REAL ANALYTICITY OF HAUSDORFF DIMENSION OF DISCONNECTED JULIA SETS OF CUBIC PARABOLIC POLYNOMIALS

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Consider a family of cubic parabolic polynomials given by $f_\lambda(z) = z(1 - z - \lambda z^2)$ for non-zero complex parameters $\lambda \in D_0$ such that for each $\lambda \in D_0$ *the polynomial* f_λ is a parabolic polynomial, that is, the polynomial f_λ has a parabolic fixed point and the Julia set of f_λ, denoted by $J(f_\lambda)$, does not contain any critical points of f_λ. We also assumed that for each $\lambda \in D_0$, *one finite critical point of the polynomial* f_λ *escapes to the super-attracting fixed point infinity*. So, the Julia sets are disconnected. The concern about the family is that the members of this family are generally not even bi-Lipschitz conjugate on their Julia sets. We have proved that the parameter set D_0 is open and contains a deleted neighborhood of the origin 0. Our main result is that the Hausdorff dimension function $D_0 \to (\frac{1}{2}, 2)$ defined by $\lambda \mapsto HD(J(f_\lambda))$ is real analytic. To prove this we have constructed a holomorphic family of holomorphic parabolic graph directed Markov systems whose limit sets coincide with the Julia sets of polynomials $\{f_\lambda\}_{\lambda \in D_0}$ up to a countable set, and hence have the same Hausdorff dimension. Then we associate to this holomorphic family of holomorphic parabolic graph directed Markov systems an analytic family, call it $\{S_\lambda\}_{\lambda \in D_0}$, of conformal graph directed Markov systems with infinite number of edges in order to reduce the problem of real analyticity of Hausdorff dimension for the given family of polynomials $\{f_\lambda\}_{\lambda \in D_0}$ to prove the corresponding statement for the family $\{S_\lambda\}_{\lambda \in D_0}$.

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CHAPTER 1

INTRODUCTION

Hausdorff dimension as a function of subsets of a given metric space usually behaves extremely irregularly. For example if $n \geq 1$ and $\mathcal{K}(\mathbb{R}^n)$ denotes the space of all non-empty compact subsets of the Euclidean space $\mathbb{R}^n$, then the function $\mathcal{K}(\mathbb{R}^n) \ni K \mapsto \mathrm{HD}(K) \in \mathbb{R}$, ascribing to the compact set K its Hausdorff dimension $\mathrm{HD}(K)$, is discontinuous at every point. It is therefore surprising indeed that the function $c \mapsto \mathrm{HD}(J_c)$ is continuous, where c belongs to M_0, the main cardioid of the Mandelbrot set $\mathcal{M}$, and J_c denotes the Julia set of the polynomial $\mathbb{C} \ni z \mapsto z^2+c$. This is a relatively straightforward consequence of (classical) Bowen's formula which states that the Hausdorff dimension of a conformal expanding repeller is given by the unique zero of the corresponding pressure function. Bowen's formula was proved in [4] for limit sets of quasi-Fuchsian groups, and it was the first application of thermodynamic formalism to fractal geometry. Its extension to conformal expanding repellers is rather straightforward; see [23] for the proof and related issues. As a matter of fact Bowen's formula can be used to prove much more, namely, that the function $M_0 \ni c \mapsto \mathrm{HD}(J_c)$ is real-analytic. This fact was proved in [25] based on considerations involving dynamical zeta-functions. Going beyond the classical (finite-to-one) conformal expanding case, real analyticity of the Hausdorff dimension was proved in [34] for Julia-Lavaurs maps and in [33] for the hyperbolic family of exponential maps. The proofs in both papers are based on a different idea than in [25]; their point is to exploit complex analyticity of the corresponding (generalized) Perron-Frobenius operators and to prove applicability of the Kato-Rellich Perturbation Theorem. Further results in this direction (for expanding systems) and simplifications of the proof can be found in [17] and [24], see also [16] and [30]. Real analyticity for still expanding though random systems is proven in [18] with the use of [27], comp. [26].

Going beyond the expanding case, up to our knowledge, the first real analyticity result is proved in [32] for analytic families of semi-hyperbolic generalized polynomial-like mappings. In this realm the Julia set is allowed to contain critical points but their forward orbit is assumed to be non-recurrent. This allows us to associate with such a family an analytic family of conformal graph directed Markov systems (in the sense of [15]) with infinite number of edges and to reduce the problem of real analyticity of Hausdorff dimension of limit sets of this family. In the current paper we investigate another important case where the expanding property breaks down, this time because of presence of parabolic points. We choose to deal with this phenomenon by working with a concrete but representative family of cubic polynomials

$$f_\lambda(z) = z(1 - z - \lambda z^2).$$

We know that (see [7]) a holomorphic endomorphism $T : J(T) \to J(T)$ is expansive if and only if $J(T)$ contains no critical point of T and an expansive holomorphic endomorphism $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is not expanding if and only if T has at least one parabolic fixed (periodic) point. It has been proved already by Fatou (see [3]) that all parabolic fixed (periodic) points for $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are contained in $J(T)$. A rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called parabolic ([7]) if its restriction to the Julia set $J(f)$ is expansive but not expanding., equivalently, if the Julia set contains no critical points but it contains at least one parabolic periodic point.

Note that each member of the family of quadratic polynomials $\{g_\lambda : z \mapsto z(1-\lambda z)\}_{\lambda\in\mathbb{C}\setminus\{0\}}$ is parabolic since for each $\lambda \in \mathbb{C} \setminus \{0\}$, the polynomial g_λ has a parabolic fixed point 0 with miltiplicity 2 and the only finite critical point of g_λ is given by $\frac{1}{2\lambda}$ which is contained in the basin of 0. The study of this family is too trivial since all its members are conjugate to $z \mapsto z^2 + \frac{1}{4}$ via Möbius transformations $h_\lambda(z) = -\lambda z + \frac{1}{2}$ and therefore all their Julia sets $J(g_\lambda)$ have the same Hausdorff dimension as $\mathrm{HD}(J(z^2 + \frac{1}{4})) \approx 1.0812$ (see [19]). Hence the Hausdorff dimension function $\lambda \mapsto \mathrm{HD}(J(g_\lambda))$ is real analytic. On the other hand, what concerns the above family $\{f_\lambda\}$, we prove that they are generally not even bi-Lipschitz conjugate on their Julia sets.

We prove in Chapter 4 (Theorem 4.5) that D_0, the set of all parameters $\lambda \in \mathbb{C}$ for which the cubic polynomial f_λ is parabolic and has no other parabolic or finite attracting periodic points, contains a deleted neighborhood of the origin 0. Our main result is that the function $D_0 \ni \lambda \mapsto \mathrm{HD}(J(f_\lambda)) \in \mathbb{R}$ is real analytic. As in [32] the general idea is to associate to the family $\{f_\lambda\}_{\lambda \in D_0}$ an analytic family, call it $\{S_\lambda\}_{\lambda \in D_0}$, of conformal graph directed Markov systems with infinite number of edges in order to reduce the problem of real analyticity of Hausdorff dimension for this family to prove the corresponding statement for the family $\{S_\lambda\}_{\lambda \in D_0}$. The basic steps of this approach are these. In Chapter 4, The Family $\mathcal{P}_3$, we prove basic facts about polynomials f_λ, $\lambda \in \mathbb{C}$. In chapter 5, Graph Directed Markov Systems, we introduce the class of parabolic graph directed Markov systems (PGDMS) and provide the reader with their basic properties. In particular we associate to each PGDMS S the canonical hyperbolic system $\hat{S}$. The concept of parabolic graph directed Markov System generalizes slightly the notion of parabolic iterated function systems introduced in [13], further investigated in [14], and treated at length in the book [15]. In Chapter 6, Analytic Families of PGDMS, we first generalize a theorem from [32] about real analyticity of the Hausdorff dimension for regularly analytic families of conformal (hyperbolic) graph directed Markov Systems. Then we introduce the concept of a holomorphic family of holomorphic parabolic graph directed Markov systems, and the central part of the chapter is a rather long proof that a holomorphic family $\{S_\lambda\}_{\lambda \in \Lambda}$ of holomorphic parabolic graph directed Markov systems gives rise to a locally regular analytic family $\{\widehat{S}^l_\lambda\}_{\lambda \in \Lambda}$ (with some $l \geq 1$) of corresponding conformal (hyperbolic) graph directed Markov Systems. These considerations are so long since they require a detailed analysis of local behavior of families of parabolic maps around their common parabolic fixed points. This permits us to conclude, see Corollary 6.14, the chapter with the theorem that the Hausdorff dimension of limit sets of a holomorphic family of holomorphic parabolic graph directed Markov systems is real analytic. In Chapter 7, PGDMS Associated With f_λ, $\lambda \in D_0$, which is the last chapter of this paper, we apply the machinery developed in the previous chapters to study the family of polynomials f_λ, $\lambda \in D_0$.

The idea is to associate to this family of polynomials a holomorphic family of holomorphic parabolic graph directed Markov systems whose limit sets coincide with the Julia sets of polynomials f_λ up to a countable set. Then to apply Corollary 6.14.

CHAPTER 2

INTRODUCTON OF GEOMETRIC STUDY OF JULIA SETS

In this chapter we introduce some of the main ideas in iteration theory and recall some relevant definitions and theorems. In the first section of the chapter we will study topological and geometrical properties of Julia sets. Since the polynomial $f_\lambda, \lambda \in D_0$ has a parabolic fixed point at 0 and of course a superattracting fixed point at ∞, in second and third sections we will discuss the behavior of points near a super-attracting fixed point and near a parabolic fixed point respectively.

2.1. Introductory Dynamical Notions

The main goal of this section is to provide relevant definitions and theorems and discuss properties of the Julia set $J(f)$ for a rational function f of degree $d \geq 2$ on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

A rational function is any function which can be written as the ratio of two polynomial functions, i.e. a function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called a *rational function* if and only if it can be written in the form

$$f(z) = \frac{P(z)}{Q(z)}, \quad z \in \widehat{\mathbb{C}},$$

where P and Q are polynomial functions, not both being the zero polynomial. If P is the zero polynomial, then f is the constant function zero. If Q is the zero polynomial, then f is the constant function ∞. The domain of f is the set of all points z for which the denominator Q is not zero, where one assumes that P and Q are coprime (that is, they have no common zeros). The degree of f is denoted by $\deg(f)$ and is defined by

$$\deg(f) = \max\{\deg(P), \deg(Q)\},$$

where $\deg(S)$ is the usual degree of the polynomial S.

If Q is the constant polynomial 1, i.e. $Q(z) = 1$ for all $z \in \widehat{\mathbb{C}}$, then f is the polynomial function P and the degree of f is the degree of the polynomial P.

Since a rational function of degree 1 is a *Möbius transformation*

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

and Möbius transformations are among the small class of functions whose iterates can be computed explicitly, we will consider the function f to have degree at least 2.

2.1.1. *The Fatou and Julia Sets*

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ on the Riemann sphere $\widehat{\mathbb{C}}$. A point ζ is a fixed point of f if $f(\zeta) = \zeta$. To each fixed point ζ of a rational function f, we associate a complex number which we call the multiplier of f at ζ and is denoted by $m(f, \zeta)$. If $\zeta \in \mathbb{C}$, the multiplier is simply the derivative $f'(\zeta)$. The multiplier of f at $\zeta = \infty$, $m(f, \infty)$, is defined by

$$m(f, \infty) = m(gfg^{-1}, g(\infty),$$

where g is a Möbius transformation with $g(\infty) \in \mathbb{C}$. A fixed point ζ of f is:

(1) an attracting fixed point if $|m(f, \zeta)| < 1$,

(2) a repelling fixed point if $|m(f, \zeta)| > 1$,

(3) an indifferent fixed point if $|m(f, \zeta)| = 1$.

Definition 2.1. The indifferent fixed point ζ is called *rationally indifferent* or *parabolic* if $f'(\zeta)$ is a root of unity and it is called *irrationally indifferent* if $|f'(\zeta)| = 1$, but $f'(\zeta)$ is not a root of unity.

A point ζ is a periodic point of a ratinal function f of period n if $\zeta, f(\zeta), \cdots, f^{n-1}(\zeta)$ are distinct but $f^n(\zeta) = \zeta$. The set of points $\{\zeta, f(\zeta), \cdots, f^{n-1}(\zeta)\}$ is called the cycle of ζ. The fixed points of f are the periodic points of period 1. More generally, a fixed point ζ of f has period n if and only if it is a fixed point of f^n but not of any lower-order iterate. A point ζ is pre-periodic under f if it is not periodic but some image of f is periodic, that is, if there exist positive integers m and n such that

$$\zeta, f(\zeta), \cdots, f^m(\zeta), f^{m+1}(\zeta), \cdots, f^{m+n-1}(\zeta)$$

are distinct but $f^{m+n}(\zeta) = f^m(\zeta)$, then ζ is pre-periodic of period n.

Definition 2.2. *Normal Families.*

A collection $\mathcal{F}$ of holomorphic functions from a Riemann surface S to a compact Riemann surface T is called a *normal family* if its closure $\bar{\mathcal{F}} \subset \mathrm{Hol}(S, T)$ is a compact set, or equivalently if every infinite sequence of functions $f_n \in \mathcal{F}$ contains a subsequence which converges locally uniformly to some limit function $g : S \to T$.

Theorem 2.3. *(Montel's Theorem) Let S be a Riemann surface and let $\mathcal{F}$ be a collection of holomorphic functions $f : S \to \widehat{\mathbb{C}}$ which omit three different values. That is, assume that there are distinct points $a, b, c \in \widehat{\mathbb{C}}$ so that $f(S) \subset \widehat{\mathbb{C}} \setminus \{a, b, c\}$ for every $f \in \mathcal{F}$. then $\mathcal{F}$ is a normal family, that is, the closure $\bar{\mathcal{F}} \subset Hol(S, \widehat{\mathbb{C}})$ is a compact set.*

Definition 2.4. *The Fatou and Julia Sets.*

Let $f^n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the n-fold iterates of f. The domain of normality for the collection of iterates $\{f^n\}$ is called the *Fatou set* for f and is denoted by $F = F(f)$. That is, if a point $z_0 \in F(f)$, then there exists some neighborhood U of z_0 so that the sequence of iterates $\{f^n\}$ restricted to U forms a normal family of functions from U to $\widehat{\mathbb{C}}$. If no such neighborhood exists, we say that z_0 in the complement of $F(f)$ and its complement is called the *Julia set*. We will use the notation $J = J(f)$ for the Julia set and write the Fatou set simply as $\widehat{\mathbb{C}} \setminus J$.

Definition 2.5. By the *grand orbit* of a point z under $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ we mean the set $GO(z, f)$ consisting of all points $z' \in \mathbb{C}$ whose orbits eventually intersect the orbit of z. Thus z and z' have the same grand orbit if and only if $f^n(z) = f^m(z')$ for some choice of $m \geq 0$ and $n \geq 0$. A point $z \in \mathbb{C}$ will be called *grand orbit finite* or *exceptional* under f if its grand orbit $GO(z, f) \subset \mathbb{C}$ is a finite set. The set of all exceptional points under f is denoted by $E(f)$.

THEOREM 2.6. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is rational of degree $d \geq 2$, then the set $E(f)$ of exceptional points can have at most two elements. These exceptional points, if they exist, must always be superattracting periodic points of f and hence must belong to the Fatou set F*

DEFINITION 2.7. The *basin of attraction* of an attracting fixed point z_0 is the set $\mathcal{A}(z_0) = \mathcal{A}(z_0, f)$ consisting of all z such that $f^n(z) \to z_0$. If $\{z_0, z_1, ..., z_m\}$ is an attracting cycle of length m, then each z_j is an attracting fixed point for f^m and we define the basin of attraction of the attracting cycle , or of z_0, to be the union of the basins of attraction $\mathcal{A}(z_j, f^m)$ of the z_j's with respect to f^m. The basin of attraction is again denoted by $\mathcal{A}(z_0)$. The *immediate basin of attraction* of the cycle, denoted by $\mathcal{A}^*(z_0)$, is the union of the m components of $\mathcal{A}(z_0)$ which contain points of the cycle.

DEFINITION 2.8. The *filled-in Julia* set $K = K(f)$ is the set of all points $z \in \mathbb{C}$ for which the orbit of z under f is bounded. Note that $J(f) \subseteq K(f)$.

Recall that a *critical point* of f is a point on the sphere where f is not locally one-to-one. These consist of solutions of $f'(z) = 0$ and of poles of f of order 2 or higher. The order of a critical point z is the integer m such that f is $(m + 1)$-to-1 in a punctured neighborhood of z. If z is not a pole, this is its multiplicity as a zero of f'.

THEOREM 2.9. *If f is a rational funtion of degree d, then there are $2d - 2$ critical points, counting with multiplicity.*

DEFINITION 2.10. By the *postcritical set* $P = P(f)$ of a rational map f we mean the union of all forward images $f^n(c)$ with $n \geq 0$, where c ranges over the critical points. The closure of the *postcritical set* $P = P(f)$ is denoted by $\bar{P}(f)$.

THEOREM 2.11. *If the postcritical set of f is finite and there are no superattracting cycles, then $J = \widehat{\mathbb{C}}$. Otherwise. J has empty interior.*

DEFINITION 2.12. A *Fatou component* for a rational map f of degree $d \geq 2$ is any connected component of the Fatou set $F(f) = \widehat{\mathbb{C}} \setminus J(f)$. A component U of $F(f)$ is called forward invariant under f if $f(U) \subseteq U$.

THEOREM 2.13. *(Fatou-Sullivan Classification of Fatou Components) If the rational function f maps the Fatou component U onto itself, then U is one of the following types:*

(1) *U is the basin for a super-attracting fixed point, that is a super-attracting component which contains a super-attracting fixed point ζ of f,*
(2) *U is the immediate basin for an attracting fixed point, that is an attracting component which contains an attracting fixed point ζ of f,*
(3) *U is the immediate basin for one petal of a parabolic fixed point which has multiplier $+1$, that is a parabolic component with a parabolic fixed point $\zeta \in \partial U$ and $f^n \to \zeta$ on U,*
(4) *U is a Siegel disk if $f : U \to U$ is analytically conjugate to a Euclidean rotaion of the unit disk onto itself,*
(5) *U is a Herman ring if $f : U \to U$ is analytically conjugate to a Euclidean rotaion of some annulus onto itself.*

THEOREM 2.14. *If f is a rational function of degree $d \geq 2$ and $\{\Omega_1, \cdots, \Omega_q\}$ be a cycle of Siegel disks or Herman rings of f, then the closure of the postcritical set of f, $\bar{P}(f)$, contains $\partial \bigcup_{j=1}^{q} \Omega_j$.*

COROLLARY 2.15. *If f is a polynomial, then the Fatou set $F(f)$ does not have a Herman ring.*

2.1.2. *Properties of Julia Set*

Here we will summerize the properties of a Julia set of a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$. Details can be found in any standard *Dynamical System* book (see [2], [20], [6]). Recall that the Fatou set $F(f)$ of f is defined as the set of all those points $z \in \widehat{\mathbb{C}}$ that admit an open neighborhood U such that the family of iterates $\{f^n|_U\}_{n\geq 1}$ is equicontinuous with respect to the spherical metric on $\widehat{\mathbb{C}}$ and the Julia set $J(f)$ is the complement of $F(f)$ in $\widehat{\mathbb{C}}$, i.e. $J(f) = \widehat{\mathbb{C}} \setminus F(f)$.

The basic properties of Julia set $J(f)$:

(a) $J(f)$ is a non-empty compact subset of $\widehat{\mathbb{C}}$.

(b) $J(f)$ is totally invariant, meaning that $f^{-1}(J(f)) = f(J(f)) = J(f)$.

(c) If $\deg(f) \geq 2$, then $J(f)$ is an infinite and perfect set and so is uncountable.

(d) For any $N \geq 1$, the Julia set of f coincides with that of f^N, i.e. $J(f) = J(f^N)$.

(e) If $\deg(f) \geq 2$, then

 (i) if z is not exceptional, then J is contained in the closure of the backward orbit of z, $O^-(z)$;

 (ii) if $z \in J$, then J is the closure of the backward orbit of z, $O^-(z)$.

(f) Any non-empty completely invariant subset of J is dense in J. If D is a union of components of F that is completely invariant, then $J = \partial D$.

(g) If $\deg(f) \geq 2$, then J is contained in the closure of the set of periodic points of f.

(h) The dynamical system $f : J(f) \to J(f)$ is topologically exact, i.e. for every non-empty open (in the relative topology) set $U \subset J(f)$ there exists an integer $n \geq 0$ such that $f^n(U) = J(f)$.

THEOREM 2.16. *The Julia set J contains all repelling fixed points and all indifferent fixed points that do not correspond to Siegel disks. The Fatou set F contains all attracting fixed points and all indifferent fixed points corresponding to the Siegel disks.*

THEOREM 2.17. *If deg(f) ≥ 2, then the Julia set is the closure of the set of all repelling periodic points of f.*

THEOREM 2.18. *If f is a polynomial of degree $d \geq 2$, then the following are equivalent.*

(1) *The unbounded component of the Fatou set $F(f)$, say F_∞ is simply connected,*
(2) *Julia set $J(f)$ is connected,*
(3) *there are no finite critical points of f in F_∞.*

THEOREM 2.19. *If z_0 is an attracting periodic point, then the basin of attraction $\mathcal{A}(z_0)$ is a union of components of the Fatou set, and the boundary of $\mathcal{A}(z_0)$ coincides with the Julia set.*

Proof. ([6]) Let U be an open neighborhood of the cycle of z_0 contained in the Fatou set. Then $\mathcal{A}(z_0)$ is the union of the backward iterates of U, an open subset of the Fatou set. If $\omega_0 \in \partial\mathcal{A}(z_0)$ and V is any neighborhood of ω_0, then the iterates $f^n(z)$ converge towards the cycle of z_0 on $V \bigcap \mathcal{A}(z_0)$, whereas they remain outside $\mathcal{A}(z_0)$ for $z \in V \setminus \mathcal{A}(z_0)$. Consequently, $\{f^n\}$ is not normal on V, and $\omega_0 \in J$. Since $\mathcal{A}(z_0)$ is completely invariant, then by the property of Julia set (1f), $J = \partial\mathcal{A}(z_0)$. □

THEOREM 2.20. *If z_0 is an attracting periodic point, then the immediate basin of attraction $\mathcal{A}^*(z_0)$ contains at least one critical point.*

Proof. ([6]) Suppose first that z_0 is an attracting fixed point. We may assume its multiplier λ satisfies $0 < |\lambda| < 1$. Let $U_0 = \Delta(z_0, \epsilon)$ be a small disk, invariant under f, on which the analytic branch h of f^{-1} satisfying $h(z_0) = z_0$ is defined. The branch h maps U_0 into $\mathcal{A}^*(z_0)$, and h is 1-to-1. Thus $U_1 = h(U_0)$ is simply connected, and $U_1 \supset U_0$. We proceed in this fashion, constructing $U_{n+1} = h(U_n) \supset U_n$ and extending h analytically to U_{n+1}. If the procedure does not terminate we obtain a sequence $h^n : U_0 \to U_n$ of analytic functions on U_0 which omits J, hence is normal on U_0. But this is impossible, since $z_0 \in U_0$ is a repelling fixed point for h. Thus we eventually reach a U_n to which we cannot extend h. There is then a critical point $p \in \mathcal{A}^*(z_0)$ such that $f(p) \in U_n$. If z_0 is a periodic point with period $n > 1$ and $|(f^n)'(z_0)| < 1$, this argument shows each component of $\mathcal{A}^*(z_0)$ contains a critical point of f^n. Since $(f^n)'(z) = \prod_{j=0}^{n} f'(f^j(z))$, $\mathcal{A}^*(z_0)$ must also contain a critical point of f. □

OBSERVATION 2.21. *Since there are only $2d-2$ critical points, counting with multiplicity, the Theorem 2.20 above shows that the number of attracting cycles is at most $2d-2$.*

2.2. Superattracting Fixed Points

The polynomial f_λ, $\lambda \in D_0$, has three fixed points 0, $-\frac{1}{\lambda}$ and ∞ which are parabolic, repelling and superattracting respectively, so the next two sections will study the dynamics of the rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ in some small neighborhoods of a superattracting and a parabolic fixed points respectively.

