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KODAI MATH. J.
34 (2011), 352–366

REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH GENERALIZED TANAKA-WEBSTER PARALLEL SHAPE OPERATOR

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Abstract

We introduce the notion of generalized Tanaka-Webster connection for hypersurfaces in complex two-plane Grassmannians $G_2(\mathbf{C}^{m+2})$ and give a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbf{C}^{m+2})$ with parallel shape operator in this connection.

Introduction

The *generalized Tanaka-Webster connection* (in short, the *g-Tanaka-Webster connection*) for contact metric manifolds has been introduced by Tanno [14] as a generalization of the well-known connection defined by Tanaka in [13] and, independently, by Webster in [15]. This connection coincides with Tanaka-Webster connection if the associated CR-structure is integrable. Tanaka-Webster connection is defined as the canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. For a real hypersurface in a Kähler manifold with almost contact metric structure (ϕ, ξ, η, g), Cho defined the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for a non-zero real number k (see [5], [6] and [7]). In particular, if a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see Proposition 7 in [7]).

Using the notion of the g-Tanaka-Webster connection, many geometers have studied some characterizations of real hypersurfaces in complex space form $\tilde{M}_n(c)$ with constant holomorphic sectional curvature c. For instance, when $c > 0$, that is, $\tilde{M}_n(c)$ is a complex projective space $\mathbf{C}P^n$, Cho [5] proved that if the

2000 *Mathematics Subject Classification*: Primary 53C40; Secondary 53C15.

Key words and phrases: Real hypersurfaces, Complex two-plane Grassmannians, Hopf hypersurface, Generalized Tanaka-Webster connection, Generalized Tanaka-Webster parallel shape operator.

The first author was supported by grant Proj. No. BSRP-2011-0013381 and the second and the third authors supported by grant Proj. No. NRF-2011-220-C00002 from National Research Foundation of Korea.

Received October 5, 2010.

shape operator A of M in $\mathbf{C}P^n$ is $\hat{\nabla}^{(k)}$-*parallel* (it means that the shape operator A satisfies $\hat{\nabla}^{(k)}A = 0$), then ξ is a principal curvature vector field and M is locally congruent to a real hypersurface of Type (A) and Type (B). (In fact, he also gave the classification of real hypersurfaces in a complex hyperbolic space $(c < 0)$ and complex Euclidean space $(c = 0)$ under the assumption $\hat{\nabla}^{(k)}$-parallel shape operator [5]). Moreover in [6] he gave the classification theorem of Levi-parallel Hopf hypersurface in $\tilde{M}_n(c)$, $c \neq 0$. Here, a real hypersurface of $\tilde{M}_n(c)$ is called *Levi-parallel* if its Levi form is parallel with respect to the g-Tanaka-Webster connection. In [8], Kon gave a characterization for real hypersurfaces of Type (A_1) in complex projective space $\mathbf{C}P^n$ under the assumption that the Ricci tensor related to the g-Tanaka-Webster connection identically vanishes.

Now let us denote by $G_2(\mathbf{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbf{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbf{C}^{m+2})$ has a remarkable geometric structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbf{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold. So, in $G_2(\mathbf{C}^{m+2})$ we have the two natural geometric conditions for real hypersurfaces M that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M (see section 2).

Here the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbf{C}^{m+2})$. The *almost contact* 3-*structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbf{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

By using two invariant conditions mentioned above and the result in Alekseevskii [1], Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbf{H}P^n$ in $G_2(\mathbf{C}^{m+2})$.*

Furthermore, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbf{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in Section 2 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

Moreover, we say that the Reeb vector field ξ on M is Killing, when the Reeb flow on M in $G_2(\mathbf{C}^{m+2})$ is *isometric*. In [4], Berndt and Suh gave some equivalent conditions of this property as follows:

THEOREM B. *Let M be a connected orientable real hypersurface in a Kähler manifold $\tilde{M}$. The following statements are equivalent*:

(1) *The Reeb flow on M is isometric*,

(2) *The shape operator A and the structure tensor field ϕ commute with each other*,

