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Real hypersurfaces in a complex projective space with constant principal curvatures

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Introduction.

This paper is a continuation of the previous one [6], in which we classified those homogeneous real hypersurfaces in a complex projective space $P_n(\boldsymbol{C})$ of complex dimension n ($\geqq 2$) which are orbits under analytic subgroups of the projective unitary group $PU(n+1)$, and gave some characterization of those hypersurfaces. We shall call each of such hypersurfaces a model space for convenience' sake. The main purpose of this paper is to give another characterization of a geodesic hypersphere in $P_n(\boldsymbol{C})$ which is one of six kinds of the model space. It can be stated as follows.

THEOREM 1. *If M is a connected complete real hypersurface in $P_n(\boldsymbol{C})$ with two constant principal curvatures, then M is a geodesic hypersphere.*

In § 1 we shall determine the principal curvatures of the model spaces. As a result of the determination we know that each model space has two or three or five constant principal curvatures. In § 3 we shall prove Theorem 1 and in § 4 give its application.

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§ 1. Principal curvatures of the model spaces.

In [6] we have seen that roughly speaking there is a one-to-one correspondence between the model spaces and the isotropy representations of various Hermitian symmetric spaces of rank two. The correspondence is given as follows. Let $(\mathfrak{u}, \theta)$ be a Hermitian effective orthogonal symmetric Lie algebra of compact type and of rank two. $\mathfrak{u}$ is a compact semisimple Lie algebra and θ is an involutive automorphism of $\mathfrak{u}$ (cf. [3]). Let $\mathfrak{u}=\mathfrak{k}+\mathfrak{p}$ be the decomposition of $\mathfrak{u}$ into the eigenspaces of θ for the eigenvalues $+1$ and -1, respectively. Then $\mathfrak{k}$ and $\mathfrak{p}$ satisfy $[\mathfrak{k}, \mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}]\subset\mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}]\subset\mathfrak{k}$. For the Killing form B of $\mathfrak{u}$ we define a positive definite inner product $\langle\,,\rangle$

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on $\mathfrak{p}$ by $\langle X, Y\rangle=-B(X, Y)$ for $X, Y\in\mathfrak{p}$. Let K be the analytic subgroup of the group of inner automorphisms of $\mathfrak{u}$ with Lie algebra $\mathrm{ad}(\mathfrak{k})$. Then K leaves the subspace $\mathfrak{p}$ of $\mathfrak{u}$ invariant and acts on $\mathfrak{p}$ as an orthogonal transformation group with respect to $\langle,\rangle$. We define a representation ρ of K on $\mathfrak{p}$ by $\rho(k)=k|\mathfrak{p}$ for $k\in K$. The differential ρ_* of ρ is an isomorphism of $\mathfrak{k}$ into the Lie algebra of the orthogonal group of $\mathfrak{p}$ and satisfies $(\rho_*X)Y=[X, Y]$ for $X\in\mathfrak{k}$ and $Y\in\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$ and S be a hypersphere in $\mathfrak{p}$ centered at the origin. For simplicity we assume that the radius of S is equal to 1. Let a be a regular element of $\mathfrak{p}$ in $S\cap\mathfrak{a}$. Since $\dim\mathfrak{a}=2$, the orbit $N=\rho(K)a$ of a under $\rho(K)$ is a hypersurface in S. It is known ([3]) that there is an element Z_0 in the center of $\mathfrak{k}$ such that

$$(\rho_*Z_0)^2=-1\,,$$
$$\langle(\rho_*Z_0)X,\ (\rho_*Z_0)Y\rangle=\langle X, Y\rangle \qquad \text{for} \quad X, Y\in\mathfrak{p}\,.$$

Thus we may regard $\mathfrak{p}$ as a complex vector $(n+1)$-space $\boldsymbol{C}^{n+1}$ with complex structure $I=\rho_*Z_0$ and Hermitian inner product $\langle,\rangle$, where $2(n+1)=\dim\mathfrak{p}$. The image $M=\pi(N)$ of N by the canonical projection π of S onto $P_n(\boldsymbol{C})$ becomes a real hypersurface in $P_n(\boldsymbol{C})$. The Riemannian metric of S induced from $\langle,\rangle$ will be denoted by g. Then g induces naturally what is called the Fubini-Study metric $\tilde{g}$ on $P_n(\boldsymbol{C})$ through π (as stated later). Then with respect to $\tilde{g}$, $P_n(\boldsymbol{C})$ has constant holomorphic sectional curvature 4. Let C denote the circle group in K generated by Z_0. Then the group $G=\rho(K)/\rho(C)$ is a compact analytic subgroup of $PU(n+1)$ which acts on M transitively as a transformation group of isometries of M, where as a Riemannian metric of M we take the one induced from $\tilde{g}$. Conversely every model space, that is, every real hypersurface in $P_n(\boldsymbol{C})$ being an orbit under an analytic subgroup of $PU(n+1)$ is congruent to a real hypersurface M obtained in this way with respect to the group of isometries of $P_n(\boldsymbol{C})$ (for the last several results, cf. [6]).

