# REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH RECURRENT SHAPE OPERATOR

SEONHUI KIM, HYUNJIN LEE AND HAE YOUNG YANG

ABSTRACT. We introduce the notion of recurrent hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and give a non-existence theorem for a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator.

## INTRODUCTION

The notion of recurrent tensor field of type $(r, s)$ on a differentiable manifold $M$ with a linear connection was well introduced in [7] and [15]. A non-zero tensor field $K$ of type $(r, s)$ on $M$ which is said to be *recurrent* if there exists a 1-form $\omega$ such that

$$\nabla K = K \otimes \omega\,.$$

Moreover, they gave some geometric interpretation of such a manifold $M$ with recurrent curvature tensor $K$ in terms of holonomy group, see also [7] and [15].

Now let us denote by $A$ the shape operator of real hypersurfaces in non flat complex space form $M_n(c)$. Recently, Hamada ([5] and [6]) applied such a notion of recurrent tensor to a shape operator or a Ricci tensor for real hypersurfaces $M$ in complex projective space $\mathbb{C}P^n$ in such a way that

$$\nabla A = \omega \otimes A$$

or

$$\nabla S = \omega \otimes S$$

for a certain 1-form $\omega$ defined on $M$, and proved the following:

**Theorem A.** *The complex projective space $\mathbb{C}P^n$ does not admit any real hypersurfaces with recurrent shape operator or recurrent Ricci tensor.*

On the other hand, Suh [9] have explained the geometrical meaning of recurrent shape operator $A$ as follows:

$$[\nabla_X A, A] = \omega(X)[A, A] = 0$$

for any tangent vector field $X$ defined on $M$. That is, *the eigenspaces of the shape operator $A$ of $M$ are parallel along any curve $\gamma$ in $M$.* Here, the eigenspaces of the shape operator $A$ are said to be *parallel* along $\gamma$ if they are invariant with respect to any parallel translation along $\gamma$.

---

[1] *2000 Mathematics Subject Classification* : Primary 53C40; Secondary 53C15.

[2] *Key words and phrases* : Real hypersurfaces, Complex two-plane Grassmannians, Hopf hypersurface, Recurrent shape operator, Recurrent hypersurfaces.

This work was supported by grant Proj. No. R17-2008-001-01001-0 from Korea Science & Engineering Foundation.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kaehler structure $J$ and a quaternionic Kaehler structure $\mathfrak{J}$ not containing $J$.

In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kaehler, quaternionic Kaehler manifold which is not a hyper-Kaehler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces $M$ that $[\,\xi\,] = \mathrm{Span}\{\,\xi\,\}$ or $\mathfrak{D}^\perp = \mathrm{Span}\{\,\xi_1,\ \xi_2,\ \xi_3\,\}$ are invariant under the shape operator $A$ of $M$. The almost contact structure vector field $\xi$ mentioned above is defined by $\xi = -JN$ is said to be a *Reeb* vector field, where $N$ denotes a local unit normal vector field of $M$ in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector field $\xi$ of $M$ in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, $\xi$ is said to be a *Hopf*. The *almost contact 3-structure* vector fields $\{\,\xi_1,\ \xi_2,\ \xi_3\,\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of $M$ in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$, $\nu = 1,\ 2,\ 3$, where $J_\nu$ denotes a canonical local basis of a quaternionic Kaehler structure $\mathfrak{J}$, such that $T_xM = \mathfrak{D}\oplus\mathfrak{D}^\perp$, $x \in M$.

When the Reeb vector field $\xi$ and the distribution $\mathfrak{D}^\perp$ is invariant by the shape operator $A$ of real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$, Berndt and Suh [2] have proved the following

**Theorem B.** *Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\,\xi\,]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of $M$ if and only if*

*(A) $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

*(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Now we introduce the notion of recurrent shape operator tensor defined in such a way that

$$(\nabla_X A)Y = \omega(X)AY \tag{0.1}$$

for a 1-form $\omega$ and any vector fields $X$ and $Y$ on $M$ in $G_2(\mathbb{C}^{m+2})$. When the shape operator $A$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ satisfies the formula (0.1), a hypersurface $M$ is said to be a *recurrent hypersurface* in $G_2(\mathbb{C}^{m+2})$.