We start by expressing our function in terms of a local uniformizing parameter z, which can be chosen so that the fixed point corresponds to $z = 0$. We can then describe the function by a power series of the form

$$f(z) = \alpha z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} \cdots,$$

where $\alpha \neq 0$ and $n \geq 1$. The series converges for $|z|$ sufficiently small. Recall that if $n = 1$, initial coefficient $\alpha = f^{'}(0)$ is called the multiplier of the fixed point $z = 0$. If $n \geq 2$, then $f^{'}(0) = 0$, and $z = 0$ is a superattracting fixed point. This section studies the case of $n > 1$.

THEOREM 2.22 (Böttcher's Theorem). *With f as above, there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(0) = 0$, which conjugates f to the n-th power map $w \mapsto w^n$ throughout some neighborhood of zero. Furthermore, ϕ is unique up to multiplication by an $(n-1)$-st root of unity.*

Thus near any superattracting fixed point, f is conjugate to a map of the form

$$\phi \circ f \circ \phi^{-1} : w \mapsto w^n,$$

with $n \geq 2$. This theorem has important applications to polynomial dynamics, since any polynomial map $\mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$ extends to a rational map of $\widehat{\mathbb{C}}$ which has a superattracting fixed point at ∞ with local degree $n = d$

Remark 2.23. Given a global holomorphic map $f : S \to S$ with a superattracting fixed point $\widehat{p}$, we can choose a local uniformizing parameter $z = z(p)$ with $z(\widehat{p}) = 0$ and construct the Böttcher coordinate $w = \phi(z(p))$ as above. To simplify the notation. we will henceforth forget the intermediate parameter z and simply write $w = \phi(p)$.

Since the holomorphic extension of the local map $p \mapsto \phi(p)$ throughout the entire basin of attraction of $\widehat{p}$ is not always possible because the n-th root function

$$p \mapsto \ (\phi(f^n(p)))^{\frac{1}{n}},$$

cannot always be defined as a single-valued function. For example, there is a trouble whenever some other point in the basin maps to exactly onto the superattracting point or whenever the basin is not simply connected. However, if we consider only the absolute value of ϕ, there is no problem.

Lemma 2.24 (Extension of $|\phi|$). *If $f : S \to S$ has a superattracting fixed point $\widehat{p}$ with basin $\mathcal{A}$, then the function $p \mapsto |\phi(p)|$, where ϕ is the associated Böttcher map of Theorem 2.22, extends uniquely to a continuous map $|\phi| : \mathcal{A} \to [0, 1)$ which satisfies the identity $|\phi(f(p))| = |\phi(p)|^n$. Furthermore, $|\phi|$ is real analytic, except at the iterated preimages of $\widehat{p}$ where it takes the value zero.*

Now, let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function with a superattracting fixed point $\widehat{p}$. Then the associated Böttcher map ϕ, which carries a neighborhood of $\widehat{p}$ biholomorphically onto a neighborhood of zero, has a local inverse ψ_ϵ mapping the ϵ-disk around zero to a neighborhood of $\widehat{p}$.

Theorem 2.25 (Critical Points in the Basin). *There exists a unique open disk $\mathbb{D}_r$ of maximal radius $0 < r \leq 1$ such that ψ_ϵ extends holomorphically to a map $\psi : \mathbb{D}_r \to \mathcal{A}^*$, where $\mathcal{A}^*$ is* *the immediate basin of $\widehat{p}$. If $r = 1$, then ψ maps the unit disk $\mathbb{D}$ biholomorphically onto $\mathcal{A}^*$ and $\widehat{p}$ is the only critical point in the basin. On the other hand, if $r < 1$, then there is at least one other critical point in $\mathcal{A}^*$, lying on the boundary of $\psi(\mathbb{D}_r)$.*

2.2.1. *Application of the Böttcher's Map to Polynomial Dynamics*

Let $f(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$, where $a_d \neq 0$, be a polynomial of degree $d \geq 2$. Without loss of generality we may assume that $a_d = 1$, otherwise, we can always choose c with $c^{d-1} = a_d$ so that the linearly conjugate polynomial $cf(\frac{z}{c})$ is monic. Since f has a superattracting fixed point at ∞, we can apply Böttcher's Theorem.

Lemma 2.26 (The Filled-in Julia Set). *For any polynomial f of degree $d \geq 2$, the filled-in Julia set $K \subset \mathbb{C}$ is compact, with connected component, with topological boundary ∂K equal to the Julia set $J = J(f)$ and with interior equal to the union of all bounded components U of the Fatou set $F = \widehat{\mathbb{C}} \setminus J$. Thus K is equal to the union of all such U, together with J itself. Any such bounded component U is necessarily simply connected.*

To better understand this filled-in Julia set K, we consider the dichotomy of Theorem 2.25 for the complementary domain $\mathcal{A}(\infty) = \widehat{\mathbb{C}} \setminus K$

Theorem 2.27 (Connected $K \iff$ Bounded Critical Orbits). *Let f be a polynomial of degree $d \geq 2$. If the filled-in Julia set $K = K(f)$ contains all of the finite critical points of f, then both K and $J = \partial K$ are connected and the complement of K in $\mathbb{C}$ is conformally isomorphic to the exterior of the closed unit disk $\bar{\mathbb{D}}$ under an isomorphism*

$$\widehat{\phi} : \mathbb{C} \setminus K \to \mathbb{C} \setminus \bar{\mathbb{D}},$$

which conjugates f on $\mathbb{C} \setminus K$ to the d-th power map $w \mapsto w^d$. On the other hand, if at least one critical point of f belongs to $\mathbb{C} \setminus K$, then both K and J have uncountably many connected components.

To study the behavior of f near ∞, we make the usual substitution $\zeta = \frac{1}{z}$ and consider the rational function

$$R(\zeta) = \frac{1}{f(\frac{1}{\zeta})}.$$

Again we may assume that f is monic. From the asymptotic equality $f(z) \sim z^d$ as $z \to \infty$, it follows that $R(\zeta) \sim \zeta^d$ as $\zeta \to 0$. Thus R has a superattracting fixed point at $\zeta = 0$ and R has a power series expansion of the form

$$R(\zeta) = \zeta^d - a_{d-1}\zeta^{d+1} + (a_{d-1}^2 - a_{d-2})\zeta^{d+2} + \cdots ,$$

for $|\zeta|$ small.

There is an associated Böttcher's map

$$\phi(\zeta) = \lim_{k\to\infty}[R^k(\zeta)]^{\frac{1}{d^k}} \in \mathbb{D},$$

which is defined and biholomorphic for $|\zeta|$ small and $\phi'(0) = 1$ since f is assumed to be monic. In practice, it is more convenient to work with the reciprocal

$$\widehat{\phi}(z) = \frac{1}{\phi(\frac{1}{z})} = \lim_{k\to\infty}[f^k(\zeta)]^{\frac{1}{d^k}} \in \mathbb{C} \setminus \bar{\mathbb{D}}.$$

Thus $\widehat{\phi}$ maps some neighborhood of ∞ biholomorphically onto a neighborhood of ∞ with $\widehat{\phi}(z) \sim z$ as $|z| \to \infty$ and $\widehat{\phi}$ conjugates the degree d polynomial map f to the d-th power map, so that $\widehat{\phi}(f(z)) = [\widehat{\phi}(z)]^d$.

Observation 2.28. *Suppose that there is at least one critical point of f in $\mathbb{C} \setminus K$. Then the conclusion of the Theorem 2.25 translates that there is a smallest number $r > 1$ so that the inverse of $\widehat{\phi}$ near ∞ extends to a conformal isomorphism*

$$\widehat{\psi} : \mathbb{C} \setminus \bar{\mathbb{D}}_r \xrightarrow{\cong} U \subset \mathbb{C} \setminus K.$$

Furthermore, the boundary ∂U of this open set $U = \widehat{\psi}(\mathbb{C} \setminus \bar{\mathbb{D}}_r)$ is a compact subset of $\mathbb{C} \setminus K$ which contains at least one critical point of f.

2.2.2. *The Green's Function of a Polynomial Map*

As in Lemma 2.24, the function $z \mapsto |\widehat{\phi}(z)|$ extends continuously throughout the attracting basin $\mathbb{C} \setminus K$, taking values $|\widehat{\phi}(z)| > 1$ for all $z \in \mathbb{C} \setminus K$. This function is finite valued, since a polynomial has no poles in the finite plane.

DEFINITION 2.29. By the *Green's function* or the *canonical potential function* associated with the filled-in Julia set K of the monic degree d polynomial f we mean the function $G : \mathbb{C} \to [0, \infty)$ which is identically zero on K and takes the value

$$G(z) = \log|\widehat{\phi}(z)| = \lim_{k\to\infty} \frac{1}{d^k} \log|f^k(z)| > 0$$

outside of K. The function G is continuous everywhere and harmonic, that is $G_{xx} + G_{yy} = 0$, where the subscripts denote the partial derivatives, with $z = x + iy$ outside of the Julia set. The curves $G =$ constant > 0 in $\mathbb{C} \setminus K$ are known as *equipotentials.* Since for all $z \in \mathbb{C} \setminus K$, $f(z) \in \mathbb{C} \setminus K$, so we have

$$G(f(z)) = \log|\widehat{\phi}(f(z))| = \log|\widehat{\phi}(z)|^d = d \cdot \log|\widehat{\phi}(z)| = d \cdot G(z),$$

which shows that f maps each equipotential to an equipotential.

2.3. Parabolic Fixed Points

This section is devoted to a brief description of the dynamics of the iterates f^n near a parabolic fixed point ([20]). Again we consider functions $f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots$ which are defined and holomorphic in some neighborhood of the origin, but in this section we suppose that the multiplier α at the fixed point is a root of unity, that is $\alpha^q = 1$ for some $q \geq 1$. Such a fixed point is said to be *parabolic* provided that f^q is not the identify map. First consider the special case $\alpha = 1$. We can write our map as

$$f(z) = z(1 + az^n + \mathcal{O}(z^{n+1})), \tag{1}$$

with $n \geq 1$ and $a \neq 0$. The integer $n+1$ is called the *multiplicity* of the fixed point. We are concerned here with fixed points of multiplicity $n+1 \geq 2$.

DEFINITION 2.30. A complex number $\mathbf{v}$ will be called a *repulsion vector* for f at the origin if the product $na\mathbf{v}^n$ is equal to $+1$, and an *attraction vector* if $na\mathbf{v}^n$ is equal to -1. The notation of the vector $\mathbf{v}$ indicates that $\mathbf{v}$ should be thought of intuitively as a tangent vector to $\mathbb{C}$ at the origin. Thus there are n equally spaced attraction vectors at the origin, separated by n equally spaced repulsion vectors. Denote them by $\mathbf{v_0}, \mathbf{v_1}, \cdots, \mathbf{v_{2n-1}}$, where $\mathbf{v_0}$ is a repulsion vector and for each $j \in \{1, 2, \cdots, 2n-1\}$

$$\mathbf{v_j} = e^{\frac{\pi i j}{n}} \mathbf{v_0} \quad \text{so that} \quad na\mathbf{v}_j^n = (-1)^j.$$

Thus $\mathbf{v_j}$ is attracting if j is odd or repelling otherwise. Note that the inverse map f^{-1} is also well-defined and holomorphic in some neighborhood of 0, and that the repulsion vectors for f are the attraction vectors for f^{-1}.

Now, consider the orbit of some point z_0, $\{z_k = f^k(z_0) : k \geq 0\}$, for the map f given by (1). We say that the orbit converges to zero *nontrivially* if $z_k \to 0$ as $k \to \infty$, but no z_k is actually equal to zero.

LEMMA 2.31. *If an orbit $\{z_k = f^k(z_0) : k \geq 0\}$ of f for some point z_0 converges to zero nontrivially, then z_k is asymptotic to $\frac{\mathbf{v}_j}{\sqrt[n]{k}}$ as $k \to \infty$ for one of the n attraction vectors $\mathbf{v}_j$. In other words, $lim_{k\to\infty} \sqrt[n]{k} z_k$ exists and is equal to one of the $\mathbf{v_j}$ with j odd. Similarly, if an orbit $\{z_k' = f^{-k}(z_0') : k \geq 0\}$ of f^{-1} for some point z_0' converges to zero nontrivially, then z_k' is asymptotic to $\frac{\mathbf{v}_j}{\sqrt[n]{k}}$ as $k \to \infty$, where $\mathbf{v}_j$ is now one of the n repulsion vectors, with j even. Any attraction or repulsion vector can occur.*

DEFINITION 2.32. If an orbit $\{z_k = f^k(z_0) : k \geq 0\}$ of f for some point z_0 converges to zero, with $z_k \sim \frac{\mathbf{v_j}}{\sqrt[n]{k}}$, then we say that this orbit $\{z_k\}_{k\geq 0}$ tends to zero from the direction $\mathbf{v_j}$.

REMARK 2.33. The array of attraction-repulsion vectors at a fixed point transforms naturally under a holomorphic change of coordinate. This array can be thought of as a geometric representation for the leading terms of the power series for f at the fixed point. More generally, consider a Riemann surface S and a map $p \mapsto f(p) \in S$ which is defined and holomorphic in some neighborhood of a fixed point $\widehat{p}$ of multiplicity $n+1 \geq 2$. Then there is a corresponding uniquely defined array of attraction-repulsion vectors in the tangent space at $\widehat{p}$ with completely analogous properties.

Now suppose that the multiplier α at a fixed point is a q-th root of unity, say $\alpha = e^{\frac{2\pi i p}{q}}$, where $\frac{p}{q}$ is a fraction in lowest terms.

LEMMA 2.34. *If the multiplier α at a fixed point $f(\hat{z}) = \hat{z}$ is a primitive q-th root of unity, then the number n of attraction vectors at $\hat{z}$ must be a multiple of q. In other words, the multiplicity $n+1$ of $\hat{z}$ as a fixed point of f^q must be congruent to* 1 *modulo q.*

DEFINITION 2.35. Consider a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with a fixed point $\widehat{p}$ of multiplier $+1$. Given an attraction vector $\mathbf{v}_j$ in the tangent space of $\widehat{\mathbb{C}}$ at $\widehat{p}$, the associated *parabolic basin of attraction* $\mathcal{A}_j = \mathcal{A}(\widehat{p}, \mathbf{v}_j)$ is defined to be the set consisting of all $p_0 \in \widehat{\mathbb{C}}$ for which the orbit $p_0 \mapsto p_1 \mapsto \cdots$ converges to $\widehat{p}$ from the direction $\mathbf{v}_j$. Evidently these basins $\mathcal{A}_1, \mathcal{A}_2, \cdots \mathcal{A}_n$ are disjoint fully invariant open sets, with the property that an orbit $p_0 \mapsto p_1 \mapsto \cdots$ under f converges to $\widehat{p}$ nontrivially if and only if it belongs to one of the $\mathcal{A}_j$. *The immediate basin* $\mathcal{A}_j^*$ is defined to be the unique connected component of $\mathcal{A}_j$ which maps into itself under f. Equivalently, $\mathcal{A}_j$ can be described as the connected component of the Fatou set $\widehat{\mathbb{C}} \setminus J(f)$ which contains p_k for large k whenever $\{p_k\}$ converges to $\widehat{p}$ from the direction $\mathbf{v}_j$.

More generally, if $\widehat{p}$ is a periodic point of period k with multiplier $\alpha = e^{\frac{2\pi i p}{q}}$ for the map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, then $\widehat{p}$ is a fixed point of multiplier $+1$ for the iterate f^{kq}. By definition, the parabolic basins for f^{kq} at $\widehat{p}$ are also called parabolic basins for f.

LEMMA 2.36. *For a holomorphic map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, each parabolic basin $\mathcal{A}_j$ is contained in the Fatou set $\widehat{\mathbb{C}} \setminus J(f)$, but each basin boundary $\partial\mathcal{A}_j$ is contained in the Julia set $J(f)$.*

DEFINITION 2.37. Let $\widehat{p} \in \widehat{\mathbb{C}}$ be a fixed point of multiplicity $n+1 \geq 2$ for a map f which is defined and univalent on some neighborhood $N \subset \widehat{\mathbb{C}}$ of $\widehat{p}$, and let $\mathbf{v}_j$ be an attraction vector at $\widehat{p}$. An open set $\mathcal{P} \subset N$ will be called an *attracting petal* for f for the vector $\mathbf{v}_j$ at $\widehat{p}$ if

(1) f maps $\mathcal{P}$ into itself, and

(2) an orbit $p_0 \mapsto p_1 \mapsto \cdots$ under f is eventually absorbed by $\mathcal{P}$ if and only if it converges to $\widehat{p}$ from the direction $\mathbf{v}_j$.

Similarly, if $f : N \overset{\cong}{\to} N'$, then an open subset$\mathcal{P} \subset N'$ will be called a *repelling petal* for the repulsion vector $\mathbf{v}_k$ if $\mathcal{P}$ is an attracting petal for the map $f^{-1} : N' \to N$ and for this vector $\mathbf{v}_k$.

THEOREM 2.38. *(Parabolic Leau-Fatou Flower Theorem) If $\hat{z}$ is a fixed point of multiplicity $n+1 \geq 2$, then within any neighborhood of $\hat{z}$ there exists simply connected petals $\mathcal{P}_j$, where the subscript j ranges over the integers modulo $2n$ and where $\mathcal{P}_j$ is either repelling or attracting according to whether j is even or odd. Furthermore, these petals can be chosen so that the union*

$$\{\hat{z}\} \cup \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{2n-1}$$

is an open neighborhood of $\hat{z}$. When $n > 1$, each $\mathcal{P}_j$ intersects each of its two immediate neighborhoods in a simply connected region $\mathcal{P}_j \cap \mathcal{P}_{j\pm 1}$ but is disjoint from the remaining $\mathcal{P}_k$.

THEOREM 2.39. *(Parabolic Linearization Theorem) For any attracting or repelling petal $\mathcal{P}$, there is one and, up to composition with a translation of $\mathbb{C}$, only one conformal embedding $\upsilon : \mathcal{P} \to \mathbb{C}$ which satisfies the* Abel functional equation

$$\upsilon(f(z)) = 1 + \upsilon(z)$$

for all $z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$.

The linearizing coordinate $\upsilon(z)$ is often referred to as a Fatou coordinate in $\mathcal{P}$.

COROLLARY 2.40. *If $\mathcal{P} \subset \widehat{\mathbb{C}}$ is an attracting petal for f, then the Fatou map $\upsilon : \mathcal{P} \to \mathbb{C}$ extends uniquely to a map $\mathcal{A} \to \mathbb{C}$ which is defined and holomorphic throughout the attracting* *basin $\mathcal{A}$ of $\mathcal{P}$, still satisfying the Abel functional equation*

$$\upsilon(f(z)) = 1 + \upsilon(z).$$

In the case of a repelling petal, the analogous statement is the following.

Corollary 2.41. *If $\mathcal{P}' \subset \widehat{\mathbb{C}}$ is a repelling patel for f, then the inverse of the Fatou map $\upsilon^{-1} : \upsilon(\mathcal{P}') \to \mathcal{P}'$ extends uniquely to a globally defined holomorphic map $\gamma : \mathbb{C} \to \widehat{\mathbb{C}}$ which satisfies the corresponding equation*

$$f(\gamma(w)) = \gamma(1 + w).$$

Theorem 2.42. *If $\widehat{z}$ is a parabolic fixed point with multiplier $\alpha = 1$ for a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, then each immediate basin for $\widehat{z}$ contains at least one critical point of f. Furthermore, each basin contains one and only one attracting petal $\mathcal{P}_{max}$ which maps univalently onto some right half-plane under υ and which is maximal with respect to this property. This preferred petal $\mathcal{P}_{max}$ always has one or more critical points on its boundary.*

The existence of the univalent map which maps the petal to some right half-plane has been studied in more refine analytically near the parabolic fixed point in Chapter 6. For a given analytic family it has been given uniformly with respect to the parameter.

As an immediate consequence of Theorem 2.42 we have the following.

Corollary 2.43. *A rational map can have at most finitely many parabolic periodic points. In fact, for a map of degree $d \geq 2$, the number of parabolic cycles plus the number of attracting cycles is at most $2d - 2$.*

CHAPTER 3

MEASURE THEORETIC PROPERTIES OF JULIA SETS OF PARABOLIC FUNCTIONS

In this chapter we review results ([7], [31]) concerning fractal and ergodic properties of Julia sets of rational functions with a rationally indifferent periodic point. Our main goal is to recall ([7]) that the Hausdorff dimension of the Julia set $J(f)$ of a parabolic rational function f can be expressed by the smallest zero of the pressure function $t \mapsto P(f_{J(f)}, -t\log|f'|)$. This result is similar to the Bowen-Manning-McCluskey formula (see [4], [12]). In the first section we recall the definition of parabolic function and then in next few sections we will provide relevant definitions and theorems in the realm of Hausdorff measures and dimensions. In the last section of this chapter we will recall the well-developed theory ([7], [31]) of parabolic functions.

3.1. Parabolic (and hyperbolic) Rational Functions

This section contains the explicit definition of parabolic rational function in the sense of M. Denker and M. Urbański ([7]). Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ and $J(f)$ be the Julia set of f. Recall that the Julia set $J(f)$ is non-empty, compact and perfect, and satisfies

$$f(J(f)) = J(f) = f^{-1}(J(f)). \tag{2}$$

By (2) f also acts on $J(f)$. Frequently we shall consider the restriction of f on $J(f)$ without special indication. Here we will discuss the expanding and expansive functions on an arbitrary compact metric space (X, d).

Definition 3.1. A continuous function $T : X \to X$ from a metric space (X, d) into itself is called *expanding* if there exist two constants $\lambda > 1$ and $\delta > 0$ such that for all pairs of points $x, y \in X$,

$$d(x,y) < \delta \Rightarrow d(T(x), T(y)) \geq \lambda d(x,y).$$

Equivalently, a rational function $f : J(f) \to J(f)$ is said to be *expanding* if there exists $\lambda > 1$ and an integer $n \geq 1$ such that

$$|(f^n)^{'}(z)| \geq \lambda \quad \text{for every } z \in J(f).$$

DEFINITION 3.2. A continuous function $T : X \to X$ from a metric space (X, d) into itself is called *expansive* if there exists $\delta > 0$ such that for all pairs of points $x, y \in X$, $x \neq y$, there exists a positive integer $n = n(x, y)$ satisfying

$$d(T^n(x), T^n(y)) \geq \delta.$$

The constant δ is called an *expansive constant* for T and T is also called δ-expansive.

Notice that T is δ-expansive if

$$\sup_{n \geq 0} d(T^n(x), T^n(y)) \leq \delta \Rightarrow x = y.$$

In other words, δ-expansiveness means that two forward T-orbits that remain forever within a distance δ from each other are originated from the same point (and are therefore only one orbit). Moreover. if T is δ-expansive, then T is $\delta^{'}$-expansive for any $0 < \delta^{'} < \delta$ and the expansiveness of T is independent of topologically equivalent metrices, although particular expansive constants generally depend on the metric chosen. That is, if two metrics d and $d^{'}$ generate the same topology on X, then T is expansive when X is equipped with the metric d if and only if T is expansive when X is equipped with the metric $d^{'}$. The expansiveness of a system may also be expressed in terms of the following "dynamical"' metrics.