Since G is an analytic subgroup of $PU(n+1)$, all principal curvatures of M is constant ([7], §1) and so let us evaluate the principal curvatures of M at a special point, say $\pi(a)$. Since the tangent space of $\boldsymbol{C}^{n+1}$ is identified with itself, a vector field on $\boldsymbol{C}^{n+1}$ can be regarded as a mapping of $\boldsymbol{C}^{n+1}$ into itself and the complex structure I of $\boldsymbol{C}^{n+1}$ as a vector field on S. Under such an identification, $\tilde{g}$ is expressed as

$$\tilde{g}(\tilde{U},\ \tilde{V})\circ\pi=g(U, V)-g(U, I)g(V, I)$$

for vector fields U, V on S and vector fields $\tilde{U}$, $\tilde{V}$ on $P_n(\boldsymbol{C})$ such that $\pi_*U=\tilde{U}$ and $\pi_*V=\tilde{V}$. When we denote by ∇ (resp. $\tilde{\nabla}$) the Riemannian connection of S (resp. $P_n(\boldsymbol{C})$) associated to g (resp. $\tilde{g}$), the following relation between ∇

and $\tilde{\nabla}$ is fundamental:

$$\tilde{g}(\tilde{\nabla}_{\tilde{W}}\tilde{U}, \tilde{V})\circ\pi = g(\nabla_W U, V) - g(\nabla_W U, I)g(V, I) - g(U, I)g(I(W), V) - g(W, I)g(I(U), V) \tag{1.1}$$

for vector fields U, V, W on S and vector fields $\tilde{U}, \tilde{V}, \tilde{W}$ on $P_n(\boldsymbol{C})$ such that $\pi_* U = \tilde{U}$, $\pi_* V = \tilde{V}$ and $\pi_* W = \tilde{W}$.

Making use of (1.1) we shall find a relation between the second fundamental form of M and N. We denote the tangent space of a manifold L at $x \in L$ by L_x. In general the second fundamental form of a submanifold M' in a Riemannian manifold for a normal vector ν at $x \in M'$ induces the symmetric linear transformation of M'_x, which is called the shape operator of M' for ν. Let b be a unit vector normal to N at a in S. Then $\{a, b\}$ is an orthonormal base of $\mathfrak{a}$ and $\pi_* b$ is a unit vector normal to M. Let N' denote the hyperplane in N_a orthogonal to the unit vector $I(a) \in N_a$. Let T (resp. $\tilde{T}$) denote the shape operator of N (resp. M) for b (resp. $\pi_* b$). For $X \in \rho_*(\mathfrak{k})$ we denote by X^* (resp. $\tilde{X}$) the Killing vector field on S (resp. $P_n(\boldsymbol{C})$) induced by X. In our notation a vector I_a^* is identified with a vector $I(a)$. It was proved in [7, pp. 471-473] that

$$g(T(X_a^*), Y_a^*) = -g(\nabla_b X^*, Y_a^*),$$

$$\tilde{g}(\tilde{T}(\tilde{X}_{\pi(a)}), \tilde{Y}_{\pi(a)}) = -\tilde{g}(\tilde{\nabla}_{\pi_* b}\tilde{X}, \tilde{Y}_{\pi(a)}),$$

and

$$T(I(a)) = -I(b)$$

for $X, Y \in \rho_*(\mathfrak{k})$ such that $X_a^*, Y_a^* \in N'$. These equations, together with (1.1), imply

$$g(T(u), v) = \tilde{g}(\tilde{T}(\pi_* u), \pi_* v), \qquad T(I(a)) = -I(b) \tag{1.2}$$

for $u, v \in N'$.

We want to express the eigenvalues of $\tilde{T}$ (they are by definition the principal curvatures of M) in terms of a root system of $\mathfrak{u}$ making use of (1.2). For a linear form α on $\mathfrak{a}$ we put

$$\mathfrak{p}_\alpha = \{X \in \mathfrak{p}\,;\ (\mathrm{ad}\,H)^2 X = -\alpha(H)^2 X \quad \text{for all} \quad H \in \mathfrak{a}\},$$

$$\mathfrak{k}_\alpha = \{X \in \mathfrak{k}\,;\ (\mathrm{ad}\,H)^2 X = -\alpha(H)^2 X \quad \text{for all} \quad H \in \mathfrak{a}\}.$$