Related to such a notion, Suh [14] has proved the non-existence for recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-invariant shape operator as follows:

**Theorem C.** *There do not exist any recurrent real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $\mathfrak{D}$ (resp. $\mathfrak{D}^\perp$)-invariant shape operator.*

On the other hand, the 1-dimensional foliation of $M$ by the integral manifolds of the Reeb vector field $\xi$ is said to be a *Hopf foliation* of $M$. We say that $M$ is a *Hopf hypersurfaces* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of $M$ is totally geodesic. By the formulas in section 2 it can be easily checked that $M$ is Hopf if and only if the Reeb vector field $\xi$ is Hopf. Such a notion of Hopf hypersurface in complex projective space $\mathbb{C}P^n$ is mainly discussed by Cecil and Ryan [4] and the invariancy of the distribution $\mathfrak{D}^\perp$ for hypersurface in quaternionic space forms was investigated in Berndt [1].

In this paper, we have considered the notion of Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ and give another non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator as follows:

**Main Theorem.** *There do not exist any Hopf recurrent hypersurfaces in complex two-plane Grassmannian,* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

## 1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2] and [3]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space $G/K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of $G$ and $K$, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $B$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since $B$ is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of $B$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $g$ such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight we will assume $m \geq 2$ from now on. Note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kaehler structure $J$ and the $\mathfrak{su}(2)$-part a quaternionic Kaehler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If $J_1$ is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and $JJ_1$ is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$.

A canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ consists of three local almost Hermitian structures $J_\nu$ in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $J_1, J_2, J_3$ of $\mathfrak{J}$ three local one-forms $q_1, q_2, q_3$ such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields $X$ on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z =& g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y \\ & - 2g(J_\nu X,Y)J_\nu Z\} + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX \\ & - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathfrak{J}$.

## 2. Some fundamental formulas in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulae and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [2], [3], [8], [10], [11], [12], [13] and [14]).

Let $M$ be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal field of $M$ and $A$ the shape operator of $M$ with respect to $N$. The Kaehler structure $J$ of $G_2(\mathbb{C}^{m+2})$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. Furthermore, let $J_1, J_2, J_3$ be a canonical local basis of $\mathfrak{J}$. Then each $J_\nu$ induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$. Using the above expression (1.2) for the curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$, the Codazzi equation becomes

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X =& \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ & + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\big\} \\ & + \sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\big\} \\ & + \sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu\,.\end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ &\phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.\end{aligned} \tag{2.1}$$

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector $X$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, where $N$ denotes a unit normal vector of $M$ in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have the following

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.2}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.3}$$

$$\begin{aligned}(\nabla_X\phi_\nu)Y = &- q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &- g(AX,Y)\xi_\nu.\end{aligned} \tag{2.4}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\ &\quad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{2.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.6}$$

## 3. A Key Lemma

Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator. Then it satisfied the condition that

$$(\nabla_X A)Y = \omega(X)AY$$

for a 1-form $\omega$ and any vector fields $X$, $Y$ defined on $M$. By the equation of Codazzi in section 2 we have that

$$\begin{aligned}(\nabla_\xi A)Y - (\nabla_Y A)\xi = \omega(\xi)AY - \omega(Y)A\xi \\ = \phi Y + \sum_{v=1}^{3}\{\, \eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi - 2g(\phi_\nu\xi,Y)\xi_\nu \,\} \\ + \sum_{\nu=1}^{3}\eta_\nu(\phi Y)\xi_\nu\,.\end{aligned} \tag{3.1}$$

