REAL HYPERSURFACES IN COMPLEX MANIFOLDS

BY

S. S. CHERN and J. K. MOSER

University of California
Berkeley, Cal., USA

New York University
New York, N. Y., USA

Introduction

Whether one studies the geometry or analysis in the complex number space $\mathbf{C}_{n+1}$, or more generally, in a complex manifold, one will have to deal with domains. Their boundaries are real hypersurfaces of real codimension one. In 1907, Poincaré showed by a heuristic argument that a real hypersurface in $\mathbf{C}_2$ has local invariants under biholomorphic transformations [6]. He also recognized the importance of the special unitary group which acts on the real hyperquadrics (cf. § 1). Following a remark by B. Segre, Elie Cartan took, up again the problem. In two profound papers [1], he gave, among other results, a complete solution of the equivalence problem, that is, the problem of finding a complete system of analytic invariants for two real analytic real hypersurfaces in $\mathbf{C}_2$ to be locally equivalent under biholomorphic transformations.

Let $z^1, ..., z^{n+1}$ be the coordinates in $\mathbf{C}_{n+1}$. We study a real hypersurface M at the origin 0 defined by the equation

$$r(z^1, ..., z^{n+1}, \bar{z}^1, ..., \bar{z}^{n+1}) = 0, \tag{0.1}$$

where r is a real analytic function vanishing at 0 such that not all its first partial derivatives are zero at 0. We set

$$z = (z^1, ..., z^n), \quad z^{n+1} = w = u + iv. \tag{0.2}$$

After an appropriate linear coordinate change the equation of M can be written as

$$v = F(z, \bar{z}, u), \tag{0.3}$$

where F is real analytic and vanishes with its first partial derivatives at 0. Our basic assumption on M is that it be nondegenerate, that is, the Levi form

This work was partially supported by the National Science Foundation, Grants GP-20096 and GP-34785X. We wish to thank the Rockefeller University for their hospitality where the first author was a visitor in the Spring of 1973.

$$\langle z, z\rangle = \sum_{1\leqslant\alpha,\beta\leqslant n} g_{\alpha\bar\beta} z^\alpha \bar z^\beta, \quad g_{\alpha\bar\beta} = \left(\frac{\partial^2 F}{\partial z^\alpha \partial \bar z^\beta}\right)_0 \tag{0.4}$$

is nondegenerate at 0. In § 2, 3 we study the problem of reducing the equation to a normal form by biholomorphic transformations of z, w. This is first studied in terms of formal power series in § 2 and their convergence to a holomorphic mapping is established in § 3. The results are stated in Theorems 2.2 and 3.5. It is worth noting that the convergence or existence proof is reduced to that of ordinary differential equations.

The normal form is found by fitting the holomorphic image of a hyperquadric closely to the given manifold. For $n=1$ this leads to 5th order osculation of the holomorphic image of a sphere at the point in question, while for $n\geqslant 2$ the approximation is more complicated. In both cases, however, the approximation takes place along a curve transversal to the complex tangent space. The family of the curves so obtained satisfies a system of second order differential equations which is holomorphically invariantly associated with the manifold. For a hyperquadric, or the sphere, these curves agree with the intersection of complex lines with the hyperquadric. For $n=1$ the differential equations can be derived from those of the sphere by constructing the osculating holomorphic image of the sphere, while for $n>1$ such a simple interpretation does not seem possible. This family of curves is clearly of basic importance for the equivalence problem. At first the differential equations for these curves are derived for real analytic hypersurfaces but they remain meaningful and invariant for five times continuously differentiable manifolds.

On the other hand, equation (0.1) implies

$$i\partial r = -i\bar\partial r, \tag{0.5}$$

which is therefore a real-valued one-form determined by M up to a non-zero factor; we will denote the common expression by θ. Let T_x and T_x^* be respectively the tangent and cotangent spaces at $x\in M$. As a basis of T_x^* we can take θ, Re (dz^α), Im (dz^α), $1\leqslant\alpha\leqslant n$. The annihilator $T_{x,C}=\theta^\perp$ in T has a complex structure and will be called the complex tangent space of M at x. Such a structure on M has been called a Cauchy-Riemann structure [8]. The assumption of the nondegeneracy of the Levi form defines a conformal hermitian structure in $T_{x,C}$. To these data we apply Cartan's method of equivalence, generalizing his work for $\mathbf{C}_2$. It turns out that a unique connection can be defined, which has the special unitary group as the structure group and which is characterized by suitable curvature conditions (Theorem 5.1). The successive covariant derivatives of the curvature of the connection give a complete system of analytic invariants of M under biholomorphic transformations. The result is, however, of wider validity. First, it suffices that the Cauchy-Riemann structure be defined abstractly on a real manifold of dimension $2n+1$. Secondly, the connection and the resulting invariants are also defined under weaker smoothness conditions, such as C^∞, although their identity will in general not insure equivalence without real analyticity. In this connection we mention the deep result of C. Fefferman [2] who showed that a biholomorphic mapping between two strictly pseudoconvex domains with smooth boundaries is smooth up to the boundary.

The equivalence problem was studied by N. Tanaka for real hypersurfaces in $\mathbf{C}_{n+1}$ called by him regular, which are hypersurfaces defined locally by the equation (0.3) where F does not involve u [7 I]. Later Tanaka stated the result in the general case [7 II], but the details, which are considerable, were to our knowledge never published.

One interesting feature of this study is the difference between the cases $\mathbf{C}_2$ and $\mathbf{C}_{n+1}$, $n \geqslant 2$. There is defined in general a tensor which depends on the partial derivatives of r up to order four inclusive and which vanishes identically when $n=1$. Thus there are invariants of order four in the general case, while for $n=1$ the lowest invariant occurs in order six. This distinction is also manifest from the normal forms.

The Cauchy-Riemann structure has another formulation which relates our study to systems of linear homogeneous partial differential equations of first order with complex coefficients. In fact, linear differential forms being covariant vector fields, the dual or annihilator of the space spanned by θ, dz^α will be spanned by the complex vector fields X_α, $1 \leqslant \alpha \leqslant n$, which are the same as complex linear homogeneous partial differential operators (cf. § 4). The question whether the differential system

$$X_\alpha w = 0, \quad 1 \leqslant \alpha \leqslant n, \tag{0.6}$$

has $n+1$ functionally independent solutions means exactly whether an abstractly given Cauchy-Riemann structure can be realized by one arising from a real hypersurface in $\mathbf{C}_{n+1}$. The answer is not necessarily affirmative. Recently, Nirenberg gave examples of linear differential operators X in three real variables such that the equation

$$Xw = 0 \tag{0.7}$$

does not have a nonconstant local solution [5].

It may be interesting to carry out this correspondence in an example. In $\mathbf{C}_2$ with the coordinates

$$z = x + yi, \quad w = u + vi, \tag{0.8}$$

consider the real hyperquadric M defined by

$$v = z\bar{z} = x^2 + y^2. \tag{0.9}$$

On M we have

$$\theta = \tfrac{1}{2}dw - i\bar{z}dz = (\tfrac{1}{2}du + xdy - ydx), \quad dz = dx + idy.$$

Solving the equations $\theta = dz = 0,$

we get $dx\colon dy\colon du = -i\colon 1\colon -2z.$

The corresponding operator, defined up to a factor, is

$$L = -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - 2z\frac{\partial}{\partial u} = -2i\left\{\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - i(x+yi)\frac{\partial}{\partial u}\right\}, \tag{0.10}$$

which is the famous operator discovered by Hans Lewy.

The spirit of our study parallels that of classical surface theory. We list the corresponding concepts as follows:

Surfaces in euclidean 3-space	Real hypersurfaces in $\mathbf{C}_{n+1}$
Group of motions	Pseudo-group of biholomorphic transformations
Immersed surface	Non-degenerate real hypersurface
Plane	Real hyperquadric
Induced riemannian structure	Induced CR-structure
Isometric imbedding	Existence of local solutions of certain systems of PDEs
Geodesics	Chains

Because of the special role played by the real hyperquadrics we will devote § 1 to a discussion of their various properties. Section 2 derives the normal form for formal power series and § 3 provides a proof that the resulting series converges to a biholomorphic mapping. These results were announced in [4]. In § 4 we solve the equivalence problem of the integrable G-structures in question in the sense of Elie Cartan. The solution is interpreted in § 5 as defining a connection in an appropriate bundle. Finally, the results of the two approaches, extrinsic and intrinsic respectively, are shown to agree with each other in § 6.

In the appendix we include results of S. Webster who derived some important consequences from the Bianchi identities.

1. The real hyperquadrics

Among the non-degenerate real hypersurfaces in $\mathbf{C}_{n+1}$ the simplest and most important are the real hyperquadrics. They form a prototype of the general non-degenerate real hypersurfaces which in turn derive their important geometrical properties from the "osculating" hyperquadrics. In fact, a main aim of this paper is to show how the geometry of a general non-degenerate real hypersurface can be considered as a generalization of that of real hyperquadrics. We shall therefore devote this section to a study of this special case.

Let z^α, z^{n+1} $(=w=u+iv)$, $1 \leqslant \alpha \leqslant n$, be the coördinates in $\mathbf{C}_{n+1}$. A real hyperquadric is defined by the equation

$$v = h_{\alpha\bar\beta} z^\alpha z^{\bar\beta}, \quad z^{\bar\beta} = \overline{z^\beta}, \tag{1.1}$$

where $h_{\alpha\bar\beta}$ are constants satisfying the conditions

$$h_{\alpha\bar\beta} = \bar h_{\beta\bar\alpha} = h_{\bar\beta\alpha}, \quad \det(h_{\alpha\bar\beta}) \neq 0. \tag{1.2}$$

Throughout this paper we will agree that small Greek indices run from 1 to n, unless otherwise specified, and we will use the summation convention. By the linear fractional transformation

$$Z^\alpha = \frac{2z^\alpha}{w+i}, \quad W = \frac{w-i}{w+i}, \tag{1.3}$$

equation (1.1) goes into

$$h_{\alpha\bar\beta} Z^\alpha Z^{\bar\beta} + W\overline{W} = 1. \tag{1.4}$$

This defines a hypersphere of dimension $2n+1$ when the matrix $(h_{\alpha\bar\beta})$ is positive definite. In general, we suppose $(h_{\alpha\bar\beta})$ to have p positive and q negative eigenvalues, $p+q=n$.

In order to describe a group which acts on the hyperquadric Q defined by (1.1), we introduce homogeneous coordinates ζ^A, $0 \leqslant A \leqslant n+1$, by the equations

$$z^i = \zeta^i/\zeta^0, \quad 1 \leqslant i \leqslant n+1. \tag{1.5}$$

$\mathbf{C}_{n+1}$ is thus imbedded as an open subset of the complex projective space $\mathbf{P}_{n+1}$ of dimension $n+1$. In homogeneous coordinates Q has the equation

$$h_{\alpha\bar\beta} \zeta^\alpha \zeta^{\bar\beta} + \frac{i}{2}(\bar\zeta^0 \zeta^{n+1} - \zeta^0 \bar\zeta^{n+1}) = 0. \tag{1.6}$$

For two vectors in $\mathbf{C}_{n+2}$:

$$Z = (\zeta^0, \zeta^1, \ldots, \zeta^{n+1}), \quad Z' = (\zeta'^0, \zeta'^1, \ldots, \zeta'^{n+1}), \tag{1.7}$$

we introduce the hermitian scalar product

$$(Z, Z') = h_{\alpha\bar\beta} \zeta^\alpha \bar\zeta'^\beta + \frac{i}{2}(\zeta^{n+1}\bar\zeta'^0 - \zeta^0 \bar\zeta'^{n+1}). \tag{1.8}$$

This product has the following properties:

(1) (Z, Z') is linear in Z and anti-linear in Z';

(2) $\overline{(Z, Z')} = (Z', Z)$;

(3) Q is defined by

$$(Z, Z) = 0. \tag{1.6a}$$

Let $SU(p+1, q+1)$ be the group of unimodular linear homogeneous transformations on ζ^A, which leave the form (Z, Z) invariant. Then Q is a homogeneous space with the group $SU(p+1, q+1)$ as its group of automorphisms. Its normal subgroup K of order $n+2$, consisting of the transformations

$$\zeta^{*A} = \varepsilon\zeta^A, \quad \varepsilon^{n+2} = 1, \quad 0 \leqslant A \leqslant n+1, \tag{1.9}$$

leaves Q pointwise fixed, while the quotient group $SU(p+1, q+1)/K$ acts on Q effectively.

By a Q-frame is meant an ordered set of $n+2$ vectors $Z_0, Z_1, \ldots, Z_{n+1}$ in $\mathbf{C}_{n+2}$ satisfying

$$(Z_\alpha, Z_\beta) = h_{\alpha\bar\beta}, \quad (Z_0, Z_{n+1}) = -(Z_{n+1}, Z_0) = -\frac{i}{2}, \tag{1.10}$$

while all other scalar products are zero, and

$$\det(Z_0, Z_1, \ldots, Z_{n+1}) = 1. \tag{1.11}$$

For later use it will be convenient to write (1.10) as

$$(Z_A, Z_B) = h_{A\bar B}, \quad 0 \leqslant A, B \leqslant n+1, \tag{1.10a}$$

where

$$h_{0,\overline{n+1}} = -h_{n+1,\bar 0} = -\frac{i}{2}, \tag{1.10b}$$

while all other h's with an index 0 or $n+1$ are zero. There is exactly one transformation of $SU(p+1, q+1)$ which maps a given Q-frame into another. By taking one Q-frame as reference, the group $SU(p+1, q+1)$ can be identified with the space of all Q-frames. In fact, let Z_A, Z_A^* be two Q-frames and let

$$Z_A^* = a_A^B Z_B. \tag{1.12}$$

The linear homogeneous transformation on $\mathbf{C}_{n+2}$ which maps the frame Z_A to the frame Z_A^* maps the vector $\zeta^A Z_A$ to

$$\zeta^A Z_A^* = \zeta^A a_A^B Z_B. \tag{1.13}$$

If we denote the latter vector by $\zeta^{*B} Z_B$, we have

$$\zeta^{*B} = a_A^B \zeta^A, \tag{1.14}$$

which is the most general transformation of $SU(p+1, q+1)$ when Z_A^* runs over all Q-frames.

Let H be the isotropy subgroup of $SU(p+1, q+1)$, that is, its largest subgroup leaving a point Z_0 of Q fixed. The most general change of Q-frames leaving the point Z_0 fixed is

$$\left.\begin{aligned} Z_0^* &= tZ_0, \\ Z_\alpha^* &= t_\alpha Z_0 + t_\alpha{}^\beta Z_\beta, \\ Z_{n+1}^* &= \tau Z_0 + \tau^\beta Z_\beta + \bar{t}^{-1} Z_{n+1}, \end{aligned}\right\} \tag{1.15}$$

where

$$\left.\begin{aligned} t_\alpha &= -2\, i t t_\alpha{}^\varrho \overline{\tau^\sigma} h_{\varrho\bar\sigma} = -2\, i t t_\alpha{}^\varrho \tau_\varrho, \\ t\bar{t}^{-1} &\det(t_\alpha{}^\beta) = 1, \\ t_\alpha{}^\varrho \bar{t}_\beta{}^\sigma h_{\varrho\bar\sigma} &= h_{\alpha\bar\beta}, \\ h_{\varrho\bar\sigma}\tau^\varrho \overline{\tau^\sigma} + \frac{i}{2}(\bar\tau\bar{t}^{-1} - \tau t^{-1}) &= 0. \end{aligned}\right\} \tag{1.16}$$

In the first equation of (1.16) we have used $h_{\varrho\bar\sigma}$ to raise or lower indices. Observe that the last equation of (1.16) means that the point Z_{n+1}^* lies on Q, as does Z_{n+1}; the equation can also be written

$$\operatorname{Im}(\tau t^{-1}) = -h_{\varrho\bar\sigma}\tau^\varrho\overline{\tau^\sigma}. \tag{1.17}$$

H is therefore the group of all matrices

$$\begin{pmatrix} t & 0 & 0 \\ t_\alpha & t_\alpha{}^\beta & 0 \\ \tau & \tau^\beta & \bar{t}^{-1} \end{pmatrix} \tag{1.18}$$

with the conditions (1.16) satisfied. Its dimension is n^2+2n+2. By (1.14) the corresponding coordinate transformation is

$$\left.\begin{aligned} \zeta^{*0} &= t\zeta^0 + t_\alpha\zeta^\alpha + \tau\zeta^{n+1}, \\ \zeta^{*\beta} &= t_\alpha{}^\beta\zeta^\alpha + \tau^\beta\zeta^{n+1}, \\ \zeta^{*n+1} &= \bar{t}^{-1}\zeta^{n+1}, \end{aligned}\right\} \tag{1.19}$$

or, in terms of the non-homogeneous coordinates defined in (1.5),

$$\left.\begin{aligned} z^{*\beta} &= (t_\alpha{}^\beta z^\alpha + \tau^\beta w)\, t^{-1}\delta^{-1} \\ w^* &= |t|^{-2} w\delta^{-1}, \end{aligned}\right\} \tag{1.20}$$

where

$$\delta = 1 + t^{-1}t_\alpha z^\alpha + t^{-1}\tau w. \tag{1.21}$$

We put

$$C_\alpha{}^\beta = t^{-1}t_\alpha{}^\beta, \quad C_\alpha{}^\beta a^\alpha = t^{-1}\tau^\beta, \quad \varrho = |t|^{-2}. \tag{1.22}$$

Then (1.20) can be written

$$\left.\begin{aligned} z^{*\beta} &= C_\alpha{}^\beta(z^\alpha + a^\alpha w)\,\delta^{-1}, \\ w^* &= \varrho w\delta^{-1}. \end{aligned}\right\} \tag{1.23}$$

By (1.16) the coefficients in (1.23) satisfy the conditions

$$C_\alpha^{\ \lambda} C_{\bar\beta}^{\ \bar\sigma} h_{\lambda\bar\sigma} = \varrho h_{\alpha\bar\beta}, \tag{1.24}$$

and the coefficients in δ satisfy

$$\left.\begin{aligned} t^{-1} t_\alpha &= -2 i a_\alpha = -2 i h_{\alpha\bar\beta} a^{\bar\beta}, \\ \operatorname{Im}(t^{-1}\tau) &= -h_{\alpha\bar\beta} a^\alpha a^{\bar\beta}. \end{aligned}\right\} \tag{1.25}$$

Equations (1.23) give the transformations of the isotropy group H in non-homogeneous coordinates.

Incidentally, the hyperquadric Q can be viewed as a Lie group. To see this we consider the isotropy subgroup leaving Z_{n+1} fixed. The relevant formulae are obtained from (1.19) by the involution $\zeta^0 \to \zeta^{n+1}$, $\zeta^{n+1} \to -\zeta^0$, $\zeta^\alpha \to \zeta^\alpha$ $(\alpha = 1, 2, ..., n)$:

$$\left.\begin{aligned} \zeta^{*0} &= t^{-1}\zeta^0 \\ \zeta^{*\beta} &= -\tau^\beta \zeta^0 + t_\alpha^{\ \beta}\zeta^\alpha \\ \zeta^{*n+1} &= -\tau\zeta^0 + t_\alpha \zeta^\alpha + t\zeta^{n+1} \end{aligned}\right\} \tag{1.26}$$

with the same restrictions (1.16) on the coefficients. We consider the subgroup obtained by choosing

$$t_\alpha^{\ \beta} = \delta_\alpha^{\ \beta}, \qquad t = 1 \tag{1.27}$$

and hence, by (1.16),

$$t_\alpha = -2 i h_{\alpha\bar\beta} \overline{\tau^\beta}, \quad \operatorname{Im} \tau + h_{\alpha\bar\beta}\tau^\alpha \overline{\tau^\beta} = 0.$$

In non-homogeneous coordinates we obtain

$$\left.\begin{aligned} z^{*\alpha} &= a^\alpha + z^\alpha \\ w^* &= b + 2 i h_{\alpha\bar\beta} z^\alpha \overline{a^\beta} + w \end{aligned}\right\} \tag{1.28}$$

where

$$a^\alpha = -\tau^\alpha, \quad b = -\tau, \quad \operatorname{Im} b = h_{\alpha\bar\beta} a^\alpha \overline{a^\beta}.$$

Thus the point with the coordinates $(a^1, a^2, ..., a^n, b)$ can be viewed as a point on Q. If we take the point $(z^1, z^2, ..., z^n, w)$ also in Q then (1.28) defines a noncommutative group law on Q, making Q a Lie group. Moreover, the $(n+2)^2 - 1$ dimensional group $SU(p+1, q+1)/K$ is generated by the subgroup (1.26) satisfying (1.27) and the isotropy group H.