(3) *The Reeb vector field ξ is a principal curvature vector of M everywhere and the principal curvature spaces contained in the maximal complex subbundle $\mathscr{D}$ of TM are complex subspaces.*

Also in [4], a characterization of real hypersurfaces of Type (A) in Theorem A was given in terms of the Reeb flow on M as follows:

THEOREM C. *Let M be a connected orientable real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$.*

Recently, Lee and Suh [9] gave a new characterization of real hypersurfaces of Type (B) in $G_2(\mathbf{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

THEOREM D. *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbf{H}P^n$ in $G_2(\mathbf{C}^{m+2})$, where $m = 2n$.*

In particular, if the shape operator A of M in $G_2(\mathbf{C}^{m+2})$ satisfies $(\nabla_X A)Y = 0$ for any vector fields X, Y on M, we say that the shape operator A is *parallel with respect to the Levi-Civita connection*. Using this notion, Suh [11] proved the non-existence theorem of real hypersurfaces in $G_2(\mathbf{C}^{m+2})$ with parallel shape operator. Moreover, in [12], he also considered a generalized condition weaker than $\nabla A = 0$, which is said to be $\mathfrak{F}$-parallel, and proved that there does not exist any real hypersurface with $\mathfrak{F}$-parallel shape operator. Here, a shape operator A of M in $G_2(\mathbf{C}^{m+2})$ is said to be $\mathfrak{F}$-*parallel* if the shape operator A satisfies $(\nabla_X A)Y = 0$ for any tangent vector fields $X \in \mathfrak{F}$ and $Y \in T_xM$, where the subdistribution $\mathfrak{F}$ is defined by $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$ (see [12]).

Now in this paper we consider a new parallel shape operator for real hypersurface M in $G_2(\mathbf{C}^{m+2})$. Here the shape operator A is called *generalized Tanaka-Webster parallel* (in short, *g-Tanaka-Webster parallel*) if the shape operator A is parallel with respect to the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$, that is, $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any vector fields X, Y on M. If we consider such a notion in complex two-plane Grassmannians $G_2(\mathbf{C}^{m+2})$, its situation is quite different from the case in complex space forms $\tilde{M}_n(c)$.

From such a point of view, in this paper we give a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbf{C}^{m+2})$ with parallel shape operator in the generalized Tanaka-Webster connection as follows:

Main Theorem. *There does not exist any Hopf hypersurface,* $\alpha \neq 2k$, *in complex two-plane Grassmannians* $G_2(\mathbf{C}^{m+2})$, $m \geq 3$, *with parallel shape operator in the generalized Tanaka-Webster connection.*

1. Riemannian geometry of $G_2(\mathbf{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbf{C}^{m+2})$, for details we refer to [2], [3] and [4]. By $G_2(\mathbf{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbf{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbf{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbf{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbf{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_o G_2(\mathbf{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbf{C}^{m+2})$. In this way $G_2(\mathbf{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbf{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbf{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbf{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbf{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbf{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbf{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbf{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbf{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\tilde{\nabla}$ of $(G_2(\mathbf{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\tilde{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbf{C}^{m+2})$.

The Riemannian curvature tensor $\tilde{R}$ of $G_2(\mathbf{C}^{m+2})$ is locally given by

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbf{C}^{m+2})$

In this section we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbf{C}^{m+2})$ (see [3], [4], [9], [10], [11] and [12]).

