

REAL AND COMPLEX ZEROS OF RIEMANNIAN RANDOM WAVES

STEVE ZELDITCH

ABSTRACT. We show that the expected limit distribution of the real zero set of a Gaussian random linear combination of eigenfunctions with frequencies from a short interval (‘asymptotically fixed frequency’) is uniform with respect to the volume form of a compact Riemannian manifold (M, g) . We further show that the complex zero set of the analytic continuations of such Riemannian random waves to a Grauert tube in the complexification of M tends to a limit current.

This article is concerned with the real and complex zero sets of *Riemannian random waves* on a real analytic Riemannian manifold (M, g) . To define Riemannian random waves, we fix an orthonormal basis $\{\varphi_{\lambda_j}\}$ of real-valued eigenfunctions of the Laplacian Δ_g of (M, g) ,

$$\Delta_g \varphi_{\lambda_j} = \lambda_j^2 \varphi_{\lambda_j}, \quad \langle \varphi_{\lambda_j}, \varphi_{\lambda_k} \rangle = \delta_{jk},$$

and define Gaussian ensembles of random functions $f = \sum_j c_j \varphi_{\lambda_j}$ of the following two types:

- The asymptotically *fixed frequency* ensemble \mathcal{H}_{I_λ} , where $I_\lambda = [\lambda, \lambda + 1]$ and where \mathcal{H}_{I_λ} is the vector space of linear combinations

$$f_\lambda = \sum_{j: \lambda_j \in [\lambda, \lambda + 1]} c_j \varphi_{\lambda_j}, \tag{1}$$

of eigenfunctions with λ_j (the frequency) in an interval $[\lambda, \lambda + 1]$ of fixed width. (Note that it is the square root of the eigenvalue of Δ , not the eigenvalue, which is asymptotically fixed).

- The *cut-off* ensembles $\mathcal{H}_{[0, \lambda]}$ where the frequency is cut-off at λ :

$$f_\lambda = \sum_{j: \lambda_j \leq \lambda} c_j \varphi_{\lambda_j}, \tag{2}$$

By *random*, we mean that the coefficients c_j are independent Gaussian random variables with mean zero and with the variance defined so that the expected L^2 norm of f equals one. Equivalently, the real vector spaces $\mathcal{H}_{[0, \lambda]}$, resp. \mathcal{H}_{I_λ} are endowed with the inner product $\langle u, v \rangle = \int_M u v dV_g$ (where dV_g is the volume form of (M, g)) and random means that we equip the vector spaces with the induced Gaussian measure. Our main results given the asymptotic distribution of real and complex zeros of such Riemannian random waves in the high frequency limit $\lambda \rightarrow \infty$.

The real zeros are straightforward to define. For each $f_\lambda \in \mathcal{H}_{[0, \lambda]}$ or \mathcal{H}_{I_λ} we associated to the zero set $Z_{f_\lambda} = \{x \in M : f_\lambda(x) = 0\}$ the positive measure

$$\langle |Z_{f_\lambda}|, \psi \rangle = \int_{Z_{f_\lambda}} \psi d\mathcal{H}^{n-1}, \tag{3}$$

Date: March 30, 2008.

Research partially supported by NSF grant DMS-0603850.

where $d\mathcal{H}^{m-1}$ is the induced (Hausdorff) hypersurface measure. Our first result (Theorem 1) shows that the normalized expected limit distribution $\frac{1}{\lambda}\mathbf{E}|Z_{f_\lambda}|$ of zeros of random Riemannian waves tends to the volume form dV_g as $\lambda \rightarrow \infty$. The result is the same in the cutoff and fixed frequency ensembles, but the proof is simpler in the former. Further, in the fixed frequency case, if one chooses a random sequence f_{λ_N} , i.e. the elements are chosen independently from the frequency intervals $[N, N+1]$ and at random from $\mathcal{H}_{[N, N+1]}$, then almost surely, $\frac{1}{N} \sum_{k=1}^N \frac{1}{\lambda_k} |Z_{f_{\lambda_k}}| \rightarrow dV_g$.

The *complex zeros* are defined by analytic continuation. Since (M, g) is assumed to be real analytic, it admits a complexification $M_{\mathbb{C}}$ and the eigenfunctions can be analytically continued to $M_{\mathbb{C}}$. Their real zero hypersurfaces extend to complex nodal hypersurfaces in $M_{\mathbb{C}}$. Our second result (Theorem 3) determines the limit distribution of these random complex nodal hypersurfaces, and shows that the limit distribution is the same as the one determined in [Z3] for complex zeros of analytic continuations of ergodic eigenfunctions, e.g. eigenfunctions of Δ_g when the geodesic flow of (M, g) is ergodic. This corroborates the random wave hypothesis as applied to complex nodal sets, i.e. the conjecture that eigenfunctions of chaotic systems resemble random waves. Note that in the real domain, the distribution of nodal lines of eigenfunctions in the ergodic case is very much an open problem. The motivation to study complex zeros of analytic continuations of eigenfunctions comes from the fact one has more control over complex zeros than real zeros of (deterministic) eigenfunctions.

0.1. Statement of results on real zeros. We now state our results more precisely. Let us first consider the real zero sets of eigenfunctions of Δ for the standard sphere (S^m, g_0) . Let $\mathcal{H}_N \subset L^2(S^m)$ denote the real d_N -dimensional inner product space of spherical harmonics of degree N . The eigenvalue is given by

$$(\lambda_N^{S^m})^2 = N(N+m-1) = (N + \frac{\beta}{4})^2 - (\frac{\beta}{4})^2$$

where $\beta = \frac{m-1}{2}$ is the common Morse index of the 2π periodic geodesics and

$$d_N = \binom{m+N-1}{N} - \binom{m+N-3}{N-2}.$$

We choose an orthonormal basis $\{\varphi_{Nj}\}_{j=1}^{d_N}$ for \mathcal{H}_N . For instance, on S^2 one can choose the real and imaginary parts of the standard Y_m^N 's. In the fixed frequency ensemble, we endow the real vector space \mathcal{H}_N with the Gaussian probability measure γ_N defined by

$$\gamma_N(f) = \left(\frac{d_N}{\pi}\right)^{d_N/2} e^{-d_N|c|^2} dc, \quad f = \sum_{j=1}^{d_N} c_j \varphi_{Nj}, \quad d_N = \dim \mathcal{H}_N. \quad (4)$$

Here, dc is d_N -dimensional real Lebesgue measure. The normalization is chosen so that $\mathbf{E}_{\gamma_N} \langle f, f \rangle = 1$, where \mathbf{E}_{γ_N} is the expected value with respect to γ_N . Equivalently, the d_N real variables c_j ($j = 1, \dots, d_N$) are independent identically distributed (i.i.d.) random variables with mean 0 and variance $\frac{1}{2d_N}$; i.e.,

$$\mathbf{E}_{\gamma_N} c_j = 0, \quad \mathbf{E}_{\gamma_N} c_j c_k = \frac{1}{2d_N} \delta_{jk}.$$

We note that the Gaussian ensemble is equivalent to picking $f_N \in \mathcal{H}_N$ at random from the unit sphere in \mathcal{H}_N with respect to the L^2 inner product. The latter description is more intuitive but it is technically more convenient to work with Gaussian measures. In the cutoff ensemble, we put the product Gaussian measure $\prod_{n=1}^N \gamma_n$ on $\bigoplus_{n=1}^N \mathcal{H}_n$.

We now consider the analogous constructions on a general compact Riemannian manifold (M, g) of dimension m . As mentioned above, and as defined more precisely in §1, the analogue of the space \mathcal{H}_N of spherical harmonics of degree N is played by the space \mathcal{H}_{I_N} of linear combinations of eigenfunctions (1) with frequencies in an interval $I_N := [N, N + 1]$. The precise decomposition of \mathbb{R} into intervals is not canonical on a generic Riemannian manifold and the results do not depend on the choice. We choose $I_N = [N, N + 1]$ only for notational simplicity. In the special case of Zoll manifolds (all of whose geodesics are closed), there is a canonical choice (an eigenvalue cluster decomposition) which is described in §1). Henceforth we abbreviate $\mathcal{H}_N = \mathcal{H}_{I_N}$ on general Riemannian manifolds. We continue to denote by $\{\varphi_{Nj}\}_{j=1}^{d_N}$ an orthonormal basis of \mathcal{H}_N where $d_N = \dim \mathcal{H}_N$. We equip it with the Gaussian measure (4) and again denote the expected value with respect to $(\mathcal{H}_N, \gamma_N)$ by \mathbf{E}_{γ_N} .

Our main result on real Riemannian random waves is to determine the expected value $\mathbf{E}_{\gamma_N} |Z_{f_N}|$. It is a positive measure satisfying

$$\langle \mathbf{E}_{\gamma_N} |Z_{f_N}|, \psi \rangle = \mathbf{E}_{\gamma_N} X_\psi^N, \tag{5}$$

where X_ψ^N is a ‘linear statistic’, i.e. the random variable

$$X_\psi^N(f_N) = \langle \psi, |Z_{f_N}| \rangle, \quad \psi \in C(M) \tag{6}$$

considered in (3).

THEOREM 1. *Let (M, g) be a compact Riemannian manifold, let $\mathcal{H}_{[0, \lambda]}$ be the cutoff ensemble and let $(\mathcal{H}_N, \gamma_N)$ be the ensemble of Riemannian waves of asymptotically fixed frequency. Then in either ensemble:*

- (1) For any $C^\infty (M, g)$, $\lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{E}_{\gamma_N} \langle |Z_{f_N}|, \psi \rangle = \int_M \psi dV_g$.
- (2) For a real analytic (M, g) , $\text{Var}(\frac{1}{N} X_\psi^N) \leq C$.

We restrict to real analytic metrics in (2) for the sake of brevity. In that case, the variance estimate follows easily from a result of Donnelly-Fefferman on volumes of real nodal hypersurfaces of real analytic (M, g) .

In the case of the standard metric g_0 on S^m , it is obvious a priori that $\mathbf{E}_{\gamma_N} |Z_{f_N}| = C_N dV_{g_0}$, the constant C_N being the expected volume of the zero sets. The expected volume was first determined by P. Bérard by a different method. The theorem above shows that asymptotically the same result holds on any compact Riemannian manifold.