DEFINITION 3.3. Let $T : (X, d) \to (X, d)$ be a dynamical system. For every $n \geq 0$, let $d_n : X \times X \to [0, \infty)$ be the metric

$$d_n(x,y) = \max\{d(T^j(x), T^j(y)) : 0 \leq j < n\}.$$

The metrics d_n arise from the dynamics of the system T and are called *dynamical metrics.*

Observe that for each $x, y \in X$ we have $d_n(x,y) \geq d_m(x,y)$ whenever $n \geq m$ and also that $d_0 = d$. Moreover, the metrics d_n, for each $n \geq 0$, are topologically equivalent. Thus, we call a continuous function $T : X \to X$ from a metric space (X, d) into itself expansive if there exists $\delta > 0$ such that for all pairs of points $x, y \in X$, $x \neq y$, there exists $n \geq 0$ such that $d_n(x,y) \geq \delta$.

Definition 3.4. A rational function $f : J(f) \to J(f)$ is called *hyperbolic* if there exists $n \geq 1$ such that

$$\inf\{|(f^n)'(z)| : z \in J(f)\} > 1.$$

Note that a rational function is hyperbolic if and only if it is expanding. We also have the following ([23]).

Proposition 3.5. *Every distance expanding dynamical system is expansive.*

The following topological characterization of hyperbolicity is well-known (see [2] and [6] for example).

Theorem 3.6. *A rational function $f : J(f) \to J(f)$ is hyperbolic if and only if*

$$\bar{P}(f) \cap J(f) = \emptyset, \textit{ where } \bar{P}(f) \textit{ is the closure of the postcritical set of } f.$$

The next theorem has been proven by M. Denker and M. Urbański ([7]).

Theorem 3.7. *A rational function $f : J(f) \to J(f)$ is expansive if and only if the Julia set $J(f)$ contains no critical points of f.*

We recall that a periodic point z of f, say of period $q \geq 1$, is called parabolic if the derivative $(f^q)'(z)$ is a root of unity. It is well-known (see [2] and [6]) that the set Ω of all parabolic periodic points is finite and contained in the Julia set. It follows from Theorems 3.6 and 3.7 that a rational function is expansive but not hyperbolic if and only if the Julia set contains no critical points of f but it intersects the closure of the postcritical set of f. Now, if we have a rational function f whose Julia set $J(f)$ contains no critical points of f, then it follows from the Fatou Sullivan's classification of connected components of Fatou set (see Theorem 2.13) and from the fact that the boundaries of Siegel disks and Herman rings intersect the closure of the postcritical set of f (see Theorem 2.14) that if f is not hyperbolic, then there must exists a parabolic point in the Julia set. On the other hand, the existence of such a point obviously rules out hyperbolicity. Hence we get the following theorem ([7]).

THEOREM 3.8. *A rational function $f : J(f) \to J(f)$ is expansive but not hyperbolic (expanding) if and only if the Julia set $J(f)$ contains no critical points of f but contains at least one parabolic point.*

DEFINITION 3.9. A rational function $f : J(f) \to J(f)$ which is expansive but not hyperbolic, i.e. whose Julia set $J(f)$ contains no critical points of f but contains at least one parabolic point is called *parabolic rational function.*

3.2. Topological Entropy

In this section, we present definitions of topological entropy and topological pressure for a dynamical system $T : X \to X$ on a compact metric space (X, d). Here we will present Bowen's definition of topological entropy. Since Bowen's definition only makes sense in a compact metric space, we will as usual assume that X is a compact metric space.

3.2.1. *Bowen's Definition of Topological Entropy*

Let $T : X \to X$ be a dynamical system on a compact metric space (X, d). Recall the Definition 3.3 of dynamical metric, also called Bowen's metric, d_n, for each $n \geq 0$:

$$d_n(x, y) = \max\{d(T^j(x), T^j(y)) : 0 \leq j < n\}.$$

Henceforth, the open ball centered at x of radius r induced by the metric d_n shall be denoted by $B_n(x, r)$. As $d_0 = d$, we shall denote $B_0(x, r)$ simply by $B(x, r)$. Observe that

$$B_n(x, r) := \{y \in X : d_n(x, y) < r\} = \cap_{j=0}^{n-1} T^{-j} \left(B(T^j(x), r) \right).$$

In other words, the ball $B_n(x, r)$ consists of points whose orbits stay within a distance r from the orbit of x until at least time $n-1$, that is, the ball $B_n(x, r)$ is the set of points whose orbits are r-shadowed by the orbit of x until at least time $n-1$.

DEFINITION 3.10. A subset F of X is said to be (n, ϵ)-*separated* if F is ϵ-separated with respect to the metric d_n, which is to say that $d_n(x, y) \geq \epsilon$ for all $x, y \in F$ with $x \neq y$.

DEFINITION 3.11. A subset F of X is called a *maximal* (n, ϵ)-*separated* if for any (n, ϵ)-separated $F^{'}$ with $F^{'} \subseteq F$, we have $F^{'} = F$. In other words, no proper subset of F is (n, ϵ)-separated.

DEFINITION 3.12. A subset E of X is said to be an (n, ϵ)-*spanning* set if

$$\cup_{x \in E} B_n(x, \epsilon) = X.$$

That is, the orbit of every point in the space is ϵ-shadowed by the orbit of a point of E until at least time $n-1$.

DEFINITION 3.13. A subset E of X is called a *minimal* (n, ϵ)-*spanning* set if for any (n, ϵ)-spanning set $E^{'}$ with $E \supseteq E^{'}$, we have $E = E^{'}$. In other words, no proper subset of E is (n, ϵ)-spanning.

The next theorem gives Bowen's definition of topological entropy of the system.

THEOREM 3.14. *For all* $\epsilon > 0$ *and all* $n \in \mathbb{N}$, *let* $F_n(\epsilon)$ *be a maximal* (n, ϵ)*-separated set in* X *and* $E_n(\epsilon)$ *be a minimal* (n, ϵ)*-spanning set in* X*. Then the topological entropy of* T, $h_{top}(T)$ *is:*

$$\begin{aligned} h_{top}(T) &= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \frac{1}{n} \log \# F_n(\epsilon) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \inf \frac{1}{n} \log \# F_n(\epsilon) \\ &= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \frac{1}{n} \log \# E_n(\epsilon) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \inf \frac{1}{n} \log \# E_n(\epsilon). \end{aligned}$$

3.2.2. *Invariant Measure*

Here we collect some of the relevant definitions from standard measure theory. The following is intended only as an aid to memory; we assume that the reader has a basic knowledge of measure theory.

DEFINITION 3.15. If $(X, \mathcal{B})$ is a measurable space and Y is a topological space, then a function $f : X \to Y$ is said to be *measurable* provided that $f^{-1}(G) \in \mathcal{B}$ for every open set G in Y.

DEFINITION 3.16. Let X be a set and $\mathcal{B}$ be a σ-algebra on X. A function $\mu : \mathcal{B} \to [0, \infty]$ is said to be *measurable* on X provided that

(i) $\mu(\emptyset) = 0$

(ii) μ is countably additive, that is, for each sequence $(A_n)_{n\geq 1}$ of pairwise disjoint sets belonging to $\mathcal{B}$, we have that

$$\mu\left(\cup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n).$$

If, in addition, $\mu(X)$ is finite, the measure μ is said to be a *finite measure*. If $\mu(X) = 1$, we say that μ is a *probability measure*. If the σ-algebra $\mathcal{B}$ is smallest σ-algebra that contains the open sets of X, this is, the σ-algebra of Borel sets, the μ is called a *Borel measure*.

Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces, and let $T : (X, \mathcal{B}) \to (Y, \mathcal{C})$ be a measurable transformation. If the domain of T is endowed with a probability measure μ, then the measurable transformation T induces a probability measure on its codomain. Indeed, the set function $\mu \circ T^{-1}$, where

$$(\mu \circ T^{-1})(B) := \mu(T^{-1}(B)), \quad \forall B \in \mathcal{C},$$

defines a probability measure on $(Y, \mathcal{C})$. This measure is sometimes called the push-down of the measure μ under the transformation T.

Definition 3.17. Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be probability spaces, and let $T : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ be a measurable transformation. Then T is said to be measure preserving transformation if $\mu \circ T^{-1} = \nu$. If $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure preserving transformation, that is, $\mu \circ T^{-1} = \mu$, then T is called a measure preserving endomorphism.

Proving the equality $\mu \circ T^{-1}(B) = \nu(B)$ for all elements of the σ-algebra $\mathcal{C}$ is generally an onerous task. The following result can sometimes be a tremendous assistance.

Theorem 3.18. *Let $T : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ be a* measurable transformation. *If $\mathcal{C} = \sigma(\mathcal{S})$ is a σ-algebra generated by the semi-algebra $\mathcal{S}$ on Y, then T is measure preserving if and only if $\mu \circ T^{-1}(S) = \nu(S)$ for all $S \in \mathcal{S}$.*

Definition 3.19. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure preserving endomorphism of a probability space $(X, \mathcal{B}, \mu)$. Then the probability measure μ on $(X, \mathcal{B})$ is called T*-invariant* or invariant with respect to T.

Note that if a measurable transformation $T : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ is invertible and its inverse T^{-1} is measurable, then $\mu(T^{-1}(B)) = \nu(B)$ for every $B \in \mathcal{C}$ if and only if $\nu(T(A)) = \mu(A)$ for every $A \in \mathcal{B}$. In particular, if $X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu)$, then μ is T-invariant if and only if μ is T^{-1} invariant. This justifies the following definition.

Definition 3.20. If $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure preserving endomorphism which is invertible and whose inverse is measurable is called a *measure preserving automorphism*.

Definition 3.21. Given a measurable transformation $T : (X, \mathcal{B}) \to (X, \mathcal{B})$, we shall denote the set of all T-invariant probability measure on the σ-algebra $\mathcal{B}$ by $M(T, \mathcal{B})$. If $\mathcal{B}$ is a Borel σ-algebra on X, we write simply $M(T) := M(T, \mathcal{B}(X))$.

3.2.3. *Ergodic Measure*

Definition 3.22. Let $T : X \to X$ be a measure preserving endomorphism of a probability space $(X, \mathcal{B}, \mu)$ Then T is said to be *ergodic* if all sets $B \in \mathcal{B}$ such that $T^{-1}(B) = B$ have the property that $\mu(B) = 0$ or $\mu(B) = 1$. Alternatively, we say that the measure μ is ergodic with respect to T.

In other words, a measure-theoretic dynamical system is ergodic if and only if it does not admit any non-trivial measure-theoretic subsystem. Let $T : (X, \mathcal{B}) \to (X, \mathcal{B})$ be a measurable transformation. The set of all T-invariant probability measures that are ergodic with respect to T is denoted by $E(T, \mathcal{B})$. If we are in the case where $\mathcal{B}$ s a Borel σ-algebra on X, we write simply $E(T) := E(T, \mathcal{B}(X))$.

Recall that the *symmetric difference* of two sets A and B is denoted by $A\Delta B$ and is the set of all points that belong to one and only one of the sets, that is,

$$A\Delta B := (A \setminus B) \cup (B \setminus A) = (A \cup B(\setminus(A \cap B).$$

A set A is μ-a.e. T-invariant means that it satisfies $\mu(T^{-1}(A)\Delta A) = 0$. Of course, any T-invariant set A is μ-a.e. T-invariant, since $T^{-1}(A)\Delta A = \emptyset$ for such a set.

PROPOSITION 3.23. *Let $T : X \to X$ be a measure preserving endomorphism of a probability space $(X, \mathcal{B}, \mu)$. Then T is ergodic if and only if all sets $B \in \mathcal{B}$ such that $\mu(T^{-1}(B)\Delta B) = 0$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$.*

It is easy to see that the family $\{B \in \mathcal{B} : T^{-1}(B) = B\}$ of all T-invariant sets forms a sub-σ-algebra of $\mathcal{B}$. So does the family $\{B \in \mathcal{B} : \mu(T^{-1}(B)\Delta B) = 0\}$ for all μ-a.e. T-invariant sets ([23]). We shall denote by $\mathcal{I}_\mu := \{B \in \mathcal{B} : \mu(T^{-1}(B)\Delta B) = 0\}$ the collection for all μ-a.e. T-invariant sets. The family $\mathcal{I}_\mu$ is a σ-algebra (see [23]).

DEFINITION 3.24. Let $T : (X, \mathcal{B}) \to (X, \mathcal{B})$ be a measurable map, let μ be a probability measure on $(X, \mathcal{B})$ and let $\phi : (X, \mathcal{B}) \to \mathbb{R}$ be a measurable function. Then

(a) The function $\phi : (X, \mathcal{B}) \to \mathbb{R}$ is said to be *T-invariant* if $\phi \circ T = \phi$.

(b) The function ϕ is said to be *μ-a.e. T-invariant* if $\phi \circ T = \phi$, μ-almost everywhere. In other words, ϕ is μ-a.e. T-invariant if the measurable set

$$\Delta\phi := \{x \in X : \phi(T(x)) \neq \phi(x)\}$$

is a null set. Equivalently, $\mu(X \setminus \Delta\phi) = \mu(\{x : \phi(T(x)) = \phi(x)\}) = 1$.

DEFINITION 3.25. Let $T : X \to X$ be a self-map and let $\phi : X \to \mathbb{R}$ be a real-valued function. Then the *n-th Birkhoff sum* of ϕ at a point $x \in X$ is

$$S_n\phi(x) = \sum_{j=0}^{n-1} \phi(T^j(x)).$$

This is the sum of the values of the function ϕ at the first n points in the orbit of x. Sometimes these are also referred to as ergodic sums.

DEFINITION 3.26. Measure-Theoretic Entropy

Let $T : X \to X$ be a continuous map of a compact metric space (X, d) and that μ is a Borel T-invariant probability measure on X. Let $B_n(x, r)$ be the open ball in the metric d_n centered in x and with radius r. If measure μ is ergodic, then (see [5]) for μ-a.e. point $x \in X$ the limit

$$\lim_{r\to 0}\lim_{n\to\infty} - \frac{\log\mu(B_n(x,r))}{\log n}$$

exists, is called the *entropy* of the system T with respect to the measure μ and is denoted by $h_\mu(T)$.

Usually a different, more classical approach is undertaken to define the entropy $h_\mu(T)$ (see [5] and [36]), the one chosen here is probably the fastest and reflecting well the nature of entropy.

3.3. Topological Pressure

Topological pressure is a generalisation of topological entropy. Indeed, the topological entropy of a dynamical system coincides with the topological pressure of that system when this latter is subject to the constant potential 0, that is, in the absence of any potential. Now we will give the definition of topological pressure.

For every $Y \subseteq X$ and $n \in \mathbb{N}$, define

$$S_n\phi(Y) := \sup_{y \in Y} S_n\phi(y).$$

In fact, the supremum is a maximum since X is compact. Now let $\mathcal{U}$ be an open cover of X. Define the natural number $Z_n(\mathcal{U})$ by

$$Z_n(\mathcal{U}) := \min\{\#\mathcal{V} : \mathcal{V} \text{ is a subcover of } \mathcal{U}^n\},$$

where

$$\mathcal{U}^n := \bigvee_{j=0}^{n-1} T^{-j}\mathcal{U} = \mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-(n-1)}\mathcal{U}.$$

$Z_n(\mathcal{U})$ denotes the minimum number of elements of $\mathcal{U}^n$ necessary to cover X. A subcover of $\mathcal{U}^n$ whose cardinality equals the minimum number is called a *minimal subcover* of $\mathcal{U}^n$. Notice that

$$Z_n(\mathcal{U}) = Z_1(\mathcal{U}^n) = \exp(H(\mathcal{U}^n)),$$

where $H(\mathcal{U}) := \log Z_1(\mathcal{U})$ is the entropy of the open cover $\mathcal{U}$ of X. Since $0 \leq H(\mathcal{U}) < \infty$, we get that $1 \leq Z_n(\mathcal{U}) < \infty$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define the n-th partition of $\mathcal{U}$ with respect to the potential ϕ by

$$Z_n(\phi, \mathcal{U}) := \inf \left\{ \sum_{V \in \mathcal{V}} \exp(S_n\phi(V)) : \mathcal{V} \text{ is a subcover of } \mathcal{U}^n \right\}.$$

It is sufficient to take the infimum over all finite subcovers since the exponential function takes only positive values. However, this infimum may not be achieved if $\mathcal{U}$ is infinite. Given an open cover $\mathcal{U}$ of X, the sequence $(Z_n(\phi, \mathcal{U}))_{n=1}^{\infty}$ is submultiplicative (see [23]) and hence the sequence $(\log Z_n(\phi, \mathcal{U}))_{n=1}^{\infty}$ is subadditive. Since $(\log Z_n(\phi, \mathcal{U}))_{n=1}^{\infty}$ is a subadditive sequence of real numbers, then the sequence $\left(\frac{1}{n} \log Z_n(\phi, \mathcal{U})\right)_{n=1}^{\infty}$ converges (see [23]) and

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, \mathcal{U})_{n=1}^{\infty} = \inf_{n\in\mathbb{N}} \frac{1}{n} \log Z_n(\phi, \mathcal{U})_{n=1}^{\infty}. \tag{3}$$

Definition 3.27. The topological pressure of ϕ with respect to $\mathcal{U}$, denoted by $P(\phi, \mathcal{U})$, is defined by

$$P(\phi, \mathcal{U}) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, \mathcal{U}) = \inf_{n\in\mathbb{N}} \frac{1}{n} \log Z_n(\phi, \mathcal{U}).$$

We can also define the *topological pressure* of the function ϕ with respect to T using the sequence of maximal (n,ϵ)-separated sets $F_n(\epsilon)$ of X as follows

$$P(T,\phi) = \lim_{\epsilon\to 0}\lim_{n\to\infty}\sup\frac{1}{n}\log\left(\sum_{x\in F_n(\epsilon)}\exp\sum_{j=0}^{n-1}\phi\circ T^j(x)\right). \tag{4}$$

Since the Julia set $J(f)$ of a expansive rational function $f: J(f)\to J(f)$ contains no critical points, the function

$$J(f)\ni z\mapsto -\log|(f^j)^{'}(z)|$$

is continuous, and therefore the function

$$[0,\infty)\ni t\mapsto P(t)=P(f|_{J(f)},-t\log|f^{'}|), \tag{5}$$

called the pressure function, is well-defined.

From (4) we see that, in case when $\phi=0$, the topological pressure is the topological entropy (see Theorem 3.14). It has been proved ([10]) that topological entropy of any rational function is equal to the logarithm of its degree. Topological pressure belongs to topological dynamics and metric entropy is a notion of measure-preserving endomorphism. The link joining them is given by the following formula called *variational principle* (see [35]).

$$P(f|_{J(f)},-t\log|f^{'}|)=\sup\{h_\mu(f)+\int fd\mu\},$$

where the supremum is taken over all Borel probability T-invariant (ergodic) measure of f.

DEFINITION 3.28. Let $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ be a rational function and μ be a f-invariant ergodic Borel probability measure on $J(f)$. Then the *Lyapunov exponent* $\chi_\mu(f)$ of $f: J(f)\to J(f)$ with respect to measure μ is defined as

$$\chi_\mu(f)=\int\log|f^{'}|d\mu.$$

$\chi_\mu(f)\le\log||f'||<\infty$ and it was proved in ([22]) that $\chi_\mu(f)\ge 0$. The following theorem was proved in ([11]) (comp. also Chapters 8-10 of [23]).

THEOREM 3.29. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function and μ is a Borel probability ergodic invariant measure on $\widehat{\mathbb{C}}$ such that $\chi_\mu(f) > 0$, then*

$$HD(\mu) = \frac{h_\mu(f)}{\chi_\mu(f)}.$$

We would like to notice that ergodicity of f and the fact that the Lyapunov exponent is positive imply that μ is supported on the Julia set. We would also like to add that due to Ruelle's inequality $h_\mu(f) \le 2 \cdot \chi_\mu(f)$, inequality $\chi_\mu(f) > 0$ is implied by inequality $h_\mu(f) > 0$.

The basic properties of the pressure function given by (5) are collected in the following theorem (see [7]).

THEOREM 3.30. *Suppose that $f : J(f) \to J(f)$ is expansive. Then*

(i) *The pressure function is convex and therefore continuous.*

(i) *The pressure function is non-increasing.*

(iii) *If the mapping f is parabolic then there exists a number $s(f) > 0$ such that $P(t) > 0$ for all $0 \le t < s(f)$, $P(t) = 0$ for all $t \ge s(f)$ and $P|_{[0,s(f)]}$ is strictly decreasing.*

(iv) *If the mapping f is hyperbolic, that is, f does not have any parabolic points, then the pressure function is strictly decreasing and $\lim_{t\to\infty} P(t) = -\infty$.*

3.4. Hausdorff Measure and Hausforff Dimension of Julia Set

Here we recall few definitions relevant to fractal properties of a Julia set, and we begin the section with the definitions of Hausdorff measure and Hausdorff dimension.

DEFINITION 3.31. Given a non-decreasing function $g : (0, \epsilon) \to (0, \infty)$ for some $\epsilon > 0$, the *g-dimensional outer Hausdorff measure* $H_g(A)$ of the set A is defined as

$$H_g(A) = \sup_{\epsilon>0} \inf\{\sum_{i=1}^{\infty} g(\text{diam}(A_i))\},$$

where the infimum is taken over all countable covers $\{A_i : i \ge 1\}$ of A by arbitrary sets whose diameters do not exceed ϵ.

If $g(x) = x^t$ for $t \in (0, \epsilon)$ for some $\epsilon > 0$, instead of writing $H_g = H_{x^t}$ we write

$$H_t(A) = \sup_{\epsilon > 0} \inf\{\sum_{i=1}^{\infty} (\text{diam}(A_i))^t\},$$

and speak about t-dimensional outer Hausdorff measure. In this case one will get comparable numbers (in the sense that ratios are bounded away from 0 and ∞) if instead of covering A by arbitrary sets one considers only open balls centered at points of A.

DEFINITION 3.32. The *Hausdorff dimension of* A is denoted by $\text{HD}(A)$ and is given by

$$\text{HD}(A) = \inf\{t : H_t(A) = 0\} = \sup\{t : H_t(A) = \infty\}.$$

DEFINITION 3.33. Given a Borel probability measure μ on a compact metric space X, then the *Hausdorff dimension of the measure* μ is denoted by $\text{HD}(\mu)$ and is defined by the number

$$\text{HD}(\mu) = \inf\{\text{HD}(Y) : \mu(Y) = 1\},$$

where infimum is taken over all subsets $Y \subset X$.

DEFINITION 3.34. The *dynamical dimension* $\text{DD}(J(f))$ is defined as

$$\text{DD}(J(f)) = \sup\{\text{HD}(\mu)\},$$

where the supremum is taken over all ergodic invariant measure of positive entropy.

Our main tool to understand fractal properties of a Julia set, i.e. its dimension and measure will be the concept of conformal measure introduced in the case of Fuchsian groups by S. Patterson in ([21]) and adopted to the case of rational functions by D. Sullivan in ([28]). Its definition is the following.

DEFINITION 3.35. Let $t \geq 0$. A measure m on $J(f)$ is said to be *t-conformal* for the rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ if $m(J(f)) = 1$ and

$$m(f(A)) = \int_A |f'|^t dm$$

for every Borel set $A \subset J(f)$ such that $f|_A$ is injective. A t-conformal measure for some $t \geq 0$ will be called a conformal measure. Since these measures are concentrated on the set $J(f)$, they also will be called conformal for $f : J(f) \to J(f)$ or conformal for f.

In ([29]) Sullivan proved that for every rational function f there exists a t-comformal measure for some $t \geq 0$. The construction for the existence of t-conformal measure does not tell too much about the exponent t of conformal measure. In order to get such information, M. Denker and M. Urbański have proposed in ([9]) a general scheme of constructing generalizations of conformal measures and in ([8]) they have applied it to the case of (Sullivan's) conformal measures. M. Denker and M. Urbański's approach is a modification of Paterson's and Sullivan's and the main difference is that they start with a point in the Julia set and not in the Fatou set.

Definition 3.36. $\delta(f)$ is the *minimal exponent* for which a conformal measure exists.

