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REAL DIVISORS ON REAL CURVES

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Let $\mathcal{C}$ be an affine or projective smooth real algebraic curve, that is, a smooth complex curve in $\mathbb{A}^n_{\mathbb{C}}$ or $\mathbb{P}^n_{\mathbb{C}}$ defined by real equations, having a non empty real part. Then every divisor E on $\mathcal{C}$, which is linearly equivalent to its conjugate E^c, is also equivalent to a divisor supported on a set of real points of $\mathcal{C}$.

1. Introduction.

Let $\mathcal{C}$ be a smooth connected algebraic curve in the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ (or in the complex affine space $\mathbb{A}^n_{\mathbb{C}}$); we say that $\mathcal{C}$ is a real curve if it is invariant by conjugation that is, if it can be defined by real equations. If the set of the real points $\mathcal{C}(\mathbb{R})$ of $\mathcal{C}$ is not empty, then $\mathcal{C}(\mathbb{R})$ is actually a curve in $\mathbb{P}^n(\mathbb{R})$ (or $\mathbb{A}^n(\mathbb{R})$), since isolated points of $\mathcal{C}(\mathbb{R})$ must be singular points of $\mathcal{C}$.

In the present paper we study linear equivalence between divisors on $\mathcal{C}$ with regard to the action of conjugation and in particular to divisors supported on $\mathcal{C}(\mathbb{R})$. By the term "divisors" we will always mean Weil divisors; but we could consider Cartier divisors or linear vector bundles as well, because of the natural isomorphism between the groups of classes modulo linear equivalence $Cl(\mathcal{C})$, $CaCl(\mathcal{C})$ and $Pic(\mathcal{C})$ (see [6], Ch.II, n.6).

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Looking at the real structure of $\mathcal{C}$, we can consider the following subgroups of $Cl(\mathcal{C})$:

$Cl(\mathcal{C})^+$ classes of divisors D equivalent to the conjugate D^c,

$Cl(\mathcal{C})^c$ classes of real divisors i.e. divisors $D = D^c$,

$Cl(\mathcal{C})_{\mathbb{R}}$ classes of divisors supported on $\mathcal{C}(\mathbb{R})$,

with the natural inclusions: $Cl(\mathcal{C}) \supseteq Cl(\mathcal{C})^+ \supseteq Cl(\mathcal{C})^c \supseteq Cl(\mathcal{C})_{\mathbb{R}}$.

In our hypotheses, the two intermediate groups are equal (see [5], Proposition 2.2 or [3], Proposition 2.7.4); the main result of the present paper (Theorem 5.1) states that, moreover, the fourth group is equal to the two former ones.

In order to prove this, we construct linear systems on $\mathcal{C}$, that map $\mathcal{C}$ onto plane curves which satisfy suitable conditions of reality, that is, meeting enough real lines in many real points. Those linear systems allow us to find a large set of real divisors equivalent to divisors supported on $\mathcal{C}(\mathbb{R})$; finally, by means of topological properties of $\mathcal{C}$ as a Riemann surface, we conclude that every real divisor D on $\mathcal{C}$ is linearly equivalent to a divisor E supported on $\mathcal{C}(\mathbb{R})$.

We want to stress that such a divisor E is not in general an effective divisor, even if D is. In fact, if we consider the (smallest) effective real divisors $P + P^c$ for every point P on $\mathcal{C} - \mathcal{C}(\mathbb{R})$, they are equivalent to pairs of real points $R_1 + R_2$ if and only if $\mathcal{C}$ is a rational or an elliptic curve. If $\mathcal{C}$ is hyperelliptic, then we have only one g^1_2 (linear system of dimension 1 and degree 2: see [6], Ch. IV, Proposition 5.3) and then we can have a linear equivalence as above, only for some, but not for all, points P on $\mathcal{C} - \mathcal{C}(\mathbb{R})$.