A much better variance estimate for S^m was obtained by J. Neuheisel in his (unpublished) Hopkins PhD thesis [Ne], which shows that the variance tends to zero at least at a rate $N^{-\delta}$ for a certain $\delta > 0$. It is very likely that one could prove the same (or a better) variance estimate on a general $C^\infty (M, g)$, but that would require a study of the pair correlation function of zeros which would take us too far afield from our main purpose. We plan to carry them out on a different occasion. Relatively sharp variance estimates for eigenfunctions on arithmetic tori are given in [RW, ORW].

An immediate consequence, by the Kolmogorov strong law of large numbers, is a limit law for random sequences of random real Riemannian waves. By a random sequence, we mean

an element of the product probability space

$$\mathcal{H}_\infty = \prod_{N=1}^\infty \mathcal{H}_N, \quad \gamma_\infty = \prod_{N=1}^\infty \gamma_N. \quad (7)$$

COROLLARY 2. *Let (M, g) be a compact real analytic Riemannian manifold, and let $\{f_N\}$ be a random sequence in (7). Then*

$$\frac{1}{N} \sum_{n=1}^N \frac{1}{\lambda_n} |Z_{f_n}| \rightarrow dV_g \quad \text{almost surely w.r.t. } (\mathcal{H}_\infty, \gamma_\infty).$$

It is natural to conjecture that $\frac{1}{\lambda_N} |Z_{f_N}| \rightarrow dV_g$ almost surely without averaging in N , but the proof would again require a stronger variance estimate than we currently possess.

0.2. Statement of results on complex zeros. We now turn to results on complex zeros of analytic continuations of eigenfunctions. By a theorem of Bruhat-Whitney [BW], an analytic manifold M admits a complexification $M_{\mathbb{C}}$ into which M embeds as a totally real submanifold. Associated to g is a plurisubharmonic exhaustion function $\rho(\zeta)$ which measures the square of the distance to the real subset M . The sublevel set $M_\tau = \{\zeta \in M_{\mathbb{C}} : \sqrt{\rho(\zeta)} < \tau\}$ is known as the Grauert tube of radius τ (cf. [Gr, GS1, GS2, LS1]).

It was observed by Boutet de Monvel [Bou] that eigenfunctions can be analytically continued to the maximal Grauert tube as holomorphic functions $\varphi_{\lambda_j}^{\mathbb{C}}$. Thus, we can complexify the Gaussian random waves as

$$f_N^{\mathbb{C}} = \sum_{j=1}^{d_N} c_{Nj} \varphi_{Nj}^{\mathbb{C}}.$$

We note that the coefficients c_{Nj} are real and that the Gaussian measure on the coefficients remains the real Gaussian measure γ_N .

Our next result determines the expected limit current of complex zeros of $f_N^{\mathbb{C}}$. The current of integration over the complex zero set

$$Z_{f_N^{\mathbb{C}}} = \{\zeta \in M_{\mathbb{C}} : f_N^{\mathbb{C}} = 0\}$$

is the $(1, 1)$ current defined by

$$\langle [Z_{f_N^{\mathbb{C}}}], \psi \rangle = \int_{Z_{f_N^{\mathbb{C}}}} \psi, \quad \psi \in \mathcal{D}^{m-1, m-1}(M_{\mathbb{C}}),$$

for smooth test forms of bi-degree $(m-1, m-1)$. In terms of scalar functions ψ we may define $Z_{f_N^{\mathbb{C}}}$ as the measure,

$$\langle [Z_{f_N^{\mathbb{C}}}], \psi \rangle = \int_{Z_{f_N^{\mathbb{C}}}} \psi \omega_g^{m-1} / (m-1)!,$$

where $\omega_g = i\partial\bar{\partial}\rho$ is the Kähler metric adapted to g .

THEOREM 3. *Let (M, g) be a real analytic compact Riemannian manifold. Then for either of the ensembles of Theorem 1, we have*

$$\mathbf{E}_{\gamma_N} \left(\frac{1}{N} [Z_{f_N^{\mathbb{C}}}] \right) \rightarrow \frac{i}{\pi} \partial\bar{\partial}|\xi|_g, \quad \text{weakly in } \mathcal{D}'^{(1,1)}(B_\varepsilon^* M).$$

As mentioned above, this result shows that the complex zeros of the random waves have the same expected limit distribution found in [Z3] for real analytic compact Riemannian manifolds with ergodic geodesic flow.

0.3. The key objects in the proof. The principal objects (for the asymptotically fixed frequency ensembles) are the two point functions

$$\Pi_{I_N}(x, y) = \mathbf{E}_{\gamma_N}(f_N(x)f_N(y)) = \sum_{j:\lambda_j \in I_N} \varphi_{\lambda_j}(x)\varphi_{\lambda_j}(y), \tag{8}$$

i.e. the spectral projections kernel for $\sqrt{\Delta}$, and their analytic extensions to the totally real anti-diagonal in $M_{\mathbb{C}} \times M_{\mathbb{C}}$ defined by

$$\Pi_{I_N}(\zeta, \bar{\zeta}) = \sum_{j:\lambda_j \in I_k} |\varphi_j^{\mathbb{C}}(\zeta)|^2. \tag{9}$$

A key point is that the latter kernels are very much off the diagonal for non-real ζ , so that the kernels grow at an exponential rate. In the cutoff ensemble, the spectral projections kernels are replaced by

$$\Pi_{[0,N]}(x, y) = \mathbf{E}_{\gamma_N}(f_N(x)f_N(y)) = \sum_{j:\lambda_j \in [0,N]} \varphi_{\lambda_j}(x)\varphi_{\lambda_j}(y), \tag{10}$$

and similarly for the complexification.

In the real domain, the distribution of zeros of random Riemannian waves is obtained by using formalism of [BSZ1, BSZ2] to express the density of zeros in terms of the kernels $\Pi_{I_N}(x, y)$. We then use the spectral asymptotics of these kernels and their derivatives to derive the limit distribution of zeros. A more in-depth analysis of their off-diagonal decay could give bounds on the variance, but as mentioned above we postpone that to a later occasion. In the complex domain, the spectral asymptotics have not been studied before. The asymptotics are more difficult than in the real domain and have an independent interest. On the other hand, the link between these kernels and the zero distribution is simpler, and we use the Poincaré-Lelong method of [BSZ3] rather than [BSZ1, BSZ2].

0.4. Discussion. This is the first article discussing zeros of Riemannian random waves of asymptotically fixed energy. We digress to compare our definitions and results on Riemannian waves to other definitions and results in the literature.

The subject of random polynomials and Fourier series and their zeros is classical; see [PW] for one of the classics. For a contemporary treatment of Gaussian random functions in a geometric setting, see [AT]. Aside from its pure mathematical interest, Gaussian random waves have been important in various branches of physics. In particular, a somewhat vague heuristic principle due to M.V. Berry [B] asserts that random waves should be a good model for quantum chaotic eigenfunctions ([Z1] contains rigorous results in this direction). Random waves in [B] and in much of the physics literature are random Euclidean plane waves of fixed energy. For a mathematician, they are defined by putting a Gaussian measure on the infinite dimensional space \mathcal{E}_{λ} of Euclidean eigenfunctions of fixed eigenvalue λ^2 on \mathbb{R}^m . The Gaussian measure satisfies $E\|f\|^2 = 1$, where $\|\cdot\|^2$ is the inner product invariant under the eigenspace representation of the Euclidean motion group.

On a compact Riemannian manifold, the closet analogue to random Euclidean plane waves of fixed frequency is that of random spherical harmonics of fixed degree on S^m , where one now puts the $SO(m+1)$ -invariant normalized Gaussian measure. In both cases, the measure is defined on an eigenspace. For a generic Riemannian manifold, the eigenspaces are of dimension one, so one cannot define an interesting Gaussian measure on the eigenspaces. The alternative proposed here and in [Z1] is to replace eigenspaces by the spaces \mathcal{H}_N spanned by eigenfunctions with asymptotically constant frequency. From the viewpoint of microlocal (semi-classical) analysis, the analogy is obvious since the spectral projections kernels have, to leading order, the same asymptotics as those for spherical harmonics. In [Z1] the emphasis was on random orthonormal bases as models for an orthonormal basis of ergodic eigenfunctions; while here we only study individual random waves. Riemannian random waves of asymptotically fixed energy seem to be a natural global model for random waves on a Riemannian manifold without boundary, and the set-up extends naturally to Riemannian manifolds with boundary and with fixed boundary conditions on Δ . When studying local behavior, Riemannian waves resemble the Euclidean plane waves of the same frequency. More precisely, the scaling limit of Riemannian random waves on length scales λ^{-1} should give back the Euclidean plane wave model with eigenvalue 1. This would be the Riemannian analogue of the universality result of [BSZ1]. Thus, the natural role of Euclidean plane waves seems to be to capture the random behavior of small length scales of order of λ^{-1} .

Further motivation to study nodal lines of Riemannian random waves has arisen in recent conjectures that nodal lines of random two-dimensional Euclidean plane waves of fixed frequency (and chaotic eigenfunctions) tend to SL_6 curves [BS, FGS, BGS, SS]. This behavior should be sufficiently universal that it should hold for Riemannian random waves of asymptotically fixed frequency on general surfaces. A related conjecture asserts that random nodal lines of partial sums of the Gaussian free field (i.e. random Riemannian waves in the cutoff ensemble) on a two-dimensional Riemannian surface tend to SLE_4 curves [SS]. Thus, one expects different behavior of random Riemannian waves for (almost) fixed frequency and for long frequency intervals; although the SLE connection is far outside the scope of this article, it does motivate us to consider both ensembles. For recent and deep results on nodal lines of random spherical harmonics which are related to conjectures in [BS, BGS] we refer to [NS].

The rationale for studying complex zeros of Riemannian random waves is that only in the complex domain can we rigorously compare the nodal sets of ergodic eigenfunctions and those of random waves. In addition, the complex zeros of complexified Riemannian random waves is a higher dimensional generalization of the classical ensembles of Kac-Hammersley of complexified random real polynomials. They may be viewed as sums of complexified eigenfunctions on a circle, although the spectral intervals are $[0, N]$ rather than $[N-1, N]$. Another recent study of complex zeros of complexified real polynomials is the thesis of B. MacDonald [Mc]; however, a significant difference is that the polynomials there are orthonormalized in the complex domain rather than the real domain.