The Maurer-Cartan forms of $SU(p+1, q+1)$ are given by the equations

$$dZ_A = \pi_A^{\ B} Z_B. \tag{1.29}$$

They are connected by relations obtained from the diffentiation of (1.10a) which are

$$\pi_{A\bar B} + \pi_{\bar B A} = 0, \tag{1.30}$$

where the lowering of indices is relative to $h_{A\bar{B}}$. For the study of the geometry of Q it will be useful to write out these equations explicitly, and we have

$$\left.\begin{aligned} \pi_{\alpha\bar{\beta}} + \pi_{\bar{\beta}\alpha} &= 0, \\ \pi_0{}^{n+1} - \bar{\pi}_0{}^{n+1} = \pi_{n+1}{}^0 - \bar{\pi}_{n+1}{}^0 &= 0, \\ \bar{\pi}_0{}^0 + \pi_{n+1}{}^{n+1} &= 0, \\ \tfrac{1}{2} i\bar{\pi}_\alpha{}^0 + \pi_{n+1}{}^\beta h_{\beta\bar{\alpha}} &= 0, \\ \bar{\pi}_\alpha{}^{n+1} + 2i\pi_0{}^\beta h_{\beta\bar{\alpha}} &= 0. \end{aligned}\right\} \tag{1.30 a}$$

Another relation between the π's arises from the differentiation of (1.11). It is

$$\pi_A{}^A = 0, \tag{1.31}$$

or, by (1.30 a),

$$\pi_\alpha{}^\alpha + \pi_0{}^0 - \bar{\pi}_0{}^0 = 0. \tag{1.31 a}$$

The structure equations of $SU(p+1, q+1)$ are obtained by the exterior differentiation of (1.29) and are

$$d\pi_A{}^B = \pi_A{}^C \wedge \pi_C{}^B, \qquad 0 \leqslant A, B, C \leqslant n+1. \tag{1.32}$$

The linear space $T_{\mathbf{C}}$ spanned by $Z_0, Z_1, \ldots, Z_n$ is the complex tangent space of Q at Z_0. It is of complex dimension n, in contrast to the real tangent space of real dimension $2n+1$ of Q, which is defined in the tangent bundle of P_{n+1}, and not in P_{n+1} itself. The intersection of Q by a complex line transversal to $T_{\mathbf{C}}$ is called a *chain*. One easily verifies that a complex line intersecting $T_{\mathbf{C}}$ transversally at some point of Q is transversal to $T_{\mathbf{C}}$ at every other point of intersection with Q. Without loss of generality, suppose the complex line be spanned by Z_0, Z_{n+1}. The line Z_0, Z_{n+1} being fixed, it follows that along a chain dZ_0, dZ_{n+1} are linear combinations of Z_0, Z_{n+1}. Hence the chains are defined by the system of differential equations

$$\pi_0{}^\alpha = \pi_{n+1}{}^\alpha = 0. \tag{1.33}$$

Through every point of Q and any preassigned direction transversal to $T_{\mathbf{C}}$ there is a unique chain. Since the complex lines in P_{n+1} depend on $4n$ real parameters, the chains on Q depend on $4n$ real parameters. The notion of a chain generalizes to an arbitrary real hypersurface of $\mathbf{C}_{n+1}$.

§ 2. Construction of a normal form

(a) In this section we consider the equivalence problem from an extrinsic point of view. Let

$$r(z^1, z^2, \ldots, z^{n+1}, \bar{z}^1, \ldots, \bar{z}^{n+1}) = 0$$

denote the considered hypersurface M in $\mathbf{C}^{n+1}$, where r is a real analytic function whose first derivatives are not all zero at the point of reference. Taking this point to be the origin we subject M to transformations holomorphic near the origin and ask for a simple normal form. At first we will avoid convergence questions by considering merely formal power series postponing the relevant existence problem to the next section.

We single out the variables

$$z^{n+1} = w = u + iv, \quad \overline{z^{n+1}} = u - iv$$

and assume that we have

$$r_{z^\alpha} = 0, \quad \alpha = 1, \ldots, n$$

$$r_w = -r_{\bar{w}} \neq 0$$

at the origin. This can be achieved by a linear transformation. Solving the above equation for v we obtain

$$v = F(z, \bar{z}, u) \tag{2.1}$$

where F is a real analytic function in the $2n+1$ variables $z, \bar{z}, u$, which vanishes at the origin together with its first derivatives. This representation lacks the previous symmetry but has the advantage that F is uniquely determined by M.

We subject this hypersurface to a holomorphic transformation

$$z^* = f(z, w), \quad w^* = g(z, w), \tag{2.2}$$

where f is n-vector valued holomorphic, g a holomorphic scalar. Moreover, f, g are required to vanish at the origin and should preserve the complex tangent space (2.1) at the origin: $w = 0$. Thus we require

$$f = 0, \quad g = 0, \quad \frac{\partial g}{\partial z} = 0 \quad \text{at } z = w = 0. \tag{2.3}$$

The resulting hypersurface M^* will be written

$$v^* = F^*(z^*, \overline{z^*}, u^*).$$

Our aim is to choose (2.2) so as to simplify this representation of M^*.

From now on we drop the assumption that F is real analytic but consider it as a formal power series in $z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n$, and u with the reality condition

$$\overline{F(z, \bar{z}, u)} = F(\bar{z}, z, u).$$

Moreover, F is assumed to have no constant or linear terms. This linear space of formal power series will be denoted by $\mathfrak{F}$. Similarly, we consider transformations (2.2) given by formal power series f, g in $z^1, \ldots, z^n, w$ without constant term and—according to (2.3)—

no terms linear in z for g. These formal transformations constitute a group under composition which we call $\mathcal{G}$. Often we combine f and g to a single element h.

For the following it is useful to decompose an element $F \in \mathfrak{F}$ into semihomogeneous parts:

$$F = \sum_{\nu=2}^{\infty} F_\nu(z, \bar{z}, u)$$

where $F_\nu(tz, t\bar{z}, t^2u) = t^\nu F_\nu(z, \bar{z}, u)$ for any $t > 0$. Thus we assign u the "weight" 2 and z, $\bar{z}$ the "weight" 1. To simplify the terms of weight $\nu = 2$ we observe that they do not contain u—since F contains no linear terms—so that

$$F_2 = Q(z) + \overline{Q(z)} + H(z, z)$$

where Q is a quadratic form of z and H a hermitian form. The transformation

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ w - 2iQ(z) \end{pmatrix}$$

removes the quadratic form, so that we can and will assume that $F_2 = H(z, z)$ is a hermitian form. This form, the Levi form, will be of fundamental importance in the following. In the sequel we will require that this form which we denote by

$$\langle z, z \rangle = F_2$$

is a *nondegenerate* hermitian form. If $\langle z, z \rangle$ is positive the hypersurface M is strictly pseudoconvex. With $\langle z_1, z_2 \rangle$ we denote the corresponding bilinear form, such that

$$\langle \lambda z_1, \mu z_2 \rangle = \lambda \bar{\mu} \langle z_1, z_2 \rangle$$

With this simplification M can be represented by

$$v = \langle z, z \rangle + F \tag{2.4}$$

where

$$F = \sum_{\nu=3}^{\infty} F_\nu$$

contains terms of weight $\nu \geq 3$ only. Now we have to restrict the transformation (2.2) by the additional requirement that $\partial^2 g / \partial z^\alpha \partial z^\beta$ vanishes at the origin.

(b) Normal forms. To determine a formal transformation in $\mathcal{G}$ simplifying M^* we write it in the form

$$z^* = z + \sum_{\nu=2}^{\infty} f_\nu, \quad w^* = w + \sum_{\nu=3}^{\infty} g_\nu, \tag{2.5}$$

where

$$f_\nu(tz, t^2w) = t^\nu f_\nu(z, w), \quad g_\nu(tz, t^2w) = t^\nu g_\nu(z, w),$$

and call ν the "weight" of these polynomials f_ν, g_ν. Inserting (2.5) into

$$v^* = \langle z^*, z^*\rangle + F^*$$

and restricting the variables z, w to the hypersurface (2.4) we get the transformation equations, in which z, $\bar{z}$, u are considered as independent variables. Collecting the terms of weight μ in the relation we get

$$F_\mu + \operatorname{Im} g_\mu(z, u + i\langle z, z\rangle) = 2 \operatorname{Re} \langle f_{\mu-1}, z\rangle + F_\mu^* + \dots$$

where the dots indicate terms depending on $f_{\nu-1}$, g_ν, F_ν, F_ν^* with $\nu < \mu$. In F_μ the arguments are z, $w = u + i\langle z, z\rangle$. We introduce the linear operator L mapping $h = (f, g)$ into

$$Lh = \operatorname{Re}\{2\langle z, f\rangle + ig\}_{w=u+i\langle z,z\rangle} \tag{2.6}$$

and write the above relation as

$$Lh = F_\mu - F_\mu^* + \dots \quad \text{for } h = (f_{\mu-1}, g_\mu) \tag{2.7}$$

and note that L maps $f_{\mu-1}$, g_μ into terms of weight μ.

In order to see how far one can simplify the power series F_μ^* one has to find a complement of the range of the operator L which is a matter of linear algebra. More precisely we will determine a linear subspace $\mathcal{N}$ of $\mathcal{F}$ such that $\mathcal{N}$ and the range of L span $\mathcal{F}$; i.e., if $\mathcal{V}$ denotes the space of $h = (f, g)$ with $f = \sum_{\nu=2}^{\infty} f_\nu$; $g = \sum_{\nu=3}^{\infty} g_\nu$, then we require that

$$\mathcal{F} = L\mathcal{V} + \mathcal{N} \quad \text{and} \quad \mathcal{N} \cap L\mathcal{V} = (0). \tag{2.8}$$

Thus $\mathcal{N}$ represents a complement of the range of L.

Going back to equation (2.7) it is clear that we can require that F_μ^* belongs to $\mathcal{N}$ and solve the resulting equation for h. Using induction it follows that (2.5) can be determined such that the function F^* belongs to $\mathcal{N}$. We call such a hypersurface M^* with $F^* \in \mathcal{N}$ in "normalform". It is of equal importance to study how much freedom one has in transforming (2.4) into normal form which clearly depends on the null space of L. Thus we have reduced the problem of finding a transformation into normal form of M to the determination of a complement of the range $\mathcal{N}$ and the null space of the operator L. Our goal will be to choose $\mathcal{N}$ such that the elements N in $\mathcal{N}$ vanish to high order at the origin so that the hypersurface M^* can be approximated to high degree by the quadratic hypersurface $v = \langle z, z\rangle$.

(c) Clearly a transformation into a normal form can be unique only up to holomorphic mappings preserving the hyperquadric $v = \langle z, z\rangle$ as well as the origin. These mappings form the $(n+1)^2 + 1$ dimensional isotropic group H studied in § 1 and given by (1.23). We will make use of H to normalize the holomorphic mapping transforming M into normal form.

After the above preparation we may consider the group $\mathcal{G}_1$ of all formal transformations preserving the family of formal hypersurfaces

$$v = \langle z, z\rangle + \{\text{weight} \geqslant 3\},$$

as well as the origin. One verifies easily that the elements of $\mathcal{G}_1$ are of the form

$$z^* = Cz + \{\text{weight} \geqslant 2\}; \quad w^* = \varrho w + \langle\text{weight} \geqslant 3\rangle,$$

where $\langle Cz, Cz\rangle = \varrho\langle z, z\rangle$. Using the form (1.23) one sees that any $\phi \in \mathcal{G}_1$ can be factored uniquely as

$$\phi = \psi \circ \phi_0$$

with $\phi_0 \in H$ and ψ a formal transformation of the form (2.5) with

$$f_2(0, w) = 0, \quad \operatorname{Re} \frac{\partial^2}{\partial w^2} g_4(0, w) = 0 \quad at \ w = 0.$$

The first term can be normalized by choice of a^α ($\alpha = 1, ..., n$) in (1.23) and the second by $\operatorname{Re}(t^{-1}\tau)$. We summarize the normalization conditions for ψ by requiring that the series

$$\left.\begin{array}{l} f, \dfrac{\partial}{\partial z^\alpha} f, \quad \dfrac{\partial}{\partial w} f \\[2ex] g, \dfrac{\partial}{\partial z^\alpha} g, \quad \dfrac{\partial}{\partial w} g, \quad \dfrac{\partial^2 g}{\partial z^\alpha \partial z^\beta}, \quad \operatorname{Re}\left(\dfrac{\partial^2 g}{\partial w^2}\right) \end{array}\right\} \tag{2.9}$$

all have no constant term.

From now on we may restrict ourselves to transformations (2.5) with the normalization (2.9). The submanifold of power series $h = (f, g)$ with the condition (2.9) will be called $\mathfrak{V}_0$. Similarly, we denote the restriction of the operator L to $\mathfrak{V}_0$ by L_0. We will see that L_0: $\mathfrak{V}_0 \to \mathfrak{F}$ is injective. This implies, in particular, that the most general formal power series mapping preserving $v = \langle z, z\rangle$ and the origin belongs to the isotropic group H.

(d) The operator L introduced above is of basic importance. To describe it more conceptually we interpret $h = (f, g)$ as a holomorphic vector field

$$X = \sum_\alpha f^\alpha \frac{\partial}{\partial z^\alpha} + g \frac{\partial}{\partial w} + \sum \bar{f}^\alpha \frac{\partial}{\partial \bar{z}^\alpha} + \bar{g} \frac{\partial}{\partial \bar{w}}$$

near the manifold, M. We describe the manifold $v = \langle z, z\rangle$ by

$$r(z, \bar{z}, w, \bar{w}) = \langle z, z\rangle - \frac{1}{2i}(w - \bar{w}) = 0.$$

Then
$$Lh = \mathfrak{L}_X r|_{r=0}$$
is the Lie-derivative of r along the holomorphic vector field X restricted to $r=0$. Of course, L is meaningful only up to a nonvanishing real factor.

For example, if we represent the manifold $r=0$ by
$$Q = \langle Z, Z\rangle + W\overline{W} = 1$$
we can associate with a holomorphic vector field
$$X = \sum_{\alpha} A^{\alpha}\frac{\partial}{\partial Z^{\alpha}} + \overline{A^{\alpha}}\frac{\partial}{\partial \overline{Z}^{\alpha}} + B\frac{\partial}{\partial W} + \overline{B}\frac{\partial}{\partial \overline{W}}$$
the Lie derivative of the above quadratic form
$$\mathfrak{L}_X Q = 2\,\mathrm{Re}\,\{\langle A, Z\rangle + B\overline{W}\}$$
restricted to $Q=1$.

For the following we will determine the kernel and a complement of the range for L in the original variables z, w. To formulate the result we order the elements F in terms of powers of z, $\bar{z}$ with coefficients being power series in u. Thus we write
$$F = \sum_{k,l\geqslant 0} F_{kl}$$
where
$$F_{kl}(\lambda z, \mu \bar{z}, u) = \lambda^k \mu^l F_{kl}(z, \bar{z}, u)$$
for all complex numbers λ, μ, and call (k, l) the "type" of F_{kl}.

The basic hermitian form will be written as
$$\langle z, z\rangle = \sum_{\alpha,\bar{\beta}} h_{\alpha\bar{\beta}} z^{\alpha}\bar{z}^{\bar{\beta}}, \quad \bar{h}_{\alpha\bar{\beta}} = h_{\beta\bar{\alpha}}.$$

Using the notation of tensor calculus we define the contraction $\mathrm{tr}\,(F_{kl}) = G_{k-1,l-1}$ of
$$F_{kl} = \sum a_{\alpha_1\ldots\alpha_k\bar{\beta}_1\ldots\bar{\beta}_l} z^{\alpha_1}\ldots z^{\alpha_k}\bar{z}^{\bar{\beta}_1}\ldots\bar{z}^{\bar{\beta}}$$
where we assume that the coefficients $a_{\alpha_1\ldots\bar{\beta}_l}$ are unchanged under permutation of $\alpha_1, \ldots, \alpha_k$ as well as of $\bar{\beta}_1, \ldots, \bar{\beta}_l$. We define for $k, l \geqslant 1$
$$\mathrm{tr}\,(F_{kl}) = \sum b_{\alpha_1\ldots\alpha_{k-1}\bar{\beta}_1\ldots\bar{\beta}_{l-1}} z^{\alpha_1}\ldots z^{\alpha_{k-1}}\bar{z}^{\bar{\beta}_1}\ldots\bar{z}^{\bar{\beta}_{l-1}} \tag{2.10}$$
where
$$b_{\alpha_1\ldots\alpha_{k-1}\bar{\beta}_1\ldots\bar{\beta}_{l-1}} = \sum_{\alpha_k,\bar{\beta}_l} h^{\alpha_k\bar{\beta}_l} a_{\alpha_1\ldots\alpha_k\bar{\beta}_1\ldots\bar{\beta}_l}.$$
Here $h^{\alpha\bar{\beta}}$ is defined as usual by
$$h^{\alpha\bar{\beta}} h_{\gamma\bar{\beta}} = \delta^{\alpha}{}_{\gamma}$$
being the Kronecker symbol.

For the description of a complement of the range of L_0 we decompose the space $\mathcal{F}$ of real formal power series as

$$\mathcal{F} = \mathcal{R} + \mathcal{N}$$

where $\mathcal{R}$ consists of series of the type

$$R = \sum_{\min(k,l)\leqslant 1} R_{kl} + G_{11}\langle z, z\rangle + (G_{10} + G_{01})\langle z, z\rangle^2 + G_{00}\langle z, z\rangle^3$$

G_{jm} being of type (j, m), and where

$$\mathcal{N} = \{N \in \mathcal{F};\ N_{kl} = 0 \ \min(k, l) \leqslant 1; \quad \operatorname{tr} N_{22} = (\operatorname{tr})^2 N_{32} = (\operatorname{tr})^3 N_{33} = 0\}. \tag{2.11}$$

This constitutes a decomposition of $\mathcal{F}$, i.e. any F can uniquely be written as $F = R + N$ with $R \in \mathcal{R}$, $N \in \mathcal{N}$. Thus $PF = R$ defines a projection operator with range $\mathcal{R}$ and null space $\mathcal{N}$. One computes easily that

$$PF = \sum_{\min(k,l)\leqslant 1} F_{kl} + G_{11}\langle z, z\rangle + (G_{10} + G_{01})\langle z, z\rangle^2 + G_{00}\langle z, z\rangle^3 \tag{2.12}$$

where

$$G_{11} = \frac{4}{n+2}\operatorname{tr}(F_{22}) - \frac{2}{(n+1)(n+2)}(\operatorname{tr})^2(F_{22})\langle z, z\rangle$$

$$G_{10} = \frac{6}{(n+1)(n+2)}(\operatorname{tr})^2 F_{32}$$

$$G_{00} = \frac{6}{n(n+1)(n+2)}(\operatorname{tr})^3 F_{33}$$

In particular, for $n = 1$

$$PF = \sum_{\min(k,l)\leqslant 1} F_{kl} + F_{22} + F_{23} + F_{32} + F_{33}. \tag{2.13}$$

While for $n > 1$ it is a requirement that $\langle z, z\rangle^l$ divides F_{kl} $(k \geqslant l)$, this is automatically satisfied for $n = 1$.

Evidently this decomposition is invariant under linear transformations of z which preserve the hermitian form $\langle z, z\rangle$.

The space $\mathcal{N}$ turns out to be an ideal in $\mathcal{F}$ under multiplication with real formal power series. We will not use this fact, however, and turn to the main result about the kernel and corange of L_0:

LEMMA 2.1. *L_0 maps $\mathcal{V}_0$ one to one onto $\mathcal{R}_3 = P\mathcal{F}_3$, where $\mathcal{F}_3$ denotes the space of those $F \in \mathcal{F}$ containing terms of weight $\geqslant 3$ only, and $\mathcal{V}_0$ is the space of formal power series satisfying* (2.9).

Before proving this lemma we draw the crucial conclusion from it: For any $F \in \mathfrak{F}_3$ the equation

$$L_0 h = F \pmod{\mathcal{N}}$$

can uniquely be solved for h in $\mathfrak{V}_0$, since this equation is equivalent to $PL_0 h = PF$. Thus $\mathcal{N}$ represents a complement of the range of L_0 and applying our previous considerations on normal forms we obtain

THEOREM 2.2. *A formal hypersurface M can be transformed by a formal transformation*

$$z^* = z + f(z, w), \quad w^* = w + g(z, w)$$

normalized by (2.9) *into a normal form*

$$v^* = \langle z^*, z^* \rangle + N \quad \text{with } N \in \mathcal{N}.$$

Moreover, this transformation is unique.

COROLLARY. *The only formal power series transformations which preserve $v = \langle z, z \rangle$ and the origin are given by the fractional linear transformations* (1.23) *constituting the group H.*

(e) Obviously it suffices to show that the equation

$$Lh = F \pmod{\mathcal{N}}$$

possesses a unique solution $h \in \mathfrak{V}_0$. Here F is a formal power series containing terms of weight $\geqslant 3$ only. Collecting terms of equal type we have to solve the equations

$$(Lh)_{kl} = F_{kl} \quad \text{for } \min(k, l) \leqslant 1$$

$$(Lh)_{kl} = F_{kl} \pmod{\mathcal{N}} \quad \text{for } (k, l) = (2, 2),\ (3, 2),\ (3,3).$$

For this purpose we calculate $(Lh)_{kl}$ for the above types (k, l); because of the real character of F we may take $k \geqslant l$. We will use the identity

$$f(z, u + i\langle z, z\rangle) = \sum_{\nu=1}^{\infty} \left(\frac{\partial}{\partial w}\right)^{\nu} f(z, u) \frac{i^{\nu} \langle z, z\rangle^{\nu}}{\nu!}.$$

Expanding $f(z, w), g(z, w)$ in powers of $z, \bar{z}$ we write

$$f = \sum_{k=0}^{\infty} f_k, \qquad g = \sum_{k=0}^{\infty} g_k$$

where $$f_k(tz, w) = t^k f_k(z, w), \quad g_k(tz, w) = t^k g_k(z, w).$$

This notation should not be confused with the previous one which combined terms of equal weight, and which will no longer be needed.