Theorem 3.37. *$DD(J(f)) = \delta(f)$.*

The following theorem which is classical in the hyperbolic case and called Bowen-Manning-McCluskey formula ([4]) has been proved in the parabolic case in ([7])

Theorem 3.38. *If a rational function $f : J(f) \to J(f)$ is expansive, then we have*

$$\delta(f) = s(f) = \mathrm{HD}(J(f)),$$

where $s(f)$ is the smallest zero of the pressure function $P(t)$.

The following theorems has been proved in ([1]).

Theorem 3.39. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a parabolic mapping, then $\mathrm{HD}(J(f)) < 2$.*

Recall that the set Ω of all parabolic fixed points is finite and contained in the Julia set. Suppose now that $\omega \in \Omega$ is a fixed point of f. Looking at the Taylor series expansion of f around ω and at Leau-Fatou Flower Theorem 2.38 one can deduce (see [1]) that there exists $\sigma > 0$ such that for every sufficiently small $\delta > 0$ and every point $z \in B(\omega, \delta) \setminus \{\omega\}$ all continuous inverse branches $f_\omega^{-n} : B(z, \sigma|z - \omega|) \to \mathbb{C}$, $n \geq 1$, of f^n moving point z

towards ω are well defined. Looking again at the Taylor series expansion of f around ω we can conclude (see [1]) that for every $x \in B(z, \sigma|z-\omega|)$

$$|(f_\omega^{-n})'(x)| \asymp n^{-\frac{p+1}{p}} \tag{6}$$

and

$$|f_\omega^{-n}(x) - \omega| \asymp n^{-\frac{1}{p}}, \tag{7}$$

where $p = p(\omega)$ is the number of petals associated with the parabolic point ω and where the comparability constants depend only on the distance of z and ω. Relying on (6) and following the proof of Theorem 8.5 in ([1]), M. Urbański has proved the following ([31]).

THEOREM 3.40. *If a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has parabolic points ω_i with $p(\omega_i)$ petals, then*

$$HD(J(f)) > \max\left\{\frac{p(\omega_i)}{p(\omega_i)+1} \;:\; \omega_i \text{ is parabolic}\right\}.$$

CHAPTER 4

THE FAMILY $\mathcal{P}_3$

By definition, the family $\mathcal{P}_3$ consists of all cubic polynomials of the form

$$f_\lambda(z) = z(1 - z - \lambda z^2),\ \ \lambda \in \mathbb{C} \setminus \{0\}. \tag{8}$$

Note that

$$f_\lambda(0) = 0 \tag{9}$$

and

$$f'_\lambda(z) = 1 - 2z - 3\lambda z^2. \tag{10}$$

Hence $f'_\lambda(0) = 1$, and therefore (looking also at (8)), we get the following.

Proposition 4.1. *The number* 0 *is a parabolic fixed point of* f_λ *with multiplicity equal to* 1 *and with one petal. The ray* $[0, +\infty)$ *forms its attracting direction and the ray* $(-\infty, 0]$ *forms its repelling direction.*

The other finite fixed point of f_λ is the non-zero solution to the equation $1 - z - \lambda z^2 = 1$, that is

$$z = -\frac{1}{\lambda}.$$

We have

$$f'_\lambda\left(-\frac{1}{\lambda}\right) = 1 + \frac{2}{\lambda} - \frac{3}{\lambda} = 1 - \frac{1}{\lambda}. \tag{11}$$

Since any two polynomials bi-Lipschitz conjugate on their Julia sets have the same moduli of multipliers at corresponding periodic points, (11) yields the followings.

Theorem 4.2. *If $\lambda, \gamma \in \mathbb{C} \setminus \{0\}$ such that $|1 - \frac{1}{\lambda}| \neq |1 - \frac{1}{\gamma}|$, then f_λ and f_γ are not bi-Lipschitz conjugate on their Julia sets. In particular, if $\lambda, \gamma \in (0, 1)$ and $\lambda = \gamma$, then f_λ and f_γ are not bi-Lipschitz conjugate on their Julia sets.*

Theorem 4.3. *If $g(z) = \bar{z}$, then $f_\lambda(z) = (g \circ f_{\bar{\lambda}} \circ g^{-1})(z)$, that is, f_λ and $f_{\bar{\lambda}}$ are bi-Lipschitz conjugate and their dynamics are symmetric about the real axis. Also, they share same topological and geometrical properties.*

The critical points of f_λ are the solutions to the equation $1 - 2z - 3\lambda z^2 = 0$, i.e.

$$c_\lambda^{(1)} = \frac{-1 + \sqrt{1 + 3\lambda}}{3\lambda} \quad \text{and} \quad c_\lambda^{(2)} = \frac{-1 - \sqrt{1 + 3\lambda}}{3\lambda}, \tag{12}$$

and we take the convention that $\sqrt{1} = 1$. We shall prove the following.

Lemma 4.4. *For all $\lambda \in \mathbb{C} \setminus \{0\}$ sufficiently small in modulus, $\lim_{n \to \infty} f_\lambda^n(c_\lambda^{(2)}) = \infty$.*

Proof. It follows from (12) that for all $\lambda \in \mathbb{C} \setminus \{0\}$ sufficiently small in modulus, say

$$\lambda \in B^*(0, R_1) := B(0, R_1) \setminus \{0\}$$

$$\frac{7}{12|\lambda|} \leq |c_\lambda^{(2)}| \leq \frac{3}{4|\lambda|}.$$

So,

$$\begin{aligned}
\left| f_\lambda^n(c_\lambda^{(2)}) \right| &= \left| c_\lambda^{(2)} \right| \left| 1 - c_\lambda^{(2)} - \lambda (c_\lambda^{(2)})^2 \right| \\
&\geq |c_\lambda^{(2)}| \left(|c_\lambda^{(2)}| - |\lambda| |c_\lambda^{(2)}|^2 - 1 \right) \\
&\geq \frac{7}{12|\lambda|} \left(\frac{7}{12|\lambda|} - \frac{9}{16|\lambda|} - 1 \right) = \frac{7}{12|\lambda|} \left(\frac{28}{48|\lambda|} - \frac{27}{48|\lambda|} - 1 \right) \\
&= \frac{7}{12|\lambda|} \left(\frac{1}{48|\lambda|} - 1 \right) \geq \frac{7}{12|\lambda|} \frac{1}{96|\lambda|} \\
&\geq \frac{7}{1200} \frac{1}{|\lambda|^2} \geq \frac{1}{2^9 |\lambda|^2},
\end{aligned} \tag{13}$$

where writing the second last inequality we assumed that $R_1 \leq \frac{1}{96}$ which implies that,

$$\frac{1}{48|\lambda|} - 1 - \frac{1}{96|\lambda|} = \frac{1}{96|\lambda|} - 1 \geq \frac{1}{96 R_1} - 1 \geq 0.$$

Now, if $|z| \geq (2^9|\lambda|^2)^{-1}$ and $\lambda \in B^*(0, R_2)$ with $0 < R_2 \leq R_1$ sufficiently small, we get

$$
\begin{aligned}
|f_\lambda(z)| &= |z||1 - z - \lambda z^2| \geq |z|(|\lambda||z^2| - |z| - 1) \\
&= |z|\left(|z|(|\lambda||z| - 1) - 1\right) \\
&\geq |z|((2^9|\lambda|^2)^{-1}\left((2^9|\lambda|)^{-1} - 1\right) - 1\big) \\
&\geq |z|(2^9|\lambda|^2)^{-1}\left((2^{10}|\lambda|)^{-1} - 1\right) \\
&\geq |z|(2^9|\lambda|^2)^{-1}(2^{11}|\lambda|)^{-1} \\
&= 2^{-20}|\lambda|^{-3}|z| \geq 2|z|
\end{aligned}
\tag{14}
$$

Combining this with (13), we get by a straight forward induction, for all $\lambda \in B^*(0, R_2)$ that

$$|f_\lambda^{n+1}(c_\lambda^{(2)})| \geq 2^n(2^9|\lambda|^2)^{-1}.$$

We are therefore done. □

Let D_0 be the set of parameters $\lambda \in \mathbb{C} \setminus \{0\}$ defined by

$$D_0 = \{\lambda \in \mathbb{C} \setminus \{0\} : f_\lambda \text{ has no non-zero parabolic or finite attracting periodic points and one finite critical point of } f_\lambda \text{ escapes to } \infty\}$$

and let

$$\mathcal{P}_3^0 = \{f_\lambda : \lambda \in D_0\}.$$

The existence of a positive $R > 0$ in the theorem bellow follows from Theorems 2.20, 2.42 and Lemma 4.4 together with Theorem 2.13 (Fatou-Sullivan Classification of Fatou Components).

THEOREM 4.5. *There exists $R > 0$ such that $B^*(0, R) \subseteq D_0$.*

From Theorems 2.16 and 2.42 we have the following.

COROLLARY 4.6. *For each $\lambda \in D_0$, the polynomial f_λ is a cubic parabolic polynomial.*

The following theorem immediately follows from Theorem 2.18.

THEOREM 4.7. *For each $\lambda \in D_0$ the Julia set $J(f_\lambda)$ is disconnected.*

Define the set of all finite critical points of f_λ for all $\lambda \in D_0$ by

$$\mathrm{Crit}(f_\lambda) = \{c \in \mathbb{C} : f_\lambda^{'}(c) = 0 \quad \forall \lambda \in D_0 \subset \mathbb{C} \setminus \{0\}\}.$$

LEMMA 4.8. *The set of parameters D_0 is open.*

Proof. To prove the Lemma, first we will prove that $\forall$ compact set $K \subset \mathbb{C}$, $\exists$ a neighborhood $\widehat{U}$ of ∞ such that $\forall\ \lambda \in K, f_\lambda(\widehat{U}) \subseteq \widehat{U}$.

Consider the function $h(z) = \frac{1}{z}$ and define $g_\lambda = h^{-1} \circ f_\lambda \circ h$. Then

$$g_\lambda(z) = \frac{1}{f_\lambda(\frac{1}{z})} = \frac{z^3}{z^2 - z - \lambda}$$

and

$$g_\lambda^{'} = \frac{z^4 - 2z^3 - 3\lambda z^2}{(z^2 - z - \lambda)^2}.$$

We have $g_\lambda(0) = 0$ and $g_\lambda^{'}(0) = 0$. For some small $0 < \epsilon < 1$, define $\mathcal{U} = \{(\lambda, z) \in K \times \widehat{\mathbb{C}} : |g_\lambda^{'}(z)| < 1 - \epsilon\}$. Then $\mathcal{U} \neq \emptyset$ since $K \times \{0\} \subseteq \mathcal{U}$, that is $\mathcal{U}$ is a non-empty open set. There exist $U, V \subset \mathbb{C}$ such that $K \subset U$, $\{0\} \subset V$ with $U \times V \subset \mathcal{U}$. Since V is an open set containing 0, without loss of generality we may assume that $V = B(0, \delta)$ for some $\delta > 0$. By Mean Value Inequality, for all $z \in V = B(0, \delta)$

$$|g_\lambda(z) - g_\lambda(0)| \leq (1 - \epsilon)|z - 0| \Rightarrow |g_\lambda(z)| < (1 - \epsilon)\delta < \delta$$

and so we have $g_\lambda(V) \subseteq V$ for all $\lambda \in K$.

Now h is an open map, implies $h(V)$ is open and $h(0) = \infty \in h(V)$. Denote $\infty \in h(V) = \widehat{U}$, an open neighborhood of ∞. Since $g_\lambda = h^{-1} \circ f_\lambda \circ h$, for all $\lambda \in K$ we have

$$f_\lambda(\widehat{U}) = f_\lambda(h(V)) = h(g_\lambda(V)) \subseteq h(V) = \widehat{U}.$$

Now we will prove the Lemma 4.8.

Since $\lambda = -\frac{1}{3} \Rightarrow c_{-\frac{1}{3}} = 1$ is a double critical point and $\lim_{n\to\infty} f_{-\frac{1}{3}}^n(1) = 0$, so $\lambda = -\frac{1}{3} \notin D_0$. Fix $\lambda_0 \in D_0$. Consider the compact set $K = \bar{B}(\lambda_0, 1)$. Then there exists a neighborhood $\widehat{U}$ of ∞ such that

$$\forall\ \lambda \in K,\ f_\lambda(\widehat{U}) \subseteq \widehat{U}.$$

Since $\lambda_0 \in D_0$, there exists $c_{\lambda_0} \in \mathrm{Crit}(f_\lambda)$ and a positive integer $N = N(\lambda_0)$ such that

$$f^n_{\lambda_0}(c_{\lambda_0}) \in \widehat{U} \quad \forall\, n \geq N(\lambda_0).$$

Take a small neighborhood of λ_0, say $B(\lambda_0, r)$, where r is small enough so that $-\frac{1}{3} \notin B(\lambda_0, r) \subseteq \bar{B}(\lambda_0, 1)$. Define

$$X = \{(\lambda, c) : \lambda \in B(\lambda_0, r) \text{ and } c \in \mathrm{Crit}(f_\lambda)\},$$

and define a function

$$\rho : X \to B(\lambda_0, r) \quad \text{by } \rho((\lambda, c)) = \lambda.$$

Then ρ is analytic and since the gradient of the function ρ is $\nabla\rho = 1$ at any point $(\lambda, c) \in X$, ρ does not have any critical point and hence there is no critical value of ρ in $B(\lambda_0, r)$. Then by Inverse Function Theorem, there exist a neighborhood N of (λ_0, c) and a neighborhood M of $\lambda_0 = \rho((\lambda_0, c))$ such that $\rho : N \to M$ is a bijection and $\rho^{-1} : M \to N$ is analytic. We can choose $r > 0$ small enough so that $M = B(\lambda_0, r)$. Then $\rho^{-1} : B(\lambda_0, r) \to X$, $\lambda \mapsto (\lambda, c)$, is analytic. Thus there exists a map $\rho^{-1} : B(\lambda_0, r) \to N$ given by $\lambda \mapsto c_\lambda \in \mathrm{Crit}(f_\lambda)$ which is analytic.

Define

$$\phi : B(\lambda_0, r) \to \widehat{U} \text{ by } \phi(\lambda) = f_\lambda^{N(\lambda_0)}(c_\lambda) \in \widehat{U}.$$

Then ϕ is continuous and $\phi(\lambda_0) \in \widehat{U}$, which imply $\phi^{-1}(\widehat{U})$ is open and $\lambda_0 \in \phi^{-1}(\widehat{U}) \subseteq K$. It is enough to show that $\phi^{-1}(\widehat{U}) \subseteq D_0$.

Let $\gamma \in \phi^{-1}(\widehat{U})$ be arbitrary. Then $\phi(\gamma) \in \widehat{U}$ implies

$$f_\gamma^{N(\lambda_0)}(c_\gamma) \in \widehat{U} \Rightarrow f^n_\gamma(f_\gamma^{N(\lambda_0)}(c_\gamma)) \in f^n_\gamma(\widehat{U}) \subseteq \widehat{U}.$$

Since $\infty \in h(B(0, \delta)) = \widehat{U}$ and h is a Möbius transformation, we have $\widehat{U} = B(\infty, \frac{1}{\delta})$. Now $f_\lambda : B(\infty, \frac{1}{\delta}) \to B(\infty, \frac{1}{\delta})$ is an analytic function with $f_\lambda(\infty) = \infty$ and $f_\lambda(\phi(\lambda)) \in B(\infty, \frac{1}{\delta})$ for all $\lambda \in \phi^{-1}(B(\infty, \frac{1}{\delta}))$, then by Schwarz Lemma

$$\lim_{n\to\infty} f^n_\gamma(\widehat{U}) = \lim_{n\to\infty} f^n_\gamma(B(\infty, \frac{1}{\delta})) = \infty,$$

hence $\gamma \in D_0$ and so D_0 is open. $\square$

Since f_λ does not have any finite attracting or any non-zero parabolic periodic point for each $\lambda \in D_0$ and the Julia set $J(f_\lambda)$ does not contain any critical point, so by Theorems 2.13, 2.14 and Corollary 2.15 we get

COROLLARY 4.9. *The only non-zero finite fixed point of f_λ, $-\frac{1}{\lambda}$, is a repelling fixed point with multiplier $|1 - \frac{1}{\lambda}| > 1$.*

By Corollary 4.9, since each $\lambda \in D_0$ safisfies $|1 - \frac{1}{\lambda}| > 1$, we see that D_0 is contained in some left half-plane.

COROLLARY 4.10. *The set $D_0 \subsetneq \{\lambda = a + ib \in \mathbb{C} \setminus \{0\} : a < \frac{1}{2} \quad \text{and} \quad b \in \mathbb{R}\}$.*

The main result of this paper is the real analyticity of the Hausdorff dimension function $D_0 \to (\frac{1}{2}, 2) : \lambda \mapsto \mathrm{HD}(J(f_\lambda))$. The proof is given in Chapter (7). The theory of parabolic and hyperbolic graph directed Markov systems with infinite number of edges has been used in the proof.

OBSERVATION 4.11. *Consider the family of cubic polynomials $\{f_{a,b}\}_{a,b \in \mathbb{C} \setminus \{0\}}$, where*

$$f_{a,b}(z) = z(1 - az - bz^2).$$

We can conjugate $f_{a,b}$ to $g_{a,b}$ via the Möbius transformation $h(z) = \frac{1}{a}z$, where

$$g_{a,b}(z) = z(1 - z - \frac{b}{a^2}z^2).$$

If $f_{a,b}$ is parabolic, $g_{a,b}$ has a parabolic fixed point at 0 with multiplicity 2 and $-\frac{a^2}{b}$ is a repelling fixed point with multiplier $1 - \frac{a^2}{b}$. Define the function

$$\kappa : \mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) \to \mathbb{C} \setminus \{0\}$$

$$\kappa((a, b)) = \frac{b}{a^2}.$$

Also define $\widetilde{D}_0 = \kappa^{-1}(D_0)$. Then the Hausdorff dimension function $\widetilde{D}_0 \ni (a, b) \mapsto HD(J(f_{a,b}))$ is real analytic.

CHAPTER 5

GRAPH DIRECTED MARKOV SYSTEMS

The theory of parabolic and hyperbolic graph directed Markov systems with infinite number of edges is used in the proof of real analyticity of the Hausdorff dimension function. This chapter is a review of the relavent definitions and theorems of Graph Directed Markov Systems (GDMS). The first section introduces the basic ideas of graph directed Markov systems. Definitions of hyperbolic and parabolic graph directed Markov systems are introduced in sections two and three respectively.

5.1. Introduction

Graph directed Markov systems are based upon a directed multigraph and an associated incidence matrix, (V, E, i, t, A). The multigraph consists of a finite set V of vertices and a countable (either finite or infinite) set of directed edges E and two functions $i, t : E \to V$. For each edge e, $i(e)$ is the initial vertex of the edge e and $t(e)$ is the terminal vertex of e. The edge goes from $t(e)$ to $i(e)$. Also, a function $A : E \times E \to \{0, 1\}$ is given, called an *incidence matrix*. The matrix A is an edge matrix. A determines which edges may follow a given edge. So, the incidence matrix has the property that are given with the property that if $A_{ab} = 1$, then $t(a) = i(b)$. But, in this paper we are considering $A_{ab} = 1$ if and only if $t(a) = i(b)$. We will consider finite and infinite walks through the vertex set consistent with the incidence matrix. Thus, we define the set of infinite A-admissible words

$$E_A^\infty = \{\omega \in E^\infty : A_{\omega_i \omega_{i+1}} = 1 \quad \text{for all} \quad i \leq 1\},$$

by E_A^n we denote the set of all subwords of E_A^∞ of length $n \geq 1$, and by E_A^* we denote the set of all finite subwords of E_A^∞. Given $\omega \in E_A^\infty$ by $|\omega|$ we denote the length of the word ω, i.e. the unique n such that $\omega \in E_A^n$. If $\omega \in E_A^\infty$ and $n \geq 1$, then

$$\omega|_n = \omega_1 \cdots \omega_n.$$

A Graph Directed Markov System (GDMS) consists of a directed multigraph and an incidence matrix together with a finite set of non-empty compact metric spaces $\{X_v\}_{v\in V}$, a number s, $0 < s < 1$, and for every $e \in E$, a 1-to-1 contraction $\phi_e : X_{t(e)} \to X_{i(e)}$ with a Lipschitz constant $\leq s$. That is, the set

$$S = \{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$$

is called a *GDMS*.

5.1.1. *Metric and Shift Map*

Given $\omega, \tau \in E_A^\infty$, we define $\omega \wedge \tau \in E_A^\infty \cup E_A^n$ to be the longest initial block common to both ω and τ. For each $\alpha > 0$, we define a *metric* d_α on E_A^∞ by setting $d_\alpha(\omega, \tau) = e^{-\alpha|\omega\wedge\tau|}$. These metrics are all equivalent and induce the same topology and Borel sets. A function is uniformly continuous with respect to one of these metrics if and only if it is uniformly continuous with respect to all. Also, a function is Hölder continuous with respect to one of these metrics if and only if it is Hölder continuous with respect to all; of course the Hölder order depends on the metric. If no metric is specifically mentioned, we take it to be d_1.

We will consider the *left shift* map $\sigma : E_A^\infty \to E_A^\infty$ defined by dropping the first entry of $\omega \in E_A^\infty$. Sometimes we also consider the shift as defined on words of finite length.

5.1.2. *Limit Set of the System S*

The main object of interest in this paper is the limit set of the system S and objects associated to this set. We now describe the limit set. Assume that $\phi_e(X_{t(e)}) \subseteq X_{i(e)}$ for all $e \in E$. Now for each $\omega \in E_A^*$, say $\omega \in E_A^n$, we consider the map coded by $\omega = \omega_1\omega_2\cdots\omega_n$:

$$\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)},$$

where $t(\omega) = t(\omega_n)$ and $i(\omega) = i(\omega_1)$.

Lemma 5.1. *For all $\omega \in E_A^\infty$, the intersection $\bigcap_{n\geq 0} \phi_{\omega|n}(X_{t(\omega_n)})$ is a singleton.*

proof. For every $e \in E$, $\mathrm{diam}(\phi_e(X_{t(e)})) \leq s \cdot \mathrm{diam}(X_{t(e)})$ for $0 < s < 1$. So for every $\omega \in E_A^\infty$, the sets $\{\phi_{\omega|n}(X_{t(\omega_n)})\}_{n\geq 1}$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n\geq 0} \phi_{\omega_{|n}}(X_{t(\omega_n)}) \neq \emptyset$. Since for every $n \geq 1$,

$$\mathrm{diam}(\phi_{\omega_{|n}}(X_{t(\omega_n)})) \leq s^n \cdot \mathrm{diam}(X_{t(\omega_n)}) \leq s^n \cdot \max\{\mathrm{diam}(X_v) : v \in V\},$$

we conclude that the intersection $\bigcap_{n\geq 0} \phi_{\omega_{|n}}(X_{t(\omega_n)})$ is a singleton. $\square$

Lemma 5.2. *$lim_{n\to\infty} sup_{\omega\in E_A^n}\{diam(\phi_\omega(X_{t(\omega)}))\} = 0$.*

Define the coding map $\pi : E_A^\infty \to X := \bigcup_{v\in V} X_v$ by

$$\pi(\omega) = \bigcap_{n\geq 0} \phi_{\omega_{|n}}(X_{t(\omega_n)}), \quad \omega \in E_A^\infty.$$

The map π is uniformly continuous with respect to the metric d_α for some fixed α. The set

$$J = J_S = \pi(E_A^\infty)$$

is called the limit set of the GDMS S. It satisfies the equation

$$J = \bigcup_{e\in E} \phi_e(J \cap X_{t(e)}).$$

If the set of vertices V is a singleton, then the GDMS is called an *Iterated Function System.*

Definition 5.3. A Graph Directed Markov System S is called *Conformal Graph Directed Markov System (CGDMS)* if the following conditions are satisfied.