1. BACKGROUND ON DENSITIES AND CORRELATIONS OF ZEROS OF REAL GAUSSIAN RANDOM WAVES

In this section, we apply the formalism in [BSZ2] to give explicit formulae for the densities of zeros of Riemannian random waves. The same formalism also could be used to give formulae for correlations between zeros.

1.1. **Assumptions on (M, g) .** We will assume the geodesic flow G^t of (M, g) is of one of the following two types:

- (1) *aperiodic*: The Liouville measure of the closed orbits of G^t , i.e. the set of vectors lying on closed geodesics, is zero; or
- (2) *periodic = Zoll*: $G^T = id$ for some $T > 0$; henceforth T denotes the minimal period. The common Morse index of the T -periodic geodesics will be denoted by β .

In the real analytic case, (M, g) is automatically one of these two types, since a positive measure of closed geodesics implies that all geodesics are closed. We only need to assume (M, g) is real analytic when considering complex zeros. In the C^∞ case, it is simple to construct examples with a positive but not full measure of closed geodesics (e.g. a pimped sphere).

The two-term Weyl laws counting eigenvalues of $\sqrt{\Delta}$ are very different in these two cases.

- (1) In the *aperiodic* case, Ivrii's two term Weyl law states

$$N(\lambda) = \#\{j : \lambda_j \leq \lambda\} = c_m \text{Vol}(M, g) \lambda^m + o(\lambda^{m-1})$$

where $m = \dim M$ and where c_m is a universal constant.

- (2) In the *periodic* case, the spectrum of $\sqrt{\Delta}$ is a union of eigenvalue clusters C_N of the form

$$C_N = \left\{ \left(\frac{2\pi}{T} \right) \left(N + \frac{\beta}{4} \right) + \mu_{Ni}, i = 1 \dots d_N \right\}$$

with $\mu_{Ni} = o(N^{-1})$. The number d_N of eigenvalues in C_N is a polynomial of degree $m - 1$.

We refer to [Ho, SV, Z1] for background and further discussion.

1.2. **Definition of Riemannian random waves.** To define Riemannian random waves, we partition the spectrum of $\sqrt{\Delta_g}$ into certain intervals I_N of width one and denote by Π_{I_N} the spectral projections for $\sqrt{\Delta_g}$ corresponding to the interval I_N . The choice of the intervals I_N is rather arbitrary for aperiodic (M, g) and as mentioned above we assume $I_N = [N, N + 1]$. But the choice has to be made carefully for Zoll manifolds.

In the Zoll case, we center the intervals around the center points $\frac{2\pi}{T}N + \frac{\beta}{4}$ of the N th cluster C_N . We call such a choice of intervals a cluster decomposition. We denote by d_N the number of eigenvalues in I_N and put $\mathcal{H}_N = \text{ran} \Pi_{I_N}$ (the range of Π_{I_N}). Thus, \mathcal{H}_N consists of linear combinations $\sum_{j: \lambda_j \in I_N} c_j \varphi_{N,j}$ of the eigenfunctions $\{\varphi_{N,j}\}$ of $\sqrt{\Delta_g}$ with eigenvalues in I_N .

The formalism is simpler in the cutoff ensemble and only requires small modifications from the asymptotically fixed frequency ensembles, so we only explain at the end how to modify the results in that case.

1.3. **Density of real zeros.** The formula for the density of zeros of random elements of \mathcal{H}_N can be derived from the general formalism of [BSZ1, BSZ2, BSZ3].

As above, we let $|Z_f|$ denote the Riemannian $(m - 1)$ -volume on Z_f . By the general formula of [BSZ1, BSZ2],

$$\mathbf{E}|Z_{f_N}| = K_1^N(z) dV_g, \quad K_1^N(x) = \int D(0, \xi, x) \|\xi\| d\xi. \quad (11)$$

We digress to connect this formula with the discussions in [BSZ1, BSZ2, Ne]. In these articles $\|\xi\|$ is written $\sqrt{\det(\xi\xi^*)}$. However, $\det(\xi\xi^*) = \|\xi\|^2$ in the codimension one case. Indeed, let df_x^* be the adjoint map with respect to the inner product g on T_xM . Let $df_x \circ df_x^* : T_xM \rightarrow \mathbb{R}$ be the composition. By $\det df_x \circ df_x^*$ is meant the determinant with respect to the inner product on T_xM ; it clearly equals $|df|^2$ in the codimension one case.

The formulae of [BSZ1, BSZ2] (the ‘Kac-Rice’ formulae) give that

$$D(0, \xi; z) = Z_n(z)D_\Lambda(\xi; z), \quad (12)$$

where

$$D_\Lambda(\xi; z) = \frac{1}{\pi^m \sqrt{\det \Lambda}} \exp(-\langle \Lambda^{-1} \xi, \xi \rangle) \quad (13)$$

is the Gaussian density with covariance matrix

$$\Lambda = C - B^* A^{-1} B = (C_{q'}^q - B_q A^{-1} B_{q'}), \quad (q = 1, \dots, m) \quad (14)$$

and

$$Z(x) = \frac{\sqrt{\det \Lambda}}{\pi \sqrt{\det \Delta}} = \frac{1}{\pi \sqrt{A}}. \quad (15)$$

In the case at hand,

$$\begin{aligned} \Delta^N(x) &= \begin{pmatrix} A^N & B^N \\ B^{N*} & C^N \end{pmatrix}, \\ (A^N) &= \mathbf{E}(X^2) = \frac{1}{d_N} \Pi_{I_N}(x, x), \\ (B^N)_q &= \mathbf{E}(X \Xi_q) = \frac{1}{d_N} \frac{\partial}{\partial y_q} \Pi_{I_N}(x, y)|_{x=y}, \\ (C^N)_{q'}^q &= \mathbf{E}(\Xi_q \Xi_{q'}) = \frac{1}{d_N} \frac{\partial^2}{\partial x_q \partial y_{q'}} \Pi_{I_N}(x, y)|_{x=y}, \\ &q, q' = 1, \dots, m. \end{aligned}$$

Making a simple change of variables in the integral (11), we have

PROPOSITION 1.1. [BSZ1] *On a real Riemannian manifold of dimension m , the density of zeros of a random Riemannian wave is*

$$K_1^N(x) = \frac{1}{\pi^m (\sqrt{d_N^{-1}} \Pi_{I_N}(x, x))} \int_{\mathbb{R}^m} \|\Lambda^N(x)^{1/2} \xi\| \exp(-\langle \xi, \xi \rangle) d\xi, \quad (16)$$

where $\Lambda^N(x)$ is a symmetric form on T_xM . For the asymptotically fixed frequency ensembles, it is given by

$$\Lambda^N(x) = \frac{1}{d_N} \left(d_x \otimes d_y \Pi_{I_N}(x, y)|_{x=y} - \frac{1}{\Pi_{I_N}(x, y)} d_x \Pi_{I_N}(x, y)|_{x=y} \otimes d_y \Pi_{I_N}(x, y)|_{x=y} \right).$$

In the cutoff ensemble the formula is the same except that Π_{I_N} is replaced by $\Pi_{[0, N]}$.

2. ZEROS OF RANDOM REAL RIEMANNIAN WAVES: PROOF OF THEOREM 1

We begin the proof with the simplest case of the round metric on S^m . Throughout this article, C_m denotes a constant depending only on the dimension. It may change from line to line.

2.1. Random spherical harmonics. To prove Theorem (2) on a round S^m , we first need to evaluate the matrix above when $\Pi_N(z, w)$ is the orthogonal projection onto spherical harmonics of degree N .

PROPOSITION 2.1. *Let $\Pi_N : L^2(S^m) \rightarrow \mathcal{H}_N$ be the orthogonal projection. Then:*

- (A) $\Pi_N(x, x) = \frac{1}{\text{Vol}(S^m)} d_N$;
- (B) $d_x \Pi_N(x, y)|_{x=y} = d_y \Pi_N(x, y)|_{x=y} = 0$;
- (C) $d_x \otimes d_y \Pi_N(x, y)|_{x=y} = \frac{1}{m \text{Vol}(S^m)} \lambda_N^2 d_N g_x$.

Proof. Statement (B) follows from statement (A) since

$$0 = d_x \Pi_N(x, x) = d_x \Pi_N(x, y)|_{x=y} + d_y \Pi_N(x, y)|_{x=y}$$

and because $d_x \Pi_N(x, y)|_{x=y} = d_y \Pi_N(x, y)|_{x=y}$. Statement (C) holds because $d_x \otimes d_y \Pi_N(x, y)|_{x=y} = C_N g_x$ by $SO(m+1)$ symmetry. To evaluate C_N we use that

$$0 = \Delta \Pi_N(x, x) = 2 \text{Tr} d_x \otimes d_y \Pi_N(x, y)|_{x=y} - 2 \lambda_N^2 \Pi_N(x, x).$$

Here, $\text{Tr} d_x \otimes d_y \Pi_N(x, y)|_{x=y}$ denotes the contraction. It equals $\sum_{j=1}^{d_N} \|d\varphi_{Nj}(x)\|^2$ where $\{\varphi_{Nj}\}$ is an orthonormal basis. Thus, $m C_N = \lambda_N^2 \Pi_N(x, x)$ and the formula of (C) follows from (A). □

The expected density of random nodal hypersurfaces is given as follows

PROPOSITION 2.2. *In the case of S^m ,*

$$K_1^N(x) = C_m \lambda_N \sim C_m N, \tag{17}$$

where $C_m = \frac{1}{\pi^m} \int_{\mathbb{R}^m} |\xi| \exp(-\langle \xi, \xi \rangle) d\xi$.