Since we assumed that $M$ is Hopf, (3.1) gives

$$\omega(\xi)AY = \alpha\omega(Y)\xi + \phi Y + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi + 3\eta_\nu(\phi Y)\xi_\nu\,\}. \tag{3.2}$$

Now we assert the key Lemma as following :

**Lemma 3.1.** *Let $M$ be a recurrent hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector $\xi$ is principal, then $\xi$ belongs to either the distribution $\mathfrak{D}$ or to the distribution $\mathfrak{D}^\perp$.*

*Proof.* To prove this Lemma we put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit vector $X_0 \in \mathfrak{D}$. Here we notice that $\eta(X_0)$ and $\eta(\xi_1)$ are not zero. In (3.2) by putting $Y = \xi_1$ we have

$$\begin{aligned} \omega(\xi)A\xi_1 &= \alpha\omega(\xi_1)\xi + \phi_1\xi + \sum_{\nu=1}^{3}\{\, 3\eta_\nu(\phi_1\xi)\xi_\nu - \eta_\nu(\xi_1)\phi_\nu\xi \,\} \\ &= \alpha\omega(\xi_1)\xi + \phi_1\xi - 3\eta_3(\xi)\xi_2 + 3\eta_2(\xi)\xi_3 - \phi_1\xi \\ &= \alpha\omega(\xi_1)\xi\,. \end{aligned} \tag{3.3}$$

We get also the following equations by putting $Y = \xi_2$ and $Y = \xi_3$ in (3.2), similarly.

$$\begin{aligned} \omega(\xi)A\xi_2 &= \alpha\omega(\xi_2)\xi - 2\eta_1(\xi)\xi_3\,, \\ \omega(\xi)A\xi_3 &= \alpha\omega(\xi_3)\xi + 2\eta_1(\xi)\xi_2\,. \end{aligned} \tag{3.4}$$

From these equations, taking an inner product with $\xi$, it follows that

$$\begin{aligned} \alpha\{\omega(\xi_1) - \omega(\xi)\eta_1(\xi)\} &= 0\,, \\ \alpha\omega(\xi_2) &= 0\,, \\ \alpha\omega(\xi_3) &= 0\,. \end{aligned} \tag{3.5}$$

Thus we can consider two cases that the first is $\alpha = 0$ and the second is not. For the first case $\alpha = 0$, by the lemma due to Pérez and Suh [8] we know that $\xi$ belongs to either the distribution $\mathfrak{D}$ or to the distribution $\mathfrak{D}^\perp$.

Now let us consider the remaining case, $\alpha \neq 0$. From (3.5), we have

$$\begin{aligned} \omega(\xi_1) &= \omega(\xi)\eta_1(\xi)\,, \\ \omega(\xi_2) = 0\,, &\quad \omega(\xi_3) = 0\,. \end{aligned} \tag{3.6}$$

Substituting these equations into (3.3) and (3.4) gives

$$\begin{aligned} \omega(\xi)A\xi_1 &= \alpha\omega(\xi)\eta_1(\xi)\xi, \\ \omega(\xi)A\xi_2 &= -2\eta_1(\xi)\xi_3, \\ \omega(\xi)A\xi_3 &= 2\eta_1(\xi)\xi_2. \end{aligned} \tag{3.7}$$

From this, we consider the following two subcases:

**Subcase II-1 .** $\omega(\xi) = 0$.

Then (3.7) gives $\eta_1(\xi) = 0$. This implies $\xi \in \mathfrak{D}$.