Then $\mathfrak{p}_{-\alpha} = \mathfrak{p}_\alpha$, $\mathfrak{k}_{-\alpha} = \mathfrak{k}_\alpha$, $\mathfrak{p}_0 = \mathfrak{a}$ and $\mathfrak{k}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Moreover ad a maps $\mathfrak{k}_\alpha$ onto $\mathfrak{p}_\alpha$, and $\mathfrak{p}_\alpha$ onto $\mathfrak{k}_\alpha$ isomorphically. A root of $\mathfrak{u}$ with respect to $\mathfrak{a}$ is by definition a linear form α on $\mathfrak{a}$ such that $\mathfrak{k}_\alpha \neq \{0\}$. Select a suitable ordering in the dual space of $\mathfrak{a}$ and denote by Δ the set of positive roots of $\mathfrak{u}$ with respect to $\mathfrak{a}$. Then we have the orthogonal direct decompositions of $\mathfrak{p}$ and $\mathfrak{k}$ with respect to B:

$$\mathfrak{k}=\mathfrak{k}_0+\sum_{\alpha\in\Delta}\mathfrak{k}_\alpha, \qquad \mathfrak{p}=\mathfrak{a}+\sum_{\alpha\in\Delta}\mathfrak{p}_\alpha.$$

It is known that there exists the subset $\Delta'=\{\lambda, \mu\}$ of Δ consisting of strongly orthogonal roots, that is, none of $\pm\lambda\pm\mu$ is contained in Δ and $\dim\mathfrak{p}_\lambda=\dim\mathfrak{p}_\mu=1$. By [4, Proposition 3.10] an element Z_0 of the center of $\mathfrak{k}$ belongs to $\mathfrak{k}_0+\mathfrak{k}_\lambda+\mathfrak{k}_\mu$. It follows that the vector space N' is spanned by $\mathfrak{p}_\alpha$ $(\alpha\in\Delta-\Delta')$ and $I(b)$. On the other hand, we proved in [7] that for each $\alpha\in\Delta$, $\kappa_\alpha=-\alpha(b)/\alpha(a)$ is an eigenvalue of T and $\mathfrak{p}_\alpha$ is contained in the eigenspace of T for κ_α. It follows from (1.2) that for each $\alpha\in\Delta-\Delta'$, κ_α is an eigenvalue of $\tilde{T}$ and $\pi_*\mathfrak{p}_\alpha$ is contained in the eigenspace of $\tilde{T}$ for κ_α. Hence $\pi_*I(b)$ is an eigenvector of $\tilde{T}$ for certain eigenvalue of $\tilde{T}$, say κ. Then again from (1.2) we see that the set of all eigenvalues of T coincides with $\{\kappa_\alpha:\alpha\in\Delta-\Delta'\}\cup\{y\,;y^2-\kappa y-1=0\}$, from which we have $\kappa=\kappa_\lambda+\kappa_\mu$. Thus we proved

THEOREM 2. *Let the notation be in above. Then the principal curvatures of the model space M corresponding to $(\mathfrak{u}, \theta)$ are given by*

$$-\alpha(b)/\alpha(a) \qquad (\alpha\in\Delta-\Delta')$$

and

$$-\lambda(b)/\lambda(a)-\mu(b)/\mu(a) \qquad (\Delta'=\{\lambda, \mu\}).$$

By virtue of Theorem 2 we can read the distinct principal curvatures $\xi_1, \cdots, \xi_r$ of M and their multiplicities $m(\xi_1), \cdots, m(\xi_r)$ from the table given by S. Araki [1]. It is easily checked that $\kappa_\alpha\neq\kappa_\beta$, $\kappa_\alpha\neq\kappa$ and $m(\kappa_\alpha)=\dim\mathfrak{p}_\alpha$ for all $\alpha, \beta\in\Delta-\Delta'$ with $\alpha\neq\beta$, and so $m(\kappa)=1$. The values $\xi_1, \cdots, \xi_r$ depend on a position of a as well as $(\mathfrak{u}, \theta)$ and hence they include a parameter t. We have the following Table