Proof. By Propositions 1.1 and 2.1, we have

$$K_1^N(x) = \frac{\sqrt{\text{Vol}(S^m)}}{\pi^m} \int_{\mathbb{R}^m} \|\Lambda^N(x)^{1/2} \xi\| \exp(-\langle \xi, \xi \rangle) d\xi, \tag{18}$$

where

$$\Lambda^N(x) = \frac{1}{d_N} \left(\frac{1}{m \text{Vol}(S^m)} \lambda_N^2 d_N g_x \right).$$

□

2.2. Random Riemannian waves: proof of Theorem 1. We now generalize the result to any compact C^∞ Riemannian manifold (M, g) which is either aperiodic or Zoll. As in the case of S^m , the key issue is the asymptotic behavior of derivatives of the spectral projections

$$\Pi_{I_N}(x, y) = \sum_{j: \lambda_j \in I_N} \varphi_{\lambda_j}(x) \varphi_{\lambda_j}(y). \quad (19)$$

PROPOSITION 2.3. *Assume (M, g) is either aperiodic and $I_N = [N, N + 1]$ or Zoll and I_N is a cluster decomposition. Let $\Pi_{I_N} : L^2(M) \rightarrow \mathcal{H}_N$ be the orthogonal projection. Then:*

- (A) $\Pi_{I_N}(x, x) = \frac{1}{\text{Vol}(M, g)} d_N (1 + o(1))$;
- (B) $d_x \Pi_{I_N}(x, y)|_{x=y} = d_y \Pi_{I_N}(x, y)|_{x=y} = o(N^m)$;
- (C) $d_x \otimes d_y \Pi_{I_N}(x, y)|_{x=y} = \frac{1}{\text{Vol}(M, g)} \lambda_N^2 d_N g_x (1 + o(1))$.

In the aperiodic case,

- (1) $\Pi_{[0, \lambda]}(x, x) = C_m \lambda^m + o(\lambda^{m-1})$;
- (2) $d_x \otimes d_y \Pi_{[0, \lambda]}(x, y)|_{x=y} = C_m \lambda^{m+2} g_x + o(\lambda^{m+1})$.

In the Zoll case, one adds the complete asymptotic expansions for Π_{I_N} over the N clusters to obtain expansions for Π_N .

Proof. Asymptotic formulae of type (A) are standard in spectral asymptotics, and we refer to [DG, Ho] for background. Asymptotics of type (C) were worked out in [Z2] (Theorem 2) in the special case of a Zoll metric. However, much of the calculation goes through for any compact Riemannian manifold. It does not appear however that (B) has been stated before or that (C) has been previously discussed on general Riemannian manifolds, although the techniques are standard. These asymptotics are dual to the heat kernel asymptotics in [BBG], but they are sharper because we are using spectral intervals for $\sqrt{\Delta}$ of fixed width rather than intervals of the form $[0, \lambda]$, which are dual to heat kernel asymptotics. Thus, we need two term asymptotics for long intervals in order to obtain asymptotics on short intervals.

We follow the standard Tauberian method of [DG], Proposition 2.1, for studying the spectral asymptotics. We consider the spectral measure

$$d_\lambda \Pi_{[0, \lambda]}(x, y) = \sum_j \delta(\lambda - \lambda_j) \varphi_{\lambda_j}(x) \varphi_{\lambda_j}(y), \quad (20)$$

whose integral over the interval I_N equals $\Pi_{I_N}(x, y)$, and the derived measures

$$\left\{ \begin{array}{l} \text{(a)} \quad d_\lambda \Pi_{[0, \lambda]}(x, x) = \sum_j \delta(\lambda - \lambda_j) \varphi_{\lambda_j}(x)^2 \\ \text{(b)} \quad d_\lambda d_x \Pi_{[0, \lambda]}(x, x) = 2 \sum_j \delta(\lambda - \lambda_j) \varphi_{\lambda_j}(x) d\varphi_{\lambda_j}(x) \\ \text{(c)} \quad d_\lambda d_x \otimes d_y \Pi_{[0, \lambda]}(x, y)|_{x=y} = \sum_j \delta(\lambda - \lambda_j) d\varphi_{\lambda_j}(x) \otimes d\varphi_{\lambda_j}(x). \end{array} \right. \quad (21)$$

We now introduce a cutoff function $\rho \in \mathcal{S}(\mathbb{R})$ with $\hat{\rho} \in C_0^\infty$ supported in sufficiently small neighborhood of 0. We also assume $\hat{\rho} \equiv 1$ in a smaller neighborhood of 0. Then there exists an expansion in inverse powers of λ with coefficients smooth in (x, y) :

$$\rho * d_\lambda \Pi_{[0, \lambda]}(x, x) = \sum_j \rho(\lambda - \lambda_j) \varphi_{\lambda_j}^2(x) \sim \sum_{k=0}^{\infty} \lambda^{m-1-k} \omega_k(x), \quad (22)$$

where ω_k are smooth in x , and $\omega_0 = 1$.

We briefly recall the proof of (22) and of (A): We have

$$\rho * d_\lambda \Pi_{[0,\lambda]}(x, y) = \int_{\mathbb{R}} \hat{\rho}(t) e^{it\lambda} U(t, x, y) dt, \quad (23)$$

where $U(t, x, y)$ is the Schwartz kernel of the wave group $U(t) = e^{-it\sqrt{\Delta}}$. We use a small-time parametrix for $U(t, x, y)$ near the diagonal of the form

$$U(t, x, y) = \int_{T_y^* M} e^{-it|\xi|_{g_y}} e^{i\langle \xi, \exp_y^{-1}(x) \rangle} A(t, x, y, \xi) d\xi \quad (24)$$

where $|\xi|_{g_x}$ is the metric norm function at x , and where $A(t, x, y, \xi)$ is a polyhomogeneous amplitude of order 0 which is supported near the diagonal. Setting $x = y$ gives

$$\rho * d_\lambda \Pi_{[0,\lambda]}(x, x) = \int_{\mathbb{R}} \int_{T_x^* M} \hat{\rho}(t) e^{it\lambda} e^{-it|\xi|_{g_x}} A(t, x, x, \xi) d\xi dt. \quad (25)$$

As in [DG], we pass to polar coordinates $r = |\xi|_g$, change variables $\theta \rightarrow \lambda\theta$ and apply the stationary phase method to the $dr dt$ integral to obtain (22).

For (C), we apply the method of [Z2] (see (3.6) - (3.7)). We denote the phase of $U(t, x, y)$ by

$$\varphi(t, x, y, \xi) = \langle \xi, \exp_y^{-1}(x) \rangle - t|\xi|_{g_y}.$$

Then applying $d_x \otimes d_y|_{x=y}$ to the integral produces a highest order term given by a universal constant times

$$d_x \varphi(t, x, y, \xi) \otimes d_y \varphi(t, x, y, \xi)|_{x=y} = \xi \otimes \xi,$$

since $a_0(t, x, x, \xi) = 1$ (cf. [DG]). It also produces lower order terms (i.e. of order ≤ 1) in which at least one derivative falls on the amplitude. If we put the $d\xi$ - integral in polar coordinates $\xi = r\omega$, we obtain an expansion

$$\rho * d_\lambda d_x \otimes d_y \Pi_{[0,\lambda]}(x, y)|_{x=y} \sim \sum_{k=0}^{\infty} \lambda^{m+1-k} B_k(x), \quad (26)$$

with the leading coefficient

$$B_0(x) = \frac{C_m}{\text{Vol}(M, g)} \int_{S_x^* M} \omega \otimes \omega d\mu_x(\omega) = \frac{C_m}{\text{Vol}(M, g)} g_x. \quad (27)$$

Here, $d\mu_x$ is the Euclidean area element induced by g on $S_x^* M$.

We now draw the conclusions for (A) - (C). In the Zoll case, (A) and (C) are already proved in detail in [Z2]. In the Zoll case, there exist complete asymptotic expansions for the spectral sums and we may deduce (B) as well from the smoothed expansion by a modification of the proof of [Z2], Theorem 2. In the aperiodic case, the measures (A) and (C) are positive and we may apply the Fourier Tauberian theorems of [Ho, SV] (see the Appendix in §6), or alternatively the remainder estimate of Ivrii, to obtain two-term expansions:

- (1) $\Pi_{[0,\lambda]}(x, x) = C_m \lambda^m + o(\lambda^{m-1})$;
- (2) $d_x \otimes d_y \Pi_{[0,\lambda]}(x, y)|_{x=y} = C_m \lambda^{m+2} g_x + o(\lambda^{m+1})$.

We then subtract the expansions across the interval I_N to obtain the stated result. We note that the drop in degree is encoded in $d_N \sim N^{m-1}$.

We still need to analyze (B). Since (B) equals $d_x \Pi_{I_N}(x, x)$, we could take the derivative in x of (25) to obtain

$$\rho * d_\lambda d_x \Pi_{[0, \lambda]}(x, x) = \int_{\mathbb{R}} \int_{T_y^* M} \hat{\rho}(t) e^{it\lambda} e^{-it|\xi|_{g_y}} d_x A(t, x, x, \xi) d\xi dt \quad (28)$$

We then have an expansion similar to that of (22) except that the amplitude is now the one-form $d_x A$. However, the Tauberian theorems do not apply to (B) since $d_\lambda d_x \Pi_{[0, \lambda]}(x, x)$ is not a positive measure. In the Zoll case, we have a complete asymptotic expansion of (A) and its x -derivative gives that of (B). But on a general Riemannian manifold one cannot use this approach.

Henceforth we assume (M, g) is aperiodic. In this case, we use the fact that, for each k , $B_k = \frac{\partial}{\partial x_k} \Delta^{-1/2}$ is a bounded pseudo-differential operator and

$$\frac{\partial}{\partial x_k} \Pi_{[0, \lambda]}(x, y)|_{x=y} = \sum_{j: \lambda_j \leq \lambda} \lambda_j (B_k \varphi_{\lambda_j}(x)) \varphi_{\lambda_j}(x).$$

We write

$$2 (B_k \varphi_{\lambda_j}(x)) \varphi_{\lambda_j}(x) = ((I + B_k) \varphi_{\lambda_j}(x))^2 - (B_k \varphi_{\lambda_j}(x))^2 - \varphi_{\lambda_j}^2.$$

We then substitute the right side into the summatory function in λ_j to obtain three asymptotic expansions to which the Tauberian theorems apply. We start with

$$\begin{aligned} & \rho * d_\lambda ((I + B_k)_x \otimes (I + B_k)_y \sqrt{\Delta} \Pi_{[0, \lambda]}(x, y)|_{x=y}) \\ &= ((I + B_k)_x \otimes (I + B_k)_y \int_{\mathbb{R}} \hat{\rho}(t) e^{-it\lambda} \sqrt{\Delta} U(t, x, y)|_{x=y} dt \\ &= ((I + B_k)_x \otimes (I + B_k)_y \int_{\mathbb{R}} \hat{\rho}(t) e^{it\lambda} \frac{\partial}{i\partial t} U(t, x, y)|_{x=y} \\ &= ((I + B_k)_x \otimes (I + B_k)_y \left(\int_{\mathbb{R}} \int_{T_y^* M} (i\hat{\rho}'(t) - \lambda\hat{\rho}) e^{it\lambda} e^{-it|\xi|_{g_y}} e^{i\langle \xi, \exp_x^{-1}(y) \rangle} A(t, x, x, \xi) d\xi dt \right) |_{x=y} \end{aligned} \quad (29)$$