(1) For every vertex $v \in V$, X_v is a compact connected subset of a Euclidean space $\mathbb{R}^d$ (the dimension d is common for all $v \in V$) and $X_v = \overline{\mathrm{Int} X_v}$.
(2) (Open Set Condition) (OSC) $\varphi_a(\mathrm{Int}X_{t(a)}) \bigcap \varphi_b(\mathrm{Int}X_{t(b)}) = \emptyset$ for all $a, b \in E$ and $a \neq b$.
(3) For every vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ such that for every $e \in E$ with $t(e) = v$, the map ϕ_e extends to a C^1-conformal diffeomorphism of W_v into $W_{i(e)}$.

(4) (Cone Property) There exists $\gamma, l > 0$, $\gamma \leq \frac{\pi}{2}$, such that for every $v \in V$ and for every $x \in X_v \subset \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}X_v$ with vertex x, central angle of measure γ, and altitude l which may depend on x.

(5) There are two constants $\alpha > 0$ and $L \geq 1$ such that

$$||\varphi_e'(y)| - |\varphi_e'(x)|| \leq L \cdot ||(\varphi_e')^{-1}||^{-1}||y - x||^\alpha$$

for all $e \in E$ and all $x, y \in W_{t(e)}$, where $|\varphi_\omega'(x)|$ means the norm of the derivative.

5.2. Parabolic Graph Directed Markov Systems

In this section we will introduce the definitions of parabolic edges and parabolic graph directed Markov systems. Assume that there exists a non-empty finite subset $\Omega \subset E$ such that $t(e) = i(e)$ for every $e \in \Omega$. Call a word $\omega \in E_A^*$ *hyperbolic* if either $\omega_{|\omega|} \notin \Omega$ or $\omega_{|\omega|-1} \neq \omega_{|\omega|}$ and $\omega_{|\omega|} \in \Omega$.

Definition 5.4. A Graph Directed Markov System S is called *Parabolic (Conformal) Graph Directed Markov System (PGDMS)* if the following conditions are satisfied.

(1) For every vertex $v \in V$, X_v is a compact connected subset of a Euclidean space $\mathbb{R}^d$ (the dimension d is common for all $v \in V$) and $X_v = \overline{\mathrm{Int}X_v}$.

(2) $\varphi_e(X_{t(e)}) \subseteq X_{i(e)}$ for all $e \in E$. this enables us to define for every $\omega \in E_A^*$, say $\omega \in E_A^n$, the map $\varphi_\omega := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)}$. Put also $t(\omega) = t(\omega_n)$ and $i(\omega) = i(\omega_1)$.

(3) (Open Set Condition) (OSC) $\varphi_a(\mathrm{Int}X_{t(a)}) \bigcap \varphi_b(\mathrm{Int}X_{t(b)}) = \emptyset$ for all $a, b \in E$ and $a \neq b$.

(4) (Cone Property) There exists $\gamma, l > 0$, $\gamma \leq \frac{\pi}{2}$, such that for every $v \in V$ and for every $x \in X_v \subset \mathbb{R}^d$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subset \mathrm{Int}X_v$ with vertex x, central angle of measure γ, and altitude l which may depend on x.

(5) If $\omega \in E_A^*$ is a hyperbolic word, then $\varphi_\omega : X_{t(\omega)} \to X_{i(\omega)}$ extends to a C^2-conformal map from $W_{t(\omega)}$ to $W_{i(\omega)}$. This conformal map is defined by the same symbol φ_ω.

(6) There are constants $\alpha > 0$ and $L \geq 1$ such that

$$||\varphi_e'(y)| - |\varphi_e'(x)|| \leq L \cdot ||\varphi_e'||||y - x||^\alpha$$

for all $e \in E$ and all $x, y \in W_{t(e)}$.

(7) (Bounded Distortion Property) There exists $K \geq 1$ such that for every hyperbolic word $\omega \in E_A^*$ and all $x, y \in W_{t(\omega)}$,

$$\frac{|\varphi_\omega'(y)|}{|\varphi_\omega'(x)|} \leq K.$$

(Here and in the sequel for any conformal mapping φ, $|\varphi'(z)|$ denotes the similarity factor (equivalently its norm as a linear map from $\mathbb{R}^d$ into $\mathbb{R}^d$) of the differential $\varphi'(z) : \mathbb{R}^d \to \mathbb{R}^d$. In addition, if $\varphi : W_v \to \mathbb{R}^d$ for some $v \in V$, then

$$||\phi'|| := \sup\{|\phi'(x)| : x \in W_v\}.)$$

(8) $\exists \quad s < 1$ such that for every hyperbolic word $\omega \in E_A^*$, $||\varphi_\omega'|| \leq s$.

(9) For every $e \in \Omega$, $t(e) = i(e)$ and there exists a unique fixed point x_e of the map $\varphi_e : X_{t(e)} \to X_{i(e)}$. In addition, $|\varphi_e'(x_e)| = 1$.

(10) For every $e \in \Omega$,

$$\lim_{n\to\infty} \text{diam}(\varphi_{e^n}(X_{t(e)})) = 0.$$

This implies that

$$\bigcap_{n=0}^{\infty} \varphi_{e^n}(X_{t(e)}) = \{x_e\}.$$

The set Ω is referred to as the set of parabolic edges, the maps φ_e, $e \in \Omega$, are called parabolic maps, and x_e, $e \in \Omega$, are called parabolic fixed points. If $\Omega = \emptyset$, then the system $S = \{\varphi_e : e \in E\}$ is called *hyperbolic or CGDMS.*

We could have in principle provided a somewhat less restrictive definition of a PGDMS allowing finitely many parabolic periodic points (fixed points of φ_ω, $\omega \in E_A^*$) that are not necessarily fixed points, but then passing to a sufficiently large iterate $S^n = \{\varphi_\omega : \omega \in E_A^n\}$ we would end up in a parabolic system as described above. Notice also that our assumptions imply each map $\varphi_\omega : X_{t(\omega)} \to X_{i(\omega)}$ such that $i(\omega) = t(\omega)$ to have a unique fixed point, call it x_ω, and that the diameters $\mathrm{diam}(\varphi_\omega^n(X_{t(\omega)}))$ converge to zero exponentially fast unless $\omega \in \Omega^* = \{e^k : \quad \text{for some} \quad k > 1 \quad \text{and} \quad e \in \Omega\}$.

Definition 5.5. The incidence matrix A is called *finitely irreducible* if there exists a finite set $\Lambda \subseteq E_A^*$ such that for all $\alpha, \beta \in E_A^*$ there exists $\gamma \in \Lambda$ such that $\alpha\gamma\beta \in E_A^*$. The system S is called finitely irreducible if the matrix A is finitely irreducible.

Definition 5.6. The incidence matrix A is called *finitely primitive* if there exists a finite set $\Lambda \subseteq E_A^*$ consists of the words with the same length such that for all $\alpha, \beta \in E_A^*$ there exists $\gamma \in \Lambda$ such that $\alpha\gamma\beta \in E_A^*$. The system S is called finitely primitive.

Definition 5.7. The incidence matrix A is called *properly finitely (pf) irreducible* if there exists a finite set $\Lambda \subseteq E_A^*$ such that for all $\alpha, \beta \in E_A^*$ there exists $\gamma \in \Lambda$ such that $\alpha\gamma\beta \in E_A^*$ and for every two letters $a, c \in E \setminus \Omega$ there exists $\beta \in \Lambda$ with $\beta_1, \beta_{|\beta|} \in E \setminus \Omega$ such that $a\beta c \in E_A^*$.

5.2.1. *Topological Pressure for PGDMS*

Here we will recall the definition of pressure function for a PGDMS. Define the function $\zeta : E_A^\infty \to \mathbb{R}$ by the formula

$$\zeta(\omega) = -\log|\varphi'_{\omega_1}(\pi(\sigma\omega))|.$$

It can be proved in the same way as Proposition 8.2.1 in [15] that the function ζ is acceptable in the sense of [15]. This implies that for every $t \in \mathbb{R}$ the topological pressure $P(\sigma, t\zeta)$ makes a meaningful sense as introduced in [15], and all versions of the variational principle established in [15] hold. One can also define the *topological pressure* without involving symbolic dynamics. Namely, see Lemma 2.1.2 in [15], for all $t \geq 0$

$$P(\sigma, t\zeta) = P(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in E_A^\infty} ||\varphi'_\omega||_\infty^t.$$

Observe that if the set of edges E is infinite and the matrix A contains sufficiently many 1s, for example, if A is finitely irreducible, then $P(0) = +\infty$ and it may happen that $P(t) = +\infty$

for some positive t. It is therefore natural to introduce the parameter

$$\theta = \theta_s := \inf\{t \geq 0 : P(t) < +\infty\}.$$

We call θ the *finiteness parameter* of the system S.

DEFINITION 5.8. Given an exponent $t \geq 0$, a Borel probability measure m on X is said to be t-*conformal* provided that $m(J) = 1$ and the following two conditions are satisfied.

(i) $m\left(\varphi_a(X_{t(a)}) \cap \varphi_b(X_{t(b)})\right) = 0$ for all $a, b \in E$ with $a \neq b$.

(ii) $m(\varphi_e(A)) = \int_A |\varphi_e'|^t dm$ for every Borel set $A \subseteq X_{t(e)}$ and $e \in E$.

It is easy to prove by induction that conditions (i) and (ii) above continue to hold with E replaced by E_A^*. The following theorem ([15]) proves that the image of all shift-invariant measure satisfies a measure theoretic open set condition for a PGDMS. Note that this result holds for a CGDMS (hyperbolic systems) (see [15]).

THEOREM 5.9. *If μ is a shift-invariant Borel probability measure on E_A^∞, then*

$$\mu \circ \pi^{-1}(\phi_\omega(X_{t(\omega)}) \cap \phi_\tau(X_{t(\tau)})) = 0$$

for all incomparable words $\omega, \tau \in E_A^\infty$.

Recall that if μ is a Borel probability measure supported on X, we denote the Hausdorff dimension of μ by $\mathrm{HD}(\mu)$ and defined by the infimum of the Hausdorff dimensions of sets with μ measure 1. Let $\alpha = \{[e] : e \in E\}$ be the partition of E_A^∞ into initial cylinders of length 1. We let $H_\mu(\alpha)$ denote the entropy of the partition α with respect to μ. The following theorem relating the Hausdorff dimension of a measure and the ratio of the entropy to the Lyapunov exponent was proved for PGDMS in ([13]).

THEOREM 5.10. *If μ is a shift-invariant ergodic Borel probability measure on E_A^∞, such that $H_\mu(\alpha) < \infty$, $\chi_\mu(\sigma) = \int \zeta d\mu < \infty$ and either $\chi_\mu(\sigma) > 0$ or $h_\mu(\sigma) > 0$ ($h_\mu(\sigma) > 0$ implies $\chi_\mu(\sigma) > 0$), then*

$$\mathrm{HD}(\mu \circ \pi^{-1}) = \frac{h_\mu(\sigma)}{\chi_\mu(\sigma)}.$$

Assume from now on that the system S is finitely irreducible, i.e. the incidence matrix A is finitely irreducible. Let

$$h = h_s := \mathrm{HD}(J_s) \text{ and let } \beta = \beta_S = \sup\{\mathrm{HD}(\mu \circ \pi^{-1})\},$$

where the supremum is taken over all ergodic invariant probability shift-invariant measures on E_A^∞, and let e be the minimum of all exponents t of all t-conformal measures on J_s. With only minor modifications one can prove in the same way as Theorem 8.3.6 in ([15]), the following version of Bowen's formula.

THEOREM 5.11. *$h = \beta = e =$ the minimal zero of the pressure function $t \mapsto P(t)$.*

In order to get a better appreciation of the right-hand side of this theorem, let us formulate the following proposition describing the shape of the graph of the pressure function. Its proof, up to minor modifications, is the same as the proof of the Proposition 8.2.5 in ([15]).

PROPOSITION 5.12. *The pressure function $P(t)$ has the following properties.*

(1) $P(t) \geq 0$ *for all* $t \geq 0$,

(2) $P(t) > 0$ *for all* $0 \leq t < h$,

(3) $P(t) = +\infty$ *for all* $0 \leq t < \theta$,

(4) $P(t) < +\infty$ *for all* $t > \theta$,

(5) $P(t) = 0$ *for all* $t \geq h$,

(6) $P(t)$ *is non-increasing,*

(7) $P(t)$ *is strictly decreasing on* $[\theta, h]$,

(8) $P(t)$ *is convex and continuous on* (θ, ∞).

REMARK 5.13. It is possible that $h = \beta = \theta$. We will call such systems "strange". Also, although it can happen that $\theta = 0$, we always have $P(0) \geq \log 2$ and therefore $h > 0$.

5.3. Associated Hyperbolic Conformal Graph Directed Markov System

In this section we describe how to associate to our parabolic system a new system which is hyperbolic and we apply its properties to study the original PGDMS (see [13]). Following

([13]) and Section 8.4 from ([15]) we will do it now. So, given a PGDMS S, the corresponding hyperbolic system $\widehat{S}$ is defined as follows.

The set of vertices $\widehat{V} = V$. The set of edges

$$\widehat{E} = \{a^n b : n \geq 1, a \in \Omega, b \neq a, A_{ab} = 1\} \cup (E \setminus \Omega) .$$

The incidence matrix $\widehat{A} : \widehat{E} \times \widehat{E} \to \{0, 1\}$ is naturally defined by requiring that $\widehat{A}_{st} = 1$ if and only if $A_{s_{|s|}t_1} = 1$, where $|s|$ and t_1 are understood here in the sense of the set of edges E. The functions t and i are defined on $\widehat{E}$ by their restrictions to $\widehat{E}$ treated as a subset of E_A^* and by the same procedure the maps $\varphi_e, e \in \widehat{E}$, are defined. Recall that a finitely irreducible parabolic system S is called properly finitely (pf) irreducible if and only if for every two letters $a, c \in E \setminus \Omega$ there exists $\beta \in \Lambda$, Λ resulting from finite irreducibility of S, such that $a\beta c \in E_A^*$ and $\beta_1, \beta_{|\beta|} \in E \setminus \Omega$. Two basic facts about the system

$$\widehat{S} = \{\phi_{a^n b} : n \geq 1, a \in \Omega, b \neq a, A_{ab} = 1\} \cup \{\phi_k : k \in E \setminus \Omega\}$$

that make them useful in study the system S are followings.

Theorem 5.14. *If S is a PGDMS, then $\widehat{S}$ is a (hyperbolic) CGDMS. If S is pf-irreducible, then $\widehat{S}$ is finitely irreducible.*

Proof. The proof that $\widehat{S}$ is a CGDMS is a minor modification of the proof of Theorem 8.4.2 in [15]. So, suppose that S is pf-irreducible and let Λ be the corresponding finite set contained in E_A^*. Shortening the words of Λ if necessary, we may assume without loss of generality that no word of Λ contains a subword of the form $e^2, e \in E$. Call all such words reduced. If a word in E_A^* can be split into blocks such that it becomes a member of $\widehat{E}_{\widehat{A}}^*$, slightly abusing terminology, we say that this word is in $\widehat{E}_{\widehat{A}}^*$. Now notice that any reduced word $\omega \in E_A^*$ with $\omega_{|\omega|} \in E \setminus \Omega$ is in $\widehat{E}_{\widehat{A}}^*$. Notice also that for every reduced word $\gamma \in E_A^*$ at least one of the words γ or $\gamma|_{|\gamma|-1}$ is in $\widehat{E}_{\widehat{A}}^*$. In order to show that $\widehat{S}$ is finitely irreducible, consider arbitrary two elements $\alpha, \beta \in \widehat{E}$. If both $\alpha, \beta \in E \setminus \Omega$, then by pf-irreducibility of S there exists a word $\gamma \in \Lambda$ such that $\alpha\gamma\beta \in E_A^*$ and $\gamma_{|\gamma|} \in E \setminus \Omega$. But then, by the first of the above observations $\gamma \in \widehat{E}_{\widehat{A}}^*$, and we are done in this case. So, suppose that $\beta = a^n b$, where

$n \geq 1, a \in \Omega$, and $b \neq a$ and α is any word in $\widehat{E}$. By finite irreducibility of S there exists $\gamma \in \Lambda$ such that $\alpha\gamma a \in E_A^*$. If γ ends with $a^q, q \geq 1$, remove from γ the last block a^q. If $\gamma \in \widehat{E}_{\widehat{A}}^*$, then we are done. Otherwise, $\gamma = \hat{\gamma}c$, where $\hat{\gamma} \in \widehat{E}_{\widehat{A}}^*$ and $c \in \Omega \setminus \{a\}$. But then $\alpha\hat{\gamma}(ca)a^n b \in \widehat{E}_{\widehat{A}}^*$ and $\hat{\gamma}(ca) \in \widehat{E}_{\widehat{A}}^*$. So, we are also done in this case. In order to end the proof notice that all the words in $\widehat{E}_{\widehat{A}}^*$ we have constructed above to join all α and β in $\widehat{E}$ led from Λ to a finite set, say $\hat{\Lambda}$. □

Let $J_{\widehat{S}}$ be the limit set generated by the system $\widehat{S}$. It is obvious that $J_{\widehat{S}} \subset J_S$. The only infinite words generated by S but not generated by $\widehat{S}$ are of the form ωe^∞, where ω is a finite word and e is a parabolic edge. Since E is a countable set, the set

$$\{\omega e^\infty \in E_A^\infty : |\omega| < \infty \text{ and } e \in \Omega\}$$

is countable. So, we have the following.

Theorem 5.15. *The limit sets J_S and $J_{\widehat{S}}$ of the systems S and $\widehat{S}$ respectively differ only by a countable set and hence have the same Hausdorff dimension. In fact, $J_{\widehat{S}} \subset J_S$ and $J_S \setminus J_{\widehat{S}} \subseteq \pi_S\left(\{\omega e^\infty \in E_A^\infty : |\omega| < \infty \text{ and } e \in \Omega\}\right)$.*

Definition 5.16. A parabolic system S is called *finite* if and only if the set of edges E is finite and it is called *holomorphic* if $d = 2$ and all maps φ_e, $e \in E$, are holomorphic, and $\varphi_e'(x_e) = 1$ for all $e \in \Omega$.

Definition 5.17. A CGDMS is said to be *regular* if there is some $t \geq 0$ such that $P(t) = 0$ and the system is said to be *strongly regular* if there exists $t \geq 0$ such that $0 < P(t) < \infty$.

Definition 5.18. A family $\{\phi_e\}_{e\in F}$ is said to be a *cofinite subsystem* of a system $\{\phi_e\}_{e\in E}$ if $F \subset E$ and the difference $E \setminus F$ is finite.

Definition 5.19. A CGDMS $S = \{\phi_e : e \in E\}$ is said to be *cofinitely regular* if each cofinite subsystem $\{\phi_e\}_{e\in F\subset E}$ of S is regular. The system S is hereditarily regular if and only if it is cofinitely regular.

Now consider a holomorphic parabolic system S. Then for every $e \in \Omega$, we have the following power series expansion about x_e. Namely,

$$\varphi_e(z) = z + a_e(z - x_e)^{1+p_e} + \sum_{n=2}^{\infty} a_n(e)(z - x_e)^{n+p_e}, \quad p_e \geq 1. \tag{15}$$

Hence (see [14]),

$$|\varphi'_{e^n}(z)| \asymp n^{-\frac{p_e+1}{p_e}}$$

uniformly on compact subsets of $X_{t(e)} \setminus \{x_e\}$. So, looking at the series,

$$\sum_{n=1}^{\infty} \|\varphi'_{a^n b}\|^t \asymp \sum_{n=1}^{\infty} n^{-\frac{p_e+1}{p_e}t}, \; a \in \Omega, \; b \neq a,$$

we immediately get the following.

Theorem 5.20. *If S is a finite holomorphic parabolic graph directed system, then the associated hyperbolic system $\widehat{S}$ is cofinitely (= hereditarily) regular.*

CHAPTER 6

ANALYTIC FAMILIES OF PGDMS

First we want to recall from ([32]) a result about real analyticity of Hausdorff dimension of limit sets. The key idea is the concept of regularly analytic families of conformal graph directed Markov systems. We also want to weaken the assumptions of Section 4 from ([32]) at some important points. Let $\Lambda \subseteq \mathbb{C}$ and let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a family of CGDMS with the same set of vertices V, the same set of edges E, the same finitely irreducible matrix A, and the same sets $\{W_v\}_{v\in V}$ with all $W_v \subseteq \mathbb{C}$. Unlike ([32]), we do not assume the compact spaces $\{X_v^\lambda\}_{\lambda\in\Lambda}$ to be all equal. Fix $\lambda_0 \in \Lambda$ and for every $\omega \in E_A^\infty$ consider the function $\psi_\omega : \Lambda \to \mathbb{C}$ given by the formula

$$\psi_\omega(\lambda) = \frac{\left(\varphi_{\omega_1}^{\lambda}\right)'(\pi_\lambda(\sigma\omega))}{\left(\varphi_{\omega_1}^{\lambda_0}\right)'(\pi_{\lambda_0}(\sigma\omega))},$$

where $\pi_\lambda := \pi_{S_{(\lambda)}} : E_A^\infty \to J_{S_\lambda}$ is the coding map induced by the CGDMS S_λ.

Definition 6.1. The family $\{S_\lambda\}_{\lambda\in\Lambda}$ is said to be *analytic* if For every $e \in E$ and every $x \in W_{t(e)}$ the function $\Lambda \ni \lambda \mapsto \varphi_e^\lambda(x) \in W_{t(e)} \subseteq \mathbb{C}$ is holomorphic.

Definition 6.2. The family $\{S_\lambda\}_{\lambda\in\Lambda}$ is called *regularly analytic* if it is analytic and satisfy following conditions:

(a) the system $\{S_{\lambda_0}\}$ is strongly regular,

(b) there exists a constant $D > 0$ such that

$$\sup\{|\psi_\omega(\lambda)| : \omega \in E_A^\infty, \lambda \in \Lambda\} \le D.$$

The basic fact resulting from this kind of analyticity is provided by the following.

Lemma 6.3. *If $\{S_\lambda\}_{\lambda\in\Lambda}$ is an analytic family, then the family $\{\Lambda \ni \lambda \mapsto \pi_\lambda(\omega) \in \mathbb{C} : \omega \in E_A^\infty\}$ consists of holomorphic maps and is normal.*

Proof. For every $v \in V$ choose a point $x_v \in \cup_{v\in V} X_v^\lambda$. Since all the maps $\Lambda \times W_{t(e)} \ni (\lambda, z) \mapsto \phi_e^\lambda(z)$, $e \in E$, are holomorphic, all the maps $\Lambda \ni \lambda \mapsto \phi_\omega^\lambda(x_{t(\omega)})$, $\omega \in E_A^*$, are also holomorphic. Since their ranges are all contained in the bounded subset of $\bigcup_{v\in V} W_v$, the family $\{\Lambda \ni \lambda \mapsto \phi_\omega^\lambda(x_{t(\omega)})\}_{\omega\in E_A^*}$ is normal. Therefore, since for every $\omega \in E_A^\infty$, the sequence of functions $\left(\Lambda \ni \lambda \mapsto \phi_{\omega|_n}^\lambda(x_{t(\omega|_n)})\right)_{n=1}^\infty$ converges pointwise to $\pi_\lambda(\omega)$, we conclude that each function $\Lambda \ni \lambda \mapsto \pi_\lambda(\omega)$, is holomorphic. Since the range of all these functions is contained in the bounded subset of $\bigcup_{v\in V} W_v$, the family $\{\Lambda \ni \lambda \mapsto \pi_\lambda(\omega)\}_{\omega\in E_A^\infty}$ is normal. We are done. $\square$

As an immediate consequence of this Lemma (6.3) and Hartog's Theorem, we get the following.