Stronger results hold in dimension greater than 1. Let V be a complex variety, non singular in codimension 1, invariant by conjugation and such that its real part $V(\mathbb{R})$ is Zariski dense in V. If V is affine, then every effective divisor $D = D^c$ on V is linearly equivalent to an effective irreducible divisor E such that $E \cap V(\mathbb{R})$ is Zariski dense in E; if V is projective, then every effective divisor $D = D^c$ is equivalent to a divisor of the type $E - kH$ (k integer), where E and H are irreducible, $E \cap V(\mathbb{R})$ and $H \cap V(\mathbb{R})$ are Zariski dense in E and H respectively and H is cut by a hyperplane (see [8], Proposition 2.5 (proof), Theorem 4.3 and Corollary 4.6).

The main difference between dimension 1 and greater than 1 arises exactly from some good properties of the hyperplane sections, true in dimension greater than 1, but not for curves (see also Remark 5.2). If $\dim(V) > 1$, by Bertini's theorem, a general real hyperplane section passing through a general point of $V(\mathbb{R})$ cuts on V an irreducible divisor, having a dense real part. If $V = \mathcal{C}$ is a curve, which is not a line, then all its hyperplane sections are reducible; furthermore, it may happen that no hyperplane section has a dense real part: in Example 5.3 we produce a smooth real curve $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{C}}$ of degree 4, with a non empty real part, that meets every line in less than 4 real points.

In the last section of the paper we extend the previous result to the divisor class group $Cl(\mathbb{R}[\mathcal{C}])$ of an integrally closed domain $\mathbb{R}[\mathcal{C}]$ of real polynomial functions on $\mathcal{C}$.

Observe that $\mathbb{R}[\mathcal{C}]$ is also the ring of polynomial functions on $\mathcal{C}(\mathbb{R})$. Several rings of functions defined over a real variety $V(\mathbb{R})$ have been studied, namely real global analytic, Nash, rational regular and, finally, polynomial functions. In the paper [2] the divisor class groups of analytic, Nash and rational regular functions are well described in geometrical terms, by homology classes, while very little was known about $Cl(\mathbb{R}[V])$, because of its "less geometrical" nature (see [2], section 7). So, in [8] we studied the divisor class group $Cl(\mathbb{R}[V])$ for a real variety V of dimension greater than 1 using algebraic tools. On the other hand, in the present paper we essentially use geometrical tools, but our geometrical object is the complex curve $\mathcal{C}$ (or the complex variety V) defined over $\mathbb{R}$, while in [2] the geometrical object was a real manifold that is the real part $V(\mathbb{R})$ of V.

2. Notations.

We will denote the fields of the real numbers and of the complex numbers by $\mathbb{R}$ and $\mathbb{C}$ respectively and by x^c, $\Re(x)$ and $\Im(x)$ the conjugation, the real part and the imaginary part of x respectively.

Let $\mathcal{C}$ be a projective curve in $\mathbb{P}^n_{\mathbb{C}}$, defined by real equations and let $\mathcal{C}(\mathbb{R})$ be its real part, that is, the collection of the real points of $\mathcal{C}$. In this paper we will always suppose $\mathcal{C}$ smooth and $\mathcal{C}(\mathbb{R})$ not empty: so $\mathcal{C}(\mathbb{R})$ is a "true" curve in the real projective space $\mathbb{P}^n(\mathbb{R})$.

We will denote by A and B respectively the coordinate ring of $\mathcal{C}$ as a variety defined over $\mathbb{R}$ and over $\mathbb{C}$, so that $B = A \otimes_{\mathbb{R}} \mathbb{C}$.

In a natural way, A and B are graded rings and we will always consider homogeneous elements in them, without further notices. In particular, $k(A)$ will be the degree zero quotient field of A, that is, the real rational functions field of $\mathcal{C}$.