**Subcase II-2 .** $\omega(\xi) \neq 0$.

Then (3.7) give the following

$$\begin{aligned} A\xi_1 &= \alpha\eta_1(\xi)\xi\,, \\ A\xi_2 &= -\frac{2\eta_1(\xi)}{\omega(\xi)}\xi_3\,, \\ A\xi_3 &= \frac{2\eta_1(\xi)}{\omega(\xi)}\xi_2\,. \end{aligned} \tag{3.8}$$

From this, taking an inner product with $\xi_3$ to the second formula of (3.8), it follows that

$$g(A\xi_2,\xi_3)=g(-\frac{2\eta_1(\xi)}{\omega(\xi)}\xi_3,\xi_3)=-\frac{2\eta_1(\xi)}{\omega(\xi)}. \tag{3.9}$$

On the other hand, from the third formula of (3.8) we have

$$g(A\xi_2,\xi_3)=g(A\xi_3,\xi_2)=g(\frac{2\eta_1(\xi)}{\omega(\xi)}\xi_2,\xi_2)=\frac{2\eta_1(\xi)}{\omega(\xi)}.$$

From this, together with (3.9), it follows $\frac{\eta_1(\xi)}{\omega(\xi)}=0$. That is, $\eta_1(\xi)=0$, which gives $\xi\in\mathfrak{D}$. This complete the proof of our Lemma 3.1. □

## 4. Recurrent hypersurfaces for $\xi\in\mathfrak{D}^\perp$

In this section by Lemma 3.1 we consider the case that $\xi\in\mathfrak{D}^\perp$. That is, we consider a Hopf hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ with recurrent shape operator and $\xi\in\mathfrak{D}^\perp$. Accordingly, we may put $\xi=\xi_1$. Then (3.2) implies the following

**Lemma 4.1.** *Let $M$ be a Hopf recurrent hypersurface in $G_2(\mathbb{C}^{m+2})$. If $\xi\in\mathfrak{D}^\perp$, then $g(A\mathfrak{D},\mathfrak{D}^\perp)=0$.*

*Proof.* Since we have assumed $\xi\in\mathfrak{D}^\perp$, we may put $\xi=\xi_1$. Then from (3.3) and (3.4) we know that

$$\begin{aligned}\omega(\xi_1)A\xi_1&=\alpha\omega(\xi_1)\xi_1,\\ \omega(\xi_1)A\xi_2&=\alpha\omega(\xi_2)\xi_1-2\xi_3,\\ \omega(\xi_1)A\xi_3&=\alpha\omega(\xi_3)\xi_1+2\xi_2.\end{aligned} \tag{4.1}$$

From this, if we take an inner product with $X\in\mathfrak{D}$, then we have

$$\omega(\xi_1)g(A\xi_\nu,X)=0,\quad \nu=1,2,3. \tag{4.2}$$

So, for the case where $\omega(\xi_1)\neq 0$ in (4.2) we have our assertion. Now let us consider the case that $\omega(\xi_1)=0$. Then (4.1) gives the following

$$\alpha\omega(\xi_2)\xi_1=2\xi_3\quad\text{and}\quad\alpha\omega(\xi_3)\xi_1=-2\xi_2, \tag{4.3}$$

which makes a contradiction. So, we complete the proof of Lemma 3.1. □

Now in order to complete the proof of our main theorem we recall a proposition due to Berndt and Suh [2] as follows:

**Proposition A.** *Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D}\subset\mathfrak{D}$, $A\xi=\alpha\xi$, and $\xi$ is tangent to $\mathfrak{D}^\perp$. Let $J_1\in\mathfrak{J}$ be the almost Hermitian structure such that $JN=J_1N$. Then $M$ has three (if $r=\pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha=\sqrt{8}\cot(\sqrt{8}r)\,,\ \beta=\sqrt{2}\cot(\sqrt{2}r)\,,\ \gamma=-\sqrt{2}\tan(\sqrt{2}r)\,,\ \mu=0$$

*with some $r\in(0,\pi/\sqrt{8})$. The corresponding multiplicities are*

$$m(\alpha)=1\ ,\ m(\beta)=2\ ,\ m(\gamma)=2m-2=m(\mu),$$

*and the corresponding eigenspaces are*

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\
T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\
T_\gamma &= \{X | X \perp \mathbb{H}\xi, JX = J_1X\}, \\
T_\mu &= \{X | X \perp \mathbb{H}\xi,\ JX = -J_1X\}
\end{aligned}$$

*where* $\mathbb{R}\xi$, $\mathbb{C}\xi$ *and* $\mathbb{H}\xi$ *respectively denotes real, complex and quaternionic span of the structure vector* $\xi$ *and* $\mathbb{C}^\perp\xi$ *denotes the orthogonal complement of* $\mathbb{C}\xi$ *in* $\mathbb{H}\xi$.

