is a segment; any choice of c_2 determines c_4, c_6, etc., up to sign. If Q' is a circle, the same holds true, since the cycle of relations among c_2, c_4, etc., is itself consistent by condition (5) among the conditions for α to be fragile. In this case that Q' is a circle, condition (5) also has a more visible interpretation – that of 'symmetry' of the vector field C. Starting with any point in the interval between z_1 and z_2, say, reflect this point into the interval from z_2 to z_3. Specifically, the point z goes to $\tau_2^{-1}(-\tau_2 z)$. By lemma 4.5, this reflection is a natural property of C. Continue reflecting through z_3, z_4, etc., until returning to the initial segment. Condition (5) is equivalent to the condition that the final point coincide with the initial point.

From now on, assume that non-zero values of c_2, c_4, etc., have been chosen so that ϕ^2 is analytic through z_1, z_3, etc. In many cases, these can be chosen so that ϕ itself is analytic through z_1, z_3, etc. In these cases we could set

$$U_\varepsilon = C, \qquad V_\varepsilon = \varepsilon\phi C, \qquad u_\varepsilon = 0, \quad \text{and} \quad v_\varepsilon = 1$$

to achieve (i)–(iv). In fact, the idea of the proof in the general case is to conjugate the resulting continuous action by a carefully chosen homeomorphism of M so as to obtain an action which is C^ω. However, since there are cases in which ϕ cannot be made analytic through z_1, z_3, etc. – namely, those in which some m_i is odd (hence Q' is a segment and $z_i \in \partial Q'$) or else Q' is a circle and the number of m_i's divisible by four is odd – we do not attempt any 'best possible' choice of c_2, c_4, etc.

LEMMA 4.6. *There is a continuous real valued function w defined on $\mathbb{R} \times Q'$ such that*:

(a) w is C^∞ on $\mathbb{R} \times (Q' \backslash \{z_1, z_3, \ldots\})$ *and* C^ω *on*

$$(\mathbb{R}\backslash\{0\}) \times (Q'\backslash\{z_1, z_3, \ldots\});$$

(b) *if i is even, then for each $\varepsilon \in \mathbb{R}$ the first four derivatives of w_ε vanish at z_i*;

(c) *the function $F_\varepsilon(z) = w_\varepsilon(z) + \arctan(\varepsilon\phi(z))$ is C^∞ on*

$$\mathbb{R} \times (Q'\backslash\{z_2, z_4, \ldots\})$$

and C^ω on

$$(\mathbb{R}\backslash\{0\}) \times (Q'\backslash\{z_2, z_4, \ldots\});$$

(d) *if i is odd, then for each $\varepsilon \in \mathbb{R}$ the first $m_i + 2$ derivatives of F_ε vanish at z_i*;

(e) $w(0, z) = 0$ *for all $z \in Q'$*;

(f) *if $z_i \in \partial Q'$ and $x_i \in \operatorname{int}(M)$, then for each $\varepsilon \in \mathbb{R}$ the expression for F_ε in the coordinates τ_i has the property that all odd order derivatives vanish at the origin.*

Proof. First, suppose that Q' is a circle. Let f be a C^∞ real valued function on

$$Q'\backslash\{z_2, z_4, \ldots, z_p\}$$

which is zero in a neighbourhood of z_i for each odd index i and agrees with ϕ in a neighbourhood of z_i for each even index i. Consider the mapping

$$h: (\mathbb{R}\backslash\{0\}) \times Q' \to \mathbb{R}/\pi\mathbb{Z}$$
$$h(\varepsilon, z) = [\arctan(\varepsilon f(z))].$$

This is C^∞ and in fact extends to a C^∞ mapping defined on

$$(\mathbb{R} \times Q')\backslash\{(0, z_2), (0, z_4), \ldots, (0, z_p)\}$$

which is zero along

$$\{0\} \times (Q'\backslash\{z_2, z_4, \ldots, z_p\}).$$

Furthermore, the difference

$$h(\varepsilon, z) - [\arctan(\varepsilon\phi(z))]$$

extends to a continuous mapping defined on all of $\mathbb{R} \times Q'$ which vanishes along $\{0\} \times Q'$ and is C^∞ on

$$\mathbb{R} \times (Q'\backslash\{z_1, z_3, \ldots, z_{p-1}\}).$$

Let h^ω be a C^ω approximation to h in the strong C^∞ topology ([**3**, p. 35]) on mappings from $(\mathbb{R}\backslash\{0\}) \times Q'$ into $\mathbb{R}/\pi\mathbb{Z}$. If the approximation is close enough, then h^ω has the same properties of extension as does h, and we assume this degree of approximation.