To leading order, application of $((I + B_k)_x \otimes (I + B_k)_y$ under the integration sign multiplies the leading term of the amplitude by $(1 + b_k)(x, d\varphi_t)$ where φ_t is the phase (cf. e.g. the ‘fundamental asymptotic expansion’ of [T]). We then apply the stationary phase method and due to the extra factor of λ in the amplitude obtain

$$\sum_{j: \lambda_j \leq \lambda} \lambda_j ((I + B_k) \varphi_{\lambda_j}(x))^2 = C_m \lambda^{m+1} \int_{S_x^* M} (1 + b_k(x, \omega))^2 d\mu(\omega) + o(\lambda^m). \quad (30)$$

The $o(\lambda^m)$ remainder holds as in the scalar case because the geodesic flow is aperiodic [DG, Ho, SV]. We then repeat the calculation for B_k and for I and subtract. We clearly cancel the leading term, leaving the remainder $o(\lambda^m)$. When we subtract the interval $[0, N]$ from $[0, N + 1]$ we obtain $o(N^m)$. □

2.3. Proof of Theorem 1. The generalization of Proposition 2.2 to a general Riemannian manifold is the following:

PROPOSITION 2.4. *For the asymptotically fixed frequency ensemble, and for any $C^\infty(M, g)$ which is either Zoll or aperiodic (and with I_N as in Proposition 2.3), we have*

$$\begin{aligned} K_1^N(x) &= \frac{1}{\pi^m (\lambda_N)^{m/2}} \int_{\mathbb{R}^m} \|\xi\| \exp\left(-\frac{1}{\lambda_N} \langle \xi, \xi \rangle\right) d\xi + o(1) \\ &\sim C_m N, \end{aligned} \quad (31)$$

where $C_m = \frac{1}{\pi^m} \int_{\mathbb{R}^m} \|\xi\| \exp(-\langle \xi, \xi \rangle) d\xi$. The same formula holds for the cutoff ensemble.

Proof. Both on a sphere S^m or on a more general (M, g) which is either Zoll or aperiodic, we have by Propositions 2.1 resp. 2.3 and the general formula for Δ^N in §1.3 that

$$\Delta^N(z) = \frac{1}{\text{Vol}(M, g)} \begin{pmatrix} (1 + o(1)) & o(1) \\ o(1) & N^2 g_x(1 + o(1)) \end{pmatrix}, \quad (32)$$

It follows that

$$\Lambda^N = C^N - B^{N*} (A^N)^{-1} B^N = \frac{1}{\text{Vol}(M, g)} N^2 g_x + o(N). \quad (33)$$

Thus, we have

$$\begin{aligned} K_1^N(x) &\sim \frac{\sqrt{\text{Vol}(M, g)}}{\pi^m} \int_{\mathbb{R}^m} \|\Lambda^N(x)^{1/2} \xi\| \exp(-\langle \xi, \xi \rangle) d\xi \\ &= \frac{N}{\pi^m} \int_{\mathbb{R}^m} \|(I + o(1))(x)^{1/2} \xi\| \exp(-\langle \xi, \xi \rangle) d\xi, \end{aligned} \quad (34)$$

where $o(1)$ denotes a matrix whose norm is $o(1)$. The integral tends to $\int_{\mathbb{R}^m} \|\xi\| \exp(-\langle \xi, \xi \rangle) d\xi$ as $N \rightarrow \infty$, completing the proof. \square

So far, we have only determined the expected values of the nodal hypersurface measures. To complete the proof of Theorem 1, we need to prove:

PROPOSITION 2.5. *If (M, g) is real analytic, then the variance of $\frac{1}{\lambda_N} X_\psi^N$ is bounded.*

Proof. By a theorem of Donnelly-Fefferman [DF], for real analytic (M, g) ,

$$c_1 \lambda \leq \mathcal{H}^{m-1}(Z_{\varphi_\lambda}) \leq C_2 \lambda, \quad (\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda; c_1, C_2 > 0). \quad (35)$$

Hence for any $f_N \in \mathcal{H}_{I_N}$, $\frac{1}{\lambda_N} Z_{f_N}$ has bounded mass. Hence, the random variable $\frac{1}{\lambda_N} X_\psi^N$ is bounded, and therefore so is its variance. \square

Remark:

The variance of $\frac{1}{\lambda_N} X_\psi^N$ is given by

$$\text{Var}\left(\frac{1}{\lambda_N} X_\psi^N\right) = \frac{1}{\lambda_N^2} \int_M \int_M (K_2^N(x, y) - K_1^N(x) K_1^N(y)) \psi(x) \psi(y) dV_g(x) dV_g(y), \quad (36)$$

where $K_2^N(x, y) = \mathbf{E}_{\gamma_N}(Z_{f_N}(x) \otimes Z_{f_N}(y))$ is the pair correlation function for zeros. Hence, boundedness would follow from

$$\frac{1}{\lambda_N^2} \int_M \int_M K_2^N(x, y) dV_g(x) dV_g(y) \leq C. \quad (37)$$

There is a formula similar to that for the density in Proposition 1.1 for $K_2^N(x, y)$ and it is likely that it could be used to prove boundedness of the variance for any C^∞ Riemannian manifold. But we leave this for the future. In the Kähler case, asymptotic formulae for the variance of smooth linear statistics are given in [SZ2, SZ3], but the method does not apply in the real case.

2.4. Random sequences and proof of Corollary 2. We recall that the set of random sequences of Riemannian waves of increasing frequency is the probability space $\mathcal{H}_\infty = \prod_{N=1}^\infty \mathcal{H}_{I_N}$ with the measure $\gamma_\infty = \prod_{N=1}^\infty \gamma_N$. An element in \mathcal{H}_∞ will be denoted $\mathbf{f} = \{f_N\}$. We have,

$$\left| \left(\frac{1}{\lambda_N} Z_{f_N}, \psi \right) \right| \leq \frac{1}{\lambda_N} \mathcal{H}^{n-1}(Z_{f_N}) \|\psi\|_{C^0}.$$

By a density argument it suffices to prove that the linear statistics $\frac{1}{\lambda_N} (Z_{f_N}, \psi) - \frac{1}{\text{Vol}(M, g)} \int_M \psi dV_g \rightarrow 0$ almost surely in \mathcal{H}_∞ . From Theorem 1, we have:

COROLLARY 2.6. (i) $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k \leq N} \mathbf{E} \left(\frac{1}{\lambda_k} X_\psi^k \right) = \frac{1}{\text{Vol}(M, g)} \int_M \psi dV_g$;
(ii) $\text{Var} \left(\frac{1}{\lambda_N} X_\psi^N \right)$ is bounded on \mathcal{H}_∞ .

Since $\frac{1}{\lambda_N} X_\psi^N$ for $\{N = 1, 2, \dots\}$ is a sequence of independent random variables in \mathcal{H}_∞ with bounded variances, the Kolmogorov strong law of large numbers implies that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k \leq N} \left(\frac{1}{\lambda_k} X_\psi^k \right) = \frac{1}{\text{Vol}(M, g)} \int_M \psi dV_g$$

almost surely.

3. ANALYTIC CONTINUATION OF EIGENFUNCTIONS AND SPECTRAL PROJECTIONS

We now turn to complex zeros of analytic continuations of random Riemannian waves. Before getting into the statistics of zeros, we need to recall the basic results on analytic continuation of eigenfunctions and to introduce the basic two-point kernels.

As mentioned in the introduction, for each analytic metric g there exists a unique plurisubharmonic exhaustion function ρ on $M_{\mathbb{C}}$ inducing a Kähler metric $\omega_g = i\partial\bar{\partial}\rho$ which agrees with g along M . We recall $\sqrt{\rho}(\zeta) = \frac{1}{2i} r_{\mathbb{C}}(\zeta, \bar{\zeta})$ where $r(x, y)$ is the distance function and $r_{\mathbb{C}}$ is its holomorphic extension to a small neighborhood of the anti-diagonal $(\zeta, \bar{\zeta})$ in $M_{\mathbb{C}} \times M_{\mathbb{C}}$; $\sqrt{\rho}$ is a solution of the homogeneous complex Monge-Ampère equation $(\partial\bar{\partial}\sqrt{\rho})^m = 0$ away from the real points. We refer to [GS1, GS2, LS1, GLS, Z3] for further background on Grauert tubes and adapted complex structures on cotangent bundles of analytic Riemannian manifolds.

The eigenfunctions φ_j admit holomorphic extensions $\varphi_j^{\mathbb{C}}$ to the maximal Grauert tube [Bou, GS2]. As a result, one can holomorphically extend the spectral measures $d\Pi_{[0, \lambda]}(x, y) =$

$\sum_j \delta(\lambda - \lambda_j) \varphi_j(x) \varphi_j(y)$ of $\sqrt{\Delta}$. The complexified diagonal spectral projections measure is defined by

$$d_\lambda \Pi_{[0, \lambda]}^{\mathbb{C}}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) |\varphi_j^{\mathbb{C}}(\zeta)|^2. \quad (38)$$

Henceforth, we generally omit the superscript and write the kernel as $\Pi_{[0, \lambda]}^{\mathbb{C}}(\zeta, \bar{\zeta})$. This kernel is not a tempered distribution due to the exponential growth of $|\varphi_j^{\mathbb{C}}(\zeta)|^2$. Since many asymptotic techniques assume spectral functions are of polynomial growth, we simultaneously consider the damped spectral projections measure

$$d_\lambda P_{[0, \lambda]}^\tau(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) e^{-2\tau \lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2, \quad (39)$$

which is a temperate distribution as long as $\sqrt{\rho}(\zeta) \leq \tau$. When we set $\tau = \sqrt{\rho}(\zeta)$ we omit the τ and put

$$d_\lambda P_{[0, \lambda]}(\zeta, \bar{\zeta}) = \sum_j \delta(\lambda - \lambda_j) e^{-2\sqrt{\rho}(\zeta) \lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2. \quad (40)$$