Without loss of generality we may put $\xi = \xi_1$. Now let us put $Y = \xi_2$ in $T_\beta$ in (3.2). Then by using (3.5) we have

$$\begin{aligned}
\omega(\xi_1)A\xi_2 &= \alpha\omega(\xi_2)\xi_1 + \phi\xi_2 + \phi_1\xi_2 + 3\sum_{\nu=1}^{3}\eta_\nu(\phi Y)\xi_\nu - \phi_2\xi \\
&= \alpha\omega(\xi_2)\xi_1 - \xi_3 + \xi_3 - 3\xi_3 + \xi_3 \\
&= \sqrt{8}\cot(\sqrt{8}r)\omega(\xi_2)\xi_1 - 2\xi_3.
\end{aligned}$$

On the other hand, by Proposition $A$ we know that

$$A\xi_2 = \beta\xi_2 = \sqrt{2}\cot(\sqrt{2}r)\xi_2.$$

Then summing up these two formulas, we have

$$\sqrt{2}\cot(\sqrt{2}r)\omega(\xi_1)\xi_2 = \sqrt{8}\cot(\sqrt{8}r)\omega(\xi_2)\xi_1 - 2\xi_3\,. \tag{4.4}$$

If we take the scalar product of (4.4) and $\xi_3$ then we derive a contradiction. So we assert the following:

**Theorem 4.2.** *There do not exist any Hopf recurrent hypersurfaces in* $G_2(\mathbb{C}^{m+2})$ *satisfying* $\xi \in \mathfrak{D}^\perp$.

## 5. Recurrent hypersurfaces for $\xi \in \mathfrak{D}$

Now by Lemma 3.1 we consider the case that the Reeb vector belongs to $\mathfrak{D}$. In this section, we give a complete classification of Hopf recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\xi\in\mathfrak{D}$. Thus we assert the following

**Lemma 5.1.** *Let* $M$ *be a Hopf recurrent hypersurface in* $G_2(\mathbb{C}^{m+2})$ *. If the Reeb vector* $\xi$ *belongs to the distribution* $\mathfrak{D}$*, then* $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

*Proof.* By using $\xi \in \mathfrak{D}$ in (3.3) and (3.4) we have the following for $\nu = 1, 2, 3$,

$$\omega(\xi)A\xi_\nu = \alpha\omega(\xi_\nu)\xi, \quad \nu = 1, 2, 3.$$

From this, by taking an inner product with $\xi$, it follows that

$$0 = \alpha\omega(\xi)\eta_\nu(\xi) = \alpha\omega(\xi_\nu).$$

That is, $\omega(\xi)A\xi_\nu = 0$. Thus we consider the following two cases:

**Case I.** $A\xi_\nu = 0$.

Then naturally we have $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

**Case II.** $\omega(\xi) = 0$.

Let us take an inner product of the equation of Codazzi with $\xi$ and using the differentiation of $A\xi = \alpha\xi$. Then we get

$$\begin{aligned} & -2g(\phi X, Y) + 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ & \qquad = g((\nabla_X A)Y, \xi) - g((\nabla_Y A)X, \xi) \\ & \qquad = g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ & \qquad = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y)\,. \end{aligned} \tag{5.1}$$

From this, if we put $X = \xi$, then

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)\,. \tag{5.2}$$

In this case the recurrence of Hopf hypersurfaces and $\omega(\xi) = 0$ give

$$(\nabla_\xi A)Y = \omega(\xi)AY = 0$$

for any tangent vector field $Y$ on $M$. This gives

$$\nabla_\xi(AY) = A(\nabla_\xi Y).$$

Then by putting $Y = \xi$ and using $M$ is Hopf, we have

$$0 = \nabla_\xi(A\xi) = \nabla_\xi(\alpha\xi) = (\xi\alpha)\xi.$$

So, we see that $\xi\alpha = 0$. From this and using (5.2) and $\xi \in \mathfrak{D}$, it follows that

$$Y\alpha = 0 \tag{5.3}$$

for any tangent vector field $Y$ on $M$. This means that the function $\alpha$ is constant on $M$.