We write Lh in the form

$$Lh = \operatorname{Re}\{2\langle f, z\rangle + ig\}_{w=u+i\langle z,z\rangle} = \langle f + f' i\langle z,z\rangle + \dots, z\rangle + \frac{i}{2}(g + g' i\langle z,z\rangle + \dots) + \text{complex conj.}$$

where the arguments of $f, f', \dots, g, g', \dots$ are z, u, and the prime indicates differentiation with respect to u. Now we collect terms of equal type (k, l). For example, if $k \geqslant 2$ the terms of type $(k, 0)$ and $(k+1, 1)$, respectively are

$$\frac{i}{2} g_k, \quad \langle f_{k+1}, z\rangle - \tfrac{1}{2} g_k' \langle z, z\rangle$$

so that we have

$$\left.\begin{aligned} ig_k &= 2F_{k0} \\ 2\langle f_{k+1}, z\rangle - g_k'\langle z,z\rangle &= 2F_{k+1,1} \end{aligned} \quad (k \geqslant 2).\right\} \tag{2.14 a}$$

For $k=1$ one gets additional terms and an easy calculation shows

$$\left.\begin{aligned} &ig_1 + 2\langle z, f_0\rangle = 2F_{10} \\ &-g_1'\langle z,z\rangle + 2\langle f_2, z\rangle - 2i\langle z, f_0'\rangle\langle z,z\rangle = 2F_{21} \\ &-\frac{i}{2} g_1''\langle z,z\rangle^2 + 2i\langle f_2', z\rangle\langle z,z\rangle - \langle z, f_0''\rangle\langle z,z\rangle^2 = 2F_{32} \quad (\operatorname{mod} \mathcal{N}). \end{aligned}\right\} \tag{2.14 b}$$

Finally, for $k=0$ one obtains four real equations,

$$\left.\begin{aligned} &-\operatorname{Im} g_0 = F_{00} \\ &\tfrac{1}{2}\operatorname{Im} g_0''\langle z,z\rangle^2 - 2\operatorname{Im}\langle f_1', z\rangle\langle z,z\rangle = F_{22} \quad (\operatorname{mod} \mathcal{N}) \\ &-\operatorname{Re} g_0'\langle z,z\rangle + 2\operatorname{Re}\langle f_1, z\rangle = F_{11} \\ &\tfrac{1}{6}\operatorname{Re} g_0'''\langle z,z\rangle^3 - \operatorname{Re}\langle f_1'', z\rangle\langle z,z\rangle^2 = F_{33} \quad (\operatorname{mod} \mathcal{N}). \end{aligned}\right\} \tag{2.14 c}$$

Thus we obtain three groups of decoupled systems of differential equations; actually the last system (2.14c) decouples into two groups.

The solution of these systems is elementary: Equations (2.14a) can be solved uniquely for f_{k+1}, g_k $(k \geqslant 2)$. Equations (2.14b) are equivalent to

$$\begin{aligned} &ig_1 + 2\langle z, f_0\rangle = 2F_{10} \\ &-g_1'\langle z,z\rangle + 2\langle f_2, z\rangle - 2i\langle z, f_0'\rangle\langle z,z\rangle = 2F_{21} \\ &-4\langle z, f_0''\rangle\langle z,z\rangle^2 = 2F_{32} - 2iF_{21}'\langle z,z\rangle - F_{10}''\langle z,z\rangle^2 \quad (\operatorname{mod} \mathcal{N}). \end{aligned}$$

Since the last equation has to be solved (mod $\mathcal{N}$) only we replace the right-hand side by its projection into $\mathcal{R}$, which we call $G_{10}\langle z,z\rangle^2$ so that

$$-4\langle z, f_0''\rangle = G_{10}.$$

With such a choice of f_0 one solves the first equation for g_1 and then the second for f_2. Here f_0 is fixed up to a linear function in w; but by our normalization (2.9), f_0 and hence g_1, f_2 are uniquely determined.

Finally we have to solve (2.14c): Since

$$F_{22} = G_{11}\langle z, z\rangle + N_{22}, \quad N_{22} \in \mathcal{N}$$

the second equation takes the form

$$\tfrac{1}{2} \operatorname{Im} g_0''\langle z, z\rangle - 2 \operatorname{Im} \langle f_1', z\rangle = G_{11}$$

which can be solved with the first for $\operatorname{Im} g_0$ and $\operatorname{Im} \langle f_1', z\rangle = (d/du) \operatorname{Im} \langle f_1, z\rangle$. Since f_1 vanishes for $u=0$ we determine $\operatorname{Im} g_0$, $\operatorname{Im} \langle f_1, z\rangle$ uniquely in this way.

The last two equations of (2.14c) are equivalent to

$$\begin{aligned} -\operatorname{Re} g_0'\langle z,z\rangle + 2 \operatorname{Re} \langle f_1, z\rangle &= F_{11} \\ -\tfrac{1}{3} \operatorname{Re} g_0''' \qquad\qquad\qquad &= G_{00} \end{aligned}$$

where we used that

$$F_{33} + \tfrac{1}{2} F_{11}''\langle z, z\rangle^2 = G_{00}\langle z, z\rangle^3 \pmod{\mathcal{N}}.$$

Clearly, the last equation can be solved for $\operatorname{Re} g_0'''$ and then the first for $\operatorname{Re} \langle f_1, z\rangle$. Thus g_0 is determined up to $aw + bw^2$, a, b real. But by our normalization both $a=0$ and $b=0$, and $\operatorname{Re} g_0$, $\operatorname{Re} \langle f_1, z\rangle$ are uniquely determined.

Thus, summarizing, all equations can be satisfied by f_k, g_k satisfying the normalization (2.9) and uniquely so. This concludes the proof of the Lemma 2.1 and hence of Theorem 2.2.

§ 3. Existence theorems

(a) So far we considered only formal series and now turn to the case of real analytic hypersurfaces M. We will show that the formal series transforming M into normal form are, in fact, convergent and represent holomorphic mappings. In the course of the proof we will obtain a geometrical interpretation of the condition

$$\operatorname{tr} N_{22} = 0, \quad (\operatorname{tr})^2 N_{32} = 0, \quad (\operatorname{tr})^3 N_{33} = 0$$

describing the normal form.

We begin with a transformation into a partial normal form: Let M be a real analytic hypersurface and γ a real analytic arc on M which is transversal to the complex tangent space of M. Moreover, we give a frame of linear independent vectors $e_\alpha \in T_{\mathbf{C}}$ ($\alpha = 1, \ldots, n$), also real analytic along the curve γ. All these data γ, e_α are given locally near a distinguished point p on γ.

THEOREM 3.1. *Given a real analytic hypersurface M with the above data γ, e_α there exists a unique holomorphic mapping ϕ taking p into the origin $z=w=0$, γ into the curve $z=0$, $w=\xi$, where ξ is a real parameter ranging over an interval, and e_α into $\phi_*(e_\alpha)=\partial/\partial z^\alpha$ and the hypersurface into $\phi^*(M)$ given by*

$$v=F_{11}(z,\bar{z},u)+\sum_{\min(k,l)\geq 2} F_{kl}(z,\bar{z},u). \tag{3.1}$$

Proof. We may assume that the variables $z=(z^1,\dots,z^n)$ and w are so introduced that p is given by $z=0$, $w=0$ and the complex tangent space of M by $w=0$. If γ is given by

$$z=p(\xi),\quad w=q(\xi)$$

where $\xi=0$ corresponds to $z=0$, $w=0$ then $q'(0)\neq 0$. The transformation

$$z=p(w^*)+z^*,\quad w=q(w^*)$$

is holomorphic and takes the curve γ into $z^*=0$, $w^*=\xi$. Changing the notation and dropping the star we can assume that the hypersurface is given by

$$v=F(z,\bar{z},u)$$

and γ by $z=0$, $w=\xi$, so that $F(0,0,u)=0$.

The function $F(z,\bar{z},u)$ is given by convergent series and is real. In the variables x^α, y^α given by $z^\alpha=x^\alpha+iy^\alpha$, $\bar{z}^\alpha=x^\alpha-iy^\alpha$ the function $F(z,\bar{z},u)$ is real analytic. The space of these functions, real analytic in some neighborhood of the origin and vanishing at the origin will be denoted by $\mathfrak{F}^\omega$. In the following it will be a useful observation that z, $\bar{z}$ can be considered as independent variables for $F\in\mathfrak{F}^\omega$.

LEMMA 3.2. *If $F\in\mathfrak{F}^\omega$ and $F(0,0,u)=0$ then there exists a unique holomorphic transformation*

$$z^*=z;\quad w^*=w+g(z,w);\quad g(0,w)=0$$

taking

$$v=F(z,\bar{z},u)$$

into

$$v^*=F^*(z^*,\bar{z}^*,u^*)$$

where

$$F^*_{k0}=F^*_{0k}=0\quad\text{for } k=1,2,\dots. \tag{3.2}$$

Proof. The conditions (3.2) can be expressed by

$$F^*(z^*,0,u)=0 \tag{3.3}$$

and a second equation which follows on account of the real character of F^*. The transformation formula gives

$$F^*(z^*, \bar{z}^*, u^*) = \frac{1}{2i}(g(z, w) - \overline{g(z, w)}) + F(z, \bar{z}, u)$$

where
$$u^* = u + \tfrac{1}{2}(g(z, w) + \overline{g(z, w)}), \quad w = u + iF(z, \bar{z}, u).$$

Keeping in mind that $z, \bar{z}, u$ can be viewed as independent variables, we set $\bar{z} = 0$ in the above equations. Observing that $\overline{g(z,w)} = 0$ for $\bar{z} = 0$, since $g(0, w) = 0$, we obtain with (3.3)

$$0 = \frac{1}{2i} g(z, u + iF(z, 0, u)) + F(z, 0, u) \tag{3.4}$$

as condition for the function g. To solve this equation we set

$$s = u + iF(z, 0, u).$$

Since, by assumption, $F(z, 0, u)$ vanishes for $z = 0$ we can solve this equation for u:

$$u = s + G(z, s) \quad \text{where } G(0, s) = 0.$$

Equation (3.4) takes the form

$$0 = \frac{1}{2i} g(z, s) + \frac{1}{i}(s - u)$$

or
$$u = s + \tfrac{1}{2} g(z, s).$$

Thus $g(z, w) = 2G(z, w)$ is the desired solution which vanishes for $z = 0$. It is clear that the steps can be reversed, and Lemma 3.2 is proven.

Thus we may assume that M is of the form

$$v = F(z, \bar{z}, u) = \sum_{\min(k,l) \geq 1} F_{kl}(z, \bar{z}, u),$$

and the curve γ is given by $z = 0$, $w = \xi$. Now we will require that $F_{11}(z, \bar{z}, 0)$ is a nondegenerate hermitian form.

LEMMA 3.3. *If* $F \in \mathfrak{F}^\omega$ *and*

$$F_{k0} = 0 = F_{0k} \quad \text{for } k = 0, 1, \ldots$$

and $F_{11}(z, \bar{z}, 0)$ *nondegenerate then there exists a holomorphic transformation*

$$z^* = z + f(z, w); \quad w^* = w \tag{3.5}$$

with $f(0, w) = 0$, $f_z(0, w) = 0$ *and such that* $v = F(z, \bar{z}, u)$ *is mapped into*

$$v^* = F_{11}^*(z^*, \bar{z}^*, u^*) + \sum_{\min(k,l) \geq 2} F_{kl}^*. \tag{3.6}$$

Proof. By $O_{\varkappa\lambda}$ we will denote a power series in $z, \bar{z}$ containing only terms of type (k, l) with $k \geqslant \varkappa$ and $l \geqslant \lambda$. Thus $F(z, \bar{z}, u)$ can be written as

$$F(z. \bar{z}, u) = F_{11}(z, \bar{z}, u) + \sum_{\alpha=1}^{n} z^\alpha A_\alpha(\bar{z}, u) + \sum_{\alpha=1}^{n} \overline{z^\alpha}\, \overline{A_\alpha(\bar{z}, u)} + O_{22}$$

where
$$A_\alpha(\bar{z}, u) = \frac{\partial}{\partial z^\alpha}(F - F_{11})\big|_{z=0} = O_{02}.$$

We restrict u to such a small interval in which the Levi form

$$F_{11}(z, \bar{z}, u) = \sum h_{\alpha\bar{\beta}}(u)\, z^\alpha \overline{z^\beta}$$

is nondegenerate. If $(h^{\alpha\bar{\beta}})$ is the inverse matrix of $(h_{\alpha\bar{\beta}})$ and the holomorphic vector function $f(z, w)$ is defined by

$$\overline{f^\beta(z, u)} = \sum_\alpha h^{\alpha\bar{\beta}}(u)\, A_\alpha(\bar{z}, u) \in O_{02} \tag{3.7}$$

then
$$\begin{aligned} F_{11}(z + f, \bar{z} + \bar{f}, u) &= F_{11}(z, \bar{z}, u) + \sum z^\alpha A_\alpha + \sum \overline{z^\alpha}\, \overline{A_\alpha} + O_{22} \\ &= F(z, \bar{z}, u) + O_{22} \end{aligned}$$

so that $v = F(z, \bar{z}, u)$ is transformed by (3.5), defined by (3.7), into

$$v^* = F_{11}(z^*, \bar{z}^*, u^*) + O_{22}.$$

Note also that, by (3.6), $f(z, u) \in O_{20}$ which finishes the proof.

With these two lemmas we see that the coordinates can be so chosen that γ is given by the u-axis: $z = 0$, $w = \xi$ and M given by (3.6). Actually the coordinates are not uniquely fixed by these requirements but the most general holomorphic transformation preserving the parametrized curve γ: $z = 0$, $w = \xi$ and the form (3.6) of M is given by

$$z^* = M(w)z, \quad w^* = w$$

where $M(w)$ is a nonsingular matrix depending holomorphically on w. This matrix can be used to transform the frame e_α into $\partial/\partial z^\alpha$ which, in turn, fixes $M(w)$ uniquely. This completes the proof of Theorem 3.1.

In order to make the hermitian form $F_{11}(z, \bar{z}, u)$ independent of u we perform a linear transformation

$$z^* = C(w)z, \quad w^* = w$$

and determine C such that

$$F_{11}(C(u)z, \overline{C(u)z}, 0) = F_{11}(z, \bar{z}, u).$$

The choice of $C(u)$ becomes unique if we require that $C(u)$ be hermitian with respect to the form

$$F_{11}(z, \bar{z}, 0) = \langle z, z \rangle$$

i.e.

$$F_{11}(Cz, \bar{z}, 0) = F_{11}(z, \bar{C}\bar{z}, 0)$$

Denoting the matrix $(h_{\alpha\beta}(u))$ by $H(u)$ these requirements amount to the two matrix equations

$$\left. \begin{aligned} C^*(u) H(0) C(u) &= H(u) \\ H(0) C(u) &= C^*(u) H(0). \end{aligned} \right\} \tag{3.8}$$

Eliminating $C^*(u)$ we obtain

$$C^2(u) = H(0)^{-1} H(u).$$

Since the right-hand side is close to the identity matrix for small u there exists a unique matrix $C(u)$ with $C(0) = I$. This solution depends analytically on u and, morever, satisfies automatically the relation (3.8). Indeed, if $C(u)$ is a solution so is $H^{-1}(0) C^*(u) H(0)$ which also reduces to the identity for $u = 0$. By uniqueness it agrees with $C(u)$ yielding (3.8).

Thus we can assume that the hypersurface is represented by

$$v = \langle z, z \rangle + \sum_{\min(k,l) \geqslant 2} F_{kl}(z, \bar{z}, u) \tag{3.9}$$

and γ is given by $z = 0$, $w = \xi$. The freedom in the change of variables preserving γ and the above form of M is given by linear map $z \to U(w) z$, $w \to w$ which preserve the form $\langle z, z \rangle$. In other words we can prescribe an analytic frame $e_\alpha(u)$ $(\alpha = 1, ..., n)$ along the u-axis which is normalized by

$$\langle e_\alpha, e_\beta \rangle = h_{\alpha\bar{\beta}} \text{ where } \langle z, z \rangle = \sum h_{\alpha\bar{\beta}} z^\alpha \overline{z^\beta}.$$

The coefficients of $F_{kl}(z, \bar{z}, u)$ in (3.9) can be viewed as functionals depending on the curve γ: $z = p(\xi)$, $w = q(\xi)$. These are, of course, *local* functionals and more precisely we have

LEMMA 3.4. *The coefficients of F_{kl} in (3.9) depend analytically on p, q, $\bar{p}$, $\bar{q}$ and their derivatives of order $\leqslant k + l$. More precisely, these coefficients depend rationally on the derivatives p', $\bar{p}'$, q', etc.*

Proof. Let $v = G(z, \bar{z}, u)$ represent the given hypersurface containing the curve $z = p(\xi)$, $w = q(\xi)$ where $\operatorname{Re} q'(0) \neq 0$. The condition that this curve be transversal to the complex tangent space amounts to

$$\operatorname{Re} \{q' - 2iG_z p' - iG_u q'\} \neq 0 \tag{3.10}$$

which we require for $\xi = 0$. First we subject the hypersurface to the transformation

$$z = p(w^*) + z^*, \quad w = q(w^*)$$

and study how the resulting hypersurface depends on p, q. This hypersurface is given implicitly by

$$\frac{1}{2i}\{q - \bar{q}\} - G(p + z^*, \bar{p} + \overline{z^*}, \tfrac{1}{2}(\bar{q} + \bar{q})) = 0 \tag{3.11}$$

where the arguments in p, q are w^*. Under the assumption (3.10) we can solve this equation for v^* to obtain the desired representation. Since the given curve was assumed to lie on the given hypersurface we have $v^* = 0$ as a solution of (3.11) if $z^* = 0$, $\overline{z^*} = 0$ Therefore the solution of (3.11)

$$v^* = F^*(z^*, \overline{z^*}, u^*) \tag{3.12}$$

vanishes for $z^* = 0, \overline{z^*} = 0$. We expand the terms in (3.11) in powers of z^*, $\overline{z^*}$, v^* and investigate the dependence of the coefficients on $p(u^*)$, $q(u^*)$ and their derivatives.

To simplify the notation we drop the star and denote the left-hand side of (3.11) by

$$\Phi(z, \bar{z}, u, v) = \sum_{\zeta + \nu > 0} \Phi_{\zeta\nu}$$

where $\Phi_{\zeta\nu}$ is a polynomial in $z, \bar{z}, v$, homogeneous of degree ζ in $z, \bar{z}$ and of degree ν in v. The equation (3.11) takes the form

$$Av + \Phi_{10} + \sum_{\zeta + \nu \geq 2} \Phi_{\zeta\nu} = 0 \tag{3.13}$$

where $$Av = \Phi_{01} = \operatorname{Re}\{q' - 2iG_z(p, \bar{p}, \tfrac{1}{2}(q + \bar{q}))p' - iG_u q'\}v$$

Thus A is an analytic function of p, $\bar{p}$, q, $\bar{q}$ and their derivatives, in fact, depending linearly on the latter. Moreover, by (3.10), we have $A \neq 0$ for small $|u|$.

Similarly, the coefficients of $\Phi_{\zeta\nu}$ are analytic functions of $p, \bar{p}, q, \bar{q}$ at $\xi = u$ and their derivatives of order $\leq \nu$. This becomes clear if one replaces $q(u + iv)$ by $q(u) + q'(u)iv + \ldots$ and similarly for $p(u + iv)$ in (3.11) and rewrites the resulting expressions as the series Φ in $z, \bar{z}, v$. In fact, the coefficients of Φ will depend polynomially on $p, \bar{p}', q'$ etc. Finally to obtain the same property for the coefficients of F^* in (3.12) we solve (3.13) for v as a power series in $z, \bar{z}$; let

$$v = V_1 + V_2 + \ldots$$

where V_ζ are homogeneous polynomials in z, $\bar{z}$ of degree ζ. We obtain V_ζ by comparison of coefficients in (3.13) in a standard fashion, which gives $A\ V_\zeta$ as a polynomial in $V_1, V_2, \ldots, V_{\zeta-1}$ with coefficients analytic in p, q, $\bar{p}$, $\bar{q}$ and their derivatives of order $\leq \zeta$; in dependence on the derivatives they are rational, the denominator being a power of A.

This proves the statement about the analytic behavior of the coefficients of F^* in (3.12). To complete the proof we have to subject this hypersurface to the holomorphic transformation of Lemma 3.2, 3.3 which preserve the curve $z=0$, $w=\xi$. From the proofs of these lemmas it is clear that the coefficients of the transformation as well as of the resulting hypersurface (3.6) have the stated analytic dependence on p, q. The same is true of the transformation $z\to C(w)z$, $w\to w$ which leads to (3.9).

(b) Returning to (3.9) it remains to satisfy the relations

$$\operatorname{tr} F_{22}=0, \quad (\operatorname{tr})^2 F_{32}=0, \quad (\operatorname{tr})^3 F_{33}=0$$

which give rise to a set of differential equations for the curve γ and for the associated frame.

We begin with the condition $(\operatorname{tr})^2F_{32}=0$ which gives rise to a differential equation of second order for the curve γ, where the parametrization is ignored. For this purpose we assume that the parametrization is fixed, say by $\operatorname{Re} q(\xi)=\xi$ and study the dependence of F_{32} on $p(\xi)$. According to Lemma 3.4 the coefficients of F_{32} are analytic functions of $p, \bar{p}$ and their derivatives up to order 5. But if the hypersurface is in the form (3.9) then F_{32} depends on the derivatives of order $\leqslant 2$ and is of the form

$$F_{32}=\langle z, Bp''\rangle\langle z, z\rangle^2+K_{32} \tag{3.14}$$

where K_{32}, B depend on p, $\bar{p}$, p', $\bar{p}'$ analytically, and B is a nonsingular matrix for small $|u|$.