The integral of the spectral measure over an interval I gives $\Pi_I(x, y) = \sum_{j: \lambda_j \in I} \varphi_j(x) \varphi_j(y)$. Its complexification gives the kernel (9) along the diagonal,

$$\Pi_I(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I} |\varphi_j^{\mathbb{C}}(\zeta)|^2, \quad (41)$$

and the integral of (39) gives its temperate version

$$P_I^\tau(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I} e^{-2\tau \lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2, \quad (42)$$

or in the crucial case of $\tau = \sqrt{\rho}(\zeta)$,

$$P_I(\zeta, \bar{\zeta}) = \sum_{j: \lambda_j \in I} e^{-2\sqrt{\rho}(\zeta) \lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2, \quad (43)$$

3.1. Poisson operator as a complex Fourier integral operator. The damped spectral projection measure $d_\lambda P_{[0, \lambda]}^\tau(\zeta, \bar{\zeta})$ (39) is dual under the real Fourier transform in the t variable to the restriction

$$U(t + 2i\tau, \zeta, \bar{\zeta}) = \sum_j e^{(-2\tau + it)\lambda_j} |\varphi_j^{\mathbb{C}}(\zeta)|^2 \quad (44)$$

to the anti-diagonal of the mixed Poisson-wave group. We recall that the kernel $U(t + i\tau, x, y)$ of the the Poisson-wave operator $e^{i(t+i\tau)\sqrt{\Delta}}$ admits a holomorphic extension in the x variable to the closed Grauert tube M_τ . The adjoint of the Poisson kernel $U(i\tau, x, y)$ also admits an anti-holomorphic extension in the y variable. The sum (44) are the diagonal values of the complexified wave kernel

$$\begin{aligned} U(t + 2i\tau, \zeta, \bar{\zeta}') &= \int_M U(t + i\tau, \zeta, y) E(i\tau, y, \bar{\zeta}') dV_g(x) \\ &= \sum_j e^{(-2\tau + it)\lambda_j} \varphi_j^{\mathbb{C}}(\zeta) \overline{\varphi_j^{\mathbb{C}}(\zeta')}. \end{aligned} \quad (45)$$

We obtain (45) by orthogonality of the real eigenfunctions on M .

Since $U(t+2i\tau, \zeta, y)$ takes its values in the CR holomorphic functions on ∂M_τ , we consider the Sobolev spaces $\mathcal{O}^{s+\frac{n-1}{4}}(\partial M_\tau)$ of CR holomorphic functions on the boundaries of the strictly pseudo-convex domain M_ε , i.e.

$$\mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\tau) = W^{s+\frac{m-1}{4}}(\partial M_\tau) \cap \mathcal{O}(\partial M_\tau),$$

where W_s is the s th Sobolev space and where $\mathcal{O}(\partial M_\varepsilon)$ is the space of boundary values of holomorphic functions. The inner product on $\mathcal{O}^0(\partial M_\tau)$ is with respect to the Liouville measure

$$d\mu_\tau = (i\partial\bar{\partial}\sqrt{\rho})^{m-1} \wedge d^c\sqrt{\rho}. \quad (46)$$

We then regard $U(t+i\tau, \zeta, y)$ as the kernel of an operator from $L^2(M) \rightarrow \mathcal{O}^0(\partial M_\tau)$. It equals its composition $\Pi_\tau \circ U(t+i\tau)$ with the Szegő projector

$$\Pi_\tau : L^2(\partial M_\tau) \rightarrow \mathcal{O}^0(\partial M_\tau)$$

for the tube M_τ , i.e. the orthogonal projection onto boundary values of holomorphic functions in the tube.

This is a useful expression for the complexified wave kernel, because $\tilde{\Pi}_\tau$ is a complex Fourier integral operator with a small wave front relation. More precisely, the real points of its canonical relation form the graph Δ_Σ of the identity map on the symplectic one $\Sigma_\tau \subset T^*\partial M_\tau$ spanned by the real one-form $d^c\rho$, i.e.

$$\Sigma_\tau = \{(\zeta; rd^c\rho(\zeta)), \quad \zeta \in \partial M_\tau, r > 0\} \subset T^*(\partial M_\tau). \quad (47)$$

We note that for each τ , there exists a symplectic equivalence $\Sigma_\tau \simeq T^*M$ by the map $(\zeta, rd^c\rho(\zeta)) \rightarrow (E_\mathbb{C}^{-1}(\zeta), r\alpha)$, where $\alpha = \xi \cdot dx$ is the action form (cf. [GS2]).

The following result was first stated by Boutet de Monvel (for more details, see also [GS2, Z3]).

THEOREM 3.1. [Bou, GS2] $\Pi_\varepsilon \circ U(i\varepsilon) : L^2(M) \rightarrow \mathcal{O}(\partial M_\varepsilon)$ is a complex Fourier integral operator of order $-\frac{m-1}{4}$ associated to the canonical relation

$$\Gamma = \{(y, \eta, \iota_\varepsilon(y, \eta))\} \subset T^*M \times \Sigma_\varepsilon.$$

Moreover, for any s ,

$$\Pi_\varepsilon \circ U(i\varepsilon) : W^s(M) \rightarrow \mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\varepsilon)$$

is a continuous isomorphism.

We obtain the holomorphic extension of the eigenfunctions φ_λ of eigenvalue λ^2 by applying the complex Fourier integral operator $U(i\tau)$:

$$U(i\tau)\varphi_\lambda = e^{-\tau\lambda}\varphi_\lambda^\mathbb{C}. \quad (48)$$

Combining Theorem 3.1 with a standard Sobolev estimate, we obtain:

COROLLARY 3.2. [Bou, GLS] Each eigenfunction φ_λ has a holomorphic extension to B_ε^*M satisfying

$$\sup_{\zeta \in M_\varepsilon} |\varphi_\lambda^\mathbb{C}(\zeta)| \leq C_\varepsilon \lambda^{m+1} e^{\varepsilon\lambda}.$$

The order of magnitude reflects the fact that $E(i\tau)$ smooths to order $-\frac{m-1}{4}$, since it is a bounded operator from a manifold M of real dimension m to one ∂M_τ of dimension $2m-1$.

We will also need the following Lemma from [Z3] (Lemma 3.1):

LEMMA 3.3. *Let $a \in S^0(T^*M - 0)$. Then for all $0 < \varepsilon < \varepsilon_0$, we have:*

$$U(i\varepsilon)^* \Pi_\varepsilon a \Pi_\varepsilon U(i\varepsilon) \in \Psi^{-\frac{m-1}{2}}(M),$$

with principal symbol equal to $a(x, \xi) |\xi|_g^{-\frac{(m-1)}{2}}$.

Here, $\Psi^s(M)$ is the class of pseudodifferential operators of order s and ε_0 is the radius of the maximal Grauert tube.

4. COMPLEX ZEROS OF RANDOM WAVES: PROOF OF THEOREM 3

Our first result is a formula for the expected distribution of complex zeros. We retain the same ensembles and notation from the real case.

PROPOSITION 4.1. *For any test form ψ ,*

$$\frac{1}{\lambda} \langle \mathbf{E}_{\gamma_N} [Z_{f_N^{\mathbb{C}}}], \psi \rangle = \frac{1}{N} \langle \partial \bar{\partial} \log \Pi_{I_N}(\zeta, \bar{\zeta}), \psi \rangle + O\left(\frac{1}{N}\right).$$

Proof. By the Poincaré-Lelong formula $Z_{f^{\mathbb{C}}} = \frac{i}{2\pi} \partial \bar{\partial} \log |f^{\mathbb{C}}(\zeta)|^2$, hence we have

$$\mathbf{E}_{\gamma_N} [Z_f^{\mathbb{C}}] = \frac{i}{2\pi} \mathbf{E}_{\gamma_N} \partial \bar{\partial} \log |f^{\mathbb{C}}(\zeta)|^2 = \frac{i}{2\pi} \partial \bar{\partial} \mathbf{E}_{\gamma_N} \log |f^{\mathbb{C}}(\zeta)|^2. \quad (49)$$

We write $f = \sum_{\lambda_j \in I_N} c_{Nj} \varphi_{Nj}$ and then complexify to obtain $f^{\mathbb{C}}(\zeta) = \sum_{\lambda_j \in I_N} c_{Nj} \varphi_{Nj}^{\mathbb{C}}(\zeta)$. Thus, we need to calculate $\mathbf{E}_{\gamma_N} \log |\sum_{\lambda_j \in I_N} c_{Nj} \varphi_{Nj}^{\mathbb{C}}|$.

Define

$$\Phi_N = (\varphi_{N1}, \dots, \varphi_{Nd_N}) : M \rightarrow \mathbb{C}^{d_N}$$

to be the vector of eigenfunctions from the orthonormal basis of eigenfunctions in the spectral interval. Then,

$$|\Phi_N^{\mathbb{C}}(\zeta)|^2 = \sum_{j=1}^{d_N} |\varphi_{Nj}(\zeta)|^2 = \Pi_{I_N}^{\mathbb{C}}(\zeta, \bar{\zeta}).$$

We use the notation:

$$\frac{\Phi_N^{\mathbb{C}}(\zeta)}{|\Phi_N^{\mathbb{C}}(\zeta)|} = U(\zeta) + iV(\zeta), \quad ; \quad U(\zeta), V(\zeta) \in \mathbb{R}^{d_N}, |U|^2 + |V|^2 = 1. \quad (50)$$

LEMMA 4.2. $\mathbf{E}_{\gamma_N} \log |f^{\mathbb{C}}(\zeta)|^2 = \log \Pi_{I_N}(\zeta, \bar{\zeta}) + G_N(\zeta, \bar{\zeta})$, where $G_N(\zeta, \bar{\zeta})$ is the uniformly bounded sequence of continuous functions on $M_{\mathbb{C}}$ given by

$$G_N(\zeta, \bar{\zeta}) = \Gamma'(\frac{1}{2}) + \Gamma(\frac{1}{2}) \log \max\{\|U + JV\|^2, \|U - JV\|^2\},$$

where J is the natural complex structure on the real 2-plane spanned by U, V in \mathbb{R}^{d_N} .