Table

	$\mathfrak{u}$	$\mathfrak{k}$	$\dim M$	r	ξ_i	$m(\xi_i)$
$\boldsymbol{A}_1$	$\mathfrak{su}(n+1)+\mathfrak{su}(2)$ $(n\geqq 2)$	$\mathfrak{s}(\mathfrak{u}(n)+\mathfrak{u}(1))$ $+\mathfrak{s}(\mathfrak{u}(1)+\mathfrak{u}(1))$	$2n-1$	2	$\xi_1=\cot t$ $\xi_2=2\cot 2t$	$m(\xi_1)=2(n-1)$ $m(\xi_2)=1$
$\boldsymbol{A}_2$	$\mathfrak{su}(p+1)+\mathfrak{su}(q+1)$ $(p\geqq q\geqq 2)$	$\mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(1))$ $+\mathfrak{s}(\mathfrak{u}(q)+\mathfrak{u}(1))$	$2(p+q)-3$	3	$\xi_1=\cot t$ $\xi_2=-\tan t$ $\xi_3=2\cot 2t$	$m(\xi_1)=2(p-1)$ $m(\xi_2)=2(q-1)$ $m(\xi_3)=1$
$\boldsymbol{B}$	$\mathfrak{o}(p+2)$ $(p\geqq 3)$	$\mathfrak{o}(p)+\boldsymbol{R}$	$2p-3$	3	$\xi_1=\cot(t-\pi/4)$ $\xi_2=-\tan(t-\pi/4)$ $\xi_3=2\cot 2t$	$m(\xi_1)=p-2$ $m(\xi_2)=p-2$ $m(\xi_3)=1$
$\boldsymbol{C}$	$\mathfrak{su}(p+2)$ $(p\geqq 3)$	$\mathfrak{s}(\mathfrak{u}(p)+\mathfrak{u}(2))$	$4p-3$	5	$\xi_i=\cot(t-\pi i/4)$ $(i=1, 2, 3, 4)$ $\xi_5=2\cot 2t$	$m(\xi_i)=2(p-2)$ $(i=0, 2)$ $m(\xi_i)=2$ $(i=1, 3)$ $m(\xi_5)=1$
$\boldsymbol{D}$	$\mathfrak{o}(10)$	$\mathfrak{u}(5)$	17	5	$\xi_i=\cot(t-\pi i/4)$ $(i=1, 2, 3, 4)$ $\xi_5=2\cot 2t$	$m(\xi_i)=4$ $(i=1, 2, 3, 4)$ $m(\xi_5)=1$
$\boldsymbol{E}$	E_6	$\mathfrak{o}(10)+\boldsymbol{R}$	29	5	$\xi_i=\cot(t-\pi i/4)$ $(i=1, 2, 3, 4)$ $\xi_5=2\cot 2t$	$m(\xi_i)=8$ $(i=0, 2)$ $m(\xi_i)=6$ $(i=1, 3)$ $m(\xi_5)=1$

As an example we work out the type C in detail. This corresponds to the type AIII in the table of [1] (Set p, l in AIII as 2, $p+1$ respectively). Then we know that two simple roots α, β of $\mathfrak{u}$ with respect to $\mathfrak{a}$ have the diagram $\underset{\alpha}{\circ}\Longrightarrow\underset{\beta}{\circ}$, and that $m(\alpha)=2$, $m(\beta)=2(p-2)$ and $m(2\alpha)=1$, where $m(\alpha)=\dim \mathfrak{p}_{\alpha}$. Since the Weyl group of $(\mathfrak{u}, \theta)$ is simply transitive on the set of Weyl chambers in $\mathfrak{a}$, we have $\Delta=\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 2\alpha, 2\alpha+2\beta\}$. Hence $m(2\alpha+\beta)=2$, $m(\alpha+\beta)=2(p-2)$, and $\Delta'=\{2\alpha, 2\alpha+2\beta\}$. The values $\xi_1, \cdots, \xi_5$ depend on the angle t between α and a.

The model space $M(\boldsymbol{A}_1)$ of type $\boldsymbol{A}_1$ is a geodesic hypersphere in $P_n(\boldsymbol{C})$. In fact, a regular element a of $\mathfrak{p}$ is decomposed into $a=a'+a''$, where $a' \in \mathfrak{su}(n+1) \cap \mathfrak{p}$ and $a'' \in \mathfrak{su}(2) \cap \mathfrak{p}$. Then it is easily seen that the distance between each point of $M(\boldsymbol{A}_1)$ and the point $\pi(a'')$ in $P_n(\boldsymbol{C})$ is equal to $\cot^{-1}(|a''|/|a'|)$. Thus we saw that every geodesic hypersurface in $P_n(\boldsymbol{C})$ with constant holomorphic sectional curvature 4 has two constant principal curvatures ξ, η such that the multiplicity of η is equal to one and $\xi^2-\xi\eta-1=0$.

REMARK 1.1. If we denote by $\tilde{J}$ the complex structure of $P_n(\boldsymbol{C})$ induced from I and by ν a normal vector field on an arbitrary model space then $\tilde{J}(\nu)$ is a direction of principal curvature (that is an eigenvector of $\tilde{T}$ for the eigenvalue κ) everywhere.

REMARK 1.2. Among model spaces of any type there is a minimal one because the mean curvature $(\xi_1+\cdots+\xi_r)/(2n-1)$ vanishes everywhere for some t.