LEMMA 6.4. *If $\{S_\lambda\}_{\lambda\in\Lambda}$ is a analytic family, then for every $\omega \in E_A^\infty$ the map $\Lambda \ni \lambda \mapsto (\varphi_{\omega_1}^\lambda)'(\pi_\lambda(\omega)) \in \mathbb{C}$, is holomorphic.*

Combining Lemma 6.4 and Lemma 6.3 we conclude that for every $\omega \in E_A^\infty$, the map $\Lambda \ni \lambda \mapsto \psi_\omega(\lambda) \in \mathbb{C}$, is holomorphic. We shall prove the following.

LEMMA 6.5. *Suppose $\{S_\lambda\}_{\lambda\in\Lambda}$ is a regular analytic family of holomorphic systems. Then for every $\omega \in E_A^\infty$ there is a well-defined $\log\psi_\omega : B(\lambda_0, R) \to \mathbb{C}$, the unique holomorphic branch of logarithm of ψ_ω such that $\log\psi_\omega(\lambda_0) = 0$. In addition, the family of functions $\{\log\psi_\omega\}_{\omega\in E_A^\infty}$ is bounded.*

Proof: Indeed, fix $R_2 > 0$ such that $B(\lambda_0, R_2) \subseteq \Lambda$. Fix $\omega \in E_A^\infty$. Since for all $\lambda \in B\left(\lambda_0, \frac{R_2}{2}\right)$ and all $0 < r \le \frac{R_2}{2}$, we have

$$\psi_\omega'(\lambda) = \frac{1}{2\pi i}\int_{\partial B(\lambda_0,r)} \frac{\psi_\omega(\gamma)}{(\gamma-\lambda)^2} d\gamma,$$

we thus obtain from (c) the following.

$$|\psi_\omega'(\lambda)| \le \frac{1}{2\pi}\int_{\partial B(\lambda_0,r)} \frac{D}{r^2}|d\gamma| = \frac{D}{r}.$$

Since $\psi_\omega(\lambda_0) = 1$, we therefore get for all $\lambda \in B(\lambda_0, r)$ that

$$|\psi_\omega(\lambda) - 1| = |\psi_\omega(\lambda) - \psi_\omega(\lambda_0)| = \left| \int_{\lambda_0}^{\lambda} \psi'_\omega(\gamma) d\gamma \right| \le \int_{\lambda_0}^{\lambda} |\psi'_\omega(\gamma)||d\gamma| \le \frac{D}{r}|\lambda - \lambda_0|.$$

So, if we take $r = \frac{R_2}{2}$, then for all $\lambda \in B(\lambda_0, R_3)$ with $R_3 = \frac{R_2}{8D}$, we get

$$|\psi_\omega(\lambda) - 1| \le \frac{1}{4}.$$

Hence, for each $\omega \in E_A^\infty$ there is a well-defined $\log \psi_\omega : B(\lambda_0, R_3) \to \mathbb{C}$, the unique holomorphic branch of the logarithm of ψ_ω such that $\log \psi_\omega(\lambda_0) = 0$, and the family of functions $\{\log \psi_\omega\}_{\omega \in \hat{E}_{\hat{A}}^\infty}$ is bounded. The proof of Lemma 6.5 is complete. □

Setting $\kappa(\omega_1) = 1$ in the proof of Theorem 4.2 in ([32]) and having Lemma 6.3, Lemma 6.4 and Lemma 6.5, the proof of Theorem 4.2 in ([32]) goes verbatim to result in the following.

THEOREM 6.6. *If $\{S_\lambda\}_{\lambda \in \Lambda}$ is a regular analytic family of holomorphic conformal graph directed Markov systems, then the function $\Lambda \ni \lambda \mapsto \mathrm{HD}(J(S_\lambda)) \in \mathbb{R}$, is real-analytic.*

DEFINITION 6.7. An analytic family $\{S_\lambda\}_{\lambda \in \Lambda}$ of holomorphic CGDMS is called *locally regularly analytic* if for every $\lambda_0 \in \Lambda$ there is $R_0 > 0$ such that the family $\{S_\lambda\}_{\lambda \in B(\lambda_0, R_0)}$ is regularly analytic.

As an immediate consequence of the Theorem 6.6, we obtain the following.

THEOREM 6.8. *If $\{S_\lambda\}_{\lambda \in \Lambda}$ is a locally regularly analytic family of holomorphic CGDMS, then the function $\Lambda \ni \lambda \mapsto \mathrm{HD}(J(S_\lambda))$, is real-analytic.*

Suppose E, V, A, Ω and $W_v \subset \mathbb{C}, v \in V$ are given so that all the requirements imposed on them by the definition of PGDMS are met. We assume in addition that A is pf-irreducible and that the set of edges E is finite. Suppose Λ is an open connected subset of $\mathbb{C}$.

DEFINITION 6.9. A family $\{S_\lambda\}_{\lambda \in \Lambda}$ of holomorphic PGDMS, each of which is built with the help of the above block $E, V, A, \Omega, \{W_v\}_{v \in V}$, is called *holomorphic* if and only if

(a) the functions $\Lambda \ni \lambda \mapsto x_e^\lambda$, $e \in \Omega$, are constants for all $\lambda \in \Lambda$; call their common values by $x_e, e \in \Omega$,

(b) the family $\{\widehat{S}_\lambda\}_{\lambda\in\Lambda}$ is analytic,

(c) for every $e \in \Omega$ there exists $R_e > 0$ such that $B(x_e, R_e) \subseteq W_{t(e)}$, and the map $\Lambda \times B(x_e, R_e) \ni (\lambda, z) \mapsto \varphi_e^\lambda(z)$ is holomorphic,

and

(d) for every $v \in V$ there exists a compact set $Y_v \subseteq W_v$ such that $X_v^\lambda \subseteq Y_v$ for all $\lambda \in \Lambda$.

THEOREM 6.10. *If $\{S_\lambda\}_{\lambda\in\Lambda}$ is a holomorphic family of holomorphic PGDMS, then there exists $l \geq 1$ such that the family $\{\widehat{S}^l_\lambda\}_{\lambda\in\Lambda}$ is locally regularly analytic.*

Proof. In virtue of Proposition 9.3.9 from ([15]), there exists $l \geq 1$ such that $\left(\varphi_\lambda^l\right)'(x_e) = 1$ for every $e \in \Omega$. This is the integer l claimed in our theorem. For the ease of exposition we replace S_λ by S_λ^l and assume without loss of generality that $l = 1$. The family $\{\widehat{S}_\lambda\}_{\lambda\in\Lambda}$ is analytic by assumption. Condition (b) of regular analyticity of $\{\widehat{S}_\lambda\}_{\lambda\in\Lambda}$ is satisfied by Theorem 5.20. So, we are only left to verify condition (c) of regular analyticity of the family $\{\widehat{S}_\lambda\}_{\lambda\in\Lambda}$. Towards this end a detailed analysis of parabolic maps φ_a^λ, $a \in \Omega, \lambda \in \Lambda$, is needed. If we dealt with a one single parabolic system the analysis done in [14] (comp. Section 9.3 in [15]) would suffice. But we want the big $\mathcal{O}$ constant in (9.4) in [15] to be independent of λ lying in a sufficiently small neighborhood of some arbitrarily chosen and then fixed parameter $\lambda_0 \in \Lambda$. So, fix $e \in \Omega$. Then with $R = R_e$, we have that

$$\varphi_e^\lambda(z) = z - a_e^\lambda(z - x_e)^{p+1} + \sum_{n=2}^{\infty} a_n^\lambda(e)(z - x_e)^{n+p}$$

for all $\lambda \in \Lambda$ and all $z \in B(x_e, R)$, where $p = p_e$. It follows from condition (c) that all the functions $\lambda \mapsto a_e^\lambda$ and $a_n^\lambda(e)$, $n \geq 2$, are analytic. Translating and rotating the plane, we may assume without loss of generality that $x_e = 0$ and one of the contracting directions of $\varphi_e^{\lambda_0}$ coincides with $(0, +\infty)$, the positive ray emanating from 0, meaning that $a_e^{\lambda_0} \in \mathbb{R}$ and $a_e^{\lambda_0} > 0$. Further on, making a homothetic change of variables, we may assume that

$$a_{\lambda_0} = \frac{1}{p}. \tag{16}$$

Further, rotating the plane again, we may of course assume that the contracting direction associating with $X^{\lambda_0}_{t(e)}$ coincides with $(0, +\infty)$. Since in the rest of this proof all iterates involving parabolic maps are of the form $\phi^\lambda_{a^n b}$, where $a \in \Omega$, and $b \neq a$ (b can be an empty word), enlarging the sets $X^{\lambda_0}_{t(e)}$, $Y_{t(e)}$, and $W^{\lambda_0}_{t(e)}$ appropriately, we may further assume without loss of generality that some initial segment of $[0, +\infty)$ is contained in $X^{\lambda_0}_{t(e)} \subseteq Y_{t(e)} \subseteq W^{\lambda_0}_{t(e)}$. We also skip for simplicity the dependence on e. The power series expansion above takes then the following form

$$\varphi_\lambda(z) = z - a_\lambda z^{p+1} + \sum_{n=2}^{\infty} a_n(\lambda) z^{n+p}, \quad z \in B(0, R),$$

where $a_\lambda := a^\lambda_e, \varphi_\lambda := \varphi^\lambda_e$.

Now, let $\sqrt[p]{z}$ be the holomorphic branch of the p-th radical defined on $\mathbb{C} \setminus (-\infty, 0]$ and sending 1 to 1. Define then $H : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ by the formula

$$H(z) = \frac{1}{\sqrt[p]{z}},$$

and consider the conjugate maps

$$\tilde{\varphi}_\lambda = H^{-1} \circ \varphi_\lambda \circ H : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C},$$

where $H^{-1}(w) = \frac{1}{w^p}$; in fact $\tilde{\varphi}_\lambda$ is defined on $U = H^{-1}(B(0, R)) \setminus (-\infty, 0]$. For all $z \in U$ we have

$$\begin{aligned}
\tilde{\varphi}_\lambda(z) &= H^{-1}\left(\varphi_\lambda(H(z))\right) = H^{-1}\left(H(z) - a_\lambda H(z)^{p+1} + \sum_{n=2}^{\infty} a_n(\lambda) H(z)^{n+p}\right) \\
&= H^{-1}\left(\frac{1}{\sqrt[p]{z}} - a_\lambda z^{-\frac{p+1}{p}} + \sum_{n=2}^{\infty} a_n(\lambda) z^{-\frac{p+n}{p}}\right) \\
&= H^{-1}\left(\frac{1}{\sqrt[p]{z}}\left(1 - a_\lambda z^{-1} + \sum_{n=2}^{\infty} a_n(\lambda) z^{-\frac{p+n-1}{p}}\right)\right) \\
&= \frac{z}{\left(1 - a_\lambda z^{-1} + \sum_{n=2}^{\infty} a_n(\lambda) z^{-\frac{p+n-1}{p}}\right)^p}
\end{aligned} \tag{17}$$

Set $w = H(z) = z^{-\frac{1}{p}}$, put

$$g_\lambda(w) = 1 - a_\lambda w^p + \sum_{n=2}^{\infty} a_n(\lambda) w^{p+n-1}$$

and $\hat{g}_\lambda(w) = (g_\lambda(w))^{-p}$. Then $(\lambda, w) \mapsto g_\lambda(w)$ is a holomorphic function of λ and w, and

(18)

$$\hat{g}_\lambda(0) = 1, \ \frac{\partial^k \hat{g}_\lambda(w)}{\partial w^k}|_{(\lambda,0)} = 0 \ \text{ for all } k = 1, 2, \cdots, p-1, \text{ and } \frac{\partial^p \hat{g}_\lambda}{\partial w^p}|_{(\lambda,0)} = (-1)^{p+1} \frac{a_\lambda (2p)!}{2}.$$

Therefore, we have the following power series expansion

$$\hat{g}_\lambda(w) = 1 + b_\lambda w^p + \sum_{n=1}^{\infty} b_n(\lambda) w^{p+n}$$

for $(\lambda, w) \in D_2((\lambda_0, 0); R)$ with some $R > 0$ sufficiently small, where $D_2(a; r) \subseteq \mathbb{C}^2$ is the polydisk centered at a and of radius r. Going back to the variable $z = w^{-p}$, we thus get from (17) that

(19)

$$\begin{aligned}\tilde{\varphi}_\lambda(z) &= z\left(1 + b_\lambda \frac{1}{z} + \frac{1}{z}\sum_{n=1}^{\infty} b_n(\lambda) H(z)^n\right) \\ &= z + b_\lambda + \sum_{n=1}^{\infty} b_n(\lambda) H(z)^n\end{aligned}$$

for all $\lambda \in B(\lambda_0, R)$ and all $z \in U$. Note that because of (16) and (18), $b_{\lambda_0} = (-1)^{p+1}(2p-1)!$. If the number of petals p is odd (which is the case for the function f_λ for each $\lambda \in D_0$ (see chapter 4)), then b_{λ_0} is positive, otherwise b_{λ_0} is negative. We can use the change of coordinate in order for b_{λ_0} to be positive. Without loss of generality, we have $b_{\lambda_0} = (2p-1)!$. Since the series $\sum_{n=1}^{\infty} b_n(\lambda) w^n$ converges absolutely uniformly on compact subsets of $D_2((\lambda_0, 0); R)$, the number

$$M = \sup\left\{\sum_{n=1}^{\infty} |b_n(\lambda)||w|^n : (\lambda, w) \in D_2\left((\lambda_0, 0); \frac{R}{2}\right)\right\}$$

is finite. Hence, for all $\lambda \in B\left(\lambda_0, \frac{R}{2}\right)$ and all

$$z \in U_1 := H^{-1}\left(B\left(0, \frac{R}{8}\min\{1, M^{-1}\}\right)\right) \setminus (-\infty, 0] \subseteq U$$

we get

$$\left|\sum_{n=1}^{\infty} b_n(\lambda)H(z)^n\right| \leq \sum_{n=1}^{\infty} |b_n(\lambda)||H(z)|^n \leq \sum_{n=1}^{\infty} |b_n(\lambda)||z|^{-\frac{n}{p}}$$

$$(20) \qquad = \sum_{n=1}^{\infty} |b_n(\lambda)| \left(\frac{R}{2}\right)^n \left(\left(\frac{R}{2}\right)^p |z|\right)^{-\frac{n}{p}} \leq M \left(\left(\frac{R}{2}\right)^p |z|\right)^{-\frac{1}{p}}$$

$$= \frac{2M}{R}|z|^{-\frac{1}{p}} \leq \frac{1}{4}.$$

Combining this estimate with (19), we get that if $\mathrm{Re}(z) > \left(\frac{8}{R}\max\{1, M\}\right)^p$, then

$$(21) \qquad \mathrm{Re}\left(\tilde{\varphi}_\lambda(z) - (z + b_\lambda)\right) = \mathrm{Re}\left(\sum_{n=1}^{\infty} b_n(\lambda)H(z)^n\right) \geq -\left|\sum_{n=1}^{\infty} b_n(\lambda)H(z)^n\right| \geq -\frac{1}{4}.$$

Since $b_{\lambda_0} = (2p-1)! \in \mathbb{R}$ and $(\lambda, w) \mapsto \hat{g}_\lambda(w)$ is holomorphic, there exists $R_1 \in \left(0, \frac{R}{2}\right)$ so small that $|b_\lambda - b_{\lambda_0}| < \frac{1}{2}$, that is

$$(22) \qquad b_{\lambda_0} - \frac{1}{2} < \mathrm{Re} b_\lambda < b_{\lambda_0} + \frac{1}{2} \quad \text{and} \quad |b_\lambda| \leq b_{\lambda_0} + \frac{1}{2}$$

for all $\lambda \in B(\lambda_0, R_1)$. It then follows from (21) and (22) that

$$\mathrm{Re}\left(\tilde{\varphi}_\lambda(z)\right) \geq \mathrm{Re}\left(z + b_\lambda\right) - \frac{1}{4}$$

$$(23) \qquad = \mathrm{Re}(z) + \mathrm{Re}\left(b_\lambda\right) - \frac{1}{4}$$

$$> \mathrm{Re}(z) + b_{\lambda_0} - \frac{1}{2} - \frac{1}{4}$$

$$= \mathrm{Re}(z) + b_{\lambda_0} - \frac{3}{4}$$

for all $\lambda \in B(\lambda_0, R_1)$ and all

$$z \in U_2 := \left\{z \in \mathbb{C} : \mathrm{Re}(z) > \left(\frac{8}{R}\max\{1, M\right\}\right)^p\} \subseteq U_1.$$

Note that $b_{\lambda_0} - \frac{3}{4} > 0$ since $b_{\lambda_0} \geq 1$. Analogously,

$$(24) \qquad |\tilde{\varphi}_\lambda(z)| \leq |z| + |b_\lambda| + \frac{1}{4} \leq |z| + b_{\lambda_0} + \frac{1}{2} + \frac{1}{4} = |z| + b_{\lambda_0} + \frac{3}{4}.$$

Set

$$K_1 = b_{\lambda_0} - \frac{3}{4} \quad \text{and} \quad K_2 = b_{\lambda_0} + \frac{3}{4}.$$

From (23) we get $\tilde{\varphi}_\lambda(U_2) \subseteq U_2$ for all $\lambda \in B(\lambda_0, R_1)$, and by a straightforward induction of (23) and (24), we get

$$\text{Re}(z) + nK_1 \le \text{Re}(\tilde{\varphi}_\lambda^n(z)) \le |\tilde{\varphi}_\lambda^n(z)| \le |z| + nK_2 \tag{25}$$

for all $\lambda \in B(\lambda_0, R_1)$, all $z \in U_2$ and all $n \ge 0$, where $K_1 < K_2$. It follows from (19) that

$$\tilde{\varphi}_\lambda'(z) = 1 + \sum_{n=1}^{\infty} b_n(\lambda) n H(z)^{n-1} H'(z) = 1 - \frac{1}{p} z^{-1} \sum_{n=1}^{\infty} b_n(\lambda) n H(z)^n. \tag{26}$$

Now notice that there exists a constant $Q \ge 1$ such that $n\left(\frac{R}{4}\right)^n \le Q\left(\frac{R}{2}\right)^n$ for all $n \ge 0$. Proceeding as in (20), we thus get for all $\lambda \in B(\lambda_0, R_2) \subseteq B\left(\lambda_0, \frac{R}{4}\right)$ and all $z \in U_2$ that

$$\begin{aligned} \left|\sum_{n=1}^{\infty} n b_n(\lambda) H(z)^n\right| &\le \sum_{n=1}^{\infty} |b_n(\lambda)| n \left(\frac{R}{4}\right)^n \left(\left(\frac{R}{4}\right)^p |z|\right)^{-\frac{n}{p}} \\ &\le Q \sum_{n=1}^{\infty} |b_n(\lambda)| \left(\left(\frac{R}{4}\right)^p |z|\right)^{-\frac{n}{p}} \le MQ \left(\left(\frac{R}{4}\right)^p |z|\right)^{-\frac{1}{p}} \\ &= 4MQR^{-1}|z|^{-\frac{1}{p}}, \end{aligned} \tag{27}$$

where writing the last inequality ("$\le$") sign we were assuming that $|z| \ge \left(\frac{4}{R}\right)^p$. Assume from now on that in the definition of U_2, the number $\text{Re}(z) > T > \left(\frac{4}{R}\right)^p$ for all $z \in U_2$. Inserting (27) to (26), we get that

$$|\tilde{\varphi}_\lambda'(z) - 1| \le 4MQ(pR)^{-1}|z|^{-\frac{p+1}{p}}. \tag{28}$$

Write $q_\lambda(z) = \tilde{\varphi}_\lambda'(z) - 1$. By the Chain Rule we have,

$$(\tilde{\varphi}_\lambda^n)'(z) = \prod_{j=0}^{n-1} \tilde{\varphi}_\lambda'\left(\tilde{\varphi}_\lambda^j(z)\right) = \prod_{j=0}^{n-1} \left(1 + q_\lambda\left(\tilde{\varphi}_\lambda^j(z)\right)\right). \tag{29}$$

But, with $Q_1 = 4MQ(pR)^{-1}$, combining (28) and (25), we get

$$\begin{aligned} \left|q_\lambda\left(\tilde{\varphi}_\lambda^j(z)\right)\right| &\le Q_1 |\tilde{\varphi}_\lambda^j(z)|^{-\frac{p+1}{p}} \\ &\le Q_1 (\text{Re}(z) + jK_1)^{-\frac{p+1}{p}} \\ &\le Q_1 (T + jK_1)^{-\frac{p+1}{p}}. \end{aligned}$$

Since the series $\sum_{j=0}^{\infty}(T+jK_1)^{-\frac{p+1}{p}}$ converges, taking $T>0$ sufficiently large and looking at (29), we get the following.

Lemma 6.11. *There exists a constant* $Q_2 \geq 1$ *such that*

$$Q_2^{-1} \leq |\left(\tilde{\varphi}_\lambda^n\right)'(z)| \leq Q_2$$

for all $(\lambda, z) \in B(\lambda_0, R_2) \times U_2$ *and all* $n \geq 1$.

Using the Chain Rule and the definition of $\tilde{\varphi}_\lambda$, we obtain

$$\begin{aligned}
|\left(\varphi_\lambda^n\right)'(H(z))| &= |\left(H \circ \tilde{\varphi}_\lambda^n \circ H^{-1}\right)'(H(z))| \\
&= |H'\left(\tilde{\varphi}_\lambda^n(z)\right)| \cdot |\left(\tilde{\varphi}_\lambda^n\right)'(z)| \cdot |\left(H^{-1}\right)'(H(z))| \\
&= \frac{1}{p}|\tilde{\varphi}_\lambda^n(z)|^{-\frac{p+1}{p}} |\left(\tilde{\varphi}_\lambda^n\right)'(z)| \cdot |H'(z)|^{-1} \\
&= |z|^{-\frac{p+1}{p}} |\left(\tilde{\varphi}_\lambda^n\right)'(z)| \cdot |\tilde{\varphi}_\lambda^n(z)|^{-\frac{p+1}{p}}.
\end{aligned}$$

Combining this with Lemma (6.11) and (25) yields

$$Q_2^{-1}|z|^{-\frac{p+1}{p}}\left(|z|+nK_2\right)^{-\frac{p+1}{p}} \leq |\left(\varphi_\lambda^n\right)'(H(z))| \leq Q_2|z|^{-\frac{p+1}{p}}\left(\mathrm{Re}(z)+nK_1\right)^{-\frac{p+1}{p}} \tag{30}$$

for all $(\lambda, z) \in B(\lambda_0, R_2) \times U_2$ and all $n \geq 0$.

Now, for every $\alpha \in (0, \pi)$ let

$$S_\alpha = \{z \in \mathbb{C} \setminus \{0\} : |\mathrm{Arg}(z)| < \alpha\}.$$

Then for every $\alpha \in \left(0, \frac{\pi}{p}\right)$, we have $H^{-1}(S_\alpha) = S_{\alpha p}$ and $H^{-1}(0) = \infty$. Now, fix $\alpha \in \left(0, \frac{\pi}{2p}\right)$. Since $0 < \alpha p < \frac{\pi}{2}$, we conclude from the above that there exists $r_2 > 0$ so small that $\mathrm{Re}(H^{-1}(z)) \geq T$ for all $z \in S_\alpha \cap B(0, r_2)$. We therefore get from (30) the following.

Lemma 6.12. *For every compact set* $\Gamma \subseteq S_\alpha \cap B(0, r_2)$*, where* $\alpha \in \left(0, \frac{\pi}{2p}\right)$*, there exists a constant* $Q_\Gamma \geq 1$ *such that*

$$Q_\Gamma^{-1} n^{-\frac{p+1}{p}} \leq |\left(\varphi_\lambda^n\right)'(z)| \leq Q_\Gamma n^{-\frac{p+1}{p}}$$

for all $\lambda \in B(\lambda_0, R_2)$*, all* $z \in \Gamma$*, and all* $n \geq 1$.

As the essentially last step in the process of verifying the condition (c) of the regular analyticity of the family $\{\widehat{S}_\lambda\}_{\lambda\in\Lambda}$ we prove first the following.