Let M be a real hypersurface of $G_2(\mathbf{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbf{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{2.1}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbf{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbf{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbf{C}^{m+2})$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbf{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in section 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}& \phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0, \\ & \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ & \phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ & \phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}\end{aligned} \tag{2.2}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$ in section 1 and (2.1), the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{aligned} \phi\phi_\nu X &= \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\ \eta_\nu(\phi X) &= \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi. \end{aligned} \tag{2.3}$$

On the other hand, from the Kähler structure J, that is, $\tilde{\nabla} J = 0$ and the quaternionic Kähler structure J_ν, together with Gauss and Weingarten equations it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.4}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.5}$$

$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \tag{2.6}$$

Summing up these formulas, we find the following:

$$\begin{aligned} \nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\ &\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX. \end{aligned} \tag{2.7}$$

Using the above expression (1.2) for the curvature tensor $\tilde{R}$ of $G_2(\mathbf{C}^{m+2})$, the equation of Codazzi becomes:

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ &\quad + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu. \end{aligned} \tag{2.8}$$

3. The g-Tanaka-Webster connection for real hypersurfaces

In this section, we introduce the notion of generalized Tanaka-Webster connection (see [5], [6], [7] and [8]).

As stated above, the Tanaka-Webster connection is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold (see [13], [15]). In [14], Tanno defined the g-Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR-structure is integrable.

From now on, we introduce the g-Tanaka-Webster connection due to Tanno [14] for real hypersurfaces in Kähler manifolds by natural extending of the canonical affine connection on a non-degenerate, pseudo-Hermitian CR manifold.

Now let us recall the g-Tanaka-Webster connection $\hat{\nabla}$ define by Tanno [14] for contact metric manifolds as follows:

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y$$

for all vector fields X and Y (see [14]).

By taking (2.4) into account, the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for real hypersurfaces of Kähler manifolds is defined by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{3.1}$$

for a non-zero real number k (see [5], [6] and [7]) (Note that $\hat{\nabla}^{(k)}$ is invariant under the choice of the orientation. Namely, we may take $-k$ instead of k in (3.1) for the opposite orientation $-N$).

Let us put

$$F_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y. \tag{3.2}$$

Then the torsion tensor $\hat{T}^{(k)}$ is given by $\hat{T}^{(k)}(X, Y) = F_X Y - F_Y X$. Also, by using (2.4) and (3.1) we can see that

$$\hat{\nabla}^{(k)}\eta = 0, \quad \hat{\nabla}^{(k)}\xi = 0, \quad \hat{\nabla}^{(k)} g = 0, \quad \hat{\nabla}^{(k)}\phi = 0. \tag{3.3}$$

Next the g-Tanaka-Webster curvature tensor $\hat{R}^{(k)}$ with respect to $\hat{\nabla}^{(k)}$ can be defined by

$$\hat{R}^{(k)}(X, Y)Z = \hat{\nabla}^{(k)}_X(\hat{\nabla}^{(k)}_Y Z) - \hat{\nabla}^{(k)}_Y(\hat{\nabla}^{(k)}_X Z) - \hat{\nabla}^{(k)}_{[X, Y]} Z \tag{3.4}$$

for all vector fields X, Y, Z on M. Then we have the following identities

$$\hat{R}^{(k)}(X, Y)Z = -\hat{R}^{(k)}(Y, X)Z,$$

$$g(\hat{R}^{(k)}(X, Y)Z, W) = -g(\hat{R}^{(k)}(X, Y)W, Z).$$

Here we remark that because the g-Tanaka-Webster connection is not torsion-free, the Jacobi-type and Bianchi-type identities do not hold in general. Moreover, the g-Tanaka-Webster Ricci tensor $\hat{S}$ is defined by

$$\hat{S}(Y, Z) = \text{trace of } \{X \mapsto \hat{R}(X, Y)Z\}. \tag{3.5}$$

4. Key Lemmas

Let M be a Hopf hypersurface in $G_2(\mathbf{C}^{m+2})$ with g-Tanaka-Webster parallel shape operator. First of all, we find the fundamental equation for the condition that the shape operator A is parallel with respect to $\hat{\nabla}^{(k)}$, that is, $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any tangent vector fields X and Y.