For any element f in A (or B), $div(f)$ is the Weil divisor cut on $\mathcal{C}$ by f; in a similar way, if $x \in k(A)$, that is $x = f \cdot g^{-1}$, $f, g \in A$, then $div(x) = div(f) - div(g)$.

Finally, we will denote the complex curve $\mathcal{C}$ as a real manifold with the Euclidean topology (Riemann surface) by $\mathcal{S}$. We recall that $\mathcal{S}$ is connected and compact; moreover, in our hypothesis, $\mathcal{S} - C(\mathbb{R})$ has only one or two connected components and, in the second case, they are exchanged by conjugation (see [7] or [3], Proposition 2.7.10).

3. Linear systems on $\mathcal{C}$.

In the present section we construct, for any given point P on $\mathcal{C}$, a dimension 2 linear system $\mathcal{L}_P$ satisfying suitable "conditions of reality". In more concrete language this means that we can find, for any point P on $\mathcal{C}$, a suitable birational real map ϕ, that sends $\mathcal{C}$ to a real plane curve $\mathcal{C}'$. Moreover, we can choose ϕ so that $\phi(P)$ is a smooth point of $\mathcal{C}'$ and the real line l passing through $\phi(P)$ and $\phi(P^c)$ (respectively the tangent line to $\mathcal{C}'$ at $\phi(P)$, if $P = P^c$) cuts out $deg(\mathcal{C}') - 2$ more real points on $\mathcal{C}'$. Finally, we prove that, in such a situation, every line "sufficiently close" to l cuts out on $\mathcal{C}'$ divisors of the same type.

Lemma 3.1. *Let P be a point on $\mathcal{C}$. Then there is a linear system $\mathcal{L}_P = \mathbb{P}^2_{\mathbb{C}}$, without base points, satisfying the following properties:*

- *i) for a generic point Q on $\mathcal{C}$ (in particular for $Q = P$), $Q + Q^c$ imposes two independent conditions on $\mathcal{L}_P$;*
- *ii) $\mathcal{L}_P = \{div(\lambda f + \mu g + \nu h) - H\ ,\ (\lambda, \mu, \nu) \in \mathbb{P}^2_{\mathbb{C}}\}$ where $f, g, h \in A$ and H is the common divisor of $div(f)$, $div(g)$, $div(h)$;*
- *iii) $div(f) = P + P^c + R_1 + ..R_t + H$, where the R_i are distinct real points and neither P nor each R_i are contained in H.*

Proof. Let's choose t real distinct points $R_1, .., R_t$ such that the divisor $D = P + P^c + R_1 + .. + R_t$ be very ample (see [6], Chapter IV, Corollary 3.2). Let $|D| = \mathbb{P}^k_{\mathbb{C}}$ be the complete linear system of the effective divisors equivalent to D and let x_0 be the point in $\mathbb{P}^k_{\mathbb{C}}$ corresponding to D. Since D is very ample, the divisors containing any two distinct points of D make up a $\mathbb{P}^{k-2}_{\mathbb{C}} \subset \mathbb{P}^k_{\mathbb{C}}$ which contains x_0. So we can choose a $\mathbb{P}^2_{\mathbb{C}}$ in $\mathbb{P}^k_{\mathbb{C}}$ containing x_0 and meeting these $(k-2)$-planes only in x_0; such a linear system $\mathbb{P}^2_{\mathbb{C}} = \mathcal{L}_P$ has no base points and any two points of D impose two independent conditions on it.

The map $\varphi : \mathcal{C} \longrightarrow \mathbb{P}^2_{\mathbb{C}}$ associated to $\mathcal{L}_P$ is a birational transformation, since it is of degree 1 at each point of D. Therefore, $Q + Q^c$ imposes two independent conditions to $\mathcal{L}_P$, for almost all points Q in $\mathcal{C}$. Moreover, D is a divisor in $\mathcal{L}_P$, because we have chosen a plane in $|D| = \mathbb{P}^k_{\mathbb{C}}$ containing x_0. So we have i).