Lemma 6.13. *Suppose that $\{S_\lambda\}_{\lambda\in\Lambda}$ is a holomorphic family of holomorphic PGDMS. Fix $\lambda_0 \in \Lambda$. Then there exist a constant $Q \geq 1$ and radius $\tilde{R} > 0$ such that*

$$Q^{-1}n^{-\frac{p+1}{p}} \leq |\left(\varphi^\lambda_{a^n b}\right)'(z)| \leq Qn^{-\frac{p+1}{p}}$$

for all $a \in \Omega$, all $b \in E\setminus\{a\}$ such that $A_{ab} = 1$, all $\lambda \in B\left(\lambda_0, \tilde{R}\right)$, all $z \in X^\lambda_{t(b)}$, and all $n \geq 1$.

Proof. Since the set E is finite it suffices to produce Q and $\tilde{R}$ for a fixed pair $(a, b) \in \Omega \times (E \setminus \{a\})$ such that $A_{ab} = 1$. Indeed, in virtue of Lemma 9.3.8 and Proposition 9.4.1 from ([15]), there exists $k \geq 1$ so large that $\varphi^{\lambda_0}_{a^n b}\left(X^{\lambda_0}_{t(b)}\right) \subseteq S_{\frac{\alpha}{4}} \cap B\left(x_a, \frac{r_2}{4}\right)$ for all $n \geq k$. By the Bounded Distortion Property, we may farther assume with $k \geq 1$ sufficiently large, and $r_3 \in (0, r_2]$, sufficiently small, that

$$\varphi^{\lambda_0}_{a^{n+k}b}\left(W_{t(b)}\right) \subseteq S_{\frac{\alpha}{3}} \cap B\left(x_a, \frac{r_2}{3}\right) \quad \text{and} \quad \varphi^{\lambda_0}_{a^k b}\left(W_{t(b)}\right) \cap B(x_a, 2r_3) = \emptyset$$

for all $n \geq 1$. It then follows from analyticity of the function

$$\Lambda \times W_{t(b)} \ni (\lambda, z) \mapsto \varphi^\lambda_{a^k b}(z)$$

(since the family $\{\hat{S}_\lambda\}_{\lambda\in\Lambda}$ is analytic) and from the compactness of the set $Y_{t(b)}$, along with condition (d) of analyticity of S_λ, that there exists $\tilde{R} \in (0, R_2)$ so small that

$$\varphi^\lambda_{a^k b}\left(X^\lambda_{t(b)}\right) \subseteq \varphi^\lambda_{a^k b}\left(Y_{t(b)}\right) \subseteq \left(S_{\frac{\alpha}{2}} \cap B\left(x_a, \frac{r_2}{2}\right)\right) \setminus B(x_a, r_3)$$

for all $\lambda \in B\left(\lambda_0, \tilde{R}\right)$. But then

$$\Gamma := \overline{\bigcup_{\lambda\in B\left(\lambda_0,\tilde{R}\right)} \varphi^\lambda_{a^k b}\left(X^\lambda_{t(b)}\right)} \subseteq \overline{S_{\frac{\alpha}{2}} \cap B\left(x_a, \frac{r_2}{2}\right)} \setminus B(x_a, r_3) \subseteq S_\alpha \cap B(x_a, r_2).$$

Since the middle set above is compact, so is Γ. Hence, applying Lemma 6.12, we conclude that

$$Q^{-1}_\Gamma(n+k)^{-\frac{p+1}{p}} \leq \left|\left(\varphi^\lambda_{a^{n+k}b}\right)'(z)\right| \leq Q_\Gamma(n+k)^{-\frac{p+1}{p}} \tag{31}$$

for all $z \in X^\lambda_{t(b)}$, all $n \geq 1$, and all $\lambda \in B\left(\lambda_0, \tilde{R}\right)$. Since, clearly,

$$0 < \inf\left\{\left|\left(\varphi^\lambda_{a^j b}\right)'(z)\right| : 0 \leq j \leq k,\, z \in Y_{t(b)},\, \lambda \in B\left(\lambda_0, \tilde{R}\right)\right\}$$
$$\leq \sup\left\{\left|\left(\varphi^\lambda_{a^j b}\right)'(z)\right| : 0 \leq j \leq k,\, z \in Y_{t(b)},\, \lambda \in B\left(\lambda_0, \tilde{R}\right)\right\} < +\infty,$$

and since

$$\left(\varphi^\lambda_{a^{n+k} b}\right)'(z) = \left(\varphi^\lambda_{a^n}\right)'\left(\varphi^\lambda_{a^k b}(z)\right)\left(\varphi^\lambda_{a^k b}\right)'(z),$$

using the Chain Rule, formula (31) yields the Lemma. □

Since the set E is finite and since for every $\lambda \in \Lambda$ and every $\omega \in \widehat{E}^\infty_{\widehat{A}}$, we have $\pi_\lambda(\sigma(\omega)) \in X^\lambda_{t(\omega_1)}$, Lemma 6.13 yields immediately condition (c) of regular analyticity for the family $\{\widehat{S}_\lambda\}_{\lambda \in B(\lambda_0, \tilde{R})}$. The proof of the Theorem 6.10 is complete. □

Combining Theorem 6.10 with Theorem 6.8 we get the following.

Corollary 6.14. *If $\{S_\lambda\}_{\lambda \in \Lambda}$ is a holomorphic family of holomorphic PGDMS, then the function $\Lambda \ni \lambda \mapsto \mathrm{HD}\left(J_{S_\lambda}\right)$ is real-analytic.*

CHAPTER 7

PGDMS ASSOCIATED WITH F_λ, $\lambda \in D_0$

In this chapter we will apply the machinery developed in the previous Chapters to study the family of polynomials $f_\lambda(z) = z(1 - z - \lambda z^2)$, $\lambda \in D_0$, described in Chapter 4. The idea is to associate to this family a holomorphic family of holomorphic parabolic graph directed Markov systems whose limit sets coincide with the Julia sets of polynomials f_λ up to a countable set. Then to apply Corollary 6.14. Fix $\lambda \in D_0$. Let J_λ be the Julia set of f_λ and let K_λ be the corresponding filled-in Julia set. Let $A_\lambda(\infty)$ be the basin of attraction to ∞ and let G_λ be *Green's function* for $A_\lambda(\infty)$ with the pole at ∞. It has the following properties.

$$G_\lambda\left(f_\lambda(z)\right) = 3G_\lambda(z),\ z \in \mathbb{C}, \tag{32}$$

$$G_\lambda \geq 0,$$

and

$$K_\lambda = G_\lambda^{-1}(0). \tag{33}$$

Let

$$\rho_\lambda = G_\lambda\left(c_\lambda^{(2)}\right) > 0.$$

Fix any $t_\lambda \in (\frac{1}{3}\rho_\lambda, \rho_\lambda)$. The set $G_\lambda^{-1}\left([0, t_\lambda]\right)$ consists of two connected components. Denote by $\widehat{W}_\lambda^0$ the component containing 0 and by $\widehat{W}_\lambda^1$ the other one. It follows from (32) that $\{f_\lambda^n\left(c_\lambda^{(2)}\right) : n \geq 0\} \bigcap \left(\widehat{W}_\lambda^0 \cup \widehat{W}_\lambda^1\right) = \emptyset$, and from (33) that

$$\{f_\lambda^n\left(c_\lambda^{(1)}\right) : n \geq 0\} \subseteq \widehat{W}_\lambda^0.$$

Consequently,

$$\left\{f_\lambda^n\left(c_\lambda^{(1)}\right), f_\lambda^n\left(c_\lambda^{(2)}\right) : n \geq 0\right\} \bigcap \widehat{W}_\lambda^1 = \emptyset.$$

7.1. Construction of Associated PGDMS

Starting a rather lengthy process of the definition of a PGDMS associated to f_λ. Set

$$V = \{1, 2, 3\}.$$

Let $f_{\lambda,0}^{-1}$ be the holomorphic inverse branch of f_λ defined on a sufficiently small neighborhood of 0 and sending 0 back to 0. Let Δ_λ^r be the repelling ray (emanating from 0) of f_λ. It follows from local behavior around parabolic points (see Chapter 6 for example) that there exists a triangle symmetric with respect to Δ_λ^r with one vertex 0 such that the convex hull of the triangle, Triangle $\cup$ Int(Triangle), $T_\lambda^r \subseteq \widehat{W}_\lambda^0$ and $f_{\lambda,0}^{-1}(T_\lambda^r) \subseteq T_\lambda^r$. Let ω_λ be the only point on $\Delta_\lambda^r \cap \partial T_\lambda^r$ different from 0. Then $f_{\lambda,0}^{-1}(\omega_\lambda) \in T_\lambda^r$ and let β_λ be the closed line segment (contained in T_λ^r) with end points ω_λ and $f_{\lambda,0}^{-1}(\omega_\lambda)$ such that β_λ is a transversal of the level sets of the Green's function. Since the diameter of $f_{\lambda,0}^{-n}(\beta_\lambda)$ is of magnitude n^{-2} and $\lim_{n\to\infty} f_{\lambda,0}^{-n}(\omega_\lambda) = 0$, we conclude that

$$\beta_\lambda^\infty := \{0\} \cup \bigcup_{n=0}^{\infty} f_{\lambda,0}^{-n}(\beta_\lambda)$$

is a piecewise smooth (with countably many pieces) closed topological arc with end points 0 and ω_λ. In addition β_λ^∞ is tangent to Δ_λ^r at the point 0. We notice that

$$(34)\quad \beta_\lambda^\infty \subsetneq f_\lambda(\beta_\lambda^\infty) = \{0\} \cup \bigcup_{n=0}^{\infty} f_{\lambda,0}^{-(n-1)}(\beta_\lambda) \quad \text{and} \quad f_{\lambda,0}^{-1}(\beta_\lambda^\infty) = \{0\} \cup \bigcup_{n=0}^{\infty} f_{\lambda,0}^{-(n+1)}(\beta_\lambda) \subseteq \beta_\lambda^\infty.$$

Let $A_\lambda(0)$ be the basin of immediate attraction of f_λ to the rationally indifferent fixed point 0. Like above, let Δ_λ^c be the contracting ray (emanating from 0) of f_λ. Again as above, there exists a triangle symmetric with respect to Δ_λ^c with one vertex 0 such that the convex hull of the triangle $T_\lambda^c \subseteq A_\lambda(0) \cup \{0\}$ and

$$(35)\qquad f_\lambda(T_\lambda^c) \subseteq \mathrm{Int}(T_\lambda^c) \cup \{0\}.$$

Let b_λ be the edge of the triangle T_λ^c not containing 0, i.e. the edge perpendicular to Δ_λ^c. For each $\lambda \in D_0$, T_λ^c can be chosen such that for some integer $k \geq 1$ we have

$$(36)\qquad f_\lambda^k\left(c_\lambda^{(1)}\right) \in b_\lambda \text{ and } \{c_\lambda^{(1)}, f_\lambda\left(c_\lambda^{(1)}\right), \cdots, f_\lambda^{k-1}\left(c_\lambda^{(1)}\right)\} \cap T_\lambda^c = \emptyset.$$

Take now a little open ball B^1_λ centered at 0 which contains T^r_λ (We can choose T^r_λ small enough so that it is contained in B^1_λ) and disjoint from the set $\{c^{(1)}_\lambda, f_\lambda\left(c^{(1)}_\lambda\right), \cdots, f^k_\lambda\left(c^{(1)}_\lambda\right)\}$. Take also an open topological disk $D_\lambda \supseteq B^1_\lambda \cup T^c_\lambda$ which is disjoint from the set

$$\{c^{(1)}_\lambda, f_\lambda\left(c^{(1)}_\lambda\right), \cdots, f^{k-1}_\lambda\left(c^{(1)}_\lambda\right)\}.$$

Then, for every $j = 1, \cdots, k-1$ there exists a unique holomorphic inverse branch

$$f^{-j}_{\lambda,0} : D_\lambda \to \mathbb{C}$$

sending 0 to 0. There also exists a unique holomorphic inverse branch

$$f^{-k}_{\lambda,0} : B^1_\lambda \cup \mathrm{Int}\,(T^c_\lambda) \to \mathbb{C}$$

sending 0 to 0. Note that for all $j = 1, \cdots, k$

$$f_\lambda \circ f^{-j}_{\lambda,0} = f^{-(j-1)}_{\lambda,0} \tag{37}$$

and, by (35), for all $1 \le j \le k-1$,

$$f^{-j}_{\lambda,0}\,(T^c_\lambda) \supseteq f^{-(j-1)}_{\lambda,0}\,(T^c_\lambda)\,. \tag{38}$$

If $j = k$, then

$$f^{-j}_{\lambda,0}\,(\mathrm{Int}\,(T^c_\lambda)) \supseteq f^{-(j-1)}_{\lambda,0}\,(\mathrm{Int}\,(T^c_\lambda))\,. \tag{39}$$

In particular, $f^{-(k-1)}_{\lambda,0}\left(f^k_\lambda\left(c^{(1)}_\lambda\right)\right) = f_\lambda\left(c^{(1)}_\lambda\right)$, and, as $f^{-1}_\lambda\left(f_\lambda\left(c^{(1)}_\lambda\right)\right) \cap A_\lambda(0) = \{c^{(1)}_\lambda\}$, it follows from (36) that $c^{(1)}_\lambda \in \partial f^{-k}_{\lambda,0}\,(\mathrm{Int}\,(T^c_\lambda))$. Note also that $f^{-k}_{\lambda,0} = \tilde{f}^{-1}_{\lambda,0} \circ f^{-(k-1)}_{\lambda,0}$, where $\tilde{f}^{-1}_{\lambda,0}$ is the extension of $f^{-1}_{\lambda,0}$ on $f^{-(k-1)}_{\lambda,0}\,(B^1_\lambda \cup \mathrm{Int}\,(T^c_\lambda))$. But, there also exists a second holomorphic inverse branch $\tilde{f}^{-1}_{\lambda,1}$ of f_λ defined on $f^{-(k-1)}_{\lambda,0}\,(B^1_\lambda \cup \mathrm{Int}\,(T^c_\lambda))$. Put $f^{-k}_{\lambda,1} = \tilde{f}^{-1}_{\lambda,1} \circ f^{-(k-1)}_{\lambda,0}$. As above $c^{(1)}_\lambda \in \partial f^{-k}_{\lambda,1}\,(\mathrm{Int}\,(T^c_\lambda))$. We thus have

$$f^{-k}_{\lambda,0}\,(\mathrm{Int}\,(T^c_\lambda)) \bigcap f^{-k}_{\lambda,1}\,(\mathrm{Int}\,(T^c_\lambda)) = \emptyset \quad \text{and} \quad \overline{f^{-k}_{\lambda,0}\,(\mathrm{Int}\,(T^c_\lambda))} \bigcap \overline{f^{-k}_{\lambda,1}\,(\mathrm{Int}\,(T^c_\lambda))} = \{c^{(1)}_\lambda\}.$$

Put $\alpha_\lambda = f_{\lambda,1}^{-k}(0)$. By continuity of $f_{\lambda,1}^{-k}$ we get that $\alpha_\lambda \in \tilde{f}_{\lambda,1}^{-1}((T_\lambda^c) \subseteq \overline{\tilde{f}_{\lambda,1}^{-1}(\mathrm{Int}\,(T_\lambda^c))} \subseteq \overline{A_\lambda(0)}$, and since $\alpha_\lambda \in J(f_\lambda)$, we obtain that

$$\alpha_\lambda \in \partial A_\lambda(0).$$

In virtue of (35), (36), and (37), $f_\lambda^2\left(c_\lambda^{(1)}\right) \in f_{\lambda,0}^{-(k-1)}(\mathrm{Int}\,(T_\lambda^c))$. Thus there exists a little open disk B_λ centered at $c_\lambda^{(1)}$ such that

$$f_\lambda^2(B_\lambda) \subseteq f_{\lambda,0}^{-(k-1)}(\mathrm{Int}\,(T_\lambda^c)). \tag{40}$$

Set

$$H_\lambda = f_{\lambda,0}^{-k}(\mathrm{Int}\,(T_\lambda^c)) \cup f_{\lambda,1}^{-k}(\mathrm{Int}\,(T_\lambda^c)) \cup B_\lambda \cup f_\lambda(B_\lambda) \cup f_\lambda^2(B_\lambda) \subset A_\lambda(0).$$

We have, by (37), (38), and (39), that

$$\begin{aligned}
& f_\lambda\left(f_{\lambda,0}^{-k}(\mathrm{Int}\,(T_\lambda^c)) \cup f_{\lambda,1}^{-k}(\mathrm{Int}\,(T_\lambda^c))\right) \\
&= f_\lambda\left(f_{\lambda,0}^{-k}(\mathrm{Int}\,(T_\lambda^c))\right) \cup f_\lambda\left(f_{\lambda,1}^{-k}(\mathrm{Int}\,(T_\lambda^c))\right) \\
&= f_{\lambda,0}^{-(k-1)}(\mathrm{Int}\,(T_\lambda^c)) \cup f_{\lambda,0}^{-(k-1)}(\mathrm{Int}\,(T_\lambda^c)) \\
&= f_{\lambda,0}^{-(k-1)}(\mathrm{Int}\,(T_\lambda^c)) \subseteq f_{\lambda,0}^{-k}(\mathrm{Int}\,(T_\lambda^c)) \subset H_\lambda.
\end{aligned}$$

By (40) and (39), we get

$$f_\lambda^2(B_\lambda) \subseteq f_{\lambda,0}^{-(k-1)}(\mathrm{Int}\,(T_\lambda^c)) \ \subseteq f_{\lambda,0}^{-k}(\mathrm{Int}\,(T_\lambda^c)) \subseteq H_\lambda.$$

Thus,

$$f_\lambda(H_\lambda) \subseteq H_\lambda \text{ and } f_\lambda\left(\overline{H}_\lambda\right) \subseteq \overline{H}_\lambda. \tag{41}$$

Let

$$\tilde{\beta}_\lambda^\infty = \tilde{f}_{\lambda,1}^{-1}(f_\lambda(\beta_\lambda^\infty)) \quad \text{and } \tilde{\omega}_\lambda = \tilde{f}_{\lambda,1}^{-1}(f_\lambda(\omega_\lambda)). \tag{42}$$

Then we get the following

$$f_\lambda(\tilde{\beta}_\lambda^\infty) = f_\lambda(\beta_\lambda^\infty) \ \text{ and } f_\lambda(\tilde{\omega}_\lambda) = f_\lambda(\omega_\lambda). \tag{43}$$

Set

$$s_\lambda = G_\lambda(\omega_\lambda) = G_\lambda(\tilde{\omega}_\lambda) < t_\lambda.$$

Define X_1^λ to be the connected component of $G_\lambda^{-1}([0, s_\lambda])$ not containing 0. Then X_1^λ is simply connected and, consequently, $\overline{A_\lambda(0)} \cap X_1^\lambda = \emptyset$. Let Z^λ be the other connected component of $G_\lambda^{-1}([0, s_\lambda])$, i.e. the one containing 0. The set Z^λ is connected and simply connected (closed topological disk with smooth boundary). By its construction the set $\overline{H}_\lambda$ is connected and simply connected too. Since, in addition, $\beta_\lambda^\infty \cap \overline{H}_\lambda = \{0\}$, $\tilde{\beta}_\lambda^\infty \cap \overline{H}_\lambda = \{\alpha_\lambda\}$, $\beta_\lambda^\infty \cap \tilde{\beta}_\lambda^\infty = \emptyset$, and since both β_λ^∞ and $\tilde{\beta}_\lambda^\infty$ are closed arcs, the closed set

$$F_\lambda := \beta_\lambda^\infty \cup \overline{H}_\lambda \cup \tilde{\beta}_\lambda^\infty \subseteq Z^\lambda \tag{44}$$

is connected. We have

$$F_\lambda \cap \partial Z^\lambda = \{\omega_\lambda, \tilde{\omega}_\lambda\}.$$

In consequence, the set $Z^\lambda \setminus F_\lambda$ has two connected components. Label their closures by X_2^λ and X_3^λ. By the construction,

$$X_1^\lambda \cap \overline{\cup_{n=0}^\infty f_\lambda^n\left(\{c_\lambda^{(1)}, c_\lambda^{(2)}\}\right)} = \emptyset, \tag{45}$$

$$X_2^\lambda \cap \overline{\cup_{n=0}^\infty f_\lambda^n\left(\{c_\lambda^{(1)}, c_\lambda^{(2)}\}\right)} = \{0\} \tag{46}$$

and

$$X_3^\lambda \cap \overline{\cup_{n=0}^\infty f_\lambda^n\left(\{c_\lambda^{(1)}, c_\lambda^{(2)}\}\right)} = \{0\} \tag{47}$$

and all these sets X_1^λ, X_2^λ and X_3^λ are simply connected. Hence all the three holomorphic inverse branches of f_λ are well-defined on each set X_1^λ, X_2^λ and X_3^λ. Since the polynomial f_λ is of degree 3, for each $a \in \{1, 2, 3\}$ there are three holomorphic inverse branches $f_{\lambda,(a,1)}^{-1}$, $f_{\lambda,(a,2)}^{-1}$ and $f_{\lambda,(a,3)}^{-1}$ of f_λ defined on X_a^λ. Consider first the case when $a = 1$. Then for each $b \in \{1, 2, 3\}$ the holomorphic inverse branch defined on X_1^λ satisfies

$$f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right) \subseteq G_\lambda^{-1}\left([0, \frac{s_\lambda}{3}]\right) \subseteq G_\lambda^{-1}([0, s_\lambda])$$

and $f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right)$ is a connected component. Since $X_1^\lambda \cap A_\lambda(0) = \emptyset$, there is a $b \in \{1,2,3\}$, say $b = 1$, such that $f_{\lambda,(1,1)}^{-1}\left(X_1^\lambda\right)$ does not contain 0. Since the connected component $f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right) \subseteq G_\lambda^{-1}\left([0,s_\lambda]\right)$ for each $b \in \{1,2,3\}$, we have

$$f_{\lambda,(1,1)}^{-1}\left(X_1^\lambda\right) \subseteq X_1^\lambda. \tag{48}$$

Now for $b \in \{2,3\}$, consider the holomorphic inverse branches $f_{\lambda,(1,b)}^{-1} : X_1^\lambda \to \mathbb{C}$ of f_λ. We then have

$$f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right) \subseteq Z^\lambda. \tag{49}$$

If $f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right) \cap F_\lambda \neq \emptyset$, there exists $z \in f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right) \cap F_\lambda$. Since $X_1^\lambda \cap Z^\lambda = \emptyset$ and $F_\lambda \subseteq Z^\lambda$, looking at (34), (41) and (43), we have $f_\lambda(z) \in f_\lambda(\beta_\lambda) \setminus \{\omega_\lambda\}$, that is $|\omega_\lambda| \geq |z| > |f_{\lambda,0}^{-1}(\omega_\lambda)|$ with $z \in \beta_\lambda$. Since β_λ is a closed line segment with end points ω_λ and $f_{\lambda,0}^{-1}(\omega_\lambda)$ and is a transversal of the level sets of the Green's function, then we have

$$G_\lambda(f_\lambda(z)) = 3G_\lambda(z) > 3G_\lambda\left(f_{\lambda,0}^{-1}(\omega_\lambda)\right) = 3 \cdot \frac{1}{3} G_\lambda(\omega_\lambda) = s_\lambda,$$

which implies $f_\lambda(z) \notin X_\lambda^1 \subseteq G_\lambda^{-1}\left([0,s_\lambda]\right)$ and hence we conclude that $f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right) \cap F_\lambda = \emptyset$. Along with (49) this gives that

$$f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right) \subseteq Z^\lambda \setminus F_\lambda.$$