Proof. We have,

$$\begin{aligned} \mathbf{E}_{\gamma_N} \log |\sum_{\lambda_j \in I_N} c_{Nj} \varphi_{Nj}^{\mathbb{C}}| &= \int_{\mathbb{R}^{d_N}} \log |\langle c, \Phi_N^{\mathbb{C}} \rangle| d\nu_k(a) \\ &= \log |\Phi_N^{\mathbb{C}}(\zeta)|^2 + G_N(\zeta, \bar{\zeta}), \end{aligned}$$

where

$$G_N(\zeta, \bar{\zeta}) = \int_{\mathbb{R}^{d_N+1}} e^{-|c|^2/2} \log |\langle c, \frac{\Phi_N^{\mathbb{C}}(\zeta)}{|\Phi_N^{\mathbb{C}}(\zeta)|} \rangle| dc \quad (51)$$

Hence it suffices to calculate $G_N(\zeta, \bar{\zeta})$.

The span of U, V is a real two-dimensional plane in \mathbb{R}^{d_N} when U, V are linearly independent; we will consider the other case in the remark below. We rotate coordinates in the integral for each ζ so that

$$U = (\cos \theta \cos \varphi)e_1, \quad V = (\cos \theta \sin \varphi)e_1 + \sin \theta e_2,$$

where $\{e_j\}$ is the standard basis of \mathbb{R}^m and where θ, φ are real angles depending on ζ and k . Then,

$$\left\langle a, \frac{\Phi_k^{\zeta}}{|\Phi_k^{\zeta}(\zeta)|} \right\rangle = a_1(\cos \theta \cos \varphi) + i \cos \theta \sin \varphi + i a_2 \sin \theta.$$

After rotating coordinates in the integral, the integrand depends only on a_1, a_2 , so we may integrate out the remaining variables a_3, \dots, a_{d_N} and obtain

$$\begin{aligned} G_N(\zeta, \bar{\zeta}) &= \int_{\mathbb{R}^2} e^{-|a|^2/2} \log \left| \left\langle a, \frac{\Phi_N^{\zeta}(\zeta)}{|\Phi_N^{\zeta}(\zeta)|} \right\rangle \right| da \\ &= \int_0^\infty e^{-\frac{r^2}{2}} \int_{S^1} \log \left| \langle r\omega, \frac{\Phi_N^{\zeta}(\zeta)}{|\Phi_N^{\zeta}(\zeta)|} \rangle \right| r dr d\omega = \\ &= \Gamma'(\frac{1}{2}) + \Gamma(\frac{1}{2}) \int_{S^1} \log \left| \langle \omega, \frac{\Phi_N^{\zeta}(\zeta)}{|\Phi_N^{\zeta}(\zeta)|} \rangle \right| d\omega \\ &= \Gamma'(\frac{1}{2}) + \Gamma(\frac{1}{2}) G_2^N(\zeta, \bar{\zeta}), \end{aligned}$$

where

$$\begin{aligned} G_2^N(\zeta, \bar{\zeta}) &= \int_{S^1} \log |\omega_1(\cos \theta \cos \varphi + i \cos \theta \sin \varphi) + i \omega_2 \sin \theta| d\omega \\ &= \int_0^{2\pi} \log |\alpha e^{iu} + \beta e^{-iu}| du = \int_0^{2\pi} \log |\alpha e^{i2u} + \beta| du \\ &= \max \{ \log |\alpha|, \log |\beta| \} \end{aligned} \tag{52}$$

with

$$2\alpha = \langle U + iV, e_1 - ie_2 \rangle = \cos \theta e^{i\varphi} + \sin \theta, \quad 2\beta = \langle U + iV, e_1 + ie_2 \rangle = \cos \theta e^{i\varphi} - \sin \theta.$$

In the two dimensional real space spanned by e_1, e_2 we may define a complex structure by $Je_1 = e_2, Je_2 = -e_1$ and then

$$G_2^N(\zeta, \bar{\zeta}) = \log \max \{ \|U + JV\|^2, \|U - JV\|^2 \}.$$

For any two vectors U, V with $\|U\|^2 + \|V\|^2 = 1$,

$$1 \leq \max \{ \|U + JV\|^2, \|U - JV\|^2 \} = \max \{ 1 + 2\langle U, JV \rangle, 1 - 2\langle U, JV \rangle \} \leq 3.$$

Therefore the logarithm is bounded above and below and is clearly continuous. This completes the proof of Lemma 4.2. \square

To complete the proof of Proposition 4.1, it suffices to integrate the $\partial\bar{\partial}$ by parts onto f in the G_N term, and use that G_N is uniformly bounded. \square

Remark: When $\zeta \in M$ we have $V = 0$ and $\|U\| = 1$ so the calculation is slightly different. It is possible that there exist other $\zeta \in M_{\mathbb{C}}$ such that U, V are linearly dependent. When

$V = 0$, we put $U = e_1$ and obtain the limiting case when $\varphi = 0$, $\langle a, \frac{\Phi_k^{\mathbb{C}}}{|\Phi_k^{\mathbb{C}}(\zeta)|} \rangle = a_1$. We integrate out all but a_1 and get

$$G_N(\zeta, \bar{\zeta}) = \int_{\mathbb{R}} e^{-|a|^2/2} \log |a| da, \quad \zeta \in M.$$

For sufficiently Grauert tubes, it is impossible that $U = 0$, but our analysis has not ruled out that U, V may be linearly dependent, i.e. the case $\theta = 0$. But the calculation above goes through without change in this case.

5. COMPLEXIFIED SPECTRAL PROJECTIONS AND WAVE GROUP

In view of Proposition 4.1, the remaining step in the proof of Theorem 3 is to show:

PROPOSITION 5.1.

$$\frac{1}{\lambda} \log \Pi_{[0, \lambda]}(\zeta, \bar{\zeta}) \rightarrow \sqrt{\rho}, \quad (53)$$

and

$$\frac{1}{N} \log \Pi_{I_N}(\zeta, \bar{\zeta}) \rightarrow \sqrt{\rho}. \quad (54)$$

For both the long $[0, \lambda]$ and short I_N intervals, the upper bound is rather simple to prove. The lower bound is more difficult, primarily for the short intervals. As discussed above, in the Zoll case, the eigenvalues lie in clusters of width N^{-1} around an arithmetic progression, so unless the intervals I_N are centered along this progression there need not exist any eigenvalues in I_N and the logarithm would equal $-\infty$. We have by definition centered the intervals on the arithmetic progression, but clearly this choice must play a role in the proof.

5.1. Proof of Proposition 5.1. We first give a simple comparison between the logarithmic asymptotics of Π_{I_N} to those of P_{I_N} (cf. (43)).

LEMMA 5.2. *For any $\tau = \sqrt{\rho}(\zeta) > 0$, there exists $C, c > 0$ so that*

$$\frac{c}{\lambda} + 2\sqrt{\rho}(\zeta) + \frac{1}{\lambda} \log P_{[\lambda, \lambda+1]}(\zeta, \bar{\zeta}) \leq \frac{1}{\lambda} \log \Pi_{[\lambda, \lambda+1]}(\zeta, \bar{\zeta}) \leq 2\sqrt{\rho}(\zeta) + \frac{1}{\lambda} \log P_{[\lambda, \lambda+1]}(\zeta, \bar{\zeta}) + \frac{C}{\lambda},$$

hence

$$\frac{1}{N} \log \Pi_{I_N}(\zeta, \bar{\zeta}) = 2\sqrt{\rho}(\zeta) + \frac{1}{N} \log P_{I_N}(\zeta, \bar{\zeta}) + O\left(\frac{1}{N}\right),$$

where the remainder is uniform.

Proof. Since $-1 \leq N - \lambda_{Nj} \leq 1$ for $\lambda_{Nj} \in I_N$, we have

$$ce^{2N\sqrt{\rho}(\zeta)} \sum_{j=1}^{d_N} e^{-2\sqrt{\rho}(\zeta)\lambda_j} |\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2 \leq \Pi_{I_N}(\zeta, \bar{\zeta}) \leq Ce^{2N\sqrt{\rho}(\zeta)} \sum_{j=1}^{d_N} e^{-2\sqrt{\rho}(\zeta)\lambda_j} |\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2. \quad (55)$$

□

Next we observe that the upper bound in both the cutoff ensemble

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \Pi_{[0, \lambda]}(\zeta, \bar{\zeta}) \leq \sqrt{\rho} \quad (56)$$

follows immediately from Corollary 3.2 and the fact (cf. Lemma 5.2) that $\Pi_{[0, \lambda]} \leq e^{\lambda\sqrt{\rho}} P_{[0, \lambda]}$. It is similar but simpler in the fixed frequency ensemble.

Hence, it remains to prove is the lower bounds. As mentioned above, the global nature of the geodesic flow must play a role in the proof. The route we choose uses the minimal amount of information necessary to obtain the result: we only use the behavior of the spectral projections in the real domain to deduce a lower bound in the complex domain. We follow the method of [SZ1].

Put $V_N(\zeta) := \frac{1}{N} \log \Pi_{[N, N+1]}(\zeta, \bar{\zeta})$. Then V_N is plurisubharmonic in M_ε , and we would like to show that $V_N \rightarrow \sqrt{\rho}$ in $L^1(M_\varepsilon)$. If not, we can find a subsequence $\{V_{N_k}\}$ with $\|V_{N_k} - \sqrt{\rho}\|_{L^1(M_\varepsilon)} \geq \delta > 0$. Since $\{V_N\}$ is uniformly bounded above, a standard result on subharmonic functions (see [Ho, Theorem 4.1.9]) implies that sequence $\{V_{N_k}\}$ either converges uniformly to $-\infty$ on M_ε or else has a subsequence which is convergent in $L^1(M_\varepsilon)$. The first possibility cannot occur since $V_N(x) \rightarrow 0$ on the totally real submanifold M .

Therefore there exists a subsequence, which we continue to denote by $\{V_{N_k}\}$, which converges in $L^1(M_\varepsilon)$ to some $G \in L^1(M_\varepsilon)$. By passing to a further subsequence, we may assume that $V_{N_k} \rightarrow G$ converges pointwise a.e. in M_ε . Let $V(\zeta) = \limsup_{k \rightarrow \infty} V_{N_k}(z)$ (a.e) and let

$$V^*(z) := \limsup_{w \rightarrow z} V(w)$$

be the upper-semicontinuous regularization of V . Then V^* is plurisubharmonic on M_ε , $V^* \leq \sqrt{\rho}$ (a.e.) and $V^* = V$ a.e.