On the other hand, by differentiating $A\xi = \alpha\xi$ and using (5.3) we have the following

$$\alpha\omega(X)\xi + A\phi AX = \alpha\phi AX.$$

So, it follows that for any tangent vector field $X$ on $M$

$$\alpha\omega(X)\xi = \alpha\phi AX - A\phi AX. \tag{5.4}$$

Now we consider a subdistribution $\mathfrak{D}_1$ of the distribution $\mathfrak{D}$ defined in such a way that

$$\mathfrak{D}_1 = \{X \in \mathfrak{D} \mid X \perp \xi,\ X \perp \phi_i\xi,\ i = 1, 2, 3\}.$$

Then from (3.2) and using $\xi \in \mathfrak{D}$ in Case II we have

$$0 = \alpha\omega(X)\xi + \phi X \tag{5.5}$$

for any $X \in \mathfrak{D}_1$. Then (5.4) and (5.5) give the following

$$\alpha\phi AX - A\phi AX + \phi X = 0 \tag{5.6}$$

for any $X \in \mathfrak{D}_1$.

On the other hand, (5.1) and (5.3) give the following

$$-2\phi X = \alpha(A\phi + \phi A)X - 2A\phi AX$$

for any $X \in \mathfrak{D}_1$ where we have used the fact that $\xi \in \mathfrak{D}$. From this and together with (5.6) we get

$$2\alpha\phi AX = \alpha(A\phi + \phi A)X,$$

which gives

$$\alpha\phi AX = \alpha A\phi X$$

for any $X \in \mathfrak{D}_1$. Then we have the following two subcases.

**Subcase II-1 .** $\alpha = 0$.

From (5.4) and (5.6) we have $\phi X = 0$ for any $X \in \mathfrak{D}_1$. Then by applying $\phi$ we have $X = 0$ for any $X \in \mathfrak{D}_1$. But this case can not appear.

**Subcase II-2 .** $\alpha \neq 0$.

By putting $Y = \xi$ in the Codazzi equation in section 2, we have

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - 2g(\phi_\nu X, \xi)\xi_\nu - \eta_\nu(\phi X)\xi_\nu\}.$$

From this, together with the recurrence and $\omega(\xi) = 0$, it follows that

$$\alpha\omega(X)\xi = -\phi X \tag{5.7}$$

for any $X \in \mathfrak{D}_1$. Taking an inner product with $\xi$ we have $\alpha\omega(X) = 0$ for any $X \in \mathfrak{D}_1$. This gives $\omega(X) = 0$ for any $X \in \mathfrak{D}_1$. Then (5.7) gives $\phi X = 0$ for any $X \in \mathfrak{D}_1$, which also makes a contradiction. So, Case II can not appear. □

Then by virtue of Theorem $A$ in the introduction, a Hopf recurrent hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\xi \in \mathfrak{D}$ is congruent to of type $B$, that is, a tube over a totally real quaternionic projective space $\mathbb{H}P^n$, $m = 2n$. Now for this type of hypersurface we introduce the following (See [2])

**Proposition B.** *Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and $\xi$ is tangent to $\mathfrak{D}$. Then the quaternionic dimension $m$ of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and $M$ has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\,, \quad \beta = 2\cot(2r)\,, \quad \gamma = 0\,, \quad \lambda = \cot(r)\,, \quad \mu = -\tan(r)$$