Let

$$w(\varepsilon, z) = h^\omega(\varepsilon, z) - [\arctan(\varepsilon\phi(z))].$$

Then w is continuous on $\mathbb{R} \times Q'$, C^∞ on

$$\mathbb{R} \times (Q'\backslash\{z_1, z_3, \ldots, z_{p-1}\}),$$

and C^ω on

$$(\mathbb{R}\backslash\{0\}) \times (Q'\backslash\{z_1, z_3, \ldots, z_{p-1}\}),$$

and w vanishes along $\{0\} \times Q'$. Because of the latter property, w may be taken as a mapping into $\mathbb{R}$ rather than $\mathbb{R}/\pi\mathbb{Z}$. In the lemma, all mappings are supposed to be real valued. Clearly the real valued function

$$F_\varepsilon(z) = w_\varepsilon(z) + \arctan(\varepsilon\phi(z)),$$

which is defined on

$$\mathbb{R} \times (Q'\backslash\{z_2, z_4, \ldots, z_p\}),$$

satisfies (c) since it consists of branches of the mapping h^ω over the various strips constituting its domain.

Thus we have attained (a), (c), and (e). To attain (b) and (d) as well is standard. For each odd index i and each integer k in the range $0 \le k \le m_i + 1$, choose a C^ω function r_i^k on Q' such that in the coordinates τ_i the jth derivative of r_i^k is δ_{jk} for $0 \le j \le m_i + 1$ and such that for $j \ne i$ the first four derivatives of r_i^k vanish at z_j if j is even and the first $m_j + 2$ derivatives of r_i^k vanish at z_j if j is odd. Similarly choose C^ω functions r_i^k for each even index i and each integer k, $0 \le k \le 3$, and let $\mathscr{K}$ be the vector space spanned by the entire collection $\{r_i^k\}$. If w satisfies (a), (c), and (e), then for each ε there is a unique $r_\varepsilon \in \mathscr{K}$ which for every even index i agrees with w_ε through order three at z_i and for every odd index i agrees with F_ε through

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order m_i+1 at z_i. Furthermore, the dependence of r_ε upon ε is C^∞ and is C^ω away from $\varepsilon=0$. Setting

$$\bar{w}(\varepsilon, z)=w(\varepsilon, z)-r_\varepsilon(z)$$

achieves (b) and (d) without destroying (a), (c), and (e). This proves lemma 4.6 when Q' is a circle.

If Q' is a segment, form the manifold Q'' by joining two copies of Q' at any boundary points representing the same interior orbit. Impose the obvious analytic structure, and let J be the natural involution on Q''. The function ϕ on Q' corresponds to a J-invariant function ϕ' on Q'' with the same essential properties as ϕ. The argument above provides a function w' on $\mathbb{R}\times Q''$ satisfying (a)–(e). Setting

$$w''(\varepsilon, z)=w'(\varepsilon, z)+w'(\varepsilon, Jz)$$

then provides a J-invariant function satisfying (a)–(e). The corresponding function w on $\mathbb{R}\times Q'$ satisfies (a)–(f). □

Define

$$\begin{aligned}
U_\varepsilon &= (\cos(w_\varepsilon)-\varepsilon\phi\sin(w_\varepsilon))C\\
V_\varepsilon &= (\sin(w_\varepsilon)+\varepsilon\phi\cos(w_\varepsilon))C\\
u_\varepsilon &= -\tfrac{1}{2}(\cos(w_\varepsilon)-\varepsilon\phi\sin(w_\varepsilon))\cdot C(w_\varepsilon)-\sin(w_\varepsilon)\\
v_\varepsilon &= -\tfrac{1}{2}(\sin(w_\varepsilon)+\varepsilon\phi\cos(w_\varepsilon))\cdot C(w_\varepsilon)+\cos(w_\varepsilon)
\end{aligned}$$