The complete linear system $|D|$ may be obtained as

$$\{div(\lambda_0 f_0 + .. + \lambda_k f_k) - H, \quad (\lambda_0, .., \lambda_k) \in \mathbb{P}^k_C\}$$

where $f_i \in B$ and H is the common divisor of all $div(f_i)$. We can suppose that $x_0 = (1, 0, .., 0)$, that is $div(f_0) = P + P^c + R_1 + .. + R_t + H$.

Moreover we can choose $f_0 \in A$ (for this, if $f_0 \notin A$, we multiply every f_i by f_0^c); hence $H = div(f_0) - D$ is invariant by conjugation, then H is also the common divisor of $div(\Re(f_i))$ and of $div(\Im(f_i))$, for $i = 1, .., k$. If we choose $k+1$ linearly independent elements $f_0' = f_0, f_1', \ldots, f_k'$ in the set $\{f_0, \Re(f_1), \Im(f_1), .., \Re(f_k), \Im(f_k)\}$, we get the linear system $|D|$ by means of elements of A.

Finally, we can choose the plane $\mathcal{L}_P$ in $|D| = \mathbb{P}^k_{\mathbb{C}}$ satisfying the previous properties and moreover, having real equations: for such a linear system $\mathcal{L}_P$ i) and ii) hold.

We still have to prove that the elements $f = f_0, g, h$ defining $\mathcal{L}_P$ can be chosen so that their common divisor H does not contain any point of D.

We can suppose $|H|$ very ample (for this, we multiply f, g, h by an element of A of sufficiently high degree) and take a divisor H' equivalent to H, invariant by conjugation and not containing any point of D. If $x \in k(A)$ is such that $div(x) = H' - H$, then xf, xg, xh belong to A, since $div(xf)$, $div(xg)$ and $div(xh)$ are effective; so we can get $\mathcal{L}_P$ by $div(\lambda xf + \mu xg + \nu xh) - H'$ and iii) is also satisfied. $\square$

Notation 3.2. *We will denote by $D_{(\lambda,\mu,\nu)}$ the divisor $div(\lambda f + \mu g + \nu h) - H$ of the linear system $\mathcal{L}_P = \{div(\lambda f + \mu g + \nu h) - H,\ (\lambda, \mu, \nu) \in \mathbb{P}^2_{\mathbb{C}}\}$ as defined in the previous lemma.*

Lemma 3.3. *Let $P \in \mathcal{C} - \mathcal{C}(\mathbb{R})$ and $\mathcal{L}_P$ be as in Lemma 3.1 and let (μ, ν) be any element in $\mathbb{R}^2$ such that $\|(\mu, \nu)\| < \epsilon$ for a suitable $\epsilon > 0$. Then :*

a) $D_{(1,\mu,\nu)}$ consists of a pair of non real conjugate points and of further real points;

b) if $(\mu', \nu') \neq (\mu, \nu)$ and $\|(\mu', \nu')\| < \epsilon$, then the pairs of non real conjugate points in $D_{(1,\mu,\nu)}$ and $D_{(1,\mu',\nu')}$ are distinct.

Proof. Let $M \subset \mathbb{P}^2_{\mathbb{C}} = \mathcal{L}_P$ be the closed algebraic subset of divisors containing some multiple points. So M is a proper subset of $\mathcal{L}_P$, by Bertini's theorem, and $D_{(1,0,0)} \notin M$, by construction. If we consider the affine real plane $\mathbb{A}^2(\mathbb{R}) = \{(1, \mu, \nu), \mu, \nu \in \mathbb{R}\} \subset \mathbb{P}^2_{\mathbb{C}}$, then the Euclidean distance d between $(1, 0, 0)$ and $M \cap \mathbb{A}^2(\mathbb{R})$ is strictly positive.

Therefore, if $\|(\mu, \nu)\| < d$, then $D_{(1,\mu,\nu)}$ contains exactly two non-real conjugate points and $deg(\mathcal{L}_P) - 2$ real points. In fact, when we move $(1, \mu', \nu')$ in the line interval $[(1, 0, 0), (1, \mu, \nu)]$, the number of real points in $D_{(1,\mu',\nu')}$ can change only if $D_{(1,\mu',\nu')}$ moves through a divisor containing a double real point.

By construction, $\{Q \in \mathcal{C}$ s.t. $Q + Q^c$ imposes only one condition on $\mathcal{L}_P\}$ is a finite set (see Lemma 3.1.i) and moreover $(1, 0, 0)$ doesn't belong to the lines $r_1, ..., r_s$ on $\mathbb{P}^2_{\mathbb{C}} = \mathcal{L}_P$ corresponding to these points. Thus, the distance d' between $(1, 0, 0)$ and $(\cup r_i) \cap \mathbb{A}^2(\mathbb{R})$ is strictly positive.

If we choose a positive number $\epsilon < \min\{d, d'\}$, then both a) and b) are satisfied. $\square$

4. The main result.

Let $\mathcal{S}$ be the Riemann surface corresponding to $\mathcal{C}$ with the Euclidean topology. In the present section we will consider the following equivalence among points of $\mathcal{S}$:

$P \sim_{\mathbb{R}} Q$ if the divisor $P + P^c - Q - Q^c$ is linearly equivalent to a divisor lying on $\mathcal{C}(\mathbb{R})$.

We want to show that all pairs of points on S are equivalent. So, we first prove that $\mathcal{U} \cap \mathcal{S} - C(\mathbb{R})$ is an Euclidean open set of S for every equivalence class $\mathcal{U}$, which implies, by topological properties of S, that there is an equivalence class $\mathcal{U}_1$ containing the whole $\mathcal{S} - C(\mathbb{R})$. On the other hand, the real part $\mathcal{C}(\mathbb{R})$ is completely contained in a class $\mathcal{U}_2$, thus we will conclude that $\mathcal{S} = \mathcal{U}_1 = \mathcal{U}_2$ by proving that $\mathcal{U}_1 \cap \mathcal{U}_2$ is not empty.

Lemma 4.1. *Let P be any point of $\mathcal{S} - C(\mathbb{R})$ and let $\mathcal{U}$ be the equivalence class of P modulo $\sim_{\mathbb{R}}$. Then the set $\mathcal{U} \cap \mathcal{S} - C(\mathbb{R})$ contains an Euclidean open neighbourhood of P.*

Proof. Let P, $\mathcal{L}_P$ and ϵ be as in the previous lemmas. Consider the algebraic real set $Y \subseteq \mathbb{P}^2(\mathbb{R}) \times \mathcal{S}$:

$$Y = \{((\lambda, \mu, \nu), Q) \; s.t. \; Q \in D_{(\lambda,\mu,\nu)}\}$$

and set $Z = Y \cap (\mathcal{B}_\epsilon \times (\mathcal{S} - C(\mathbb{R})))$ where $\mathcal{B}_\epsilon = \{(1, \mu, \nu)/\|(\mu, \nu)\| < \epsilon\}$; then Z is a semialgebraic real subset of Y(see [1] Ch. 2).

Now we observe that the first canonical projection $\pi_1 : Z \longrightarrow \mathcal{B}_\epsilon$ is surjective and $2 : 1$ (see Lemma 3.1); then Z is a 2-dimensional open semialgebraic real set.

Moreover, the second canonical projection $\pi_2 : Z \longrightarrow \mathcal{S}$ is injective (see Lemma 3.3 b)); so $\pi_2(Z)$ too is a 2-dimensional open semialgebraic real set, containing P and contained in $\mathcal{U}$. $\square$

Theorem 4.2. *Let D be a real divisor on $\mathcal{C}$ (that is $D = D^c$).*

Then D is linearly equivalent to a divisor E completely supported on $\mathcal{C}(\mathbb{R})$.