To prove this statement we recall that (3.9) was obtained by a transformation

$$z\to p(w)+C(w)z+\dots, \quad w\to q(w)+\dots$$

We choose $\operatorname{Re} q(u)=u$ fixing the parametrization; $\operatorname{Im} q(u)$ is determined by $p, \bar{p}$. To study the dependence of F_{32} at $u=u_0$ we subject (3.9) to the transformation

$$z=s(w^*)+z^*+\dots, \quad w=q(w^*+u_0) \tag{3.15}$$

which amounts to replacing $p(u)$ by $p^*(u^*)=p(u_0+u)+C(u_0+u)s(u)$. Considering p and p' fixed at $u=u_0$ we require $s(0)=0$, $s'(0)=0$ and investigate the dependence of F_{32} on the germ of s at $u=u_0$. We choose the higher order terms in (3.15) in such a way that the form of (3.9) is preserved as far as terms of weight $\leqslant 5$ is concerned. This is accomplished by the choice

$$z=z^*+s(w^*)+2i\langle z^*, s'(\overline{w^*})\rangle z^*$$
$$w=w^*+u_0+2i\langle z^*, s(\overline{w^*})\rangle.$$

Since the hermitian form $\langle , \rangle$ is antilinear in the second argument this transformation is holomorphic. One computes

$$v-\langle z, z\rangle = v^*-\langle z^*, z^*\rangle + 4\ \mathrm{Re}\ \langle z^*, s''(0)\rangle \langle z^*, z^*\rangle^2 + \dots$$

if z, w lies on the manifold (3.9). The dots indicate terms of weight $\geqslant 6$ in z^*, $\overline{z^*}$, u^*. Thus, for $u^*=0$ we get, setting $z^*=z$,

$$F_{32}|_{u=u_0} = F^*_{32}|_{u^*=0} + 2\langle z^*, s''(0)\rangle \langle z^*, z^*\rangle^2.$$

Hence F^*_{32} depends on s, s', s'' only, and using that

$$(C(u_0+u)s(u))'' = C(u_0)s''(0) \quad \text{for } u=0$$

we see that

$$F^*_{32} + 2\langle z^*, C^{-1}(u_0)p''(0)\rangle \langle z^*, z^*\rangle^2$$

is independent of s which proves (3.14) with $B(u) = -2C^{-1}(u_0)$. Thus $B(0) = -2I$, and $B(u)$ is nonsingular for small values of $|u|$.

Therefore the equation $(\mathrm{tr})^2 F_{32}=0$ can be written as a differential equation

$$p'' = Q(p, \bar{p}, p', \bar{p}', u)$$

with an analytic right-hand side. Thus for given $p(0)$, $p'(0)$ there exists a unique analytic solution $p(u)$ for sufficiently small $|u|$. Choosing the curve γ in this manner we have $(\mathrm{tr})^2 F_{32}=0$.

To show that this differential equation $(\mathrm{tr})^2 F_{32}=0$ is independent of the parametrization and the frame e_α we subject the hypersurface (3.9) to the most general self mapping

$$z \to \sqrt{g'(w)}\, U(w) z$$
$$w \to g(w)$$

where $\mathrm{Im}\, g(u)=0$, $g(0)=0$, $g'(0)>0$, $\langle Uz, Uz\rangle = \langle z, z\rangle$ for real w. One checks easily that under such mapping F_{32} is replaced by

$$g'^{-3/2} F_{32}(U^{-1}z, \bar{U}^{-1}\bar{z}, g^{-1}(u))$$

and the equation $(\mathrm{tr})^2 F_{32}=0$ remains satisfied for $z=0$. Thus $(\mathrm{tr})^2 F_{32}$ is a differential equation for γ irrespective of the parametrization and the frame.

Next we fix the frame e_α so that $\mathrm{tr}\, F_{22}=0$. For this purpose we subject (3.9) with $(\mathrm{tr})^2 F_{32}=0$ to a coordinate transformation

$$z^* = U(w)z, \quad w^* = w$$

with a nonsingular matrix $U(w)$ which for $\mathrm{Im}\, w=0$ preserves the form $\langle z, z\rangle = \langle Uz, Uz\rangle$. We will define U via a differential equation

$$\frac{d}{du}U = UA \quad \text{with} \quad \langle Az, z\rangle + \langle z, Az\rangle = 0 \tag{3.16}$$

and find from $U(w) = U(u) + ivU' + \ldots$ that

$$\begin{aligned}\langle z^*, z^*\rangle &= \langle (U + iU'\langle z, z\rangle + \ldots)z,\ (U + iU'\langle z, z\rangle + \ldots)z\rangle \\ &= \langle (I + iA\langle z, z\rangle + \ldots)z,\ (I + iA\langle z, z\rangle + \ldots)z\rangle \\ &= \langle z, z\rangle(1 + 2i\langle Az, z\rangle + \ldots)\end{aligned}$$

where the arguments of U, A are u and the dots indicate terms of order $\geqslant 6$ in $z, \bar{z}$. Thus

$$F^*_{22} = F_{22} + 2i\langle Az, z\rangle\langle z, z\rangle, \quad F^*_{32} = F_{32},$$

where on the left side we set $z^* = U(u)z$. Thus, since $\operatorname{tr} F_{22}$ is a hermitian form the equation $\operatorname{tr} F^*_{22} = 0$ determines $\langle iAz, z\rangle$ uniquely as a hermitian form, hence A is uniquely determined as an antihermitian matrix with respect to $\langle , \rangle$. Thus the differential equation (3.16) defines a $U(u)$, analytic in u, and preserving the form $\langle , \rangle$ if $U(0)$ does. More geometrically, (3.16) can be viewed as a first order differential equation

$$\frac{de_\alpha}{du} = \sum a^\beta_\alpha(u)\, e_\beta, \quad \langle e_\alpha, e_\beta\rangle = h_{\alpha\bar{\beta}}$$

for the frame. Note that the term F_{32} is not affected by this choice of the frame.

Finally, we are left with choosing the parametrization on the curve in such a way that $(\operatorname{tr})^3 F_{33} = 0$. For this purpose perform the transformation

$$z^* = (q'(w))^{1/2} z, \quad w^* = q(w)$$

with

$$q(0) = 0, \quad \overline{q(w)} = q(\bar{w}), \quad q'(0) > 0.$$

Thus

$$v^* = q'(u)\, v - \tfrac{1}{6} q''' v^3 + \ldots$$

$$\langle z^*, z^*\rangle = q'(u)\langle z, z\rangle - \tfrac{1}{2}\left(q''' - \frac{q''^2}{q'}\right)\langle z, z\rangle^3$$

which gives for z, w on the hypersurface

$$v^* - \langle z^*, z^*\rangle = q'(v - \langle z, z\rangle) + \left(\tfrac{1}{3} q''' - \tfrac{1}{2}\frac{q''^2}{q'}\right)\langle z, z\rangle^3 + \ldots$$

or

$$F^*_{33} = q' F_{33} + \left(\tfrac{1}{3} q''' - \tfrac{1}{2}\frac{q''^2}{q'}\right)\langle z, z\rangle^3.$$

Thus, $(\operatorname{tr})^3 F^*_{33} = 0$ gives rise to an analytic third order differential equation for the real function $q(u)$, uniquely determined by $q(0)=0$, $q'(0)>0$, $q''(0)$, which are assumed real. Thus we have a distinguished parameter ξ in the above curve which is determined up to real projective transformations $\xi\to\xi/(\alpha\xi+\beta)$, $\beta>0$.

Thus we have constructed a holomorphic transformation taking M into the normal form, and the existence proof has been reduced to that for ordinary differential equations. The choice of the initial values for $p'(0)\in C^n$, $U(0)$ and $\operatorname{Re} q'(0)$, $\operatorname{Re} q''(0)$ allows us to satisfy the normalization condition (2.9) of § 2. In fact, these $2n+n^2+1+1=(n+1)^2+1$ real parameters characterize precisely an element of the isotropic group H. Thus we have shown

THEOREM 3.5. *If M is a real analytic manifold the unique formal transformation of Theorem 2.2 taking M into a normal form and satisfying the normalization condition is given by convergent series, i.e. defines a holomorphic mapping.*

Two real analytic manifolds M_1, M_2 with distinguished points $p_1\in M_1$, $p_2\in M_2$ are holomorphically equivalent by a holomorphic mapping ϕ taking p_1 into p_2 if and only if $\langle M_k, p_k\rangle$ for $k=1, 2$ have the same normal forms for some choice of the normalization conditions. Thus the problem of equivalence is reduced to a finite dimensional one.

The arbitrary initial values for the differential equations $\operatorname{tr} F_{22}=0$, $(\operatorname{tr})^2 F_{32}=0$, $(\operatorname{tr})^3 F_{33}=0$ have a geometrical interpretation: At a fixed point $p\in M$ they correspond to

(i) a normalized frame $e_\alpha\in T_{\mathbf{C}}$, $\langle e_\alpha, e_\beta\rangle=h_{\alpha\bar\beta}$
(ii) a vector $e_{n+1}\in T_{\mathbf{R}}-T_{\mathbf{C}}$ corresponding to the tangent vector of the curve γ, and
(iii) a real number fixing the parametrization, corresponding to $\operatorname{Re} q''(0)$.

With the concepts of the following section this will be viewed as a frame in a line bundle over M.

As a consequence of these results above we see that the holomorphic mappings taking a nondegenerate hypersurface into themselves form a finite dimensional group. In fact, fixing a point the dimension of this group is at most equal to that of the isotropy group H, i.e. $(n+1)^2+1$. Adding the freedom of choice of a point gives $2n+1+(n+1)^2+1=(n+2)^2-1$ as an upper bound for the dimension of the group of holomorphic self mappings of M. This upper bound is realized for the hyperquadrics.

The above differential equations define a holomorphically invariant family of a parametrized curve γ transversal to the complex tangent bundle, with a frame e_α propagating along γ. The parameter ξ is fixed up to a projective transformation $\xi/(\alpha\xi+\beta)$ $(\beta\neq 0)$ keeping $\xi=0$ fixed. Thus cross ratios of 4 points on these curves are invariantly defined. We summarize: (i) $\operatorname{tr} F_{22}=0$ represents a first order differential equation for the frame e_α, (ii) $(\mathrm{tr})^2 F_{32}=0$ defines a second order differential equation for the distinguished curves γ, irrespective of parametrization and (iii) $(\mathrm{tr})^3 F_{33}=0$ defines a third order differential equation for the parametrization.

(c) The differential equations $\mathrm{tr}\, F_{22}=0$, $(\mathrm{tr})^2 F_{32}=0$, $(\mathrm{tr})^3 F_{33}=0$ remain meaningful for merely smooth manifolds. Indeed, if M is six times continuously differentiable one can achieve the above normal forms up to terms of order 6 inclusive, simply truncating the above series expansions. Clearly the resulting families of curves and frames are invariantly associated with the manifold under mappings holomorphic near M. Indeed since the differential equations are obtained by the expansions of § 2 up to terms of weight $\leqslant 6$ at any point one may approximate M at this point by a real analytic one and read off the holomorphic invariance of this system of differential equations. In this case the distinguished curves γ are, in general, only 3 times continuously differentiable but the normal form (see (2.11) via a holomorphic map, cannot be achieved, not even to sixth order in $z, \bar{z}$. This would require that the function $f(z, u)$, $g(z, u)$ defining the transformation and which can be taken as polynomials in z admit an analytic continuation to complex values of u. If the Levi form is indefinite one has to require an analytic continuation to both sides which can happen only in the exceptional case of analytic curves γ. If, however, the Levi-form is definite, i.e. in the pseudoconvex case one has to require only that $f(z, u)$, $g(z, u)$ admit one sided analytic continuations. However, we do not pursue this artificial question but record that the structure of differential equations for the curves γ and their associated frame is meaningful in the case of six times differentiabl manifolds.

(d) In the case $n=1$ the normal form has a simpler form since the contraction (tr) becomes redundant. For this reason F_{22}, F_{23}, F_{32}, F_{33} all vanish and the normal form can be written

$$v = z\bar{z} + c_{42} z^4 \bar{z}^2 + c_{24} z^2 \bar{z}^4 + \sum_{k+l\geqslant 7} c_{kl} z^k \bar{z}^l \tag{3.18}$$

where again $\min(k, l) \geqslant 2$. This normal form is unique only up to the 5 dimensional group H given by

$$\left.\begin{aligned} z &\to \lambda(z+aw)\,\delta^{-1}, \quad w = 1 - 2i\bar{a}z - (r + i|a|^2)\,w \\ w &\to |\lambda|^2 w\delta^{-1} \end{aligned}\right\} \tag{3.19}$$

with $0 \neq \lambda \in \mathbf{C}$, $a \in \mathbf{C}$, $r \in \mathbf{R}$. It is easily seen that the property $c_{42}(0) \neq 0$ is invariant under these transformations. If $c_{42}(0)=0$ we call the origin an umbilical point. For a non-umbilical we can always achieve $c_{42}(0)=1$ since $z \to \lambda z$ leads to $c_{42}(0) \to \lambda^3 \bar{\lambda} c_{42}(0)$. By this normalization λ is fixed up to sign.

For a nonumbilical point we can use the parameters a, r to achieve

$$c_{43}(0)=0, \quad \operatorname{Re} c_{42}'(0)=0$$

so that the so normalized hypersurface can be approximated to order 7 in $z, \bar{z}, u$ by the algebraic surface

$$v = z\bar{z} + 2 \operatorname{Re} \{z^4 \bar{z}^2 (1 + jz + iku\} \tag{3.20}$$

where $j \in \mathbf{C}$, $k \in \mathbf{R}$, and j^2, k are invariants at the origin.

The above statements follow from the fact that (3.19) with $\lambda = 1$, $r = 0$ leads to

$$c_{43}(0) \to c_{43}(0) + 2ia, \quad c_{52}(0) \to c_{52}(0) + 4i\bar{a}$$

so that $j = c_{52}(0) + 2\overline{c_{43}(0)}$ is unchanged. We fix a so that $c_{43}(0) = 0$ and consider (3.19) with $\lambda = 1$, $a = 0$ which gives rise to

$$\operatorname{Re} c_{42}'(0) \to \operatorname{Re} c_{42}'(0) + 4r$$

Choosing $\operatorname{Re} c_{42}'(0) = 0$ we obtain (3.20), where we still have the freedom to replace z by $-z$. Thus j^2 and k are indeed invariants.

The above choice (3.20) distinguishes a special frame at the origin, by prescribing a tangent vector $\partial/\partial u$ transversal to the complex tangent plane and a complex tangent vector pair $\pm \partial/\partial z$ in the complex tangent plane. These pairs of vectors can be assigned to any point of M which is non-umbilical. These considerations clearly are meaningful for seven times differentiable M.

The above vector fields, singular at umbilical points, can be viewed as analogous to the directions of principal curvature in classical differential geometry. This analogy suggests the question: Are there compact manifolds without umbilical points? Are there such manifolds diffeomorphic to the sphere S^3?

Clearly the sphere $|z|^2 + |w|^2 = 1$ consists of umbilical points only as, except for one point, this manifold can be transformed into $v = z\bar{z}$ (cf. (1.4)). Therefore we can say by (3.18): *Any 3-dimensional manifold M in C^2 can at a point be osculated by the holomorphic image of the sphere $|z|^2 + |w|^2 = 1$ up to order 5 but generally not to sixth order.* In the latter case we have an umbilical point.

For $n \geq 2$ the analogous definition of an umbilical point is different: A point p on M is called umbilical if the term F_{22} in the normal form vanishes. Again, it is easily seen that this condition is independent of the transformation (1.23) and we can say: *Any non-degenerate manifold M of real dimension $2n+1$ in $\mathbf{C}_{n+1}$ $(n \geq 2)$ can at a point be osculated by the holomorphic image of a hyperquadric $v = \langle z, z \rangle$ up to order 3, but generally not to order 4.* In case one has fourth order osculation one speaks of an umbilical point.

(e) The algebraic problems connected with the action of the isotropy group on the normal form are prohibitively complicated for large n. But for a strictly pseudoconvex 5-dimensional manifold in $\mathbf{C}_3$ we obtain an interesting invariant connected with the 4th order terms F_{22}.

We assume $n=2$ and

$$\langle z, z\rangle = \sum_{\alpha=1}^{2} z^\alpha \overline{z^\alpha}$$

and consider a quartic $F_{22}(z, \bar z)$ of type $(2, 2)$ with $\operatorname{tr} F_{22}=0$. If we subject the manifold

$$v^* = \langle z^*, z^*\rangle + F_{22}(z^*, \overline{z^*}) + \dots$$

to the transformation (1.23) of the isotropy group of Q the fourth order term is replaced by

$$F_{22}(z^*, \overline{z^*}) = N_{22}(z, \bar z) \tag{3.21}$$

where

$$z^{*\beta} = C_\alpha{}^\beta z^\alpha, \quad C_\alpha{}^\sigma C_{\bar\beta}{}^{\bar\sigma} = \varrho\delta_{\alpha\bar\beta}, \quad \varrho > 0. \tag{3.22}$$

The question arises to find invariants of N_{22} under these transformations, which are evidently multiples of unitary transformations.

It turns out, and we will show, that one can find (3.22) such that N_{22} takes the form

$$N_{22} = \lambda_1\phi_1 + \lambda_2\phi_2 + \lambda_3\phi_3$$

where ϕ_1, ϕ_2, ϕ_3 are fixed quartics and $\lambda_1, \lambda_2, \lambda_3$ are three real numbers which we may order $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ and which satisfy

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

The λ_j may still be replaced by $\varrho\lambda_j$, so that

$$\frac{\lambda_3 - \lambda_2}{\lambda_2 - \lambda_1} = \mu$$

is a numerical invariant, provided we assume that the λ_j are distinct. In this case the matrix $C_\alpha{}^\beta$ is fixed up to a complex factor by these requirements. Geometrically speaking to every λ_j corresponds a pair of complex lines—if the λ_j are distinct—so that we have altogether three pairs of complex lines in the complex tangent space holomorphically invariantly associated with the manifold. We remark that $\lambda_3 = \max_j \lambda_j = 0$ characterizes an umbilical point, i.e. $F_{22}=0$.

The λ_j are reminiscent of eigenvalues of a quadratic form and, in fact, the above problem can be reduced to the equivalence problem of a quadratic form. One verifies by computation that any quartic F_{22} with $\operatorname{tr} F_{22}=0$ is invariant under the involution

$$(z^1, z^2) \to (\overline{z^2}, -\overline{z^1}) \tag{3.23}$$

and conversely any such quartic differs from one with tr $F_{22}=0$ by a multiple of $\langle z, z\rangle^2$

The function $\langle z, z\rangle^{-2} F_{22}$ can be viewed as a function on the complex projective space $\mathbf{CP}^1$, that is on S^2. We use the familiar mapping [3], derived from the stereographic projection:

$$\xi_1 = z^1\overline{z^2} + z^2\overline{z^1}$$

$$i\xi_2 = z^1\overline{z^2} - z^2\overline{z^1}$$

$$\xi_3 = z^1\overline{z^1} - z^2\overline{z^2}$$

so that

$$\sum_{\nu=1}^{3} \xi_\nu^2 = \langle z, z\rangle^2$$

to map $\langle z, z\rangle = 1$ onto S^2. Then the above involution (3.23) goes into the antipodal maps and one verifies that F_{22} becomes a real quadratic form

$$F_{22}(z, \bar{z}) = \Phi(\xi) = \sum_{\nu,\mu=1}^{3} b_{\nu\mu}\xi_\nu\xi_\mu.$$

Moreover

$$\operatorname{tr} F_{22} = \tfrac{1}{2}\left(\sum_{\nu=1}^{3} b_{\nu\nu}\right)\langle z, z\rangle,$$

so that tr $F_{22}=0$ if and only if the trace of the quadratic form vanishes.

We subject F_{22} to the transformation (3.22). At first we take $\varrho=1$, so that $(C_\alpha{}^\beta)=C$ is unitary. We assume furthermore that det $C=1$ because of the homogeneous character of F_{22}. Then, as is well known every such C corresponds to a proper orthogonal transformation of the ξ-space, and every such orthogonal transformation belongs to two such unitary transformations, namely $\pm C$. Thus the equivalence problem is reduced to that of the quadratic form Φ under proper orthogonal transformations. Choosing this transformation so that Φ is mapped into diagonal form

$$\sum_{\nu=1}^{3} \lambda_\nu \xi_\nu^2$$

we have $\sum_{\nu=1}^{3} \lambda_\nu = 0$. Moreover, if the eigenvalues λ_ν are distinct and ordered the orthogonal transformation is up to $\xi_\nu \to \pm\xi_\nu$ uniquely determined by this requirement.

To complete the discussion we have to free ourselves from the restriction det $C=1$ and take the stretching $z\to\varrho z$ into account. Both factors are taken into account by a transformation $z\to\gamma z$, $w\to|\gamma|^2 w$ with γ a complex number which leads to $\lambda_j \to |\gamma|^2\lambda_j$.

Thus if we set

$$\phi_1 = \xi_1^2 = 2\{z^1 z^2 \overline{z^1 z^2} + \operatorname{Re}(z^1 \bar{z}^2)^2\}$$

$$\phi_2 = \xi_2^2 = -2\{z^1 z^2 \overline{z^1 z^2} - \operatorname{Re}(z^1 \bar{z}^2)^2\}$$

$$\phi_3 = \xi_3^2 = (z^1 \bar{z}^1)^2 + (z^2 \bar{z}^2)^2 - 2 z^1 z^2 \bar{z}^1 \bar{z}^2$$

Then the above assertions follow. The pairs of complex lines which correspond to an eigendirection have the form

$$a_1 z^1 + a_2 z^2 = 0$$

$$\bar{a}_2 z^1 - \bar{a}_1 z^2 = 0$$

where a_1, a_2 are not both zero, i.e. the second line is obtained from the first by the involution (3.23).