*with some $r \in (0, \pi/4)$. The corresponding multiplicities are*

$$m(\alpha) = 1\,, \quad m(\beta) = 3 = m(\gamma)\,, \quad m(\lambda) = 4n - 4 = m(\mu)$$

*and the corresponding eigenspaces are*

$$T_\alpha = \mathbb{R}\xi\,, \quad T_\beta = \mathfrak{J}J\xi\,, \quad T_\gamma = \mathfrak{J}\xi\,, \quad T_\lambda\,, \quad T_\mu\,,$$

*where*

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp\,, \quad \mathfrak{J}T_\lambda = T_\lambda\,, \quad \mathfrak{J}T_\mu = T_\mu\,, \quad JT_\lambda = T_\mu\,.$$

Then by putting $Y = \phi_1\xi$ in $T_\gamma$ in Proposition $B$, we have

$$\begin{aligned} 0 = \gamma\omega(\xi)\phi_1\xi = \omega(\xi)A\phi_1\xi &= \alpha\omega(\phi_1\xi)\xi + \phi^2\xi_1 + 3\sum_{\nu=1}^{3}\eta_\nu(\phi^2\xi_1)\xi_\nu \\ &= \alpha\omega(\phi_1\xi)\xi - \xi_1 - 3\xi_1 \\ &= \alpha\omega(\phi_1\xi)\xi - 4\xi_1, \end{aligned}$$

which gives a contradiction for $\xi\in\mathfrak{D}$. So we assert the following

**Theorem 5.2.** *There do not exist any Hopf recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\xi \in \mathfrak{D}$.*

Then summing up Theorems 4.2 and 5.2 we complete the proof of our Main Theorem in the introduction.

**Acknowledgmet** The authors would like to express their hearty thanks to Professor Young Jin Suh for his valuable suggestions and continuous encouragement during the preparation of this work.

## References

[1] J. Berndt, *Real hypersurfaces in quaternionic space forms*, J. Reine Angew. Math., **419** (1991), 9–26.
[2] J. Berndt and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians*, Monatshefte für Math., **127** (1999), 1-14.
[3] J. Berndt and Y.J. Suh, *Isometric flows on real hypersurfaces in complex two-plane Grassmannians*, Monatshefte für Math., **137** (2002), 87-98.
[4] T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc., **269** (1982), 481-499.
[5] T. Hamada, *On real hypersurfaces of a complex projective space with recurrent second fundamental form*, Ramanujan Math. Soc., **11** (1996), 103–1079.
[6] T. Hamada, *Real hypersurfaces of a complex projective space with recurrent Ricci tensor*, Glasgow Math. J., **41** (1999), 297–302.
[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I*, Wiley and Sons Inc. (New York-London, 1963).
[8] J.D. Pérez and Y.J. Suh, *The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians*, J. Korean Math. Soc., **44** (2007), 211–235.
[9] Y.J. Suh,*Real hypersurfaces in complex space forms with $\eta$-recurrent second fundamental tensors*, Math. J. Toyama, **19** (1996), 127–141.
[10] Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator*, Bull. of Austral. Math. Soc., **68** (2003), 493–502.
[11] Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with commuting shape operator*, Bull. of Austral. Math. Soc., **69** (2003), 379–393.
[12] Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with vanishing Lie derivatives*, Canadian Math. Bull., **49** (2006), 134–143.
[13] Y.J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians*, Monatshefte für Math., **141** (2006), 337–355.
[14] Y.J. Suh,*Recurrent real hypersurfaces in complex two-plane Grassmannians*, Acta Math. Hunger., **112** (2006), 89–102.
[15] Y.C. Wong, *Recurrent tensors on a linearly connected differentiable manifold*, Trans. Amer. Math. Soc., **99** (1961), 325–341.

Seonhui Kim, Hyunjin Lee and Hae Young Yang
Department of Mathematics,
Kyungpook National University,
Taegu, 702-701, KOREA
*E-mail address*: kimsh0123@hanmail.net, lhjibis@hanmail.net and yang9973@hanmail.net