From (3.1), we have

$$
\begin{aligned}
(4.1)\qquad (\hat{\nabla}_X^{(k)} A)Y &= \hat{\nabla}_X^{(k)}(AY) - A(\hat{\nabla}_X^{(k)} Y) \\
&= \nabla_X(AY) + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY \\
&\quad - A(\nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y) \\
&= (\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY \\
&\quad - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y.
\end{aligned}
$$

Under our conditions, $(\hat{\nabla}_X^{(k)} A)Y = 0$ and $A\xi = \alpha\xi$, it follows that

$$
\begin{aligned}
(4.2)\qquad & (\nabla_X A)Y + g(\phi AX, AY)\xi - \alpha\eta(Y)\phi AX - k\eta(X)\phi AY \\
&\quad - \alpha g(\phi AX, Y)\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y = 0
\end{aligned}
$$

for any tangent vector fields X and Y on M.

From the equation (4.2), we can assert following:

Lemma 4.1. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbf{C}^{m+2})$, $m \geq 3$. If M has the generalized Tanaka-Webster parallel shape operator, then the smooth function $\alpha = g(A\xi, \xi)$ is constant.*

Proof. Substituting ξ for any tangent vector field Y in (4.2) and using the notion of Hopf, that is, $A\xi = \alpha\xi$, we have

$$
(4.3)\qquad (\nabla_X A)\xi - \alpha\phi AX + A\phi AX = 0
$$

for any vector field X tangent to M.

On the other hand, taking the covariant derivative for $A\xi = \alpha\xi$ along any direction X, we get

$$
(4.4)\qquad (\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX.
$$

From (4.4), the equation (4.3) can be written by

$$
(X\alpha)\xi + \alpha\phi AX - A\phi AX - \alpha\phi AX + A\phi AX = 0,
$$

that is, we obtain for any vector field X tangent to M

$$
(4.5)\qquad (X\alpha)\xi = 0.
$$

This implies that $X\alpha = 0$ for any tangent vector field X on M. Therefore we have our assertion. $\square$

Under the assumption of $A\xi = \alpha\xi$, the Codazzi equation (2.8) becomes

$$
(\nabla_\xi A)Y - (\nabla_Y A)\xi = \phi Y + \sum_{\nu=1}^{3} \{\eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi - 3g(\phi_\nu\xi, Y)\xi_\nu\}
$$

for any tangent vector field Y on M.

From this, taking an inner product with ξ, it gives that

$$g((\nabla_\xi A)Y, \xi) - g((\nabla_Y A)\xi, \xi) = 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y).$$

On the other hand, by using (4.4), we obtain

$$\begin{aligned} g((\nabla_\xi A)Y, \xi) - g((\nabla_Y A)\xi, \xi) &= g(Y, (\nabla_\xi A)\xi) - g(\xi, (\nabla_Y A)\xi) \\ &= g(Y, (\xi\alpha)\xi) - g(\xi, (Y\alpha)\xi + \alpha\phi AY - A\phi AY) \\ &= (\xi\alpha)\eta(Y) - (Y\alpha), \end{aligned}$$

when we have used two formulas that $(\nabla_\xi A)\xi = (\xi\alpha)\xi$ and $(\nabla_Y A)\xi = (Y\alpha)\xi + \alpha\phi AY - A\phi AY$.

Consequently, we have the following

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y) \tag{4.6}$$

for any tangent vector field Y on M (see [4]).

Now we give one of Key Lemmas as follows:

Lemma 4.2. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbf{C}^{m+2})$, $m \geq 3$. If M has the parallel shape operator with respect to the generalized Tanaka-Webster connection, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. In order to prove our lemma, let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$. Since we knew that α is constant in Lemma 4.1, we have

$$\sum_{\nu=1}^{3} \eta_\nu(\xi)\phi\xi_\nu = 0, \tag{4.7}$$

when we have used the formula (4.6).

Since $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$, then the equation (4.7) can be written as

$$\sum_{\nu=1}^{3} \eta(\xi_1)\eta_\nu(\xi_1)\phi\xi_\nu = 0,$$

which gives $\eta(\xi_1)\phi\xi_1 = 0$.

On the other hand, from the fact $\phi\xi_\nu = \phi_\nu\xi$, it follows $\eta(\xi_1)\phi\xi_1 = \eta(\xi_1)\eta(X_0)\phi_1 X_0$. Thus we have

$$\eta(\xi_1)\eta(X_0)\phi_1 X_0 = 0.$$

Since $\eta(X_0)\eta(\xi_1) \neq 0$, we have $\phi_1 X_0 = 0$. But this gives a contradiction. Because $g(\phi_1 X_0, \phi_1 X_0) = g(X_0, X_0)$ and X_0 is a unit, $\phi_1 X_0$ becomes a non zero vector. So we complete the proof of our Lemma. $\square$

Before giving the proof of our Main Theorem given in the introduction, let us check whether the shape operator A of real hypersurfaces of Type (A) or of Type (B) in Theorem A is parallel with respect to the g-Tanaka-Webster connection.

First let us check for the case that M is locally congruent to a real hypersurface of Type (A), an open part of a tube around a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$. We recall a proposition due to Berndt and Suh [3] as follows:

Proposition E. *Let M be a connected real hypersurface of $G_2(\mathbf{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned}
T_\alpha &= \mathbf{R}\xi = \mathbf{R}JN = \mathbf{R}\xi_1 = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\},\\
T_\beta &= \mathbf{C}^\perp\xi = \mathbf{C}^\perp N = \mathbf{R}\xi_2 \oplus \mathbf{R}\xi_3 = \mathrm{Span}\{\xi_2, \xi_3\},\\
T_\lambda &= \{X \mid X \perp \mathbf{H}\xi, JX = J_1 X\},\\
T_\mu &= \{X \mid X \perp \mathbf{H}\xi, JX = -J_1 X\}
\end{aligned}$$

where $\mathbf{R}\xi$, $\mathbf{C}\xi$ and $\mathbf{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector field ξ and $\mathbf{C}^\perp\xi$ denotes the orthogonal complement of $\mathbf{C}\xi$ in $\mathbf{H}\xi$.

Now let us suppose that a real hypersurface of Type (A) has the parallel shape operator with respect to the g-Tanaka-Webster. Then we see that $(\hat{\nabla}^{(k)}_X A)\xi_2 = 0$ for a unit eingenvector $X \in T_\lambda$. Then it follows that

$$\begin{aligned}
(4.8) \qquad (\hat{\nabla}^{(k)}_X A)\xi_2 &= \nabla_X(A\xi_2) + g(\phi AX, A\xi_2)\xi - \eta(A\xi_2)\phi AX - k\eta(X)\phi A\xi_2\\
&\quad - A(\nabla_X\xi_2 + g(\phi AX, \xi_2)\xi - \eta(\xi_2)\phi AX - k\eta(X)\phi\xi_2)\\
&= \beta\nabla_X\xi_2 - A(\nabla_X\xi_2)\\
&= 0,
\end{aligned}$$

because $\xi \in \mathfrak{D}^\perp$, $X \in T_\lambda$ and $\xi_2 \in T_\beta$.

On the other hand, since we put $\xi = \xi_1$ from the assumption $\xi \in \mathfrak{D}^{\perp}$, we obtain that $q_2(X) = 2g(A\xi_2, X)$ and $q_3(X) = 2g(A\xi_3, X)$ for any tangent vector field X on M. Thus the equation (4.8) can be changed by

$$\beta\lambda\phi_2 X - \lambda A\phi_2 X = 0.$$

From (2.1), (2.2) and (2.3), we see that $\phi_2 X \in T_\mu$ for any $X \in T_\lambda$, that is, $A\phi_2 X = \mu\phi_2 X$. Since $\mu = 0$, we have

$$\beta\lambda\phi_2 X = 0$$

for any vector field $X \in T_\lambda$. Thus we have $\beta\lambda$ is zero and this case can not occur for some $r \in (0, \pi/2\sqrt{8})$. So we conclude a remark as follows:

Remark 4.3. The shape operator A of real hypersurfaces of Type (A) in $G_2(\mathbf{C}^{m+2})$ is not parallel with respect to the generalized Tanaka-Webster connection.

As a second, let us check whether the shape operator A of real hypersurfaces of Type (B) is parallel with respect to the g-Tanaka-Webster connection. As is well known in Berndt and Suh [3], a real hypersurface of Type (B) has five distinct constant principal curvatures as follows:

PROPOSITION F. *Let M be a connected real hypersurface of $G_2(\mathbf{C}^{m+2})$. Suppose that $A\mathfrak{D} \in \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbf{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbf{R}\xi = \operatorname{Span}\{\xi\}, \\ T_\beta &= \mathfrak{J}J\xi = \operatorname{Span}\{\xi_\nu \mid \nu = 1,2,3\}, \\ T_\gamma &= \mathfrak{J}\xi = \operatorname{Span}\{\phi_\nu\xi \mid \nu = 1,2,3\}, \\ & T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbf{HC}\xi)^{\perp}, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

Here we suppose that a real hypersurface of Type (B) has the g-Tanaka-Webster parallel shape operator. Then we see that $(\hat{\nabla}^{(k)}_X A)\xi_2 = 0$ for a unit eingenvector $X \in T_\lambda$. Then it follows that

$$
\begin{aligned}
(4.9)\qquad (\hat{\nabla}^{(k)}_X A)\xi_2 &= \nabla_X(A\xi_2) + g(\phi AX, A\xi_2)\xi - \eta(A\xi_2)\phi AX - k\eta(X)\phi A\xi_2 \\
&\quad - A(\nabla_X \xi_2 + g(\phi AX, \xi_2)\xi - \eta(\xi_2)\phi AX - k\eta(X)\phi\xi_2) \\
&= \beta\nabla_X\xi_2 - A(\nabla_X\xi_2) \\
&= 0,
\end{aligned}
$$

because $\xi \in \mathfrak{D}$, $X \in T_\lambda$ and $\xi_2 \in T_\beta$.

From (2.5) and $\xi_\nu \in T_\beta$, the equation (4.9) can be written by

$$\beta\lambda\phi_2 X - \lambda A\phi_2 X = 0.$$

Since $\mathfrak{J}Z \in T_\lambda$ for any $Z \in T_\lambda$, we see that $A\phi_2 X = \lambda X$. From these facts it follows that

$$\lambda(\beta - \lambda)\phi_2 X = 0$$

for any vector field $X \in T_\lambda$. From this, taking an inner product with $\phi_2 X$, we have

$$\lambda(\beta - \lambda) = 0.$$

Since $\lambda = \cot r$ $(0 < r < \pi/4)$ is not zero, we have $\beta = \lambda$. But this case also can not occur for some $r \in (0, \pi/4)$. In fact, since $\beta = 2\cot(2r)$ and $\lambda = \cot r$, we obtain $\beta - \lambda = -\tan r = \mu < 0$ where $r \in (0, \pi/4)$. So we also give the following remark:

Remark 4.4. The shape operator A of real hypersurfaces of Type (B) in $G_2(\mathbf{C}^{m+2})$ is not parallel with respect to the generalized Tanaka-Webster connection.

5. The proof of Main Theorem

In this section, let us M be a Hopf hypersurface M in $G_2(\mathbf{C}^{m+2})$ with the g-Tanaka-Webster parallel shape operator. Then by Lemma 4.2 we consider the following two cases:

- **Case I:** the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$,
- **Case II:** the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$.

First, let us consider the Case I, that is, $\xi \in \mathfrak{D}$. By Theorem D, we see that M is locally congruent to a real hypersurface of Type (B) under our assumption. But in section 4 we have checked that the shape operator A of real hypersurface of Type (B) is not g-Tanaka-Webster parallel (see Remark 4.4). From these facts, first we assert the following:

Theorem 5.1. *There does not exist any Hopf hypersurface in* $G_2(\mathbf{C}^{m+2})$, $m \geq 3$, *with generalized Tanaka-Webster parallel shape operator if the Reeb vector field* ξ *belongs to the distribution* $\mathfrak{D}$.

Next we consider for the case $\xi \in \mathfrak{D}^{\perp}$. Accordingly, we may put $\xi = \xi_1$. Then we have the following:

Lemma 5.2. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbf{C}^{m+2})$, $m \geq 3$ with $\xi \in \mathfrak{D}^{\perp}$. If M has the parallel shape operator in the generalized Tanaka-Webster connection and $\alpha \neq 2k$, then the structure tensor ϕ commutes with the shape operator A of M.*

Proof. Using (4.1) and $A\xi = \alpha\xi$, we have

$$(5.1) \qquad \begin{aligned} &(\hat{\nabla}_X^{(k)} A)\xi - (\hat{\nabla}_\xi^{(k)} A)X \\ &\quad = (\nabla_X A)\xi - \alpha\phi AX + A\phi AX - (\nabla_\xi A)X + k\phi AX - kA\phi X \end{aligned}$$

for any vector field $X \in T_x M$ and any point $x \in M$.

From the equation of Codazzi (2.8) we see that

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X + \sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3g(\phi_\nu X, \xi)\xi_\nu\}.$$

Moreover, since $\phi_2\xi = \phi_2\xi_1 = -\xi_3$ and $\phi_3\xi = \phi_3\xi_1 = \xi_2$, it follows that

$$(5.2) \qquad (\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X - \phi_1 X - 2\eta_3(X)\xi_2 + 2\eta_2(X)\xi_3.$$

Substituting (5.2) into (5.1), we have

$$(5.3) \qquad \begin{aligned} (\hat{\nabla}_X^{(k)} A)\xi - (\hat{\nabla}_\xi^{(k)} A)X &= -\phi X - \phi_1 X + (k - \alpha)\phi AX - kA\phi X \\ &\quad + A\phi AX - 2\eta_3(X)\xi_2 + 2\eta_2(X)\xi_3. \end{aligned}$$

Then the parallel shape operator in the g-Tanaka-Webster connection gives

$$(5.4) \quad -\phi X - \phi_1 X + (k - \alpha)\phi AX - kA\phi X + A\phi AX - 2\eta_3(X)\xi_2 + 2\eta_2(X)\xi_3 = 0$$

for any tangent vector field X on M.

Now we introduce the formula derived from $A\xi = \alpha\xi$ (see [4]) as follows:

$$(5.5) \qquad \begin{aligned} &\alpha A\phi X + \alpha\phi AX - 2A\phi AX + 2\phi X \\ &\quad = -2\sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu\xi + \eta_\nu(\phi X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X \\ &\qquad\quad - 2\eta(X)\eta_\nu(\xi)\phi_\nu\xi - 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\}. \end{aligned}$$

Since $\xi = \xi_1$, the equation (5.5) gives

$$(5.6) \quad 2A\phi AX = \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\phi_1 X + 4\eta_3(X)\xi_2 - 4\eta_2(X)\xi_3.$$

Thus from (5.4) and (5.6) we have

$$(5.7) \qquad (2k - \alpha)\phi AX - (2k - \alpha)A\phi X = 0.$$

Since $\alpha \neq 2k$, we have $(\phi A - A\phi)X = 0$ for any vector field $X \in T_xM$. It means that the shape operator A commutes with the structure tensor ϕ. □

Therefore from Theorems B and C in the introduction, we assert the following:

LEMMA 5.3. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbf{C}^{m+2})$, $m \geq 3$. If M satisfies the assumptions in Lemma 5.2, M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$.*

As mentioned in Remark 4.3, the shape operator A for real hypersurfaces of Type (A) can not parallel with respect to the g-Tanaka-Webster connection. From this we assert the following:

THEOREM 5.4. *There does not exist any Hopf hypersurface in $G_2(\mathbf{C}^{m+2})$ with parallel shape operator with respect to the generalized Tanaka-Webster connection if $\xi \in \mathfrak{D}^\perp$ and $\alpha \neq 2k$.*

Summing up Theorems 5.1 and 5.4, we give a complete proof of our Main Theorem in the introduction. □

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