4. Solution of an equivalence problem

Let G be the group of all nonsingular matrices of the form

$$\begin{pmatrix} u & 0 & 0 \\ v^\alpha & u_\beta{}^\alpha & 0 \\ v^{\bar{\alpha}} & 0 & u_{\bar{\beta}}{}^{\bar{\alpha}} \end{pmatrix}, \quad v^{\bar{\alpha}} = \overline{v^\alpha}, \quad u_{\bar{\beta}}{}^{\bar{\alpha}} = \overline{u_\beta{}^\alpha}, \tag{4.1}$$

where, as throughout this section, the small Greek indices run from 1 to n, u is real, and $v^\alpha, u_\beta{}^\alpha$ are complex. G can be considered as a subgroup of $GL(2n+1, R)$. A G-structure in a manifold M of dimension $2n+1$ is a reduction of the group of its tangent bundle to G. Locally it is given by linear differential forms $\theta, \theta^\alpha, \theta^{\bar{\alpha}}$, where θ is real and θ^α are complex, which are defined up to a transformation of G and satisfy the condition

$$\theta \wedge \theta^1 \wedge \ldots \wedge \theta^n \wedge \theta^{\bar{1}} \wedge \ldots \wedge \theta^{\bar{n}} \neq 0. \tag{4.2}$$

Let T_x and T_x^*, $x \in M$, be respectively the tangent and cotangent spaces of M at x. The multiples of θ define a line E_x in T_x^* and their totality is a real line bundle over M, to be denoted by E. The annihilator $E_x^\perp = T_{x,c}$ in T_x, called the complex tangent space, has a complex structure.

The G-structure is called integrable if the Frobenius condition is satisfied: $d\theta$, $d\theta^\alpha$ belong to the differential ideal generated by θ, θ^β. Since θ is real, this condition implies

$$d\theta \equiv i h_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}, \quad \text{mod } \theta, \tag{4.3}$$

where

$$h_{\alpha\bar{\beta}} = \bar{h}_{\beta\bar{\alpha}} = h_{\bar{\beta}\alpha}. \tag{4.4}$$

An integrable G-structure is called nondegenerate if

$$\det (h_{\alpha\bar{\beta}}) \neq 0. \tag{4.5}$$

Integrable G-structures include the special cases:

(1) Real hypersurfaces in $\mathbf{C}_{n+1}$. Let z^α, w be the coordinates of $\mathbf{C}_{n+1}$. A real hypersurface M can be locally defined by

$$r(z^\alpha, \bar{z^\alpha}, w, \bar{w}) = 0, \quad r_w \neq 0, \tag{4.6}$$

where r is a smooth real-valued function. On M a G-structure is defined by putting

$$\theta = i\partial r, \quad \theta^\alpha = dz^\alpha. \tag{4.7}$$

(2) Complex-valued linear differential operators of the first order in $\mathbf{R}_{2n+1}$. Denote the operators by P_α and suppose the following conditions be satisfied: (a) P_α, $P_{\bar{\beta}}$ are linearly independent; (b) $[P_\alpha, P_\beta]$ is a linear combination of P_γ. We interpret the operators as complex vector fields and let L be the n-dimensional linear space spanned by P_α. Its annihilator $L^\perp$ is of dimension $n+1$. Condition (a) implies that $L^\perp \cap \bar{L}^\perp$ is one-dimensional. We can choose a real one-form $\theta \in L^\perp \cap \bar{L}^\perp$ and the forms θ, θ^α to span $L^\perp$. The G-structure so defined is integrable because of condition (b).

We shall define a complete system of local invariants of nondegenerate integrable G-structures.

We consider the real line bundle E, which consists of the multiples $u\theta$, u (>0) being a fiber coordinate. In E the form

$$\omega = u\theta \tag{4.8}$$

is intrinsically defined. By (4.3) its exterior derivative has the local expression

$$d\omega = iuh_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + \omega \wedge \left(-\frac{du}{u} + \phi_0\right), \tag{4.9}$$

where θ^α, ϕ_0 are one-forms in M and ϕ_0 is real. This equation can be written

$$d\omega = ig_{\alpha\bar{\beta}}\,\omega^\alpha \wedge \omega^{\bar{\beta}} + \omega \wedge \phi, \tag{4.10}$$

where ω^α are linear combinations of θ^β, θ and $g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha}$ are constants. The nondegeneracy of the G-structure is expressed by

$$\det (g_{\alpha\bar{\beta}}) \neq 0. \tag{4.11}$$

The forms ω, $\operatorname{Re}\omega^\alpha$, $\operatorname{Im}\omega^{\bar{\alpha}}$ and ϕ constitute a basis of the cotangent space of E. The most general transformation on ω, ω^α, $\omega^{\bar{\alpha}}$, ϕ leaving the equation (4.10) (and the form ω) invariant has the matrix of coefficients

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$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ v^{\alpha} & u_{\beta}{}^{\alpha} & 0 & 0 \\ v^{\bar{\alpha}} & 0 & u_{\bar{\beta}}{}^{\bar{\alpha}} & 0 \\ s & i g_{\varrho\bar{\sigma}} u_{\beta}{}^{\varrho} v^{\bar{\sigma}} & -i g_{\varrho\bar{\sigma}} u_{\bar{\beta}}{}^{\bar{\sigma}} v^{\varrho} & 1 \end{pmatrix} \tag{4.12}$$

where s is real and $u_{\beta}{}^{\alpha}$, v^{α} are complex satisfying the equations

$$g_{\alpha\bar{\beta}} u_{\varrho}{}^{\alpha} u_{\bar{\sigma}}{}^{\bar{\beta}} = g_{\varrho\bar{\sigma}}. \tag{4.13}$$

Let G_1 be the group of all the nonsingular matrices (4.12). It follows that E has a G_1-structure. Denote by Y its principal G_1-bundle. Then we have

$$G_1 \xrightarrow{j} Y \xrightarrow{\pi} E, \tag{4.14}$$

where j is inclusion of a fiber and π is projection. The quantities s, $u_{\alpha}{}^{\beta}$, v^{α} in (4.12), considered as new variables, are local fiber coordinates of Y. Observe that we have the dimensions

$$\dim G_1 = (n+1)^2, \quad \dim E = 2(n+1), \quad \dim Y = (n+2)^2 - 1. \tag{4.15}$$

In Y there are intrinsically (and hence globally) defined forms ω, ω^{α}, $\omega^{\bar{\alpha}}$, ϕ, and we will introduce new ones by intrinsic conditions, so that the total number equals the dimension of Y and they are everywhere linearly independent.

The condition that our G-structure is integrable implies

$$d\omega^{\alpha} = \omega^{\beta} \wedge \phi_{\beta .}^{\ \alpha} + \omega \wedge \phi^{\alpha} \tag{4.16}$$

where $\phi_{\beta}{}^{\alpha}, \phi^{\alpha}$ are not completely determined. We shall study the consequences of the equations (4.10), (4.16) by exterior differentiation. To be in a slightly more general situation the $g_{\alpha\bar{\beta}}$'s are allowed to be variable. It will be convenient to follow the practice of tensor analysis to introduce $g^{\alpha\bar{\beta}}$ by the equations

$$g_{\alpha\bar{\beta}} g^{\gamma\bar{\beta}} = \delta_{\alpha}^{\gamma}, \quad g_{\alpha\bar{\beta}} g^{\alpha\bar{\gamma}} = \delta_{\bar{\beta}}^{\bar{\gamma}} \tag{4.17}$$

and to use them to raise and lower indices. It will then be important to know the location of an index and this will be indicated by a dot, thus

$$u_{\alpha .}^{\ \beta} g_{\beta\bar{\gamma}} = u_{\alpha\bar{\gamma}}, \quad u_{.\alpha}^{\beta} g^{\alpha\bar{\gamma}} = u^{\beta\bar{\gamma}}, \text{ etc.} \tag{4.18}$$

The exterior differentiations of (4.10), (4.16) give respectively

$$i(dg_{\alpha\bar{\beta}} - \phi_{\alpha\bar{\beta}} - \phi_{\bar{\beta}\alpha} + g_{\alpha\bar{\beta}}\phi) \wedge \omega^{\alpha} \wedge \omega^{\bar{\beta}} + (-d\phi + i\omega_{\bar{\beta}} \wedge \phi^{\bar{\beta}} + i\phi_{\bar{\beta}} \wedge \omega^{\bar{\beta}}) \wedge \omega = 0, \tag{4.19}$$

$$(d\phi_{\beta .}^{\ \alpha} - \phi_{\beta .}^{\ \gamma} \wedge \phi_{\gamma .}^{\ \alpha} - i\omega_{\beta} \wedge \phi^{\alpha}) \wedge \omega^{\beta} + (d\phi^{\alpha} - \phi \wedge \phi^{\alpha} - \phi^{\beta} \wedge \phi_{\beta .}^{\ \alpha}) \wedge \omega = 0. \tag{4.20}$$

LEMMA 4.1. *There exist* $\phi_{\beta\cdot}^{\ \alpha}$ *which satisfy* (4.16) *and*

$$dg_{\alpha\bar\beta}+g_{\alpha\bar\beta}\phi-\phi_{\alpha\bar\beta}-\phi_{\bar\beta\alpha}=0,\quad \phi_{\bar\beta\alpha}=\bar\phi_{\bar\beta\alpha},\tag{4.21}$$

or

$$dg^{\alpha\bar\beta}-g^{\alpha\bar\beta}\phi+\phi^{\alpha\bar\beta}+\phi^{\bar\beta\alpha}=0.\tag{4.21 a}$$

Such $\phi_{\beta\cdot}^{\ \alpha}$ *are determined up to additive terms in* ω.

In fact, it follows from (4.19) that the expression in its first parentheses is a linear combination of ω^α, $\omega^{\bar\beta}$, ω, i.e.,

$$dg_{\alpha\bar\beta}-\phi_{\alpha\bar\beta}-\phi_{\bar\beta\alpha}+g_{\alpha\bar\beta}\phi=A_{\alpha\bar\beta\gamma}\,\omega^\gamma+B_{\alpha\bar\beta\bar\gamma}\,\omega^{\bar\gamma}+C_{\alpha\bar\beta}\,\omega,\tag{4.22}$$

where

$$A_{\alpha\bar\beta\gamma}=A_{\gamma\bar\beta\alpha},\quad B_{\alpha\bar\beta\bar\gamma}=B_{\alpha\bar\gamma\bar\beta}.\tag{4.23}$$

From the hermitian property of $g_{\alpha\bar\beta}$ we have also

$$\bar A_{\alpha\bar\beta\gamma}=B_{\beta\bar\alpha\bar\gamma},\quad \bar C_{\alpha\bar\beta}=C_{\beta\bar\alpha}.\tag{4.24}$$

The forms

$$\phi'_{\alpha\bar\beta}=\phi_{\alpha\bar\beta}+A_{\alpha\bar\beta\gamma}\,\omega^\gamma+\tfrac12 C_{\alpha\bar\beta}\,\omega\tag{4.25}$$

satisfy on account of (4.23) the equations (4.16) and (4.21). The second statement in the lemma can be verified without difficulty.

From now on we will suppose (4.21) to be valid. Equation (4.19) then gives

$$d\phi=i\omega_{\bar\beta}\wedge\phi^{\bar\beta}+i\phi_{\bar\beta}\wedge\omega^{\bar\beta}+\omega\wedge\psi,\tag{4.26}$$

where ψ is a real one-form.

LEMMA 4.2. *Let* $\Phi_{\beta\cdot}^{\ \alpha}$ *be exterior quadratic differential forms, satisfying*

$$\Phi_{\beta\cdot}^{\ \alpha}\wedge\omega^\beta\equiv0,\quad \Phi_{\alpha\bar\beta}+\Phi_{\bar\beta\alpha}\equiv0,\quad \operatorname{mod}\omega.\tag{4.27}$$

Then we have

$$\Phi_{\alpha\bar\varrho}\equiv S_{\alpha\beta\bar\varrho\bar\sigma}\,\omega^\beta\wedge\omega^{\bar\sigma},\quad \operatorname{mod}\omega,\tag{4.28}$$

where $S_{\alpha\beta\bar\varrho\bar\sigma}$ *has the symmetry properties:*

$$S_{\alpha\beta\bar\varrho\bar\sigma}=S_{\beta\alpha\bar\varrho\bar\sigma}=S_{\alpha\beta\bar\sigma\bar\varrho},\tag{4.29}$$

$$S_{\alpha\beta\bar\varrho\bar\sigma}=\bar S_{\varrho\sigma\bar\alpha\bar\beta}=S_{\bar\varrho\bar\sigma\alpha\beta}.\tag{4.30}$$

Computing $\operatorname{mod}\omega$, we have, from the first equation of (4.27),

$$\Phi_{\beta\bar\alpha}\equiv\chi_{\beta\bar\alpha\gamma}\wedge\omega^\gamma,$$

where $\chi_{\beta\bar\alpha\gamma}$ are one-forms. Its complex conjugate is

$$\Phi_{\bar\beta\alpha}\equiv\chi_{\bar\beta\alpha\bar\gamma}\wedge\omega^{\bar\gamma}.$$

By the second equation of (4.27) we have

$$\chi_{\alpha\bar{\beta}\gamma} \wedge \omega^{\gamma} + \chi_{\bar{\beta}\alpha\bar{\gamma}} \wedge \omega^{\bar{\gamma}} \equiv 0, \quad \text{mod } \omega.$$

The first term, $\chi_{\alpha\bar{\beta}\gamma} \wedge \omega^{\gamma}$, is therefore congruent to zero mod ω, $\omega^{\bar{\sigma}}$. But it is obviously congruent to zero mod ω^{ϱ}. Hence we have the conclusion (4.28). The symmetry properties (4.29) and (4.30) follow immediately from (4.27). Thus Lemma 4.2 is proved.

Equation (4.20) indicates the necessity of studying the expression

$$\Pi_{\alpha}{}^{\gamma}_{\cdot} = d\phi_{\alpha}{}^{\gamma}_{\cdot} - \phi_{\alpha}{}^{\beta}_{\cdot} \wedge \phi_{\beta}{}^{\gamma}_{\cdot}. \tag{4.31}$$

Using (4.21) we have

$$\Pi_{\beta\bar{\alpha}} = g_{\gamma\bar{\alpha}} d\phi_{\beta\cdot}{}^{\gamma} - \phi_{\beta\cdot}{}^{\gamma} \wedge \phi_{\gamma\bar{\alpha}} = d\phi_{\beta\bar{\alpha}} - \phi_{\beta\bar{\alpha}} \wedge \phi - \phi_{\bar{\alpha}\gamma} \wedge \phi_{\beta\cdot}{}^{\gamma}. \tag{4.32}$$

It follows that $$\Pi_{\beta\bar{\alpha}} + \Pi_{\bar{\alpha}\beta} = d(\phi_{\beta\bar{\alpha}} + \phi_{\bar{\alpha}\beta}) - (\phi_{\beta\bar{\alpha}} + \phi_{\bar{\alpha}\beta}) \wedge \phi,$$

since $$\phi_{\beta\bar{\gamma}} \wedge \phi_{\bar{\alpha}\cdot}{}^{\bar{\gamma}} = \phi_{\beta\cdot}{}^{\gamma} \wedge \phi_{\bar{\alpha}\gamma}.$$

Using the differentiation of (4.21), we get

$$\Pi_{\beta\bar{\alpha}} + \Pi_{\bar{\alpha}\beta} = g_{\beta\bar{\alpha}} d\phi. \tag{4.33}$$

By (4.20), (4.26), (4.33), it is found that

$$\Phi_{\beta\cdot}{}^{\gamma} \equiv \Pi_{\beta\cdot}{}^{\gamma} - i\omega_{\beta} \wedge \phi^{\gamma} + i\phi_{\beta} \wedge \omega^{\gamma} + i\delta_{\beta}{}^{\gamma}(\phi_{\sigma} \wedge \omega^{\sigma}), \quad \text{mod } \omega \tag{4.34}$$

or $$\Phi_{\beta\bar{\alpha}} \equiv \Pi_{\beta\bar{\alpha}} - i\omega_{\beta} \wedge \phi_{\bar{\alpha}} + i\phi_{\beta} \wedge \omega_{\bar{\alpha}} + ig_{\beta\bar{\alpha}}(\phi_{\sigma} \wedge \omega^{\sigma}), \quad \text{mod } \omega \tag{4.34 a}$$

fulfill the conditions of Lemma 4.2. For such Φ the conclusions (4.28)–(4.30) of the Lemma are valid.

The forms $\phi_{\beta}{}^{\alpha}_{\cdot}$, ϕ^{α}, ψ fulfilling equations (4.16), (4.21), and (4.26) are defined up to the transformation

$$\left.\begin{aligned} \phi_{\beta}{}^{\alpha}_{\cdot} &= \phi'_{\beta}{}^{\alpha}_{\cdot} + D_{\beta}{}^{\alpha}_{\cdot}\omega, \\ \phi^{\alpha} &= \phi'^{\alpha} + D_{\beta}{}^{\alpha}_{\cdot}\omega^{\beta} + E^{\alpha}\omega \\ \psi &= \psi' + G\omega + i(E_{\alpha}\omega^{\alpha} - E_{\bar{\alpha}}\omega^{\bar{\alpha}}), \end{aligned}\right\} \tag{4.35}$$

where G is real and $$D_{\alpha\bar{\beta}} + D_{\bar{\beta}\alpha} = 0. \tag{4.36}$$

LEMMA 4.3. *The $D_{\beta}{}^{\alpha}_{\cdot}$ can be uniquely determined by the conditions*

$$S_{\varrho\bar{\sigma}} \underset{\text{def}}{=} g^{\alpha\bar{\beta}} S_{\alpha\varrho\bar{\beta}\bar{\sigma}} = 0. \tag{4.37}$$

To prove Lemma 4.3 it suffices to study the effect on $S_{\alpha\beta\bar{\varrho}\bar{\sigma}}$ when the transformation (4.35) is performed. We put

$$S = g^{\alpha\bar{\beta}} S_{\alpha\bar{\beta}}, \quad D = D_{\alpha.}^{\ \alpha}. \tag{4.38}$$

Since $g^{\alpha\bar{\beta}}$ and $S_{\alpha\bar{\beta}}$ are hermitian and $D_{\alpha\bar{\beta}}$ is skew-hermitian, S is real and D is purely imaginary. Denoting the new coefficients by dashes, we find

$$S_{\alpha\beta.\bar{\sigma}}^{\ \ \gamma} = S'^{\ \ \gamma}_{\alpha\beta.\bar{\sigma}} + i(D_{\alpha.}^{\ \gamma}\, g_{\beta\bar{\sigma}} + D_{\beta.}^{\ \gamma}\, g_{\alpha\bar{\sigma}} - \delta_\beta^{\ \gamma} D_{\bar{\sigma}\alpha} - \delta_\alpha^{\ \gamma} D_{\bar{\sigma}\beta}). \tag{4.39}$$

It follows that
$$S_{\varrho\bar{\sigma}} = S'_{\varrho\bar{\sigma}} + i\{g_{\varrho\bar{\sigma}} D + D_{\varrho\bar{\sigma}} - (n+1)\, D_{\bar{\sigma}\varrho}\}. \tag{4.40}$$

Since we wish to make $S'_{\varrho\bar{\sigma}} = 0$, the lemma is proved if we show that there is one and only one set of $D_{\beta.}^{\ \gamma}$ satisfying (4.36) and

$$-iS_{\varrho\bar{\sigma}} = g_{\varrho\bar{\sigma}} D + (n+2)\, D_{\varrho\bar{\sigma}}. \tag{4.41}$$

In fact, contracting (4.41), we get

$$2(n+1)\, D = -iS. \tag{4.42}$$

Substitution of this into (4.41) gives

$$(n+2)\, D_{\varrho\bar{\sigma}} = -iS_{\varrho\bar{\sigma}} + \frac{i}{2(n+1)} S g_{\varrho\bar{\sigma}}. \tag{4.43}$$

It is immediately verified that the $D_{\varrho\bar{\sigma}}$ given by (4.43) satisfy (4.36) and (4.41). This proves Lemma 4.3.