REMARK 1.3. By a theorem of Tashiro-Tachibana [8] there is no totally umbilical real hypersurface in $P_n(\boldsymbol{C})$.

§ 2. Structure equations.

Hereafter let $P_n(\boldsymbol{C})$ $(n \geqq 2)$ be a complex projective space with the metric of constant holomorphic sectional curvature $4c$ and M be a connected Riemannian real hypersurface with the induced metric. We denote by $F(M)$ the bundle of orthonormal frames of M. Then $F(M)$ is a principal fibre bundle over M with structure group $O(2n-1)$. An element u of $F(M)$ can be expressed by $u=(p: e_1, \cdots, e_{2n-1})$, where p is a point of M and $e_1, \cdots, e_{2n-1}$ is an ordered orthonormal base of M_p. We denote by θ_i, θ_{ij} and Θ_{ij}*) the canonical 1-forms, the connection forms and the curvature forms of $F(M)$ respectively. Then they satisfy

*) Hereafter the indices i, j, k, l run from 1 to $2n-1$ and the indices A, B, C, D run from 1 to $2n$.

(2.1) $$d\theta_i+\sum_j \theta_{ij}\wedge\theta_j=0\,,\qquad \theta_{ij}+\theta_{ji}=0\,,$$

(2.2) $$d\theta_{ij}+\sum_k \theta_{ik}\wedge\theta_{kj}=\Theta_{ij}\,.$$

We denote by $F(P)$ the bundle of orthonormal frames of $P_n(\boldsymbol{C})$, and by $\tilde{\theta}_A$, $\tilde{\theta}_{AB}$ and $\tilde{\Theta}_{AB}$ the canonical 1-forms, the connection forms and the curvature forms of $F(P)$ respectively. Then $\tilde{\theta}_A$ and $\tilde{\theta}_{AB}$ satisfy

(2.3) $$d\tilde{\theta}_A+\sum_B \tilde{\theta}_{AB}\wedge\tilde{\theta}_B=0\,,\qquad \tilde{\theta}_{AB}+\tilde{\theta}_{BA}=0\,,$$

and $\tilde{\Theta}_{AB}$ are given by

(2.4) $$\begin{aligned}\tilde{\Theta}_{AB}&=d\tilde{\theta}_{AB}+\sum_C \tilde{\theta}_{AC}\wedge\tilde{\theta}_{CB}\\ &=c\tilde{\theta}_A\wedge\tilde{\theta}_B+c\sum_{C,D}(\tilde{J}_{AC}\tilde{J}_{BD}+\tilde{J}_{AB}\tilde{J}_{CD})\tilde{\theta}_C\wedge\tilde{\theta}_D\,,\end{aligned}$$

where $\tilde{J}=(\tilde{J}_{AB})$ denotes the complex structure of $P_n(\boldsymbol{C})$, that is, $J(\tilde{e}_A)=\sum_B\tilde{J}_{BA}\tilde{e}_B$ at $(\tilde{p}:\tilde{e}_1,\cdots,\tilde{e}_{2n})\in F(P)$. Moreover $\tilde{J}$ satisfies

(2.5) $$\sum_C \tilde{J}_{AC}\tilde{J}_{CB}=-\delta_{AB}\,,\qquad \tilde{J}_{AB}+J_{BA}=0\,,$$

(2.6) $$d\tilde{J}_{AB}=\sum_C(\tilde{J}_{AC}\tilde{\theta}_{CB}-\tilde{J}_{BC}\tilde{\theta}_{CA})\,.$$

The equation (2.6) means that $\tilde{J}$ is parallel.

The isometric inclusion mapping ι of M into $P_n(C)$ induces three tensor fields $H=(H_{ij})$, $J=(J_{ij})$ and $f=(f_i)$ on $F(M)$ as follows. For an element $u=(p:e_1,\cdots,e_{2n-1})\in F(M)$ there exists a unique tangent vector e_{2n} of $P_n(\boldsymbol{C})$ at $\iota(p)$ such that $\tilde{u}=(\iota(p):\iota_*e_1,\cdots,\iota_*e_{2n-1},\tilde{e}_{2n})$ is an element of $F(P)$ compatible with the orientation of $P_n(\boldsymbol{C})$ determined by $\tilde{J}$. This mapping $u\to\tilde{u}$ of $F(M)$ into $F(P)$ is also denoted by the same letter ι. Then denoting by ι^* the dual mapping of ι_* we have $\theta_i=\iota^*\tilde{\theta}_i$ and $\iota^*\tilde{\theta}_{2n}=0$, from which we know $\theta_{ij}=\iota^*\tilde{\theta}_{ij}$ and $0=\iota^*d\hat{\theta}_{2n}=-\sum_i\iota^*\tilde{\theta}_{2n,i}\wedge\theta_i=0$. By Cartan's lemma we may write as