Proof. We can obtain any real divisor on $\mathcal{C}$ by summing up real points and pairs of non real conjugate points. So, without losing in generality, we may prove the assumption only for $D = P + P^c$, $P \in \mathcal{S} - C(\mathbb{R})$.

Consider the equivalence $\sim_{\mathbb{R}}$ among points of $\mathcal{S}$ and let $\mathcal{U}$ be an equivalence class such that $\mathcal{U} \cap \mathcal{S} - C(\mathbb{R})$ is not empty. We can see, from Lemma 4.1, that $\mathcal{U} \cap \mathcal{S} - C(\mathbb{R})$ is an Euclidean open subset of S, for it contains an open neighbourhood of every point. So each connected component of $\mathcal{S} - C(\mathbb{R})$ is completely contained in one of the equivalence classes $\mathcal{U}$, together with its conjugate, since each class $\mathcal{U}$ is invariant by conjugation because of its construction. Moreover $\mathcal{S} - C(\mathbb{R})$ has only one or two connected components, which are exchanged by conjugation in the second case (see [7] or [3], Proposition 2.7.10); so $\mathcal{S} - C(\mathbb{R}) \subset \mathcal{U}$ and then, for any pair of non real points P and Q, $P + P^c - Q - Q^c$ is linearly equivalent to a divisor completely supported on $\mathcal{C}(\mathbb{R})$.

We have still to prove that there is a non real point Q such that $Q + Q^c$ is linearly equivalent to a divisor supported on $\mathcal{C}(\mathbb{R})$.

Let $P \in \mathcal{C}(\mathbb{R})$ and $\mathcal{L}_P$ be as in Lemma 3.1. The linear system $\mathcal{L}_P$ gives a birational morphism $\phi : \mathcal{C} \to \mathcal{C}' \subset \mathbb{P}^2_{\mathbb{C}}$ such that $\mathcal{L}_P$ corresponds to the linear system $\mathcal{L}'$ on $\mathcal{C}'$ cut out by lines. By construction, ϕ and $\mathcal{C}'$ are defined over $\mathbb{R}$, so that a general real point on $\mathcal{C}$ corresponds to a real point on $\mathcal{C}'$ and vice versa.

Consider an affine real plane $\mathbb{A}^2(\mathbb{R}) \subset \mathbb{P}^2_{\mathbb{C}}$ containing $P' = \phi(P)$. Under a suitable coordinate transformation, we can suppose that P' is the origin $(0, 0)$, the tangent line to $\mathcal{C}'$ at P' is $y = 0$, and $\mathcal{C}' \cap \mathbb{A}^2(\mathbb{R})$ is defined by $F(x, y) = y - x^2 + bxy + cy^2 +$ (monomials of higher degree).

By the implicit function theorem applied to $F(x, y) = 0$, the curve $\mathcal{C}'(\mathbb{R}) \cap \mathbb{A}^2(\mathbb{R})$, restricted to a small open disk containing $(0, 0)$, is the graph of a real function $y = \psi(x)$. If we expand $y = \psi(x)$ near the origin, we find the second degree Taylor polynomial $T(x) = x^2$.

Thus, there are no points (x_0, y_0) on $\mathcal{C}'(\mathbb{R}) \cap \mathbb{A}^2(\mathbb{R})$ near the origin, having $y_0 < 0$.

Let's denote by L_μ the line in $\mathbb{P}^2_{\mathbb{C}}$ corresponding to the line $y - \mu = 0$ in $\mathbb{A}^2(\mathbb{R})$. If μ is sufficiently small, the line L_μ cuts on $\mathcal{C}'$ a divisor, which is sum of $k + 2$ distinct real points $R'_{1\mu}, \ldots, R'_{k\mu}$, such that

$$\lim_{\mu \to 0} (R'_{i\mu}) = \phi(R_i)$$

and of a pair of conjugate points $Q_\mu + Q^c_\mu$, such that

$$\lim_{\mu \to 0} (Q_\mu + Q^c_\mu) = 2P'.$$

If μ is negative, Q_μ and Q^c_μ can not be real and then the divisor $2P' + \phi(R'_1) + .. + \phi(R'_k) - R'_{1\mu} - .. - R'_{k\mu}$, which is completely supported on $\mathcal{C}'_{\mathbb{R}}$, is linearly equivalent to a pair of non real points $Q_\mu + Q^c_\mu$.

Finally, using the inverse of the birational map ϕ, we can conclude that the same result holds for the corresponding divisors on $\mathcal{C}$. □

5. The divisor class group.

First of all, we observe that we can rephrase Theorem 4.2 in terms of the Picard group of the curve $\mathcal{C}$ as follows: the subgroup $Pic(\mathcal{C})^+ = Pic(\mathcal{C})^c$ of $Pic(\mathcal{C})$, containing classes of real divisors, is equal to the subgroup $Pic(\mathcal{C})_{\mathbb{R}}$, containing only classes of divisors whose support is on $\mathcal{C}(\mathbb{R})$.

On the other hand, we can give a purely algebraic definition of divisors. Let $Cl(A)$ be the divisor class group of the coordinate ring A of $\mathcal{C}$, that is the free abelian group generated by height 1 prime ideals of A, modulo the principal divisors $div(x)$, where x is any element of the total quotient field $K(A)$ (see [4] or [6]: note that $Cl(A) = Cl(Spec(A))$. We emphasise that $K(A)$ is equal to $k(A)$, the field of real rational functions on $\mathcal{C}$, if $\mathcal{C}$ is affine, while, if $\mathcal{C}$ is projective, $k(A)$ contains only the degree zero elements of $K(A)$.

Following [9] and [8], we will call an ideal $\mathcal{I}$ of A a *defining ideal* if it contains all the elements of A that vanish on a point of $\mathcal{C}(\mathbb{R})$. We will denote by $Cl(A)_{\mathbb{R}}$ the subgroup of $Cl(A)$ containing only classes of defining ideals.

The following result generalises Theorem 4.2 to affine or projective curves and also to their coordinate rings, with respect to a suitable embedding.

Theorem 5.1. *Let $\mathcal{C}$ be an affine or projective smooth real curve such that $\mathcal{C}(\mathbb{R})$ is Zariski dense in $\mathcal{C}$. Then:*

1) $Cl(\mathcal{C})^c$ is generated by classes of divisors D supported on $\mathcal{C}(\mathbb{R})$ i.e. $Cl(\mathcal{C})^c = Cl(\mathcal{C})_{\mathbb{R}}$.

2) If, moreover, the real coordinate ring $A = \mathbb{R}[\mathcal{C}]$ is integrally closed, then $Cl(A)$ is generated by classes of defining ideals i.e. $Cl(A) = Cl(A)_{\mathbb{R}}$.

Proof. If $\mathcal{C}$ is projective, statement 1) follows by Theorem 4.2.

So, suppose $\mathcal{C}$ is affine. Let X be a projective smooth closure of $\mathcal{C}$ and $Z = X - \mathcal{C}$ the hyperplane section at infinity. Thus, we can easily deduce 1) by applying the previous part to X and using the canonical surjective homomorphism $Cl(X) \longrightarrow Cl(\mathcal{C})$ (see [6], II Proposition 6.5).

To prove claim 2), first we consider a projective curve $\mathcal{C}$. So A is a graded ring and $Cl(A)$ can be generated by homogeneous prime ideals of height 1 (see [4], Proposition 10.2). There is a natural 1:1 correspondence between homogeneous prime defining ideals of height 1 of A and real points on $\mathcal{C}$ and, on the other hand, between homogeneous prime non defining ideals of height 1 of A and pairs of non real conjugate points on $\mathcal{C}$.