By the condition (4.37) the $\phi_{\beta.}^{\ \gamma}$ are completely determined and we wish to compute their exterior derivatives. By (4.34) we can put

$$\Pi_{\beta.}^{\ \gamma} - i\omega_\beta \wedge \phi^\gamma + i\phi_\beta \wedge \omega^\gamma + i\delta_\beta^{\ \gamma}(\phi_\sigma \wedge \omega^\sigma) = S_{\beta\varrho.\bar{\sigma}}^{\ \ \gamma}\, \omega^\varrho \wedge \omega^{\bar{\sigma}} + \lambda_{\beta.}^{\ \gamma} \wedge \omega, \tag{4.44}$$

where $\lambda_{\beta.}^{\ \gamma}$ are one-forms. Substituting this into (4.20), we get

$$d\phi^\alpha - \phi \wedge \phi^\alpha - \phi^\beta \wedge \phi_{\beta.}^{\ \alpha} - \lambda_{\beta.}^{\ \alpha} \wedge \omega^\beta = \mu^\alpha \wedge \omega, \tag{4.45}$$

μ^α being also one-forms. From (4.44), (4.33), and (4.26), we get

$$(\lambda_{\beta\bar{\alpha}} + \lambda_{\bar{\alpha}\beta}) \wedge \omega = g_{\beta\bar{\alpha}}\, \omega \wedge \psi,$$

or
$$\lambda_{\beta\bar{\alpha}} + \lambda_{\bar{\alpha}\beta} + g_{\beta\bar{\alpha}} \psi \equiv 0, \quad \text{mod } \omega. \tag{4.46}$$

To utilize the condition (4.37) we shall take the exterior derivative of (4.44). We will need the following formulas, which follow immediately from (4.16), (4.45), (4.21):

$$d\omega_\alpha = d(g_{\alpha\bar{\beta}}\omega^{\bar{\beta}}) = -\omega^{\bar{\beta}} \wedge \phi_{\alpha\bar{\beta}} + \omega_\alpha \wedge \phi + \omega \wedge \phi_\alpha, \tag{4.47}$$

$$d\phi_\alpha = d(g_{\alpha\bar{\beta}}\phi^{\bar{\beta}}) = \phi_{\alpha\bar{\beta}} \wedge \phi^{\bar{\beta}} + \lambda_{\bar{\gamma}\alpha} \wedge \omega^{\bar{\gamma}} + \mu_\alpha \wedge \omega. \tag{4.48}$$

We take the exterior derivative of (4.44) and consider only terms involving $\omega^\varrho \wedge \omega^{\bar{\sigma}}$, ignoring those in ω. It gives

$$dS_{\beta\varrho.\bar\sigma}^{\gamma} - S_{\tau\varrho.\bar\sigma}^{\gamma}\,\phi_{\beta.}^{\tau} - S_{\beta\tau.\bar\sigma}^{\gamma}\,\phi_{\varrho.}^{\tau} + S_{\beta\varrho.\bar\sigma}^{\tau}\,\phi_{\tau.}^{\gamma} - S_{\beta\varrho.\bar\tau}^{\gamma}\,\phi_{\bar\sigma.}^{\bar\tau}$$
$$\equiv i(\lambda_{\beta.}^{\gamma}\,g_{\varrho\bar\sigma} + \lambda_{\varrho.}^{\gamma}\,g_{\beta\bar\sigma} - \delta_{\beta}^{\gamma}\,\lambda_{\bar\sigma\varrho} - \delta_{\varrho}^{\gamma}\,\lambda_{\bar\sigma\beta}) \quad \text{mod } \omega, \omega^{\alpha}, \omega^{\bar\beta} \tag{4.49}$$

and by contraction

$$dS_{\varrho\bar\sigma} - S_{\tau\bar\sigma}\phi_{\varrho.}^{\tau} - S_{\varrho\bar\tau}\phi_{\bar\sigma.}^{\bar\tau} \equiv i\{g_{\varrho\bar\sigma}\lambda_{\beta.}^{\beta} + \lambda_{\varrho\bar\sigma} - (n+1)\,\lambda_{\bar\sigma\varrho}\}, \quad \text{mod } \omega, \omega^{\alpha}, \omega^{\bar\beta}. \tag{4.50}$$

When (4.37) is satisfied, the left-hand side, and hence also the right-hand side, of (4.50) are congruent to zero. The congruence so obtained, combined with (4.46), gives

$$\lambda_{\varrho\bar\sigma} \equiv -\tfrac{1}{2}\,g_{\varrho\bar\sigma}\psi \quad \text{or} \quad \lambda_{\varrho.}^{\sigma} \equiv -\tfrac{1}{2}\,\delta_{\varrho}^{\sigma}\psi, \quad \text{mod } \omega, \omega^{\alpha}, \omega^{\bar\beta}.$$

Hence we can put

$$\lambda_{\varrho.}^{\sigma} = -\tfrac{1}{2}\,\delta_{\varrho}^{\sigma}\psi + V_{\varrho.\beta}^{\sigma}\,\omega^{\beta} + W_{\varrho.\bar\beta}^{\sigma}\,\omega^{\bar\beta} + a_{\varrho.}^{\sigma}\,\omega \tag{4.51}$$

or

$$\lambda_{\varrho\bar\sigma} = -\tfrac{1}{2}\,g_{\varrho\bar\sigma}\psi + V_{\varrho\bar\sigma\beta}\omega^{\beta} + W_{\varrho\bar\sigma\bar\beta}\omega^{\bar\beta} + a_{\varrho\bar\sigma}\omega. \tag{4.51a}$$

Substituting into (4.46), we get

$$V_{\varrho\bar\sigma\beta} + W_{\bar\sigma\varrho\beta} = 0. \tag{4.52}$$

We can therefore write (4.44) in the form

$$\Phi_{\beta.}^{\gamma} \underset{\text{def}}{=} d\phi_{\beta.}^{\gamma} - \phi_{\beta.}^{\sigma}\wedge\phi_{\sigma.}^{\gamma} - i\,\omega_{\beta}\wedge\phi^{\gamma} + i\phi_{\beta}\wedge\omega^{\gamma} + i\delta_{\beta}^{\gamma}\{\phi_{\sigma}\wedge\omega^{\sigma}\} + \tfrac{1}{2}\,\delta_{\beta}^{\gamma}\psi\wedge\omega$$
$$= S_{\beta\varrho.\bar\sigma}^{\gamma}\,\omega^{\varrho}\wedge\omega^{\bar\sigma} + V_{\beta.\varrho}^{\gamma}\,\omega^{\varrho}\wedge\omega - V_{.\beta\bar\sigma}^{\gamma}\,\omega^{\bar\sigma}\wedge\omega, \tag{4.53}$$

which is the formula for $d\phi_{\beta.}^{\gamma}$. Formula (4.53) defines $\Phi_{\beta.}^{\gamma}$ completely; it is consistent with earlier notations in Lemma 4.2 and in the subsequent discussions where $\Phi_{\beta.}^{\gamma}$ are defined only mod ω. Substituting into (4.20), we get

$$\Phi^{\alpha} \underset{\text{def}}{=} d\phi^{\alpha} - \phi\wedge\phi^{\alpha} - \phi^{\beta}\wedge\phi_{\beta.}^{\alpha} + \tfrac{1}{2}\psi\wedge\omega^{\alpha} = -V_{\beta.\gamma}^{\alpha}\,\omega^{\beta}\wedge\omega^{\gamma} + V_{.\beta\bar\sigma}^{\alpha}\,\omega^{\beta}\wedge\omega^{\bar\sigma} + \nu^{\alpha}\wedge\omega, \tag{4.54}$$

where ν^{α} are one-forms. Notice also that (4.49) simplifies to

$$dS_{\beta\varrho.\bar\sigma}^{\gamma} - S_{\tau\varrho.\bar\sigma}^{\gamma}\,\phi_{\beta.}^{\tau} - S_{\beta\tau.\bar\sigma}^{\gamma}\,\phi_{\varrho.}^{\tau} + S_{\beta\varrho.\bar\sigma}^{\tau}\,\phi_{\tau.}^{\gamma} - S_{\beta\varrho.\bar\tau}^{\gamma}\,\phi_{\bar\sigma.}^{\bar\tau} \equiv 0, \quad \text{mod } \omega, \omega^{\alpha}, \omega^{\bar\beta} \tag{4.55}$$

on account of (4.51) or (4.51a).

Consider again the transformation (4.35) with $D_{\beta.}^{\alpha}=0$. The $\phi_{\beta.}^{\alpha}$ are now completely determined. From (4.53) its effect on $V_{\beta.\varrho}^{\gamma}$ is given by

$$V_{\beta.\varrho}^{\gamma} = V_{\beta.\varrho}^{\prime\gamma} - i\{\delta_{\varrho}^{\gamma}E_{\beta} + \tfrac{1}{2}\,\delta_{\beta}^{\gamma}E_{\varrho}\}. \tag{4.56}$$

Contracting, we have

$$V_{\beta.\varrho}^{\varrho} = V_{\beta.\varrho}^{\prime\varrho} - i\{n+\tfrac{1}{2}\}\,E_{\beta}. \tag{4.57}$$

This leads to the lemma:

LEMMA 4.4. *With* (4.21) *and* (4.37) *fulfilled as in Lemmas* 4.1 *and* 4.2 *there is a unique set of* ϕ^α *satisfying*

$$V^{\varrho}_{\beta.\varrho} = 0. \tag{4.58}$$

To find an expression for $d\psi$ we differentiate the equation (4.26). Using (4.16), (4.47), and (4.54), we get

$$\omega \wedge (-d\psi + \phi \wedge \psi + 2i\phi^\beta \wedge \phi_\beta - i\omega^\beta \wedge \nu_\beta - i\nu^\beta \wedge \omega_\beta) = 0.$$

Hence we can write

$$\Psi \underset{\text{def}}{=} d\psi - \phi \wedge \psi - 2i\phi^\beta \wedge \phi_\beta = -i\omega^\beta \wedge \nu_\beta - i\nu^\beta \wedge \omega_\beta + \varrho \wedge \omega, \tag{4.59}$$

where ϱ is a one-form.

With this expression for $d\psi$ (and expressions for other exterior derivatives found above) we differentiate (4.54) mod ω and retain only terms involving $\omega^\varrho \wedge \omega^{\bar\sigma}$. By the same argument used above, we derive the formula

$$\begin{aligned} dV^{\alpha}_{.\varrho\bar\sigma} - V^{\alpha}_{.\beta\bar\sigma}\phi^{\beta}_{\varrho.} + V^{\beta}_{.\varrho\bar\sigma}\phi^{\alpha}_{\beta.} - V^{\alpha}_{.\varrho\bar\tau}\phi^{\bar\tau}_{\bar\sigma.} - V^{\alpha}_{.\varrho\bar\sigma}\phi \\ = S_{\beta\varrho.\bar\sigma}^{\alpha}\phi^\beta + ig_{\varrho\bar\sigma}\nu^\alpha + \frac{i}{2}\delta_\varrho{}^\alpha \nu_{\bar\sigma} \mod \omega, \omega^\alpha, \omega^{\bar\beta}. \end{aligned} \tag{4.60}$$

Condition (4.58) is equivalent to

$$V^{\alpha}_{.\varrho\bar\sigma} g^{\varrho\bar\sigma} = 0. \tag{4.58 a}$$

Its differentiation gives, by using (4.21 a) and (4.60),

$$\nu^\gamma \equiv 0, \quad \mod \omega, \omega^\alpha, \omega^{\bar\beta}.$$

Hence we can put

$$\nu^\gamma \equiv P_{\alpha.}^{\gamma}\omega^\alpha + Q_{\bar\beta.}^{\gamma}\omega^{\bar\beta} \quad \mod \omega. \tag{4.61}$$

Substitution into (4.54) gives

$$\Phi^\alpha = -V^{\alpha}_{\beta.\gamma}\omega^\beta \wedge \omega^\gamma + V^{\alpha}_{.\beta\bar\sigma}\omega^\beta \wedge \omega^{\bar\sigma} + P^{\alpha}_{\beta.}\omega^\beta \wedge \omega + Q^{\alpha}_{\bar\beta.}\omega^{\bar\beta} \wedge \omega. \tag{4.62}$$

For future use we also write down the formula

$$\begin{aligned} \Phi_\alpha &= d\phi_\alpha - \phi_{\alpha\bar\beta} \wedge \phi^{\bar\beta} + \tfrac{1}{2}\psi \wedge \omega_\alpha \\ &= -V_{\bar\beta\alpha\bar\gamma}\omega^{\bar\beta} \wedge \omega^{\bar\gamma} - V_{\alpha\bar\gamma\beta}\omega^\beta \wedge \omega^{\bar\gamma} + Q_{\beta\alpha}\omega^\beta \wedge \omega + P_{\bar\beta\alpha}\omega^{\bar\beta} \wedge \omega. \end{aligned} \tag{4.63}$$

Since the indeterminacy in ω can be absorbed in ϱ, substitution of (4.61) into (4.59) gives

$$\Psi = i\{Q_{\alpha\beta}\omega^\alpha \wedge \omega^\beta - Q_{\bar\alpha\bar\beta}\omega^{\bar\alpha} \wedge \omega^{\bar\beta}\} - i\tilde P_{\varrho\bar\sigma}\omega^\varrho \wedge \omega^{\bar\sigma} + \varrho \wedge \omega, \tag{4.64}$$

where

$$\tilde P_{\alpha\bar\beta} = P_{\alpha\bar\beta} + P_{\bar\beta\alpha} = \tilde P_{\bar\beta\alpha}. \tag{4.65}$$

It remains to determine ψ, which can still undergo the transformation

$$\psi = \psi' + G\omega, \tag{4.66}$$

where G is real. Denoting the new coefficients by dashes, we get, from (4.54) and (4.62),

$$P'^{\alpha}_{\beta.} = P^{\alpha}_{\beta.} + \tfrac{1}{2}\delta_{\beta}{}^{\alpha} G, \tag{4.67}$$

which gives

$$P'^{\alpha}_{\alpha.} = P^{\alpha}_{\alpha.} + \frac{n}{2} G. \tag{4.68}$$

On the other hand, from (4.65) we have

$$\tilde{P}^{\alpha}_{\alpha.} = 2\,\mathrm{Re}\,(P^{\alpha}_{\alpha.}). \tag{4.69}$$

The equation

$$\tilde{P}'^{\alpha}_{\alpha} = \tilde{P}^{\alpha}_{\alpha.} + nG$$

involves only real quantities and we have the lemma:

LEMMA 4.5. *The real form ψ is completely determined by the condition*

$$\tilde{P}^{\alpha}_{\alpha.} = 0. \tag{4.70}$$

We differentiate the equation (4.64), using th fact that Ψ is defined by (4.59). Computing mod ω and considering only the terms involving $\omega^{\varrho} \wedge \omega^{\bar{\sigma}}$, we get

$$\begin{aligned} &d\tilde{P}_{\varrho\bar{\sigma}} - \tilde{P}_{\tau\bar{\sigma}}\phi_{\varrho.}^{\tau} - \tilde{P}_{\varrho\bar{\tau}}\phi_{\bar{\sigma}.}^{\bar{\tau}} - \tilde{P}_{\varrho\bar{\sigma}}\phi \\ &\qquad \equiv 2V^{\beta}_{.\varrho\bar{\sigma}}\phi_{\beta} + 2V_{\beta\bar{\sigma}\varrho}\phi^{\beta} - g_{\varrho\bar{\sigma}}\eta, \quad \mathrm{mod}\ \omega, \omega^{\alpha}, \omega^{\bar{\beta}}. \end{aligned} \tag{4.71}$$

From (4.70) and using (4.21 a) and (4.58), we get

$$\eta \equiv 0, \quad \mathrm{mod}\ \omega, \omega^{\alpha}, \omega^{\bar{\beta}}.$$

Since Ψ is real, we can write (4.64) in the form

$$\Psi = i\{Q_{\alpha\beta}\omega^{\alpha} \wedge \omega^{\beta} - Q_{\bar{\alpha}\bar{\beta}}\omega^{\bar{\alpha}} \wedge \omega^{\bar{\beta}}\} - i\tilde{P}_{\varrho\bar{\sigma}}\omega^{\varrho} \wedge \omega^{\bar{\sigma}} + \{R_{\alpha}\omega^{\alpha} + R_{\bar{\alpha}}\omega^{\bar{\alpha}}\} \wedge \omega. \tag{4.72}$$

We summarize the discussions of this section in the theorem:

THEOREM 4.6. *Let the manifold M of dimension $2n+1$ be provided with an integrable nondegenerate G-structure. Then the real line bundle E over M has a G_1-structure, in whose associated principal G_1-bundle Y there is a completely determined set of one-forms ω, ω^{α}, ϕ, $\phi^{\beta}_{\alpha.}$, ϕ^{α}, ψ, of which ω, ϕ, ψ are real, which satisfy the equations* (4.10), (4.16), (4.21), (4.26), (4.37), (4.53), (4.54), (4.58), (4.59), (4.62), (4.70), (4.72). *The forms*

$$\omega, \omega^{\alpha}, \omega^{\bar{\alpha}}, \phi, \phi_{\alpha\bar{\beta}}, \phi^{\alpha}, \phi^{\bar{\alpha}}, \psi \tag{4.73}$$

are linearly independent.

In particular, suppose that the G-structure arises from a real analytic real hypersurface M in $\mathbf{C}_{n+1}$. Suppose there is a second real analytic hypersurface M' in $\mathbf{C}_{n+1}$ whose corresponding concepts are denoted by dashes. In order that there is locally a biholomorphic transformation of $\mathbf{C}_{n+1}$ to $\mathbf{C}'_{n+1}$ which maps M to M' it is necessary and sufficient that there is a real analytic diffeomorphism of Y to Y' under which the forms in (4.73) are respectively equal to the forms with dashes.

The necessity follows from our derivation of the forms in (4.73). To prove the sufficiency condition take the $2n+1$ local variables on M as complex variables. The ω, ω^α are linear combinations of dz^α, dw and are linearly independent over the complex numbers. From

$$\omega' = \omega, \quad \omega'^\alpha = \omega^\alpha$$

we see that the diffeomorphism has the property that dz'^α, dw' are linear combinations of dz^β, dw which implies that z'^α, w' are holomorphic functions of z^β, w.

The problem for $n=1$ was solved by E. Cartan [1]. In this case conditions (4.37), (4.58), (4.70) reduce to

$$S_{11\bar{1}\bar{1}} = V_{1\bar{1}1} = \tilde{P}_{1\bar{1}} = 0.$$

Exterior differentiation of (4.53) then gives

$$P_{1\bar{1}} = 0.$$

Our formulas reduce to those given by Cartan.

As a final remark we wish to emphasize the algebraic nature of our derivation. Most likely the theorem is a special case of a more general theorem on filtered Lie algebras.

5. The connection

(a) The flat case. We apply the results of § 4 to the special case of the nondegenerate real hyperquadrics Q discussed in § 1. The notations introduced in both sections will be used. In particular, we suppose

$$g_{\alpha\bar{\beta}} = h_{\alpha\bar{\beta}} \tag{5.1}$$

and write the equation (1.1) of Q as

$$-\frac{i}{2}(w-\bar{w}) - g_{\alpha\bar{\beta}} z^\alpha \bar{z}^{\beta} = 0. \tag{5.2}$$

By (4,7) and (4.8) we have

$$\omega = u\{\tfrac{1}{2}\,dw - i g_{\alpha\bar{\beta}} \bar{z}^{\beta}\, dz^\alpha\}. \tag{5.3}$$

On the other hand, given Q consider Q-frames Z_A such that the point Z_0 lies on Q. We write

$$Z_0 = tY,\ Y = (1, z^1, \dots, z^n, w) \tag{5.4}$$

Then
$$(dZ_0, Z_0) = \frac{i}{2}\pi_0^{n+1} = |t|^2 (dY, Y) = |t|^2 \left(\frac{i}{2} dw + g_{\alpha\bar\beta} z^{\bar\beta} dz^\alpha\right). \tag{5.5}$$

By setting
$$u = |t|^2, \tag{5.6}$$

we have
$$\omega = \tfrac{1}{2}\pi_0{}^{n+1} = -i(dZ_0, Z_0). \tag{5.7}$$

The structure equation (1.32) for $d\pi_0{}^{n+1}$ shows that we can put

$$\omega^\alpha = \pi_0{}^\alpha,\ \ \phi = -\pi_0{}^0 + \pi_{n+1}{}^{n+1} = -\pi_0{}^0 - \bar\pi_0{}^0. \tag{5.8 a}$$

In fact, by setting

$$\phi^\alpha = 2\pi_{n+1}{}^\alpha,\ \ \phi_{\alpha.}^{\ \beta} = \pi_{\alpha.}^{\ \beta} - \delta_\alpha{}^\beta \pi_0{}^0,\ \ \psi = -4\pi_{n+1}{}^0, \tag{5.8 b}$$

we find with the aid of (1.30a) that the equations (1.32) are identical to the equations given in Theorem 4.6 of § 4 with

$$S_{\alpha\beta\bar\varrho\bar\sigma} = V_{\alpha\bar\beta\gamma} = P_{\alpha\bar\beta} = Q_{\alpha\beta} = R_\alpha = 0. \tag{5.9}$$

A (nondegenerate integrable) G-structure satisfying the conditions (5.9) is called *flat*. Conversely, it follows from the Theorem of Frobenius that every real analytic flat G-structure is locally equivalent to one arisen from a nondegenerate real hyperquadric in $\mathbf{C}_{n+1}$.

Under the change of Q-frame (1.15) we have, by (5.7),

$$\omega^* = |t|^2 \omega. \tag{5.10}$$

We therefore restrict ourselves to the subgroup H_1 of H characterized by the condition $|t| = 1$. The form ω is then invariant under H_1. From (1.29) we have

$$\pi_0{}^0 = 2i(dZ_0, Z_{n+1}),\quad \pi_0{}^\alpha = g^{\alpha\bar\beta}(dZ_0, Z_\beta). \tag{5.11}$$

By (5.8a) it follows that under a change of Q-frames by H_1, we have

$$\left.\begin{aligned}
\omega^* &= \omega,\\
\omega^{*\alpha} &= t(it^\alpha \omega + t^\alpha_{.\beta}\omega^\beta),\\
\omega^{*\bar\alpha} &= t^{-1}(-it^{\bar\alpha}\omega + t^{\bar\alpha}_{.\bar\beta}\omega^{\bar\beta}),\\
\phi^* &= \mathrm{Re}\,(\tau t^{-1})\,\omega - 2it\tau_\alpha\omega^\alpha + 2it^{-1}\tau_{\bar\alpha}\omega^{\bar\alpha} + \phi.
\end{aligned}\right\} \tag{5.12}$$

The matrix of the coefficients in (5.12) belongs to the group G_1 introduced in § 4. The mapping

$$H_1 \to G_1 \tag{5.13}$$

so defined is clearly a homomorphism. In fact, if K denotes the group defined in (1.9), we have the isomorphism

$$H_1/K \cong G_1. \tag{5.13 a}$$

Since $SU(p+1, q+1)/K \supset \mathrm{H}_1/K$, we will consider G_1 as a subgroup of the former via the isomorphism (5.13a). This identification is essential in the treatment of the general case; the group $SU(p+1, q+1)$ is paramount in the whole theory.