(2.7) $$\phi_i\equiv\iota^*\tilde{\theta}_{2n,i}=\sum_j H_{ij}\theta_j\,,\qquad H_{ij}=H_{ji}\,.$$

The quadratic form $\sum_i\phi_i\theta_i$ is called the *second fundamental form* of M for $\tilde{e}_{2n}$. Put $J_{ij}=\tilde{J}_{ij}\circ\iota$ and $f_i=\tilde{J}_{2n,i}\circ\iota$. The pair (J,f) is called the *almost contact structure* of M. From (2.2), (2.4) and (2.7) we have the equation of Gauss

(2.8) $$\Theta_{ij}=\phi_i\wedge\phi_j+c\theta_i\wedge\theta_j+c\sum_{k,l}(J_{ik}J_{jl}+J_{ij}J_{kl})\theta_k\wedge\theta_l\,.$$

From (2.4) and (2.7) we have the equation of Codazzi

(2.9) $$d\phi_i+\sum_j\phi_j\wedge\theta_{ji}=c\sum_{j,k}(f_jJ_{ik}+f_iJ_{jk})\theta_j\wedge\theta_k\,.$$

Moreover (J, f) satisfies by (2.5) and (2.6)

(2.10) $$\sum_k J_{ik}J_{kj}=f_if_j-\delta_{ij}\,,\qquad \sum_j f_jJ_{ji}=0\,,$$

$$\sum_i f_i^2=1\,,\qquad J_{ij}+J_{ji}=0\,,$$

(2.11) $$dJ_{ij}=\sum_k(J_{ik}\theta_{kj}-J_{kj}\theta_{ik})-f_i\phi_j+f_j\phi_i\,,$$

$$df_i=\sum_j(f_j\theta_{ji}-J_{ji}\phi_j)\,.$$

§ 3. Proof of Theorem 1.

Assume that M has two constant principal curvatures ξ and η $(\xi\neq\eta)$. Let m be the multiplicity of η. Define the subbundle F' of $F(M)$ by

$$F'=\{u\in F(M)\,;\ \phi_a=\xi\theta_a,\ \phi_r=\eta\theta_r \text{ at } u\}^{*)}$$

and restrict all differential forms and tensor fields under consideration to F'. Hereafter we shall promise that "$f_a=0$" means "$f_a=0$ for all a on a nonempty open set of F'", and "$f_a\neq 0$" means "$f_a\neq 0$ for some a on a nonempty open set of F'", etc.

LEMMA 3.1.

(1) $$f_aJ_{bc}=0 \quad \textit{and} \quad f_rJ_{st}=0\,.$$

(2) $$(\eta-\xi)\theta_{ar}=c\sum_b(f_bJ_{ar}-f_rJ_{ab}+2f_aJ_{br})\theta_b$$
$$+c\sum_s(f_sJ_{ra}-f_aJ_{rs}+2f_rJ_{sa})\theta_s\,.$$

PROOF. By (2.1) and (2.9) the exterior derivatives of $\phi_a=\xi\theta_a$ and $\phi_r=\eta\theta_r$ give

(3.1) $$(\xi-\eta)\sum_r\theta_{ar}\wedge\theta_r=c\sum_{j,k}(f_jJ_{ak}+f_aJ_{jk})\theta_j\wedge\theta_k\,,$$

(3.2) $$(\eta-\xi)\sum_a\theta_{ar}\wedge\theta_a=c\sum_{j,k}(f_jJ_{rk}+f_rJ_{jk})\theta_j\wedge\theta_k\,.$$

Taking account of the coefficients of $\theta_b\wedge\theta_c$ in (3.1) we have

(3.3) $$f_bJ_{ac}-f_cJ_{ab}+2f_aJ_{bc}=0\,.$$

Put $c=a$ in (3.3) to get $f_aJ_{ba}=0$. Multiplying (3.3) by f_a therefore we have $f_aJ_{bc}=0$. Similarly we have $f_rJ_{st}=0$ from (3.2). We can prove (2) easily by applying a method of indeterminate coefficients to (3.1) and (3.2). Q. E. D.

LEMMA 3.2. $f_a=0$ *or* $f_r=0$.

PROOF. From (2.10) and (1) of Lemma 3.1 we have

*) Hereafter the indices a, b, c run from 1 to $2n-1-m$ and the indices r, s, t run from $2n-m$ to $2n-1$.

$$0=\sum_{a,b} f_a J_{ab} J_{br}=\sum_a f_a(-\sum_s J_{as} J_{sr}+f_a f_r)=f_r \sum_a f_a^2$$

since $\sum_a f_a J_{as}=-\sum_r f_r J_{rs}=0$. Q. E. D.

Without loss of generality we may assume $f_a=0$ and so $f_r \neq 0$ by Lemma 3.2. Then (1) of Lemma 3.1 implies $J_{rs}=0$. By (2.11) and (2) of Lemma 3.1 the derivative of $f_a=0$ gives

(3.4) $$(\xi^2-\eta\xi-c)J_{ab}=0 .$$

Similarly the derivative of $J_{rs}=0$ gives

(3.5) $$(\eta^2-\xi\eta+2c)(f_r\delta_{st}-f_s\delta_{rt})=0 .$$

LEMMA 3.3. $m=1$.

PROOF. Suppose $m \geqq 2$. Then (3.5) implies $\eta^2-\xi\eta+2c=0$. Hence from (3.4) we have $J_{ab}=0$ since $\xi^2-\eta\xi-c \neq 0$. Take the exterior derivative of (2) of Lemma 3.1 making use of (2.1), (2.2), (2.8), (2.10) and (2) of Lemma 3.1 itself to obtain

$$\begin{aligned}&(\eta-\xi)(c+\xi\eta)\theta_a \wedge \theta_r+3c\xi f_r \sum_s f_s \theta_a \wedge \theta_s\\ &+c\sum_{b,s}((2\eta-3\xi)J_{ar}J_{bs}+(\eta-3\xi)J_{as}J_{br})\theta_b \wedge \theta_s\\ &+\frac{2c^2}{\eta-\xi}\sum_{b,s,t}(f_s^2 J_{ar}J_{bt}+f_r f_t J_{as}J_{bs})\theta_b \wedge \theta_t=0 .\end{aligned}$$

Summing up the coefficients of $\theta_a \wedge \theta_r$ in above equation on r and making use of $\sum_r J_{ar}^2=\sum_r f_r^2=1$ and $2c=\eta(\xi-\eta)$ we have

$$\eta^2+m\xi\eta+(m+3)c=0 ,$$

which contradicts $\eta^2-\xi\eta+2c=0$. Q. E. D.

Let $S^{2n+1}(c)$ denote a $(2n+1)$-sphere with constant sectional curvature c and π be the canonical projection of $S^{2n+1}(c)$ onto $P_n(\boldsymbol{C})$.

PROOF OF THEOREM 1. It follows from (1.2) and Lemma 3.3 that the principal curvatures of a hypersurface $N=\pi^{-1}(M)$ in $S^{2n+1}(c)$ are given by ξ and the roots of the equation $y^2-\eta y-c=0$. On the other hand, we have $J_{ab} \neq 0$ since $f_a=0$ and $J_{ar}=0$. Hence (3.4) implies $\xi^2-\eta\xi-c=0$. Thus N has two constant principal curvatures ξ with multiplicity $2n-1$ and $-c/\xi$ with multiplicity 1. By a theorem of E. Cartan [**2**, p. 180] we see that N is congruent to a product $S^{2n-1}(\xi^2+c)\times S^1(c(\xi^2+c)/\xi^2)$ of two spheres, which is exactly an orbit in $S^{2n+1}(c)$ of type $\boldsymbol{A}_1$ in the Table in § 1. By a comment below the Table M is a geodesic hypersphere in $P_n(\boldsymbol{C})$. Q. E. D.

REMARK 2.1. The radius of above geodesic hypersphere M is equal to $(|\xi|/\sqrt{c})\cot^{-1}(|\xi|/\sqrt{c})$.

§ 4. An application of Theorem 1.

Now modifying the condition of Theorem 1 we obtain

THEOREM 3. *If a connected complete real hypersurface M in $P_n(\boldsymbol{C})$ has two principal curvatures ξ with multiplicity $2n-2$ and η with multiplicity* 1, *then M is a geodesic hypersphere.*

PROOF. Owing to Theorem 1 it suffices to prove that both ξ and η are constant. We adopt the notation in § 3. Thus the index r stands for $2n-1$. By (2.1) and (2.4) the exterior derivative of $\phi_a=\xi\theta_a$ and $\phi_r=\eta\theta_r$ give

$$\sum_b \{\delta_{ab}d\xi+c\sum_j(f_bJ_{aj}+f_aJ_{bj})\theta_j\}\wedge\theta_b + \{(\eta-\xi)\theta_{ar}+c\sum_j(f_rJ_{aj}+f_aJ_{rj})\theta_j\}\wedge\theta_r=0\,, \tag{4.1}$$

$$\sum_a \{(\eta-\xi)\theta_{ar}+c\sum_j(f_rJ_{aj}+f_aJ_{rj})\theta_j\}\wedge\theta_a + \{d\eta+2cf_r\sum_a J_{ra}\theta_a\}\wedge\theta_r=0\,. \tag{4.2}$$

It follows from Cartan's lemma that { }'s in (4.1) and (4.2) can be expressed as

$$\delta_{ab}d\xi+c\sum_j(f_bJ_{aj}+f_aJ_{rj})\theta_j=\sum_c A_{abc}\theta_c+A_{ab}\theta_r\,, \tag{4.3}$$

$$(\eta-\xi)\theta_{ar}+c\sum_j(f_rJ_{aj}+f_aJ_{rj})\theta_j=\sum_b A_{ab}\theta_b+A_a\theta_r\,, \tag{4.4}$$

$$d\eta+2cf_r\sum_a J_{ra}\theta_a=\sum_a A_a\theta_a+A\theta_r\,, \tag{4.5}$$

where $A_{abc}=A_{acb}=A_{bac}$. From (4.3) we have $A_{abc}=c(f_bJ_{ac}+f_aJ_{bc})$ and $A_{ab}=c(f_bJ_{ar}+f_aJ_{br})$ for $a\neq b$. Hence $0=A_{abc}-A_{acb}=c(f_bJ_{ac}-f_cJ_{ab}+2f_aJ_{bc})$ for $a\neq b$. From this we have $f_aJ_{bc}=0$ as in the proof of (1) of Lemma 3.1. Then putting $d\xi=\sum_a\xi_a\theta_a+\xi_r\theta_r$ and $d\eta=\sum_a\eta_a\theta_a+\eta_r\theta_r$ we have from (4.3) and (4.5)

$$A_{aa}=2cf_aJ_{ar}+\xi_r\,,\qquad A_a=-\eta_a\,,$$

Thus (4.4) was reduced to

$$(\eta-\xi)\theta_{ar}=c\sum_b(f_bJ_{ar}-f_rJ_{ab}+2f_aJ_{br})\theta_b+\xi_r\theta_a+\eta_a\theta_r\,. \tag{4.6}$$

Here we shall divide into two cases.

(1) The case where $f_a\neq 0$. Then we have $f_r=0$ as in the proof of Lemma 3.2. By (2.11) and (4.6) the derivative of $f_r=0$ gives $\eta_a=0$ and

$$(\xi^2-\eta\xi+2c)J_{ar}+f_a\xi_r=0\,. \tag{4.7}$$

Multiply (4.7) by f_a and sum up on a to obtain $\xi_r=0$. Thus (4.7) again implies $\xi^2-\eta\xi+2c=0$ since $J_{ar}\neq 0$. The derivative of $\xi^2-\eta\xi+2c=0$ gives $\xi_a=0$ and $\eta_r=0$, which shows that both ξ and η are constant.

(2) The case $f_a=0$. Then since $J_{ab}\neq 0$, the derivative of $f_a=0$ gives

$\xi_r=0$, $\eta_a=0$ and $\xi^2-\eta\xi-c=0$ as in the case (1). The derivative of $\xi^2-\eta\xi-c=0$ gives $\eta_r=0$, which implies that η is constant, and hence ξ is also constant. Q. E. D.

Following S. Tachibana and T. Kashiwada [5] we shall call a real hypersurface in $P_n(\boldsymbol{C})$ totally η-umbilic if $H_{ij}=\alpha\delta_{ij}+\beta f_i f_j$ holds good for some scalar functions α and β. For a matrix $Q=(x\delta_{\lambda\mu}+y_\lambda z_\mu)$ of degree D we see that $\det Q=x^D+(\sum_\lambda y_\lambda z_\lambda)x^{D-1}$ by induction on D and differentiation on x. Thus if M is η-umbilic hypersurface in $P_n(\boldsymbol{C})$ then each principal curvature κ of M satisfies

$$\begin{aligned}0&=\det(H-\kappa\delta_{ij})=\det((\alpha-\kappa)\delta_{ij}+\beta f_i f_j)\\&=(\alpha-\kappa)^{2n-2}(\alpha-\kappa+\beta)\end{aligned}$$

since $\sum_i f_i^2=1$. So $\kappa=\alpha$ or $\alpha+\beta$. But β does not vanish everywhere. In fact by [8, Theorem 3] the set $F'=\{u\in F(M)\,;\,\beta=0 \text{ at } u\}$ contains no non-empty open sets of $F(M)$. On the other hand, Theorem 3 shows that both α and β are constant on $F(M)-F'$ and hence $F'=\emptyset$ by continuity of β. Thus Theorem 3 again implies

COROLLARY 4. *If M is a connected complete totally η-umbilic hypersurface in $P_n(\boldsymbol{C})$ then M is a geodesic hypersphere.*

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