ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

REAL CLOSED SPACES

NIELS SCHWARTZ

Dedicated to the memory of Gus Efroymson

Let R be a fixed real closed field, $(M, \mathcal{O})$ an affine semialgebraic (= sa) space (see Delfs and Knebusch [1]) where M is a semialgebraic subset of some R^m. Through the correspondence between sa subsets of R^m and constructible subsets of $X(R^m) = X(R[X_1, \ldots, X_m])$, the real spectrum $\mathcal{O}$ of $R[X_1, \ldots, X_m]$ (see [2]) can also be considered as a sheaf on $\tilde{M}$, the constructible subset of $X(R^m)$ corresponding to M. Let $\mathscr{A}$ be the constant sheaf $R[X_1, \ldots, X_m]$ on M. Then $\mathcal{O}$ can be reconstructed from $\mathscr{A}$ by certain types of ring extensions. The same construction can be done starting from any ring A and any constructible subset K of the real spectrum $X(A)$ of A. In this way one obtains locally ringed spaces which are called affine real closed spaces (real closed since these spaces can be viewed as generalizing real closed fields). A real closed space is a locally ringed space which has an open cover by affine real closed spaces. In particular, to any sa space $(M, \mathcal{O})$ with an open affine cover $M = \bigcup M_i$ one constructs a real closed space $\tilde{M}$ by first taking the affine real closed spaces $\tilde{M}$ corresponding to the M_i as explained above and then glueing these together along the subsets corresponding to the $M_i \cap M_j$.

It is possible to develop a theory of real closed spaces very much reminiscent of and very closely related to Grothendieck's theory of schemes ([EGA]). For example, the notions of quasi-compact, quasi-separated, separated, universally closed morphisms can be defined just as in [EGA]. Here is a small sample of results stated only for the special case of real closed spaces associated to sa spaces.

THEOREM 1. *The following are equivalent*: (a) *M is separated*; (b) *$\tilde{M}$ is separated*; (c) *$R(M)$ fulfills a valuative condition* (*as in* [EGA I, 5.5.4]).

THEOREM 2. *The following are equivalent*: (a) *M is affine*; (b) *M is regular* (*see* [3]); (c) *$R(M)$ is affine*; *the space of closed points of $\tilde{M}$ is Hausdorff.*

THEOREM 3. *The following are equivalent*: (a) *M is complete*; (b) *$R(M)$ is separated and universally closed over R*; (c) *M can be embedded as a* *closed and bounded subspace into some R^m. In particular, if the equivalent conditions hold, then M is affine.*

The real closed spaces associated with sa spaces can be characterized in the following way.

THEOREM 4. *For a real closed space K over R the following are equivalent:*

a) *$K \cong \tilde{M}$ for some* sa *space M.*

b) *K is quasi-separated and of finite type over R.*

REFERENCES

1. H. Delfs and M. Knebusch, *Semialgebraic topology over a real closed field* II: *basic theory of semialgebraic spaces*, Math. Z. **178** (1981), 175–213.

2. M. Coste and M. F. Coste-Roy, *La topologie du spectre réel*, Contemporary Mathematics **8** (1982), 27–59.

3. R. Robson, *Embedding semi-algebraic spaces*, Math. Z. **183** (1983), 365–370.

MATHEMATISCHE INSTITUT, U OF MUNICH, THERESIENSTR. 39, D-8000 MUNCHEN 2, WEST GERMANY