So, the free group on real divisors on $\mathcal{C}$ is, in a natural way, isomorphic to $Div_h(A)$, the free group on the set of homogeneous height 1 prime ideals of A.

Moreover, two real divisors on $\mathcal{C}$ are linearly equivalent if and only if the corresponding elements of $Div(A)$ differ by a principal divisor $div(x)$, $x \in k(A)$, and then, if and only if they are in the same class of $Cl(A)$.

Thus we obtain a canonical surjective homomorphism $Cl(\mathcal{C})^c \longrightarrow Cl(A)$ that sends $Cl(\mathcal{C})_{\mathbb{R}}$ over $Cl(A)_{\mathbb{R}}$; so 2) follows from 1).

In the affine case, the proof is quite similar, but easier, because $Cl(\mathcal{C})^c$ and $Cl(A)$ are actually isomorphic. $\square$

Remark 5.2. Observe that the statement of Theorem 5.1 for real varieties of higher dimension has been proved in [8], Corollario 4.6 and Teorema 4.3). We consider an affine or projective complex variety V of dimension greater than 1, defined by real equations so that $V = V^c$ and such that the subset of the real points $V(\mathbb{R})$ is Zariski dense in V. Moreover we suppose that V is normal so that (choosing a suitable embedding, in the projective case) the real coordinate ring $A = \mathbb{R}[V]$ is integrally closed.

Under these assumptions, the divisor class group $Cl(V)$ of Weil divisors, modulo linear equivalence, is well defined and, if V is locally factorial, it is canonically isomorphic to the Picard group $Pic(V)$ (see [6], II,6).

The classes of real divisors, that is divisors $D = D^c$, form a subgroup $Cl(V)^c$ of $Cl(V)$. The classes of real divisors D, such that $D \cap V(\mathbb{R})$ is Zariski dense in D, form a subgroup $Cl(V)_{\mathbb{R}}$ of $Cl(V)^c$. In our previous paper we proved that $Cl(V)^c = Cl(V)_R$ and that $Cl(A)$ can be generated by classes of prime defining ideals (that is ideals that vanish on codimension 1 algebraic subsets of $V(\mathbb{R})$).

We observe that for a projective variety V, we have the exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow Cl(V)^c \longrightarrow Cl(A) \longrightarrow 0$$

(see also [6], II Exercise 6.3) where the first map sends 1 to the class of a hyperplane section.

If $dim(V) > 1$ we can always find a hyperplane H such that $H \cap V(\mathbb{R})$ is Zariski dense in $H \cap V$ (see [8], Corollario 3.5), while that is not true in general if $V = \mathcal{C}$ is a curve, as proved by the following example.

Example 5.3. Let $\mathcal{C}$ be the quartic plane curve in $\mathbb{P}^2_{\mathbb{C}}$ defined by the equation: $(x^2 + y^2)(x^2 + 2y^2 + z^2) + xz^2(y + z) = 0$. Easy calculations show that $\mathcal{C}$ is smooth and, moreover, that the real part $\mathcal{C}(\mathbb{R})$ is not empty and does not contain any inflection point. More precisely, no tangent line meets $\mathcal{C}(\mathbb{R})$ with multiplicity greater than 2. So a real line can meet every oval of $\mathcal{C}(\mathbb{R})$ in at most two real points.

Moreover, if we cut $\mathcal{C}$ by a line through the point $(0, 0, 1) \in \mathcal{C}(\mathbb{R})$, we find only one more real point (and a pair of non real conjugate points); then $\mathcal{C}(\mathbb{R})$ has only one oval.

Thus, every line intersects $\mathcal{C}(\mathbb{R})$ in at most 2 real points and then no hyperplane section of $\mathcal{C}$ is completely supported on $\mathcal{C}(\mathbb{R})$.

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