We introduce the matrix notation

$$(h) = (h_{A\bar{B}}), \tag{5.14}$$

where $h_{A\bar{B}}$ are defined in (1.10a). The Lie algebra $\mathfrak{su}$ of $SU(p+1, q+1)$ is the algebra of all matrices

$$(l) = (l_{A.}^{\ B}), \quad 0 \leqslant A, B \leqslant n+1, \tag{5.15}$$

satisfying

$$(l)(h) + (h)\,{}^t(\bar{l}) = 0, \quad \mathrm{Tr}\,(l) = 0. \tag{5.16}$$

The Lie algebra of H_1 is the subalgebra of $\mathfrak{su}$ satisfying the conditions

$$l_0^\alpha = l_0^{n+1} = \mathrm{Re}\,(l_0^0) = 0. \tag{5.17}$$

With this notation it follows from (1.30) and (5.16) that the matrix

$$(\pi) = (\pi_{A.}^{\ B}) \tag{5.18}$$

is an $\mathfrak{su}$-valued one-form on the group $SU(p+1, q+1)$. The Maurer-Cartan equations (1.32) of the latter can be written

$$d(\pi) = (\pi) \wedge (\pi). \tag{5.19}$$

Let

$${}^t(Z) = (Z_0, \ldots, Z_{n+1}) \tag{5.20}$$

be a matrix of vectors of $\mathbf{C}_{n+2}$. Then equation (1.29) can be written

$$d(Z) = (\pi)(Z), \tag{5.21}$$

and equation (1.15) for the change of Q-frames becomes

$$(Z^*) = (t)(Z), \tag{5.22}$$

where the entries in (t) are supposed to be constants. If (π^*) is defined by

$$d(Z^*) = (\pi^*)(Z^*), \tag{5.23}$$

we have

$$(\pi^*) = (t)\,(\pi)\,(t)^{-1} \underset{\mathrm{def}}{=} \mathrm{ad}\,(t)\,(\pi). \tag{5.24}$$

This equation will have an important generalization.

(b) General remarks on connections. Let Y be a principal G_1-bundle over a manifold E. Let Γ be a linear group which contains G_1 as a subgroup; in our case we will have

$$\Gamma = SU(p+1, q+1)/K \supset H_1/K \cong G_1. \tag{5.25}$$

In applications of connections it frequently occurs that one should consider in the bundle Y a connection relative to the larger group Γ. For instance, this is the case of classical Riemannian geometry, where we consider in the bundle of orthonormal frames a connection relative to the group of motions of euclidean space.

Let γ be the Lie algebra of Γ realized as a Lie algebra of matrices. Then G_1 acts on γ by the adjoint transformation

$$\text{ad}\,(t)(l) = (t)(l)(t)^{-1}, \quad (t) \in G_1,\ (l) \in \gamma. \tag{5.26}$$

A Γ-connection in the bundle Y is a γ-valued one-form (π), the connection form, such that under a change of frame by the group G_1, (π) transforms according to the formula

$$(\pi^*) = \text{ad}\,(t)(\pi), \quad (t) \in G_1. \tag{5.27}$$

Its curvature form is defined by

$$(\Pi) = d(\pi) - (\pi) \wedge (\pi) \tag{5.28}$$

and is therefore a γ-valued two-form following the same transformation law:

$$(\Pi^*) = \text{ad}\,(t)(\Pi), \quad (t) \in G_1. \tag{5.29}$$

The adjoint transformation of G_1 on γ leaves the Lie algebra $\mathfrak{g}_1$ of G_1 invariant and induces an action on the quotient space $\gamma/\mathfrak{g}_1$. The projection of the curvature form on $\gamma/\mathfrak{g}_1$ is called the torsion form.

(c) Definition of the connection. This will be a geometrical interpretation of the results of § 4. Our first problem is to write the equations listed in the Theorem 4.1 of § 4, i.e., the equations (4.10), etc. in a convenient form, making use of the group $SU(p+1, q+1)$ and its Lie algebra $\mathfrak{su}$. The $g_{\alpha\bar\beta}$ are from now on supposed to be constants and we call attention to the convention (5.1). Following the flat case we solve the equations (5.7), (5.8a), (5.8b) and put

$$\left.\begin{array}{llrl}
\pi_0{}^{n+1} = 2\omega, & & -(n+2)\,\pi_0{}^0 & = \phi_\alpha{}^\alpha + \phi, \\
\pi_0{}^\alpha = \omega^\alpha, & & \pi_\alpha{}^{n+1} & = 2i\omega_\alpha, \\
\pi_{n+1}{}^\alpha = \tfrac{1}{2}\phi^\alpha, & & \pi_\alpha{}^0 & = -i\phi_\alpha, \\
\pi_{\alpha.}{}^{\beta} = \phi_{\alpha.}{}^{\beta} + \delta_\alpha{}^\beta \pi_0{}^0, & & & \\
\pi_{n+1}{}^0 = -\tfrac{1}{4}\psi, & & \pi_{n+1}{}^{n+1} & = -\bar\pi_0{}^0.
\end{array}\right\} \tag{5.30}$$

The $\pi_A{}^B$ are one-forms in Y, and the matrix

$$(\pi) = (\pi_A{}^B), \quad 0 \leqslant A, B \leqslant n+1, \tag{5.31}$$

is $\mathfrak{su}$-valued, i.e.,

$$(\pi)(h) + (h)\,{}^t(\bar{\pi}) = 0, \quad \operatorname{Tr}(\pi) = 0. \tag{5.32}$$

Moreover, restricted to a fiber of Y, the non-zero π's give the Maurer-Cartan forms of H_1, as is already in the flat case.

As in the flat case it is immediately verified that using the form (π) the equations in the theorem of § 4 can be written

$$d(\pi) = (\pi) \wedge (\pi) + (\Pi), \tag{5.33}$$

where

$$(\Pi) = \begin{pmatrix} \Pi_0{}^0 & 0 & 0 \\ \Pi_\alpha{}^0 & \Pi_\alpha{}^\beta & 0 \\ \Pi_{n+1}^{\ 0} & \Pi_{n+1}^{\ \beta} & -\bar{\Pi}_0{}^0 \end{pmatrix} \tag{5.34}$$

and

$$\left.\begin{aligned} (n+2)\,\Pi_0^0 &= -\Phi_{\alpha\cdot}^{\alpha}, \quad \Pi_{n+1}^{\ 0} = -\tfrac{1}{4}\Psi, \\ \Pi_\alpha{}^0 &= -i\Phi_\alpha, \quad \Pi_{n+1}^{\ \beta} = \tfrac{1}{2}\Phi^\beta, \\ \Pi_{\alpha\cdot}^{\beta} &= \Phi_{\alpha\cdot}^{\beta} - \frac{1}{n+2}\delta_\alpha{}^\beta\,\Phi_{\gamma\cdot}^{\gamma}. \end{aligned}\right\} \tag{5.35}$$

where the right-hand side members are exterior two-forms in ω, ω^α, $\omega^{\bar{\beta}}$, defined in § 4. For any such form

$$\Theta \equiv a_{\alpha\bar{\beta}}\,\omega^\alpha \wedge \omega^{\bar{\beta}} + \text{terms quadratic in } \omega^\varrho \text{ or } \omega^{\bar{\sigma}}, \text{ mod } \omega, \tag{5.36}$$

we set

$$\operatorname{Tr}\Theta = g^{\alpha\bar{\beta}}\,a_{\alpha\bar{\beta}}. \tag{5.37}$$

Then equations (4.37), (4.58), (4.70) can be expressed respectively by

$$\left.\begin{aligned} &\operatorname{Tr}\Pi_\alpha{}^\beta = 0, \ \operatorname{Tr}\Pi_0^0 = 0, \\ &\operatorname{Tr}\Pi_\beta{}^0 = \operatorname{Tr}\Pi_{n+1}^{\ \alpha} = 0, \\ &\operatorname{Tr}\Pi_{n+1}^{\ 0} = 0, \end{aligned}\right\} \tag{5.38}$$

and their totality can be summarized in the matrix equation

$$\operatorname{Tr}(\Pi) = 0. \tag{5.39}$$

Under the adjoint transformation of H_1,

$$(\pi) \to \operatorname{ad}(t)(\pi),$$

$$(\Pi) \to \operatorname{ad}(t)(\Pi),$$

the condition (5.39) remains invariant. We submit ω, ω^α, $\omega^{\bar\beta}$, ϕ to the linear transformation with the matrix (4.12) and denote the new quantities by the same symbols with asterisks. Since (π) is uniquely determined by (5.39) according to theorem 4.1 in § 4 and since these conditions are invariant under the adjoint transformation by H_1, we have

$$(\pi^*) = \operatorname{ad}(t)(\pi), \quad t \in G_1. \tag{5.40}$$

Therefore (π) satisfies the conditions of a connection form and we have the theorem:

THEOREM 5.1. *Given a non-degenerate integrable G-structure on a manifold M of dimension $2n+1$. Consider the principal bundle Y over E with the group $G_1 \subset SU(p+1, q+1)/K$. There is in Y a uniquely defined connection with the group $SU(p+1, q+1)$, which is characterized by the vanishing of the torsion form and the condition* (5.39).

In terms of Q-frames Z_A which are meaningful under the group $SU(p+1, q+1)$, the connection can be written

$$DZ_A = \pi_{A.}^{\ B} Z_B. \tag{5.41}$$

These equations are to be compared with (5.21) where the differential is taken in the ordinary sense.

(d) **Chains.** Consider a curve λ which is everywhere transversal to the complex tangent hyperplane. Its tangent line can be defined by

$$\omega^\alpha = 0. \tag{5.42}$$

By (4.16) restricted to λ, we get

$$\phi^\alpha = b^\alpha \omega. \tag{5.43}$$

The curve λ is called a chain if $b^\alpha = 0$. The chains are therefore defined by the differential system

$$\omega^\alpha = \phi^\alpha = 0. \tag{5.44}$$

They generalize the chains on the real hyperquadrics in $\mathbf{C}_{n+1}$ (cf. (1.33)) and are here defined intrinsically. It is easily seen that through a point of M and tangent to a vector transversal to the complex tangent hyperplane there passes exactly one chain.

When restricted to a chain, equations (4.10), (4.26), (4.59), (4.72) give

$$d\omega = \omega \wedge \phi, \quad d\phi = \omega \wedge \psi, \quad d\psi = \phi \wedge \psi. \tag{5.45}$$

The forms ω, ϕ, ψ being real, these are the equations of structure of the group of real linear fractional transformations in one real variable. It follows that on a chain there is a preferred parameter defined up to a linear fractional transformation. In other words, on a chain the cross ratio of four points, a real value, is well defined.

6. Actual computation for real hypersurfaces

Consider the real hypersurface M in $\mathbf{C}_{n+1}$ defined by the equation (4.6). We wish to relate the invariants of the G-structure with the function $r(z^\alpha, \bar{z}^\alpha, w, \bar{w})$, and thus also with the normal form of the equation of M established in § 2, 3. This amounts to solving the structure equations listed in the theorem of § 4, with the G-structure given by (4.7); the unique existence of the solution was the assertion of the theorem. We observe that it suffices to find a particular set of forms satisfying the structure equations, because the most general ones are then completely determined by applying the linear transformation with the matrix (4.12). In actual application it will be advantageous to allow $g_{\alpha\bar{\beta}}$ to be variable, which was the freedom permitted in § 4. Our method consists of first finding a set of solutions of the structure equations, without necessarily satisfying the trace conditions (4.37), (4.58), (4.70). By successive steps we will then modify the forms to fulfill these conditions.

We set

$$\omega = \theta = i\partial r, \quad \omega^\alpha = dz^\alpha. \tag{6.1}$$

Then (4.10) becomes

$$d\theta = i g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \theta \wedge \phi. \tag{6.2}$$

It is fulfilled if

$$\left.\begin{aligned} g_{\alpha\bar{\beta}} &= -r_{\alpha\bar{\beta}} + r_w^{-1} r_\alpha r_{w\bar{\beta}} + r_{\bar{w}}^{-1} r_{\bar{\beta}} r_{\bar{w}\alpha} - (r_w r_{\bar{w}})^{-1} r_{w\bar{w}} r_\alpha r_{\bar{\beta}} \\ \phi &= -r_{\bar{w}}^{-1} r_{\bar{w}\alpha} dz^\alpha - r_w^{-1} r_{w\bar{\beta}} d\bar{z}^\beta + (r_w r_{\bar{w}})^{-1} r_{w\bar{w}} (r_\alpha dz^\alpha + r_{\bar{\beta}} d\bar{z}^\beta), \end{aligned}\right\} \tag{6.3}$$

where we use the convention

$$r_\alpha = \frac{\partial r}{\partial z^\alpha}, \quad r_{\bar{\beta}} = \frac{\partial r}{\partial \bar{z}_\beta}, \quad r_{\alpha\bar{\beta}} = \frac{\partial^2 r}{\partial z^\alpha \partial \bar{z}^\beta}, \text{ etc.} \tag{6.4}$$

Exterior differentiation of (6.2) gives

$$i(dg_{\alpha\bar{\beta}} + g_{\alpha\bar{\beta}}\phi) \wedge dz^\alpha \wedge d\bar{z}^\beta - \theta \wedge d\phi = 0. \tag{6.5}$$

This allows us to put

$$dg_{\alpha\bar{\beta}} + g_{\alpha\bar{\beta}}\phi = a_{\alpha\bar{\beta}\gamma} dz^\gamma + a_{\bar{\beta}\alpha\bar{\gamma}} d\bar{z}^\gamma + c_{\alpha\bar{\beta}}\theta, \tag{6.6}$$

$$d\phi = i c_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \theta \wedge \mu, \tag{6.7}$$

where

$$a_{\alpha\bar{\beta}\gamma} = a_{\gamma\bar{\beta}\alpha}, \quad c_{\alpha\bar{\beta}} = c_{\bar{\beta}\alpha}. \tag{6.8}$$

With θ, $g_{\alpha\bar{\beta}}$, ϕ given by (6.1), (6.3), equations (6.6), (6.7) determine completely $a_{\alpha\bar{\beta}\gamma}$, $c_{\alpha\bar{\beta}}$, and also μ, when we assume that μ is a linear combination of dz^α, $d\bar{z}^\beta$ only. The $a_{\alpha\bar{\beta}\gamma}$, $c_{\alpha\bar{\beta}}$, μ so defined involve partial derivatives of r up to order 3 inclusive.

With ω, ω^α, ϕ given by (6.1), (6.3), we see that the following forms satisfy (4.16), (4.21), and (4.26):

$$\left.\begin{aligned} \phi_{\beta.}^{\ \alpha(1)} &= a_{\beta.\gamma}^{\alpha} dz^{\gamma} + \tfrac{1}{2} c_{\beta.}^{\alpha} \theta, \\ \phi^{\alpha(1)} &= \tfrac{1}{2} c_{\beta.}^{\alpha} dz^{\beta}, \\ \psi^{(1)} &= \mu. \end{aligned}\right\} \tag{6.9}$$

Its most general solution, to be denoted by $\phi_{\beta.}^{\alpha}$, ϕ^{α}, ψ, is related to the particular solution (6.9), the "first approximation", by

$$\left.\begin{aligned} \phi_{\beta.}^{\ \alpha(1)} - \phi_{\beta.}^{\alpha} &= d_{\beta.}^{\alpha} \theta, \\ \phi^{\alpha(1)} &= \phi^{\alpha} + d_{\beta.}^{\alpha} dz^{\beta} + e^{\alpha} \theta, \\ \psi^{(1)} &= \psi + g\theta + i(e_{\alpha} dz^{\alpha} - e_{\bar\beta} dz^{\bar\beta}), \end{aligned}\right\} \tag{6.10}$$

where $d_{\beta.}^{\alpha}$ satisfy
$$d_{\alpha\bar\beta} + d_{\bar\beta\alpha} = 0 \tag{6.11}$$

and g is real; cf. (4.35), (4.36). We will determine the coefficients in (6.10) by the conditions (4.37), (4.58), (4.70).

In view of (4.53) we set

$$\begin{aligned} d\phi_{\beta.}^{\ \gamma(1)} - \phi_{\beta.}^{\ \sigma(1)} \wedge \phi_{\sigma.}^{\ \gamma(1)} - ig_{\beta\bar\sigma} dz^{\bar\sigma} \wedge \phi^{\gamma(1)} + i\phi_{\beta}^{(1)} \wedge dz^{\gamma} + i\delta_{\beta}^{\ \gamma}(\phi_{\sigma}^{(1)} \wedge dz^{\sigma}) \\ \equiv s_{\beta\varrho.\bar\sigma}^{\ \ \gamma(1)} dz^{\varrho} \wedge dz^{\bar\sigma}, \quad \bmod \theta, \end{aligned} \tag{6.12}$$

by which $s_{\beta\varrho.\bar\sigma}^{\ \ \gamma(1)}$ are completely determined. Let

$$s_{\varrho\bar\sigma}^{(1)} = g^{\alpha\bar\beta} s_{\alpha\varrho\bar\beta\bar\sigma}^{(1)}, \qquad s^{(1)} = g^{\alpha\bar\beta} s_{\alpha\bar\beta}^{(1)}. \tag{6.13}$$

By (4.43) the condition (4.37) is fulfilled if we put

$$(n+2)\, d_{\varrho\bar\sigma} = -is_{\varrho\bar\sigma}^{(1)} + \frac{i}{2(n+1)} g_{\varrho\bar\sigma} s^{(1)}. \tag{6.14}$$

This equation determines $d_{\varrho\bar\sigma}$ and we have completely determined

$$\phi_{\beta.}^{\alpha} = \phi_{\beta.}^{\ \alpha(1)} - d_{\beta.}^{\alpha} \theta = a_{\beta.\gamma}^{\alpha} dz^{\gamma} + (\tfrac{1}{2} c_{\beta.}^{\alpha} - d_{\beta.}^{\alpha})\, \theta. \tag{6.15}$$

For the determination of ϕ^{α} we introduce the "second approximation":

$$\phi^{\alpha(2)} = \phi^{\alpha(1)} - d_{\beta.}^{\alpha} dz^{\beta}. \tag{6.16}$$

Again in view of (4.53), we set

$$\begin{aligned} d\phi_{\beta.}^{\gamma} - \phi_{\beta.}^{\sigma} \wedge \phi_{\sigma.}^{\gamma} - ig_{\beta\bar\sigma} dz^{\bar\sigma} \wedge \phi^{\gamma(2)} + i\phi_{\beta}^{(2)} \wedge dz^{\gamma} + i\delta_{\beta}^{\ \gamma}(\phi_{\sigma}^{(2)} \wedge dz^{\sigma}) + \tfrac{1}{2}\delta_{\beta}^{\ \gamma}\psi^{(1)} \wedge \theta \\ = s_{\beta\varrho.\bar\sigma}^{\ \ \gamma} dz^{\varrho} \wedge dz^{\bar\sigma} + v_{\beta.\varrho}^{\ \gamma(1)} dz^{\varrho} \wedge \theta - v_{\cdot\beta\bar\sigma}^{\gamma(1)} dz^{\bar\sigma} \wedge \theta, \end{aligned} \tag{6.17}$$

which defines the coefficients $s_{\beta\varrho.\bar\sigma}^{\ \ \gamma}$, $v_{\beta.\varrho}^{\ \gamma(1)}$. The former satisfy

$$s_{\varrho\bar{\sigma}} = g^{\alpha\bar{\beta}} s_{\alpha\varrho\bar{\beta}\bar{\sigma}} = 0. \tag{6.18}$$

By (4.57) we determine e_β by

$$-i(n+\tfrac{1}{2})\, e_\beta = v_{\beta.\varrho}^{\varrho(1)}, \tag{6.19}$$

so that (4.58) will be satisfied. We have then completely determined

$$\phi^\alpha = \phi^{\alpha(2)} - e^\alpha \theta \tag{6.20}$$

and we introduce the "second approximation"

$$\psi^{(2)} = \psi^{(1)} - i(e_\alpha dz^\alpha - e_{\bar{\beta}}\, dz^{\bar{\beta}}). \tag{6.21}$$

By (4.54) and (4.62) we set

$$\begin{aligned} &d\phi^\alpha - \phi \wedge \phi^\alpha - \phi^\beta \wedge \phi_{\beta.}^{\alpha} + \tfrac{1}{2}\psi^{(2)} \wedge dz^\alpha \\ &= -v_{\beta.\gamma}^{\alpha}\, dz^\beta \wedge dz^\gamma + v_{.\beta\bar{\sigma}}^{\alpha}\, dz^\beta \wedge dz^{\bar{\sigma}} + p_{\beta.}^{\alpha(1)} dz^\beta \wedge \theta + q_{\bar{\beta}.}^{\alpha} dz^\beta \wedge \theta. \end{aligned} \tag{6.22}$$

The condition (4.70) is fulfilled by setting

$$g = -\frac{2}{n} \operatorname{Re}(p_{\alpha.}^{\alpha(1)}) \tag{6.23}$$

and

$$\psi = \psi^{(2)} - g\theta. \tag{6.24}$$

The forms $\phi_{\beta.}^{\alpha}$, ϕ^α, ψ so determined in successive steps satisfy now all the structure equations, together with the trace conditions (4.37), (4.58), (4.70). Notice that our formulas allow the computation of the invariants from the function r. The determinations $d_{\beta.}^{\alpha}$, e^α, g involve respectively partial derivatives of r up to the fourth, fifth, and sixth orders inclusive.

The procedure described above can be applied when the equation of M is in the normal form of § 2, 3. Then we have

$$r = \frac{1}{2i}(w - \bar{w}) - \langle z, z\rangle - N_{22} - N_{32} - N_{23} - N_{42} - N_{24} - N_{33} - \dots, \tag{6.25}$$

where

$$\left.\begin{aligned} N_{22} &= b_{\alpha_1\alpha_2\bar{\beta}_1\bar{\beta}_2} z^{\alpha_1} z^{\alpha_2} z^{\bar{\beta}_1} z^{\bar{\beta}_2} \\ N_{32} = \bar{N}_{23} &= k_{\alpha_1\alpha_2\alpha_3\bar{\beta}_1\bar{\beta}_2} z^{\alpha_1} z^{\alpha_2} z^{\alpha_3} z^{\bar{\beta}_1} z^{\bar{\beta}_2} \\ N_{42} = \bar{N}_{24} &= l_{\alpha_1\dots\alpha_4\bar{\beta}_1\bar{\beta}_2} z^{\alpha_1} z^{\alpha_2} z^{\alpha_3} z^{\alpha_4} z^{\bar{\beta}_1} z^{\bar{\beta}_2} \\ N_{33} &= m_{\alpha_1\alpha_2\alpha_3\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3} z^{\alpha_1} z^{\alpha_2} z^{\alpha_3} z^{\bar{\beta}_1} z^{\bar{\beta}_2} z^{\bar{\beta}_3} \end{aligned}\right\} \tag{6.26}$$

and N_{22} and N_{33} are real; the coefficients, which are functions of u, satisfy the usual symmetry relations and are completely determined by the polynomials. Moreover, we have the trace conditions

$$\operatorname{Tr} N_{22} = \operatorname{Tr}^2 N_{32} = 0, \tag{6.27}$$

$$\operatorname{Tr}^3 N_{33} = 0, \tag{6.28}$$

where the traces are formed with respect to $\langle , \rangle$.

The computation is lengthy and we will only state the following results:

(1) Along the u-curve Γ, i.e., the curve defined by

$$z^\alpha = v = 0, \tag{6.29}$$

we have $\phi^\alpha = 0$. This means that Γ is a chain. In fact, this is true whenever the conditions (6.27) are satisfied.

(2) Along Γ we find

$$s_{\alpha\beta\bar{\varrho}\bar{\sigma}} = -4 b_{\alpha\beta\bar{\varrho}\bar{\sigma}}, \tag{6.30}$$

$$v_{\alpha.\gamma}^{\beta} = -\frac{12 i}{n+2} h^{\beta\bar{\sigma}} k_{\alpha\gamma\bar{\sigma}}, \tag{6.31}$$

$$q_{\bar{\beta}.}^{\alpha} = -\frac{48}{(n+1)(n+2)} h^{\alpha\bar{\gamma}} l_{\bar{\gamma}\bar{\beta}}, \tag{6.32}$$

where the quantities are defined by

$$\langle z, z\rangle = h_{\alpha\bar{\beta}} z^\alpha z^{\bar{\beta}}. \tag{6.33}$$

$$\operatorname{Tr} N_{32} = k_{\alpha_1\alpha_2\bar{\beta}} z^{\alpha_1} z^{\alpha_2} z^{\bar{\beta}}. \tag{6.34}$$

$$\operatorname{Tr}^2 N_{24} = l_{\bar{\beta}_1\bar{\beta}_2} z^{\bar{\beta}_1} z^{\bar{\beta}_2}. \tag{6.35}$$

The situation is particularly simple for $n = 1$. Then conditions (6.27) and (6.28) imply

$$N_{22} = N_{32} = N_{33} = 0. \tag{6.36}$$

On the other hand, we have the remarks at the end of § 4; the invariant of lowest order is q_{11}. Equation (6.32) identifies it with the coefficient in N_{42}.

Appendix. Bianchi Identities

BY

S. M. WEBSTER

University of California, Berkeley, California, USA

In this appendix we will show that there are further symmetry relations on the curvature of the connection, which follow from the Bianchi identities and which simplify the structure equation.

The Bianchi identities for the connection defined in section 5c are obtained by taking the exterior derivative of the structure equation (5.33). This yields

$$0=(\Pi)\wedge(\pi)-(\pi)\wedge(\Pi)+d(\Pi).$$

To write this more explicitly it is convenient to use the formulation given in the theorem of section 4. In the G_1 bundle Y over E we have the independent linear differential forms

$$\omega,\ \omega^{\alpha},\ \omega^{\bar\beta},\ \phi,\ \phi_{\alpha}{}^{\beta},\ \phi^{\alpha},\ \phi^{\bar\beta},\ \psi,$$

the relations
$$\phi_{\alpha\bar\beta}+\phi_{\bar\beta\alpha}-g_{\alpha\bar\beta}\phi=0$$

with the $g_{\alpha\bar\beta}$ constant, and the structure equations

$$d\omega=ig_{\alpha\bar\beta}\omega^{\alpha}\wedge\omega^{\bar\beta}+\omega\wedge\phi \tag{A. 1}$$

$$d\omega^{\alpha}=\omega^{\beta}\wedge\phi_{\beta}{}^{\alpha}+\omega\wedge\phi^{\alpha} \tag{A. 2}$$

$$d\phi=i\omega_{\bar\beta}\wedge\phi^{\bar\beta}+i\phi_{\bar\beta}\wedge\omega^{\bar\beta}+\omega\wedge\psi \tag{A. 3}$$

$$d\phi_{\beta}{}^{\alpha}=\phi_{\beta}{}^{\sigma}\wedge\phi_{\sigma}{}^{\alpha}+i\omega_{\beta}\wedge\phi^{\alpha}-i\phi_{\beta}\wedge\omega^{\alpha}-i\delta_{\beta}{}^{\alpha}(\phi_{\sigma}\wedge\omega^{\sigma})-\tfrac{1}{2}\delta_{\beta}{}^{\alpha}\psi\wedge\omega+\Phi_{\beta}{}^{\alpha} \tag{A. 4}$$

$$d\phi^{\alpha}=\phi\wedge\phi^{\alpha}+\phi^{\beta}\wedge\phi_{\beta}{}^{\alpha}-\tfrac{1}{2}\psi\wedge\omega^{\alpha}+\Phi^{\alpha} \tag{A. 5}$$

$$d\psi=\phi\wedge\psi+2i\phi^{\beta}\wedge\phi_{\beta}+\Psi. \tag{A. 6}$$

The curvature forms are given by

$$\Phi_{\beta}{}^{\alpha}=S_{\beta\varrho\cdot\bar\sigma}^{\ \ \alpha}\omega^{\varrho}\wedge\omega^{\bar\sigma}+V_{\beta\cdot\varrho}^{\alpha}\omega^{\varrho}\wedge\omega-V_{\cdot\beta\bar\sigma}^{\alpha}\omega^{\bar\sigma}\wedge\omega \tag{A. 7}$$

$$\Phi^{\alpha}=-V_{\varrho\cdot\sigma}^{\alpha}\omega^{\varrho}\wedge\omega^{\sigma}+V_{\cdot\varrho\bar\sigma}^{\alpha}\omega^{\varrho}\wedge\omega^{\bar\sigma}+P_{\varrho}{}^{\alpha}\omega^{\varrho}\wedge\omega+Q_{\bar\sigma}{}^{\alpha}\omega^{\bar\sigma}\wedge\omega, \tag{A. 8}$$

$$\Psi=iQ_{\varrho\sigma}\omega^{\varrho}\wedge\omega^{\sigma}-iQ_{\bar\varrho\bar\sigma}\omega^{\bar\varrho}\wedge\omega^{\bar\sigma}-i\tilde P_{\varrho\bar\sigma}\omega^{\varrho}\wedge\omega^{\bar\sigma}+(R_{\varrho}\omega^{\varrho}+R_{\bar\sigma}\omega^{\bar\sigma})\wedge\omega, \tag{A. 9}$$

where the coefficients satisfy the relations

$$S_{\beta\varrho\bar\alpha\bar\sigma}=S_{\varrho\beta\bar\alpha\bar\sigma}=S_{\beta\varrho\bar\sigma\bar\alpha},$$

$$S_{\beta\varrho\bar\alpha\bar\sigma}=\overline{S_{\alpha\sigma\bar\beta\bar\varrho}}=S_{\bar\alpha\bar\sigma\beta\varrho},$$

$$\tilde P_{\alpha\bar\beta}=P_{\alpha\bar\beta}+P_{\bar\beta\alpha},$$

and
$$V_{\beta\cdot\varrho}^{\varrho}=g^{\beta\bar\alpha}S_{\beta\varrho\bar\alpha\bar\sigma}=g^{\alpha\bar\beta}\tilde P_{\alpha\bar\beta}=0.$$

Differentiating equations (A.1) through (A. 6) yields, respectively,

$$0=(\phi_{\alpha\bar\beta}+\phi_{\bar\beta\alpha}-g_{\alpha\bar\beta}\phi)\wedge\omega^{\alpha}\wedge\omega^{\bar\beta} \tag{A. 1'}$$

$$0=\omega^{\beta}\wedge\phi_{\beta\cdot}^{\alpha}+\omega\wedge\Phi^{\alpha} \tag{A. 2'}$$

$$0 = \omega \wedge \Psi - i(\Phi^\alpha \wedge \omega_\alpha - \Phi^{\bar\alpha} \wedge \omega_{\bar\alpha}) \tag{A. 3'}$$

$$0 = d\Phi_\beta{}^\alpha + \Phi_\beta{}^\gamma \wedge \phi_\gamma{}^\alpha - \phi_\beta{}^\gamma \wedge \Phi_\gamma{}^\alpha - i\omega_\beta \wedge \Phi^\alpha - i\Phi_\beta \wedge \omega^\alpha - \delta_\beta{}^\alpha \{i\Phi^{\bar\sigma} \wedge \omega_{\bar\sigma} + \tfrac{1}{2}\Psi \wedge \omega\} \tag{A. 4'}$$

$$0 = d\Phi^\alpha + \Phi^\beta \wedge \phi_\beta{}^\alpha - \phi^\beta \wedge \Phi_\beta{}^\alpha - \phi \wedge \Phi^\alpha - \tfrac{1}{2}\Psi \wedge \omega^\alpha \tag{A. 5'}$$

$$0 = d\Psi + 2i\Phi^\beta \wedge \phi_\beta - 2i\phi^\beta \wedge \Phi_\beta - \phi \wedge \Psi. \tag{A. 6'}$$

These are the Bianchi identities. The actual verification of these equations is rather long, but they result from differentiating and simply dropping all terms which do not contain one of the curvature forms $\Phi_\beta{}^\alpha$, Φ^α, Ψ or one of their differentials. Equations (A.1′), (A.2′), and (A.3′) are trivial because of the relations $\phi_{\alpha\bar\beta} + \phi_{\bar\beta\alpha} = g_{\alpha\bar\beta}\,\phi$ and $S_{\beta\varrho.\bar\sigma}^{\;\;\alpha\bar{}} = S_{\varrho\beta.\bar\sigma}^{\;\;\alpha}$. Substituting (A.3′) into (A.4′) gives

$$0 = d\Phi_\beta{}^\alpha + \Phi_\beta{}^\gamma \wedge \phi_\gamma{}^\alpha - \phi_\beta{}^\gamma \wedge \Phi_\gamma{}^\alpha - i\omega_\beta \wedge \Phi^\alpha - i\Phi_\beta \wedge \omega^\alpha - \frac{i}{2}\delta_\beta{}^\alpha \{\Phi^\sigma \wedge \omega_\sigma + \Phi^{\bar\sigma} \wedge \omega_{\bar\sigma}\}. \tag{A. 4''}$$

Substituting the expression (A.7) for $\Phi_\beta{}^\alpha$ into (A.4″) gives, after differentiating and lowering the index α,

$$\begin{aligned} 0 = {} & DS_{\beta\varrho\bar\alpha\bar\sigma} \wedge \omega^\varrho \wedge \omega^{\bar\sigma} + B_{\beta\bar\alpha\varrho} \wedge \omega^\varrho \wedge \omega - \overline{B_{\alpha\bar\beta\varrho}} \wedge \omega^{\bar\varrho} \wedge \omega \\ & + iC_{\beta\bar\alpha\mu\varrho\bar\sigma}\,\omega^\mu \wedge \omega^\varrho \wedge \omega^{\bar\sigma} + i\overline{C_{\alpha\bar\beta\mu\varrho\bar\sigma}}\,\omega^{\bar\mu} \wedge \omega^{\bar\varrho} \wedge \omega^\sigma \\ & + iD_{\beta\bar\alpha\varrho\sigma}\,\omega^\varrho \wedge \omega^\sigma \wedge \omega + i\overline{D_{\alpha\bar\beta\varrho\sigma}}\,\omega^{\bar\varrho} \wedge \omega^{\bar\sigma} \wedge \omega \\ & + iE_{\beta\bar\alpha\varrho\bar\sigma}\,\omega^\varrho \wedge \omega^{\bar\sigma} \wedge \omega, \end{aligned} \tag{A. 10}$$

where we define

$$\begin{aligned} DS_{\beta\varrho\bar\alpha\bar\sigma} &= dS_{\beta\varrho\bar\alpha\bar\sigma} - S_{\mu\varrho\bar\alpha\bar\sigma}\,\phi_\beta^\mu - S_{\beta\mu\bar\alpha\bar\sigma}\,\phi_\varrho^\mu - S_{\beta\varrho\bar\mu\bar\sigma}\,\phi_{\bar\alpha}^{\bar\mu} - S_{\beta\varrho\bar\alpha\bar\mu}\,\phi_{\bar\sigma}^{\bar\mu} + S_{\beta\varrho\bar\alpha\bar\sigma}\,\phi \\ &= S_{\beta\varrho\bar\alpha\bar\sigma\mu}\,\omega^\mu + S_{\beta\varrho\bar\alpha\bar\sigma\bar\mu}\,\omega^{\bar\mu} + S_{\beta\varrho\bar\alpha\bar\sigma}\,\omega, \\ DV_{\beta\bar\alpha\varrho} &= dV_{\beta\bar\alpha\varrho} - V_{\mu\bar\alpha\varrho}\,\phi_\beta^\mu - V_{\beta\bar\mu\varrho}\,\phi_{\bar\alpha}^{\bar\mu} - V_{\beta\bar\alpha\mu}\,\phi_\varrho^\mu - S_{\beta\varrho\bar\alpha\bar\sigma}\,\phi^{\bar\sigma} \\ &= V_{\beta\bar\alpha\varrho\mu}\,\omega^\mu + V_{\beta\bar\alpha\varrho\bar\mu}\,\omega^{\bar\mu} + V_{\beta\bar\alpha\varrho *}\,\omega, \\ C_{\beta\bar\alpha\mu\varrho\bar\sigma} &= V_{\beta\bar\alpha\varrho}\,g_{\mu\bar\sigma} + V_{\mu\bar\alpha\varrho}\,g_{\beta\bar\sigma} + g_{\bar\alpha\mu}V_{\beta\bar\sigma\varrho} + g_{\beta\bar\alpha}V_{\mu\bar\sigma\varrho}, \\ D_{\beta\bar\alpha\varrho\sigma} &= Q_{\varrho\beta}\,g_{\sigma\bar\alpha} + \tfrac{1}{2}g_{\beta\bar\alpha}\,Q_{\varrho\sigma}, \\ E_{\beta\bar\alpha\varrho\bar\sigma} &= g_{\beta\bar\sigma}P_{\varrho\bar\alpha} - P_{\bar\sigma\beta}\,g_{\varrho\bar\alpha} + \tfrac{1}{2}g_{\beta\bar\alpha}(P_{\varrho\bar\sigma} - P_{\bar\sigma\varrho}). \end{aligned}$$

Comparing terms of the same type in (A.10), we get the following three relations:

$$\begin{aligned} S_{\beta\varrho\bar\alpha\bar\sigma\mu} - S_{\beta\mu\bar\alpha\bar\sigma\varrho} = {} & -i\{V_{\beta\bar\alpha\varrho}\,g_{\mu\bar\sigma} - V_{\beta\bar\alpha\mu}\,g_{\varrho\bar\sigma} \\ & + (V_{\mu\bar\alpha\varrho} - V_{\varrho\bar\alpha\mu})\,g_{\beta\bar\sigma} + g_{\bar\alpha\mu}V_{\beta\bar\sigma\varrho} - g_{\bar\alpha\varrho}V_{\beta\bar\sigma\mu} + g_{\beta\bar\alpha}(V_{\mu\bar\sigma\varrho} - V_{\varrho\bar\sigma\mu})\}, \end{aligned} \tag{A. 11}$$

$$V_{\beta\bar\alpha\sigma\varrho} - V_{\beta\bar\alpha\varrho\sigma} = -i\{Q_{\varrho\beta}\,g_{\sigma\bar\alpha} - Q_{\sigma\beta}\,g_{\varrho\bar\alpha} + \tfrac{1}{2}g_{\beta\bar\alpha}(Q_{\varrho\sigma} - Q_{\sigma\varrho})\}, \tag{A. 12}$$

$$\overline{V_{\alpha\bar{\beta}\sigma\bar{\varrho}}} + V_{\beta\bar{\alpha}\varrho\bar{\sigma}} = i\{g_{\beta\bar{\sigma}}P_{\varrho\bar{\alpha}} - P_{\bar{\sigma}\beta}g_{\varrho\bar{\alpha}} + \tfrac{1}{2}g_{\beta\bar{\alpha}}(P_{\varrho\bar{\sigma}} - P_{\bar{\sigma}\varrho})\} + S_{\beta\varrho\bar{\alpha}\bar{\sigma}}. \tag{A. 13}$$

Multiplying (A.11) by $g^{\beta\bar{\alpha}}g^{\mu\bar{\sigma}}$, summing over α, β, μ, and σ, and using the relations $g^{\beta\bar{\alpha}}S_{\beta\varrho\bar{\alpha}\bar{\sigma}\mu} = V^{\varrho}_{\beta.\varrho} = 0$ gives

$$V^{\beta}_{\beta.\sigma} = 0.$$

so that contracting β and $\bar{\alpha}$ only in (A.11) gives

$$V_{\mu\bar{\sigma}\varrho} = V_{\varrho\bar{\sigma}\mu}. \tag{A.14}$$

It then follows that $g^{\beta\bar{\alpha}}V_{\beta\bar{\alpha}\sigma\varrho} = 0$, so that contracting the indices β and $\bar{\alpha}$ in (A.12) and in (A.13) gives

$$Q_{\varrho\sigma} = Q_{\sigma\varrho} \quad \text{and} \quad P_{\varrho\bar{\sigma}} = P_{\bar{\sigma}\varrho}. \tag{A.15}$$

Equations (A.5′) and (A.6′) give further relations but no further symmetries of the curvature functions $S_{\beta\varrho\bar{\alpha}\bar{\sigma}}$, $V_{\beta\bar{\alpha}\varrho}$, $P_{\varrho\bar{\sigma}}$, $Q_{\varrho\sigma}$ or R_{α}.

We can now write the curvature forms Φ^{α} and Ψ as follows:

$$\Phi^{\alpha} = V^{\alpha}_{\varrho\bar{\sigma}}\,\omega^{\varrho} \wedge \omega^{\bar{\sigma}} + P_{\varrho}{}^{\alpha}\,\omega^{\varrho} \wedge \omega + Q_{\bar{\sigma}}{}^{\alpha}\,\omega^{\bar{\sigma}} \wedge \omega, \tag{A. 8′}$$

$$\Psi = -2iP_{\varrho\bar{\sigma}}\,\omega^{\varrho} \wedge \omega^{\bar{\sigma}} + R_{\varrho}\,\omega^{\varrho} \wedge \omega + R_{\bar{\sigma}}\,\omega^{\bar{\sigma}} \wedge \omega. \tag{A. 9′}$$

Since $V^{\beta}_{\beta.\varrho} = 0$ we now have $\Phi_{\alpha}{}^{\alpha} = 0$, so that in the equation (5.35) $\Pi_0{}^0 = 0$ and $\Pi_{\alpha}{}^{\beta} = \Phi_{\alpha}{}^{\beta}$.

References

[1] Cartan, E., Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I. *Ann. Math. Pura Appl.*, (4) 11 (1932) 17–90 (or *Oeuvres* II, 2, 1231–1304); II, *Ann. Scuola Norm. Sup. Pisa*, (2) 1 (1932) 333–354 (or *Oeuvres* III, 2, 1217–1238).

[2]. Fefferman, C., The Bergman Kernel and Biholomorphic Mappings of Pseudoconvex Domains. *Invent. Math.*, 26 (1974), 1–65.

[3]. Hopf, H., Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. *Math. Ann.*, 104, 1931, 637–665, § 5.

[4]. Moser, J., Holomorphic equivalence and normal forms of hypersurfaces. To appear in *Proc. Symp. in Pure Math.*, Amer. Math. Soc.

[5]. Nirenberg, L., *Lectures on linear partial differential equations.* Regional Conf. Series in Math.. No. 17 Amer. Math. Soc. 1973.

[6]. Poincaré, H., Les fonctions analytiques de deux variables et la représentation conforme. *Rend. Circ. Mat. Palermo* (1907), 185–220.

[7]. Tanaka, N., I. On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables. *J. Math. Soc. Japan*, 14 (1962), 397–429; II. Graded Lie algebras and geometric structures, *Proc. US-Japan Seminar in Differential Geometry*, 1965, 147–150.

[8]. Wells, R. O., Function theory on differentiable submanifolds. *Contributions to Analysis*, Academic Press, 1974, 407–441.

Received May 15, 1974