Since $f_{\lambda,(1,b)}^{-1}\left(X_1^\lambda\right)$ is a connected set it must be contained in one of the two connected components of $Z^\lambda \setminus F_\lambda$, the more in the closure of one of these two components. Set $b = 2$ if this closure is X_2^λ and set $b = 3$ if it is X_3^λ. We thus have

$$f_{\lambda,(1,2)}^{-1}\left(X_1^\lambda\right) \subseteq X_2^\lambda \text{ and } f_{\lambda,(1,3)}^{-1}\left(X_1^\lambda\right) \subseteq X_3^\lambda. \tag{50}$$

Now consider the case when $a \in \{2,3\}$. Without loss of generality we may assume that $a = 2$. Since the map f_λ restricted to $A_\lambda(0)$ is of degree 2, there are two branches of f_λ^{-1} defined on X_2^λ whose images intersect $A_\lambda(0)$. Fix one of them and label it by $f_{\lambda,(2,b)}^{-1}$,

$b \in \{2,3\}$. Since

$$f^{-1}_{\lambda,(2,b)}\left(X^\lambda_2\right) \subseteq f^{-1}_\lambda\left(Z^\lambda\right) \subseteq G^{-1}_\lambda\left([0,\frac{s_\lambda}{3}]\right) \subseteq G^{-1}_\lambda\left([0,s_\lambda]\right),$$

and since $f^{-1}_{\lambda,(2,b)}(X^\lambda_2)$ is a connected set and $f^{-1}_{\lambda,(2,b)}\left(X^\lambda_2\right) \cap A_\lambda(0) \neq \emptyset$, we conclude that

$$f^{-1}_{\lambda,(2,b)}\left(X^\lambda_2\right) \subseteq Z^\lambda. \tag{51}$$

Since $\mathrm{Int}\left(X^\lambda_2\right) \cap F_\lambda = \emptyset$, and $z \in f^{-1}_{\lambda,(2,b)}\left(\mathrm{Int}\left(X^\lambda_2\right)\right) \cap F_\lambda$ implies $f_\lambda(z) \in \mathrm{Int}\left(X^\lambda_2\right) \cap f_\lambda(\beta_l) = \emptyset$, we have that $f^{-1}_{\lambda,(2,b)}\left(\mathrm{Int}\left(X^\lambda_2\right)\right) \cap F_\lambda = \emptyset$. Together with (51) this yields

$$f^{-1}_{\lambda,(2,b)}\left(\mathrm{Int}\left(X^\lambda_2\right)\right) \subseteq Z^\lambda \setminus F_\lambda.$$

The same argument as above then gives that

$$f^{-1}_{\lambda,(2,b)}\left(\mathrm{Int}\left(X^\lambda_2\right)\right) \subseteq X^\lambda_2 \quad \text{or} \quad f^{-1}_{\lambda,(2,b)}\left(\mathrm{Int}\left(X^\lambda_2\right)\right) \subseteq X^\lambda_3.$$

Thus

$$f^{-1}_{\lambda,(2,b)}\left(X^\lambda_2\right) \subseteq X^\lambda_2 \quad \text{or} \quad f^{-1}_{\lambda,(2,b)}\left(X^\lambda_2\right) \subseteq X^\lambda_3.$$

Put b equal to 2 or 3 according to whether the first or the second part of the above alternative holds, i.e.

$$f^{-1}_{\lambda,(2,2)}\left(X^\lambda_2\right) \subseteq X^\lambda_2 \quad \text{and} \quad f^{-1}_{\lambda,(2,3)}\left(X^\lambda_2\right) \subseteq X^\lambda_3. \tag{52}$$

Note that

$$f^{-1}_{\lambda,(2,2)}(0) = 0 \quad \text{and} \quad f^{-1}_{\lambda,(2,3)}(0) = \alpha_\lambda. \tag{53}$$

It is left to consider the branch $f^{-1}_{\lambda,(2,1)} : X^\lambda_2 \to \mathbb{C}$ characterized by the property that

$$f^{-1}_{\lambda,(2,1)}\left(X^\lambda_2\right) \cap A_\lambda(0) = \emptyset.$$

By (51), $f^{-1}_{\lambda,(2,2)}\left(X^\lambda_2\right) \cup f^{-1}_{\lambda,(2,3)}\left(X^\lambda_2\right) \subseteq Z^\lambda$. Therefore, since f_λ is of degree 2 on Z^λ, we have that

$$f^{-1}_{\lambda,(2,1)}\left(X^\lambda_2\right) \cap Z^\lambda = \emptyset. \tag{54}$$

But $f_{\lambda,(2,1)}^{-1}\left(X_2^\lambda\right) \subseteq G_\lambda^{-1}\left([0, s_\lambda]\right)$, and since $f_{\lambda,(2,1)}^{-1}\left(X_2^\lambda\right)$ is connected, we conclude from (54) and from the definition of Z^λ and X_2^λ, that,

$$f_{\lambda,(2,1)}^{-1}\left(X_2^\lambda\right) \subseteq X_1^\lambda. \tag{55}$$

With the same argument as above, we get the followings for $a = 3$.

$$f_{\lambda,(3,1)}^{-1}\left(X_3^\lambda\right) \subseteq X_1^\lambda, \quad f_{\lambda,(3,2)}^{-1}\left(X_3^\lambda\right) \subseteq X_2^\lambda \quad \text{and} \quad f_{\lambda,(3,3)}^{-1}\left(X_3^\lambda\right) \subseteq X_3^\lambda \tag{56}$$

with

$$f_{\lambda,(3,3)}^{-1}(0) = 0 \quad \text{and} \quad f_{\lambda,(3,2)}^{-1}(0) = \alpha_\lambda. \tag{57}$$

Now, we shall define the open simply connected sets W_1^λ, W_2^λ and W_3^λ. Fix $\xi_\lambda \in (s_\lambda, t_\lambda)$ and define W_1^λ to be the connected component of $G_\lambda^{-1}\left([0, \xi_\lambda]\right)$ not containing 0. Clearly W_1^λ is an open topological disk with smooth boundary and

$$X_1^\lambda \subseteq W_1^\lambda \quad \text{and} \quad \overline{W}_1^\lambda \cap \overline{\bigcup_{n=0}^{\infty}\{f_\lambda^n\left(c_\lambda^{(1)}\right), f_\lambda^n\left(c_\lambda^{(2)}\right)\}} = \emptyset. \tag{58}$$

Since, by (52), $f_{\lambda,(2,3)}^{-1}\left(X_2^\lambda\right) \subseteq X_3^\lambda$, and since, by (53), $0 \notin f_{\lambda,(2,3)}^{-1}\left(X_2^\lambda\right)$, it follows from (47) that there exists an open topological disk $U_2^\lambda \supseteq f_{\lambda,(2,3)}^{-1}\left(X_2^\lambda\right)$ whose closure is disjoint from $\overline{\bigcup_{n=0}^{\infty}\{f_\lambda^n\left(c_\lambda^{(1)}\right), f_\lambda^n\left(c_\lambda^{(2)}\right)\}}$. In virtue of (47) there exists an open topological disk $W_2^\lambda \subseteq \mathbb{C}$ with the following properties.

(a) $X_2^\lambda \subseteq W_2^\lambda \subseteq G_\lambda^{-1}\left([0, \xi_\lambda]\right)$,

(b) $f_\lambda\left(\{c_\lambda^{(1)}, c_\lambda^{(2)}\}\right) \cap \overline{W}_2^\lambda = \emptyset$, and if $f_{\lambda,(2,3)}^{-1}$ is a holomorphic extension of $f_{\lambda,(2,3)}^{-1}$ onto W_2^λ (which exists because of (b) and for which we keep the same symbol $f_{\lambda,(2,3)}^{-1}$), then

(c) $\overline{f_{\lambda,(2,3)}^{-1}\left(W_2^\lambda\right)} \subset U_2^\lambda$.

And from (c),

$$\overline{f_{\lambda,(2,3)}^{-1}\left(W_2^\lambda\right)} \cap \overline{\bigcup_{n=0}^{\infty}\{f_\lambda^n\left(c_\lambda^{(1)}\right), f_\lambda^n\left(c_\lambda^{(2)}\right)\}} = \emptyset. \tag{59}$$

The sets U_3^λ and W_3^λ are defined verbatim with 2 and 3 mutually interchanged.

In virtue of (58) and (a) there exists $\delta > 0$ such that

$$B\left(X_i^\lambda, \delta\right) \subseteq W_i^\lambda \tag{60}$$

for all $i = 1, 2, 3$. By (58), the family $\mathcal{F}_1^\lambda$ of all holomorphic inverse branches of all iterates of f_λ is well-defined on an open set containing $\overline{W}_1^\lambda$. Since $J(f_\lambda) \cap W_1^\lambda \neq \emptyset$ this family is normal and all its limit functions are constant. Likewise, the family $\mathcal{F}_2^{\lambda'}$ of all holomorphic inverse branches of all iterates of f_λ is well-defined on U_2^λ. Let $\mathcal{F}_2^\lambda = \{\phi \circ f_{\lambda,(2,3)}^{-1} : \phi \in \mathcal{F}_2^{\lambda'}\}$ and let $\mathcal{F}_3^\lambda$ be defined analogously. Again, since $J(f_\lambda) \cap W_2^\lambda \neq \emptyset$ and $J(f) \cap W_3^\lambda \neq \emptyset$, both families $\mathcal{F}_2^\lambda$ and $\mathcal{F}_3^\lambda$ are normal and all their limit functions are constant. Using (c) we therefore conclude that there exists $q_\lambda = q \geq 1$ such that if $\phi \in \mathcal{F}^\lambda := \mathcal{F}_1^\lambda \cup \mathcal{F}_2^\lambda \cup \mathcal{F}_3^\lambda$ is a holomorphic inverse branch of f_λ^n with $n \geq q$ (we say $\phi \in \mathcal{F}_n^\lambda$), then

$$\operatorname{diam}\left(\phi\left(W_i^\lambda\right)\right) < \delta \quad \text{and} \quad \sup\{|\phi'(z)| : z \in W_i^\lambda\} < \frac{1}{2}, \tag{61}$$

where $i = 1, 2$ or 3 according to whether $\phi \in \mathcal{F}_i^\lambda$. Note that each such element $\phi \in \mathcal{F}_i^\lambda$ forms a unique holomorphic extension of some unique element $f_{\lambda,\omega_1}^{-1} \circ f_{\lambda,\omega_2}^{-1} \circ \cdots \circ f_{\lambda,\omega_n}^{-1}$, where all $\omega_j \in \{1, 2, 3\}^2$. Let S_λ be the system determined by the set of vertices $V = \{1, 2, 3\}$, the set of edges $E_q = (\{1, 2, 3\}^2)^q$, the spaces X_v^λ and W_v^λ, $v \in V$, described above, the maps $t(a_1, b_1, a_2, b_2, \cdots, a_q, b_q) = a_q$, $i(a_1, b_1, a_2, b_2, \cdots, a_q, b_q) = b_1$, the generators $f_{\lambda,\tau}^{-q} : X_{t(\tau)}^\lambda \to X_{i(\tau)}^\lambda$, $\tau = (a_1, b_1, a_2, b_2, \cdots, a_q, b_q) \in E_q$ and $f_{\lambda,\tau}^{-q} = f_{\lambda,(a_1,b_1)}^{-1} \circ f_{\lambda,(a_2,b_2)}^{-1} \circ \cdots \circ f_{\lambda,(a_q,b_q)}^{-1}$, $\Omega = \{(2, 2, \cdots, 2), (3, 3, \cdots, 3)\}$, and the incidence matrix $A : E_q \times E_q \to \{0, 1\}$ consisting of all entries equal to 1. After all these definitions and preparations it is rather easy to prove the following

Proposition 7.1. *For every $\lambda \in D_0$, S_λ is a pf-irreducible PGDMS.*

Proof. Conditions (1) and (4) follow directly from the definition of the sets X_v^λ, $v \in V$. Condition (2) is fulfilled by (48), (50), (52), (55) and (56) and the definition of $f_{\lambda,\tau}^{-q}$, $\tau \in E_q$, given above. Condition (3) follows from the fact that the interiors $\{\operatorname{Int}\left(X_v^\lambda\right)\}_{v \in V}$ are mutually disjoint and that the generators of the system S_λ are formed by continuous inverse branches of a single map, namely f_λ^q. Let us deal with condition (5). If $\omega \in E_q^*$ is a hyperbolic word with $|\omega| = n$, then $\phi_\omega \in \mathcal{F}_n^\lambda$, and it follows from (61) that $\operatorname{diam}\left(\phi_\omega\left(W_{t(\omega)}^\lambda\right)\right) < \delta$. But $\phi_\omega\left(X_{t(\omega)}^\lambda\right) \subseteq X_{i(\omega)}^\lambda$, and using (60), we conclude that

$$\phi_\omega\left(W_{t(\omega)}^\lambda\right) \subseteq B\left(X_{i(\omega)}^\lambda, \delta\right) \subseteq W_{i(\omega)}^\lambda.$$

Condition (5) is established. Conditions (6) and (7) follow directly from Koebe Distortion Theorem. Condition (8) is established by (61). To see that conditions (9) and (10) hold, consider without loss of generality the parabolic map $\phi_{(2,2)^q}$. It is enough to note that the sets $\phi_{(2,2)^q}^n\left(2B \cap X_2^\lambda\right)$ converge to the parabolic point 0, by the local behavior of parabolic points, where B is a sufficiently small ball centered at 0, and that the family of maps $\phi_{(2,2)^q}^n$, restricted to some sufficiently small neighborhood of $X_2^\lambda \setminus B$, is well-defined and normal. In conclusion, S_λ is a parabolic graph directed Markov system. It is obvious that S_λ is a pf-system since the incidence matrix A consists of 1s only and since $E_q \setminus \Omega$ is not empty. We are done. □

Now, we shall prove the following.

Lemma 7.2. *For every $\lambda_0 \in D_0$ there exists $R_0 > 0$ such that with suitably chosen sets X_v^λ, $\lambda \in B(\lambda_0, R_0)$, the family $\{S_\lambda\}_{\lambda \in B(\lambda_0, R_0)}$ of holomorphic PGDMSs is holomorphic.*

Proof. It follows from the construction of systems S_λ and local behavior of maps f_λ around zero, that the only non-trivial task to be done is to verify that the family $\{S_\lambda\}_{\lambda \in D_0}$ satisfies conditions (b), (c), and (d) of the definition of holomorphic families of holomorphic PGDMSs. In order to do it, fix $\lambda_0 \in D_0$. Put $O_{\lambda_0} = G_{\lambda_0}^{-1}((3\rho_{\lambda_0}, +\infty])$. Then $f_{\lambda_0}(O_{\lambda_0}) \subseteq G_{\lambda_0}^{-1}((9\rho_{\lambda_0}, +\infty])$ and taking $R_1 > 0$ sufficiently small, we will have $f_\lambda(O_{\lambda_0}) \subseteq G_{\lambda_0}^{-1}((8\rho_{\lambda_0}, +\infty]) \subseteq O_{\lambda_0}$ for all $\lambda \in B(\lambda_0, R_1)$. Consequently,

$$f_\lambda^{-1}\left(G_{\lambda_0}^{-1}([0, 3\rho_{\lambda_0}])\right) \subseteq G_{\lambda_0}^{-1}([0, 3\rho_{\lambda_0}]). \tag{62}$$

We have by our construction,

$$W_1^{\lambda_0} \cup W_2^{\lambda_0} \cup W_3^{\lambda_0} \subseteq G_{\lambda_0}^{-1}([0, t_{\lambda_0}]) \subseteq G_{\lambda_0}^{-1}([0, \rho_{\lambda_0}]) \subseteq G_{\lambda_0}^{-1}([0, 3\rho_{\lambda_0}]). \tag{63}$$

There also exists $R_2 \in (0, R_1]$ so small that

$$\left(U_2^{\lambda_0} \cup U_3^{\lambda_0}\right) \cap \bigcup_{\lambda \in B(\lambda_0, R_2)} \bigcup_{n=0}^{\infty} \{f_\lambda^n\left(c_\lambda^{(1)}\right), f_\lambda^n\left(c_\lambda^{(2)}\right)\} = \emptyset.$$

From continuity of the functions $D_0 \ni \lambda \mapsto c_\lambda^{(1)}, c_\lambda^{(2)}$, and consequently, of the function, $\lambda \mapsto \rho_\lambda$, we can choose the numbers $s_\lambda < \xi_\lambda$ such that

$$\lim_{\lambda \to \lambda_0} s_\lambda = s_{\lambda_0} < \xi_{\lambda_0} = \lim_{\lambda \to \lambda_0} \xi_\lambda. \tag{64}$$

We can also choose ω_λ so that $\lim_{\lambda \to \lambda_0} \omega_\lambda = \omega_{\lambda_0}$, $\lim_{\lambda \to \lambda_0} T_\lambda^c = T_{\lambda_0}^c$, $\lim_{\lambda \to \lambda_0} \overline{B}_\lambda = \overline{B}_{\lambda_0}$ (the two latter in the sense of Hausdorff metric on compact subsets of the complex plane $\mathbb{C}$). Consequently, also $\lim_{\lambda \to \lambda_0} \beta_\lambda^\infty = \beta_{\lambda_0}^\infty$, $\lim_{\lambda \to \lambda_0} \tilde{\beta}_\lambda^\infty = \tilde{\beta}_{\lambda_0}^\infty$, and $\lim_{\lambda \to \lambda_0} \overline{H}_\lambda = \overline{H}_{\lambda_0}$. Therefore (see (44)

$$\lim_{\lambda \to \lambda_0} F_\lambda = F_{\lambda_0}. \tag{65}$$

It follows immediately from (64) that

$$\lim_{\lambda \to \lambda_0} X_1^\lambda = X_1^{\lambda_0} \text{ and } \lim_{\lambda \to \lambda_0} Z^\lambda = Z^{\lambda_0}$$

Along with (65) this implies that

$$\lim_{\lambda \to \lambda_0} X_i^\lambda = X_i^{\lambda_0} \text{ for } i = 1, 2, 3.$$

Because of this and (64) we can find for every $i = 1, 2, 3$ one open set $U_i^{\lambda_0}$ and open sets $W_i^{\lambda_0} \subseteq \tilde{W}_i^{\lambda_0}$ that all satisfy all the requirements for the sets U_i^λ and W_i^λ from the construction leading to Proposition 7.1 up to formula (60) if λ is sufficiently close to λ_0, say $\lambda \in B(\lambda_0, R_3)$, $R_3 \in (0, R_2]$. That is, we can from now on either set

$$W_i^\lambda := W_i^{\lambda_0} \quad \text{or} \quad W_i^\lambda := \widetilde{W}_i^{\lambda_0}$$

for all $i = 1, 2, 3$ and for all $\lambda \in B(\lambda_0, R_3)$. We can even find compact sets Y_1, Y_2 and Y_3 such that $X_i^\lambda \subseteq Y_i \subseteq W_i^{\lambda_0}$ for all $i = 1, 2, 3$ and all $\lambda \in B(\lambda_0, R_3)$. Hence, the condition (d) of the definition of holomorphic families of holomorphic PGDMS is satisfied. Recall that for every $\lambda \in B(\lambda_0, R_3)$ the family $\mathcal{F}^\lambda$ is bijectively parametrized by the set $\widehat{E}^*$, where $E = \{1, 2, 3\}^2$,

$\Omega = \{(2,2),(3,3)\}$, and $\widehat{E}$ is defined accordingly. In fact, in view of our construction of the sets $W_i^{\lambda_0}$, it follows from the Implicit Function Theorem and Monodromy Theorem, that for every $\omega \in \widehat{E}^*$, there exists a holomorphic function $g_\omega : B(\lambda_0, R_3) \times \widetilde{W}_{t(\omega)}^{\lambda_0} \to \mathbb{C}$ such that, abusing slightly notation, we have

$$\left\{ g_\omega\big|_{\{\lambda\}\times\widetilde{W}_i^{\lambda_0}} : \omega \in \widehat{E}^* \text{ and } t(\omega) = i \right\} = \mathcal{F}_i^\lambda$$

for all $i = 1, 2, 3$. In virtue of (62) and (63) we have,

$$g_\omega \left(B(\lambda_0, R_3) \times \widetilde{W}_{t(\omega)}^{\lambda_0} \right) \subseteq G_{\lambda_0}^{-1} \left([0, 3\rho_{\lambda_0}] \right)$$

for all $\omega \in \widehat{E}^*$. Since the set $G_{\lambda_0}^{-1}([0, 3\rho_{\lambda_0}])$ is bounded, we thus conclude that for each $i = 1, 2, 3$, the family

$$\Gamma_i = \{ g_\omega : B(\lambda_0, R_3) \times \widetilde{W}_{t(\omega)}^{\lambda_0} \to \mathbb{C} : \omega \in \widehat{E}^* \text{ and } t(\omega) = i \}$$

is normal. Since for each $\lambda \in B(\lambda_0, R_3)$ and each $i \in \{1, 2, 3\}$, $X_i^\lambda \subseteq W_i^{\lambda_0}$ and $J_\lambda \cap X_i^\lambda \neq \emptyset$, all the limit functions of the normal family $\mathcal{F}_i^\lambda$ are constant. But this means that all the limit functions of the family Γ_i depend only on the first coordinate λ. Therefore (remember that $W_i^{\lambda_0} \subset \widetilde{W}_i^{\lambda_0}$), there exists $R_4 \in (0, R_3]$ and $q \geq 1$ such that

$$\operatorname{diam} \left(g_\omega \left(B(\lambda_0, R_4) \times \widetilde{W}_i^{\lambda_0} \right) \right) < \delta$$

and

$$\sup \left\{ \left| \frac{\partial g_\omega}{\partial z}(\lambda, z) \right| : (\lambda, z) \in B(\lambda_0, R) \times W_i^{\lambda_0} \right\} < \frac{1}{2}$$

for all $i \in \{1, 2, 3\}$ and all $\omega \in \widehat{E}^*$ with $|\omega| \geq q$. Now, as in the previous chapter, we conclude from this, (60), the inclusion $\phi_\omega^\lambda \left(X_{t(\omega)}^\lambda \right) \subseteq X_{t(\omega)}^\lambda$, and equality $\phi_\omega^\lambda = g_\omega\big|_{\{\lambda\}\times B(\lambda_0, R_4)}$ (with obvious abuse of notation) that

$$\phi_\omega^\lambda \left(W_{t(\omega)}^{\lambda_0} \right) \subseteq B \left(X_{t(\omega)}^{\lambda_0}, \delta \right) \subseteq W_{t(\omega)}^{\lambda_0}$$

for all $\lambda \in B(\lambda_0, R_4)$ and all $\omega \in \widehat{E}^*$ with $|\omega| \geq q$. Now, define the systems S_λ, $\lambda \in B(\lambda_0, R_4)$, as appearing in Proposition 7.1 with the help of this same $q \geq 1$. Since obviously all the maps $B(\lambda_0, R_4) \times W_i^{\lambda_0} \ni (\lambda, z) \mapsto \phi_\omega^\lambda(z)$, $\omega \in \widehat{E}^*$ are holomorphic, the family $\{\widehat{S}_\lambda\}_{\lambda \in B(\lambda_0, R_4)}$ is analytic, meaning that condition (b) of the definition of holomorphic families of holomorphic PGDMSs is satisfied. Since clearly, all the maps $B(\lambda_0, R_4) \times B(0, R_0) \ni (\lambda, z) \mapsto \phi_e^\lambda(z)$, with e being $(2,2)^q$ or $(3,3)^q$ and $R_0 \in (0, R_4]$ sufficiently small, are holomorphic, we see that condition (c) of the definition of holomorphic families of holomorphic PGDMSs is satisfied, and we may therefore conclude that the family $\{S_\lambda\}_{\lambda \in B(\lambda_0, R_0)}$ is holomorphic. We are done. □

As an immediate consequence of this Lemma and Corollary 6.14, we get the following main result of our paper.

Theorem 7.3. *The Hausdorff dimension function* $D_0 \ni \lambda \mapsto \mathrm{HD}\,(J(f_\lambda))$ *is real-analytic.*

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