on $Q'\backslash\{z_1, z_3, \ldots\}$ and

$$\begin{aligned}
U_\varepsilon &= (1+\varepsilon^2\phi^2)^{\frac{1}{2}}\cdot\cos(F_\varepsilon)\cdot C\\
V_\varepsilon &= (1+\varepsilon^2\phi^2)^{\frac{1}{2}}\cdot\sin(F_\varepsilon)\cdot C\\
u_\varepsilon &= -\tfrac{1}{2}(1+\varepsilon^2\phi^2)^{\frac{1}{2}}\cdot\cos(F_\varepsilon)\cdot C(F_\varepsilon)-(1+\varepsilon^2\phi^2)^{-\frac{1}{2}}\cdot\sin(F_\varepsilon)\\
v_\varepsilon &= -\tfrac{1}{2}(1+\varepsilon^2\phi^2)^{\frac{1}{2}}\cdot\sin(F_\varepsilon)\cdot C(F_\varepsilon)+(1+\varepsilon^2\phi^2)^{-\frac{1}{2}}\cdot\cos(F_\varepsilon)
\end{aligned}$$

on $Q'\backslash\{z_2, z_4, \ldots\}$. These agree on their common domains, and by the lemma they are C^ω for each value of ε and vary continuously with ε in the C^∞ topology. Notice that ϕC is analytic through z_2, z_4, etc. so there are no singularities in these objects. It is easily verified that $(U_\varepsilon, V_\varepsilon, u_\varepsilon, v_\varepsilon)$ satisfy (i)–(iv) for each value of ε (in fact, 4.6(f) is included precisely to achieve (i)) and that

$$U_0=C, \quad V_0=0, \quad u_0=0, \quad \text{and} \quad v_0=1.$$

Therefore $(U_\varepsilon, V_\varepsilon, u_\varepsilon, v_\varepsilon)$ determine a curve β_ε in $\mathrm{Act}^\omega(G, M)$ with $\beta_0=\alpha$.

Suppose that i is even. The local unfolding near x_i has the property that if two perturbations (H', X', Y') and (H'', X'', Y'') of $(\alpha_*(H), \alpha_*(X), \alpha_*(Y))$ near x_i agree through terms of third order about x_i and if

$$H'(x_i)=H''(x_i)=0,$$

then the resulting models μ_i' and μ_i'' are identical. Therefore in determining $\mu_i(\beta_\varepsilon)$ we may assume that

$$U_\varepsilon=C, \quad V_\varepsilon=\varepsilon\phi C, \quad u_\varepsilon=0, \quad \text{and} \quad v_\varepsilon=1$$

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near z_i. Under this assumption, in the coordinates ϕ_i^α which normalize α_* near x_i we have

$$\beta_{\varepsilon*}(H) = -2s\frac{\partial}{\partial s} + [2t - 4\varepsilon c_i s(1+s^2)^{-2}]\frac{\partial}{\partial t}$$

$$\beta_{\varepsilon*}(X) = -s^2\frac{\partial}{\partial s} + [2st + \varepsilon c_i(1-s^2)(1+s^2)^{-2}]\frac{\partial}{\partial t}$$

$$\beta_{\varepsilon*}(Y) = \frac{\partial}{\partial s} + \varepsilon c_i(1-s^2)(1+s^2)^{-2}\frac{\partial}{\partial t}.$$

One can verify directly that $\mu_i(\beta_\varepsilon)$ is then the model in which

$$\bar{X} = -s^2\frac{\partial}{\partial s} + (2st + \varepsilon c_i)\frac{\partial}{\partial t}.$$

That is, in the terminology used in discussing $\mathscr{U}$ in the case of fragile actions,

$$\varepsilon_i = \varepsilon c_i.$$

This completes the proof of theorem 4.4 and of theorem 4.2, for we have shown that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Gamma(\beta_\varepsilon) \neq 0.$$

For completeness, we mention that in fact when i is odd, $\varepsilon_i = -\frac{1}{4}d_i\varepsilon^2$, where the expression for ϕ^2 in the coordinates τ_i is that $\phi(z)^2 = d_i\tau_i(z)^{m_i}$. The values of $u_1(\beta_\varepsilon), \ldots, u_q(\beta_\varepsilon)$ may be computed directly from $\varepsilon_1, \ldots, \varepsilon_p$.

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