Since $\|V_{N_k} - \sqrt{\rho}\|_{L^1(U')} \geq \delta > 0$, we know that $V^* \not\equiv \sqrt{\rho}$. Hence, for some $\varepsilon > 0$, the open set $U_\varepsilon = \{z \in M_\varepsilon : V^* < \sqrt{\rho} - \varepsilon\}$ is non-empty. Let U'' be a non-empty, relatively compact, open subset of U_ε . Then by Hartogs' Lemma, there exists a positive integer K such that for $\zeta \in U''$, $k \geq K$, $V_{N_k} \leq \sqrt{\rho} - \varepsilon/2$, i.e.

$$\Pi_{[N, N+1]}(\zeta, \bar{\zeta}) \leq e^{(\sqrt{\rho} - \varepsilon)N_k}, \quad \zeta \in U'', \quad k \geq K. \quad (57)$$

By Lemma 5.2, it follows that

$$P_N(\zeta, \bar{\zeta}) \leq e^{-\varepsilon N_k}, \quad \zeta \in U'', \quad k \geq K. \quad (58)$$

We now let $\chi \in C_0^\infty(U'')$ be a smooth bump function which equals one on some ball $B \subset U''$. By (58), we have

$$\int_{\partial M_\varepsilon} \chi P_N(\zeta, \bar{\zeta}) d\mu \leq e^{-\varepsilon N_k}. \quad (59)$$

We observe that

$$\begin{aligned} \int_{\partial M_\varepsilon} \chi P_N(\zeta, \bar{\zeta}) d\mu &= \sum_{j: \lambda_j \in [N, N+1]} e^{-2\varepsilon \lambda_j} \int_{\partial M_\varepsilon} \chi |\varphi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2 d\mu \\ &= \sum_{j: \lambda_j \in [N, N+1]} \langle \chi U(i\sqrt{\rho}(\zeta)) \varphi_{\lambda_j}, U(i\sqrt{\rho}(\zeta)) \varphi_{\lambda_j} \rangle_{L^2(\partial M_\varepsilon)} \\ &= \sum_{j: \lambda_j \in [N, N+1]} \langle U(i\sqrt{\rho}(\zeta))^* \chi U(i\sqrt{\rho}(\zeta)) \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle_{L^2(M)} \\ &= \text{Tr} \Pi_{I_N} U(i\sqrt{\rho}(\zeta))^* \chi U(i\sqrt{\rho}(\zeta)). \end{aligned} \quad (60)$$

By Theorem 3.1 and Lemma 3.3, $U(i\sqrt{\rho}(\zeta))^* \chi U(i\sqrt{\rho}(\zeta))$ is a pseudo-differential operator with principal symbol equal to $|\xi|^{-\frac{n-1}{2}} \chi(x, \xi)$. For intervals I_N satisfying our assumptions, it follows from a standard local Weyl law [Ho] (see also [Z1] for further discussion in the

present context) that for a zeroth order pseudodifferential operator A , and for a Laplacian which is either aperiodic or Zoll,

$$\text{Tr}\Pi_{I_N}A = C_m N^{m-1} \int_{B^*M} \chi d\mu + o(N^{m-1}). \tag{61}$$

Putting $A = \sqrt{\Delta}^{\frac{n-1}{2}} U(i\sqrt{\rho}(\zeta))^* \chi U(i\sqrt{\rho}(\zeta))$

$$\text{Tr}\Pi_{I_N}U(i\sqrt{\rho}(\zeta))^* \chi U(i\sqrt{\rho}(\zeta)) \sim C_m N^{-\frac{m-1}{2}} \int_{B^*M} \chi d\mu. \tag{62}$$

The latter asymptotics contradict the exponential decay of (59). □

6. APPENDIX ON TAUBERIAN THEOREMS

We record here the statements of the Tauberian theorems that we use in the article. Our main reference is [SV], Appendix B and we follow their notation.

We denote by F_+ the class of real-valued, monotone nondecreasing functions $N(\lambda)$ of polynomial growth supported on \mathbb{R}_+ . The following Tauberian theorem uses only the singularity at $t = 0$ of \widehat{dN} to obtain a one term asymptotic of $N(\lambda)$ as $\lambda \rightarrow \infty$:

THEOREM 6.1. *Let $N \in F_+$ and let $\psi \in \mathcal{S}(\mathbb{R})$ satisfy the conditions: ψ is even, $\psi(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, $\hat{\psi} \in C_0^\infty$, and $\hat{\psi}(0) = 1$. Then,*

$$\psi * dN(\lambda) \leq A\lambda^\nu \implies |N(\lambda) - N * \psi(\lambda)| \leq CA\lambda^\nu,$$

where C is independent of A, λ .

REFERENCES

[AT] R. J. Adler and J. E. Taylor, *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.

[B] M. V. Berry, Regular and irregular semiclassical wavefunctions. *J. Phys. A* 10 (1977), no. 12, 2083–2091.

[BBG] P. Bérard, G. Besson, and S. Gallot, Embedding Riemannian manifolds by their heat kernel. *Geom. Funct. Anal.* 4 (1994), no. 4, 373–398.

[BSZ1] Bleher, Pavel; Shiffman, Bernard; Zelditch, Steve Universality and scaling of zeros on symplectic manifolds. *Random matrix models and their applications*, 31–69, *Math. Sci. Res. Inst. Publ.*, 40, Cambridge Univ. Press, Cambridge, 2001.

[BSZ2] P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds, *Invent. Math.* 142 (2000), no. 2, 351–395. <http://xxx.lanl.gov/abs/math-ph/9904020>.

[BSZ3] P. Bleher, B. Shiffman, and S. Zelditch, Poincaré-Lelong approach to universality and scaling of zero point correlations, *Comm. Math. Phys.* 208 (2000), 771–785.

[BGS] G. Blum, S. Gnutzmann and U. Smilansky, Nodal Domains Statistics: A Criterion for Quantum Chaos, *Physical Review Letters*, 2002.

[BS] E. Bogomolny and C. Schmit, Percolation Model for Nodal Domains of Chaotic Wave Functions, *Physical Review Letters*, 2002.

[Bou] L. Boutet de Monvel, Convergence dans le domaine complexe des séries de fonctions propres. *C. R. Acad. Sci. Paris Sér. A-B* 287 (1978), no. 13, A855–A856.

[BW] F. Bruhat and H. Whitney, Quelques propriétés fondamentales des ensembles analytiques-réels. *Comment. Math. Helv.* 33 1959 132–160.

- [DF] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, *Invent. Math.* 93 (1988), 161-183.
- [DG] J. J. Duistermaat and V. W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.* 29 (1975), no. 1, 39–79.
- [FGS] G. Foltin, S. Gnutzmann and U. Smilansky, The morphology of nodal lines of random waves versus percolation, *Journal of Physics A* 37 (2004), 11363-11371.
- [GLS] F. Golse, E. Leichtnam, and M. Stenzel, Intrinsic microlocal analysis and inversion formulae for the heat equation on compact real-analytic Riemannian manifolds. *Ann. Sci. École Norm. Sup. (4)* 29 (1996), no. 6, 669–736.
- [Gr] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, *Math. Annalen* 146 (1962), 331–368.
- [GS1] V. Guillemin and M. Stenzel, Grauert tubes and the homogeneous Monge-Ampère equation. *J. Differential Geom.* 34 (1991), no. 2, 561–570.
- [GS2] V. Guillemin and M. Stenzel, Grauert tubes and the homogeneous Monge-Ampère equation. II. *J. Differential Geom.* 35 (1992), no. 3, 627–641.
- [Ho] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Volume III, Springer-Verlag Berlin Heidelberg, 1983.
- [JL] D. Jerison and G. Lebeau, Nodal sets of sums of eigenfunctions, Ch. 14 of *Harmonic analysis and partial differential equations* (Chicago, IL, 1996), 223–239, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1999.
- [LS1] L. Lempert and R. Szöke, Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds. *Math. Ann.* 290 (1991), no. 4, 689–712.
- [Mc] B. McDonald, Thesis, Johns Hopkins Univ. 2007.
- [NS] F. Nazarov and M. Sodin, On the Number of Nodal Domains of Random Spherical Harmonics (arXiv:0706.2409)
- [Ne] J. Neuheisel, The asymptotic distribution of nodal sets on spheres, Ph.D. thesis, Johns Hopkins University, 2000.
- [ORW] F. Oravecz, Z. Rudnick and I. Wigman, Nodal sets for random eigenfunctions on the torus, math-ph/0609072.
- [PW] R. Paley and N. Wiener, *Fourier transforms in the complex domain*. Reprint of the 1934 original. American Mathematical Society Colloquium Publications, 19. American Mathematical Society, Providence, RI, 1987.
- [RW] Z. Rudnick and I. Wigman, On the volume of nodal sets for eigenfunctions of the Laplacian on the torus, to appear in *Annales Henri Poincaré* (rXiv:math-ph/0702081).
- [SV] Y. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators. Translated from the Russian manuscript by the authors*. Translations of Mathematical Monographs, 155. American Mathematical Society, Providence, RI, 1997.
- [SS] O. Schramm and S. Sheffield, Contour lines of the two-dimensional discrete Gaussian free field, math.PR/0605337.
- [SZ1] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.* 200 (1999), no. 3, 661–683.
- [SZ2] B. Shiffman and S. Zelditch, Number variance of random zeros on complex manifolds, to appear in *GAFA* (arXiv: arXiv:math/0608743v2)
- [SZ3] B. Shiffman and S. Zelditch, Number variance of random zeros on complex manifolds, II: smooth statistics (arXiv:0711.1840).
- [T] M. E. Taylor, *Pseudodifferential operators*. Princeton Mathematical Series, 34. Princeton University Press, Princeton, N.J., 1981.
- [Z1] S. Zelditch, A random matrix model for quantum mixing, *Int.Math. Res.Not.* 3 (1996), 115-137.
- [Z2] S. Zelditch, Fine structure of Zoll spectra. *J. Funct. Anal.* 143 (1997), no. 2, 415–460.
- [Z3] S. Zelditch, Complex zeros of real ergodic eigenfunctions, *Invent. Math.* 167 (2007), no. 2, 419–443.

- [Z4] S. Zelditch, Pluri-subharmonic theory on Grauert tubes of real analytic Riemannian manifolds (in preparation).

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA