Real hypersurfaces in complex two-plane Grassmannians with parallel normal Jacobi operator

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October 30 , 2008

[Motivation and Problem]

J. Berndt (1991, J. Reine Angew. Math. 419)

Definition. Let M be a real hypersurface in a Riemannian manifold $\bar{M}$. If M satisfies

$$\bar{R}_N \circ A = A \circ \bar{R}_N,$$

then we call M curvature-adapted to $\bar{M}$.

- $\bar{R}_N := \bar{R}(\cdot\,, N)N \in \mathrm{End}(T_x M), x \in M$
  $\rightleftharpoons$ the normal Jacobi operator of M w. r. t. N
- A : the shape operator of M w. r. t. N

- M : a real hypersurface of quaternionic projective space QP^m
- N : a unit normal vector field on M in QP^m
- $\xi_\nu = -J_\nu N, \nu = 1, 2, 3$
- $T_x M = \mathfrak{D} \bigoplus \mathfrak{D}^\perp$, $x \in M$
    - $\mathfrak{D}^\perp = \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$

Remark.

$$M \hookrightarrow QP^m$$

$$g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0 \Longleftrightarrow \bar{R}_N \circ A = A \circ \bar{R}_N$$

J. Berndt (1991, J. Reine Angew. Math. 419)

Theorem. Let M be a connected curvature-adapted real hypersurface in QP^m ($m \geq 2$). Then M is congruent to an open part of one of the following real hypersurfaces in QP^m :
(A) a tube of some radius r, $0 < r < \frac{\pi}{2}$ around the canonically (totally geodesic) embedded quaternionic projective space QP^k for some $k \in \{0, 1, \cdots, m-1\}$.
(B) a tube of some radius r, $0 < r < \frac{\pi}{4}$ around the canonically (totally geodesic) embedded complex projective space $P_m(\mathbb{C})$.

Conversely, each of these model spaces is curvature-adapted in QP^m.

J. D. Pérez and Y.J.Suh (1997,Diff.Geom.and Its Appl.7)

Theorem. Let M be a real hypersurface of QP^m ,$m \geq 3$,satisfying $\nabla_{\xi_\nu} R = 0, \nu = 1, 2, 3$. Then M is congruent to an open subset of a tube of radius $\frac{\pi}{4}$ over the canonically (totally geodesic) embedded quaternionic projective space QP^k ,$k \in \{0, 1, \cdots, m-1\}$.

(Here , $\xi_\nu = -J_\nu N, \nu = 1, 2, 3$.)

Problem.

$$M \hookrightarrow G_2(\mathbb{C}^{m+2})$$

$$\nabla_X \bar{R}_N = 0 \Longrightarrow M \cong (?)$$

[Riemannian geometry of $G_2(\mathbb{C}^{m+2})$]

- $G_2(\mathbb{C}^{m+2})$
  : the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$
- $\bar{\nabla}$
  : the Riemannian connection of $(G_2(\mathbb{C}^{m+2}), g)$
- $G_2(\mathbb{C}^{m+2}) = G/K$
  - $G = SU(m+2)$
    $= \{A \in GL(m+2,\mathbb{C}) \mid AA^* = I, detA = 1\}$
  - $K = S(U(2) \times U(m)) \subset G$
    $= \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in GL(m+2,\mathbb{C}) \mid g_1 \in U(2), g_2 \in U(m), detg_1 detg_2 = 1 \right\}$
- $dim(G_2(\mathbb{C}^{m+2})) = 4m$

- $\mathfrak{g}$ and $\mathfrak{k}$; the Lie algebra of G and K , respectively.
  - $\mathfrak{g} = \mathfrak{su}(m+2)$
    $= \{X \in \mathfrak{gl}(m+2,\mathbb{C}) | X + X^* = 0, trX = 0\}$
    $= \mathfrak{k} \bigoplus \mathfrak{m}$
  - Put $o = eK$, where e : the identity of G.
  - $T_o G_2(\mathbb{C}^{m+2}) \approx \mathfrak{m}$
  - $$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{gl}(m+2,\mathbb{C}) \mid A \in \mathfrak{u}(2) , B \in \mathfrak{u}(m), tr(A+B) = 0 \right\}$$
    $$= \mathfrak{su}(m) \bigoplus \underset{\substack{\downarrow \\ \mathcal{J}}}{\mathfrak{su}(2)} \bigoplus \underset{\substack{\downarrow \\ J}}{\mathfrak{R}}$$
    , where $\mathfrak{R}$: the center of $\mathfrak{k}$.
- compact, irreducible, Kähler, Quaternionic Kähler (not hyper Kähler)
- a Kähler structure J

  $+$

  a quaternionic Kähler structure $\mathcal{J}$

- $\{J_\nu,\ \nu = 1, 2, 3\}$: a canonical local basis of $\mathfrak{J}$ such that
  - $J_\nu^2 = -id, \quad \nu = 1, 2, 3$
  - $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$
    where the index is taken modulo three
  - $\bar{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2}$ where $q_\nu,\ \nu = 1, 2, 3$
    ; three local one-forms

- (geometric structure)
  J_1:any almost Hermitian structure in $\mathfrak{J}$
  - $JJ_1 = J_1 J$
  - JJ_1 : a symmetric endomorphism with $(JJ_1)^2 = id$ and $tr(JJ_1) = 0$

Let $M^{4m-1} \hookrightarrow G_2(\mathbb{C}^{m+2})$

- g: the induced Riemannian metric on M
- ∇: the Riemannian connection of (M, g)
- N: a local unit normal vector field of M
- A: the shape operator of M w. r. t. N
- $JX = \phi X + \eta(X)N$

  , where $\eta(X) = g(X, \xi)$, $\forall X \in TM$

  $\xi = -JN$
  - $\eta(\xi) = 1$, $\phi\xi = 0$, $\eta(\phi X) = 0$, $\phi^2 X = -X + \eta(X)\xi$
  - $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, $\forall X, Y \in TM$

$J \to (\phi, \xi, \eta, g)$: an almost contact metric structure

- $J_\nu X = \phi_\nu X + \eta_\nu(X)N$
  , where $\eta_\nu(X) = g(X, \xi_\nu)$, $\forall X \in TM$
  $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$
  - $\eta_\nu(\xi_\nu) = 1$, $\phi_\nu \xi_\nu = 0$, $\eta_\nu(\phi_\nu X) = 0$, ${\phi_\nu}^2 X = -X + \eta_\nu(X)\xi_\nu$
  - $g(\phi_\nu X, \phi_\nu Y) = g(X, Y) - \eta_\nu(X)\eta_\nu(Y)$, $\forall X, Y \in TM$

$J_\nu(\nu = 1, 2, 3) \rightarrow (\phi_\nu, \xi_\nu, \eta_\nu, g)$: an almost contact metric structure

- $T_x M = \mathfrak{D} \bigoplus \mathfrak{D}^\perp$ where $\mathfrak{D}^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$

[Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$]

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &-g(JX,Z)JY - 2g(JX,Y)JZ + \sum_{\nu=1}^{3} g(J_\nu Y,Z)J_\nu X \\ &- \sum_{\nu=1}^{3}\{g(J_\nu X,Z)J_\nu Y + 2g(J_\nu X,Y)J_\nu Z\} \\ &+ \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathcal{J}$.

- $JX = \phi X + \eta(X)N \quad , \ \forall X \in TM$
  $\xi = -JN$
- $J_\nu X = \phi_\nu X + \eta_\nu(X)N \ , \forall X \in TM$
  $\xi_\nu = -J_\nu N \ , \ \nu = 1, 2, 3$
- $\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N$: the Gauss formula
  $,\forall X, Y \in TM$
- $\bar{\nabla}_X N = -AX$: the Weingarten formula
  $,\forall X \in TM$

$$\begin{aligned}\bar{R}(X, Y)Z &= \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \\ &= R(X, Y)Z - g(AY, Z)AX + g(AX, Z)AY \\ &+g((\nabla_X A)Y, Z)N - g((\nabla_Y A)X, Z)N\end{aligned}$$

$$
\begin{aligned}
R(X,Y)Z = \; & g(Y,Z)X - g(X,Z)Y \\
& + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\
& + \sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\} \\
& + \sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\
&- \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y, Z) - \eta(Y)g(\phi_\nu\phi X, Z)\}\xi_\nu \\
&+ g(AY, Z)AX - g(AX, Z)AY\,,
\end{aligned}
$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

$$
\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = \;& \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
& + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\
& + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\
& + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu \,.
\end{aligned}
$$

J. Berndt and Y. J. Suh (1999, Monatshefte fu̇r Math., 127)

Proposition. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Suppose that both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M. Then ξ is tangent to $\mathfrak{D}$ or to $\mathfrak{D}^\perp$ everywhere.

- $[\xi]$ is invariant under the shape operator A of M, that is, $A\xi = \alpha\xi$.
- $\mathfrak{D}^\perp$ is invariant under the shape operator A of M, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

J. Berndt and Y. J. Suh (1999, Monatshefte fur Math., 127)

Theorem A. Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if
(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or
(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic QP^n in $G_2(\mathbb{C}^{m+2})$.

[The normal Jacobi operator $\bar{R}_N$]

$$\begin{aligned}\bar{R}_N(X) :=& \bar{R}(X,N)N \\ =& X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ & - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu)\} \\ & + \sum_{\nu=1}^{3}\eta_\nu(\phi X)\phi_\nu\xi\end{aligned}$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$.

I. Jeong, J. D.Péreze and Y. J. Suh (2007, Acta Math. Hungarica, 117)

- The commuting condition $A \circ \bar{R}_N = \bar{R}_N \circ A$ gives

$$
\begin{aligned}
&3\sum_{\nu=1}^{3}\eta_\nu(X)A\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\xi)A\phi_\nu\phi X + \sum_{\nu=1}^{3}\eta(X)\eta_\nu(\xi)A\xi_\nu \\
&\qquad + \sum_{\nu=1}^{3}\eta_\nu(\phi X)A\phi_\nu\xi + 3\eta(X)A\xi \\
&= 3\sum_{\nu=1}^{3}\eta_\nu(AX)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\phi AX + \sum_{\nu=1}^{3}\eta_\nu(\xi)\eta(AX)\xi_\nu \\
&\qquad + \sum_{\nu=1}^{3}\eta_\nu(\phi AX)\phi_\nu\xi + 3\eta(AX)\xi.
\end{aligned}
$$

Theorem. Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting normal Jacobi operator. Then M is Hopf provided with $\xi\in\mathfrak{D}$ or $\xi\in\mathfrak{D}^\perp$.

- The commuting condition $\phi \circ \bar{R}_N = \bar{R}_N \circ \phi$ gives

$$
\begin{aligned}
3\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi\phi_\nu\phi X - \eta(X)\phi\xi_\nu) \\
- \eta_\nu(\phi X)\phi\phi_\nu\xi\} \\
= 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_\nu\phi^2 X - \eta(\phi X)\xi_\nu) \\
- \eta_\nu(\phi^2 X)\phi_\nu\xi\}.
\end{aligned}
$$

Theorem. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying $\bar{R}_N \circ \phi = \phi \circ \bar{R}_N$, then $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.

[Result-I]

I. Jeong, J. D.Péreze and Y. J. Suh (2007, Acta Math. Hungarica, 117)

Theorem 1. Let M be a $\mathfrak{D}^{\perp}$-invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then M is Hopf if the normal Jacobi operator $\bar{R}_N$ both commutes with the shape operator A and the structure tensor ϕ.

Theorem 2. Let M be a $\mathfrak{D}^{\perp}$-invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$. If the normal Jacobi operator $\bar{R}_N$ commutes with the structure tensor and the shape operator, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

I. Jeong and Y. J. Suh (2008,J.of Korean Math. Soc.,45)

$$\begin{aligned}(\mathcal{L}_\xi \bar{R}_N)X =& \mathcal{L}_\xi(\bar{R}_N X) - \bar{R}_N(\mathcal{L}_\xi X)\\ =& [\xi,\ \bar{R}_N X] - \bar{R}_N [\xi,\ X]\\ =& \nabla_\xi (\bar{R}_N X) - \nabla_{\bar{R}_N X}\xi - \bar{R}_N \nabla_\xi X + \bar{R}_N \nabla_X \xi\\ =& (\nabla_\xi \bar{R}_N)X - \phi A\bar{R}_N X + \bar{R}_N \phi AX.\end{aligned}$$

(Here , $\nabla_X \xi = \phi AX, \forall X \in TM$)

$$
\begin{aligned}
(\mathcal{L}_\xi \bar{R}_N)X =& 3g(\phi A\xi\,, X)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, X)\xi_\nu \\
&+ 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi_\nu A\xi \\
&- \sum_{\nu=1}^{3} [\xi(\eta_\nu(\xi))(\phi_\nu\phi X - \eta(X)\xi_\nu \\
&+ \eta_\nu(\xi)\{-q_{\nu+1}(\xi)\phi_{\nu+2}\phi X \\
&+ q_{\nu+2}(\xi)\phi_{\nu+1}\phi X + \eta_\nu(\phi X)A\xi \\
&- g(A\xi, \phi X)\xi_\nu + \eta(X)\phi_\nu A\xi
\end{aligned}
$$

$$
\begin{aligned}
&- g(A\xi, X)\phi_\nu\xi - g(\phi A\xi, X)\xi_\nu \\
&- \eta(X)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \phi_\nu A\xi)\} \\
&- g(\phi_\nu A\xi, \phi X)\phi_\nu\xi - \eta(X)\eta_\nu(A\xi)\phi_\nu\xi \\
&+ g(A\xi, X)\eta_\nu(\xi)\phi_\nu\xi \\
&- \eta_\nu(\phi X)\{\eta_\nu(\xi)A\xi - g(A\xi, \xi)\xi_\nu + \phi_\nu\phi A\xi\}\Big] \\
&- 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi A\phi_\nu\phi X-\eta(X)\phi A\xi_\nu)\\
&-\eta_\nu(\phi X)\phi A\phi_\nu\xi\}+3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu\\
&+\sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu AX-\eta_\nu(AX)\phi_\nu\xi\}\\
&=0
\end{aligned}
$$

Definition. A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ has *commuting shape operator* on the distribution $\mathfrak{D}^\perp$ if the shape operator A of M commutes with the structure tensor ϕ on $\mathfrak{D}^\perp$, that is, $A\phi\xi_\nu = \phi A\xi_\nu$, $\nu = 1, 2, 3$.

Lemma. Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \bar{R}_N = 0$. If M has commuting shape operator on the distribution $\mathfrak{D}^\perp$, then the structure vector ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

[Result - II]

I. Jeong and Y. J. Suh (2008,J.of Korean Math. Soc.,45)

Theorem 1. There do not exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \bar{R}_N = 0$ and $\xi \in \mathfrak{D}^\perp$.

Theorem 2. There do not exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \bar{R}_N = 0$ and $\xi \in \mathfrak{D}$.

Theorem 3. There do not exist any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \bar{R}_N = 0$ and commuting shape operator on the distribution $\mathfrak{D}^\perp$.

[Step 1] Parallel normal Jacobi operator $\bar{R}_N$

$$
\begin{aligned}
(\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N Y) - \bar{R}_N(\nabla_X Y) \\
&= 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX \\
&+3\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+1}(X)\eta_{\nu+2}(Y) \\
&+g(\phi_\nu AX, Y)\Big\}\xi_\nu \\
&+3\sum_{\nu=1}^{3}\eta_\nu(Y)\Big\{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{\nu=1}^{3}\Bigg[\Big\{q_{\nu+2}(X)\eta_{\nu+1}(\xi)-q_{\nu+1}(X)\eta_{\nu+2}(\xi)\\
&+2\eta_\nu(\phi AX)\Big\}(\phi_\nu\phi Y-\eta(Y)\xi_\nu)\\
&+\eta_\nu(\xi)\Big\{-q_{\nu+1}(X)\phi_{\nu+2}\phi Y+q_{\nu+2}(X)\phi_{\nu+1}\phi Y\\
&+\eta_\nu(\phi Y)AX-g(AX,\phi Y)\xi_\nu\\
&+\eta(Y)\phi_\nu AX-g(AX,Y)\phi_\nu\xi-g(\phi AX,Y)\xi_\nu\\
&-\eta(Y)(q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&-\Big\{q_{\nu+2}(X)\eta_{\nu+1}(\phi Y) - q_{\nu+1}(X)\eta_{\nu+2}(\phi Y) \\
&+g(\phi_\nu AX, \phi Y)\Big\}\phi_\nu\xi \\
&-\Big\{\eta(Y)\eta_\nu(AX) - g(AX, Y)\eta_\nu(\xi)\Big\}\phi_\nu\xi \\
&-\eta_\nu(\phi Y)\Big\{q_{\nu+2}(X)\phi_\nu\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi \\
&+\phi_\nu\phi AX - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX\Big\}\Bigg] \\
&= 0
\end{aligned}
$$

Lemma. Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator. Then the Reeb vector ξ belongs to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ unless the geodesic Reeb flow is non-vanishing.

Remark. Consider the geodesic Reeb flow is vanishing, that is $\alpha = 0$. In this case, by differentiating $A\xi = 0$ and using the same method as in Berndt and Suh(2002,Monatshefte fur Math.,137), Pérez and Suh(2007,J.Korean Math.,44) proved that the Reeb vector ξ belongs to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

Theorem 1. Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator. Then ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$.

$\diamond$ In step 2 and 3 we respectively prove a non-existence theorem for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator , $m \geq 3$, when the Reeb vector ξ belongs to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$.

[Step 2] Parallel normal Jacobi operator for $\xi \in \mathfrak{D}^{\perp}$

Lemma A. Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator and $\xi \in \mathfrak{D}^{\perp}$. Then $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.

Remark. From above Lemma A and together with Theorem A (Berndt and Suh, 1999) we know that any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator are congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

J. Berndt and Y. J. Suh (1999, Monatshefte fur Math., 127)

Proposition A. Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathcal{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three(if $r = \pi/2\sqrt{8}$) or four(otherwise) distinct constant principal curvatures

$$\alpha = \sqrt{8}cot(\sqrt{8}r)\ ,\ \beta = \sqrt{2}cot(\sqrt{2}r)\ ,$$

$$\lambda = -\sqrt{2}tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 2\ ,\ m(\lambda) = 2m - 2 = m(\mu),$$

and for the corresponding eigenspaces we have

$$T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1,$$
$$T_\beta = \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3,$$

$$T_\lambda = \{X | X \perp \mathbb{H}\xi, JX = J_1 X\},$$
$$T_\mu = \{X | X \perp \mathbb{H}\xi, JX = -J_1 X\}.$$

, where $\mathbb{R}\xi$,$\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

$\diamond$ From the proof of Lemma A we obtained $\eta_3(AX) = 0$, for any vector field X on M. Then, by putting $X = \xi_3$ and using Proposition A we have

$$
\begin{aligned}
0 &= \eta_3(A\xi_3) \\
&= g(A\xi_3, \xi_3) \\
&= \beta g(\xi_3, \xi_3) \\
&= \beta
\end{aligned}
$$

But $r \in (0, \frac{\pi}{\sqrt{8}})$, on which we know $\beta = \sqrt{2}\cot\sqrt{2}r \neq 0$. This makes a contradiction.

$\diamond$ A real hypersurface of type (A) in Theorem A(Berndt and Suh, 1999) do not satisfy parallel normal Jacobi operator for $\xi \in \mathfrak{D}^{\perp}$.

Theorem 2. There do not exist any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator and $\xi \in \mathfrak{D}^{\perp}$.

[Step 3] Parallel normal Jacobi operator for $\xi \in \mathfrak{D}$

Lemma B. Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator and $\xi \in \mathfrak{D}$. Then $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.

Remark. From above Lemma and together with Theorem A (Berndt and Suh, 1999) we know that any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator are congruent to a tube over a totally geodesic QP^n in $G_2(\mathbb{C}^{m+2})$,$m = 2n$.

J. Berndt and Y. J. Suh (1999, Monatshefte fur Math., 127)

Proposition B. Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures

$$\alpha = -2\tan(2r)\ ,\ \beta = 2\cot(2r)\ ,$$

$$\gamma = 0\ ,\ \lambda = \cot(r)\ ,\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 3 = m(\gamma)\ ,$$

$$m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi\ ,\ T_\beta = \mathcal{J}J\xi\ ,\ T_\gamma = \mathcal{J}\xi\ ,\ T_\lambda\ ,\ T_\mu\ ,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp \ ,$$

$$\mathcal{J}T_\lambda = T_\lambda \ , \ \mathcal{J}T_\mu = T_\mu \ , \ JT_\lambda = T_\mu \ .$$

◇ From the proof of Lemma B we obtained

$$0 = 3\phi AX + 5\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi.$$

Then by putting $X = \xi_\mu$ and using $A\phi_\nu\xi = 0$ we have

$$\begin{aligned} 0 &= 3\phi A\xi_\mu + 5\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\phi_\nu\xi \\ &= 3\phi A\xi_\mu + 5\sum_{\nu=1}^{3} g(\phi A\xi_\mu, \xi_\nu)\xi_\nu + \sum_{\nu=1}^{3} g(A\xi_\mu, \xi_\nu)\phi_\nu\xi \\ &= 3\beta\phi\xi_\mu + \beta\phi_\mu\xi \\ &= 4\beta\phi\xi_\mu. \end{aligned}$$

Then it follows that $\beta = 0$. But $r \in (0, \frac{\pi}{4})$, on which we know $\beta = 2\cot 2r \neq 0$. This makes a contradiction.

$\diamond$ A real hypersurfaces of type (B) in Theorem A(Berndt and Suh, 1999) do not satisfy parallel normal Jacobi operator for $\xi \in \mathfrak{D}$.

Theorem 3. There do not exist any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator and $\xi \in \mathfrak{D}$.

[Result -III]

Theorem 1. Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator. Then ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$.

Theorem 2. There do not exist any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m{\geq}3$, with parallel normal Jacobi operator and $\xi{\in}\mathfrak{D}^{\perp}$.

Theorem 3. There do not exist any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m{\geq}3$, with parallel normal Jacobi operator and $\xi{\in}\mathfrak{D}$.

Theorem 4. There do not exist any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator.