Proceedings of The Fourteenth International
Workshop on Diff. Geom. 14(2010) 203-215

Real hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster parallel shape operator

Imsoon Jeong
Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea
e-mail : imsoon.jeong@gmail.com

Hyunjin Lee
Graduate School of Electrical Engineering and Computer Science, Kyungpook National University, Taegu 702-701, Korea
e-mail : lhjibis@hanmail.net

Young Jin Suh
Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea
e-mail : yjsuh@knu.ac.kr

(2010 Mathematics Subject Classification : 53C40, 53C15.)

Abstract. We introduce the notion of generalized Tanaka-Webster connection for hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and give a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator in this connection.

1 Introduction

The *generalized Tanaka-Webster connection* (in short, the *g-Tanaka-Webster connection*) for contact metric manifolds has been introduced by Tanno [15] as a generalization of the well-known connection defined by Tanaka in [14] and, independently, by Webster in [16]. This connection coincides with Tanaka-Webster connection if the associated CR-structure is integrable. Tanaka-Webster connection is defined as the canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. For a real hypersurface in a Kähler manifold with almost contact

Key words and phrases: Real hypersurfaces, Complex two-plane Grassmannians, Hopf hypersurface, Generalized Tanaka-Webster connection, Generalized Tanaka-Webster parallel shape operator.

* This work was supported by grant Proj. No. BSRP-2010-0020931 from National Research Foundation of Korea.

metric structure (ϕ, ξ, η, g), Cho defined the g-Tanaka-Webster connection $\widehat{\nabla}^{(k)}$ for a non-zero real number k (see [5], [6] and [7]). In particular, if a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the g-Tanaka-Webster connection $\widehat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see Proposition 7 in [7]).

Using the notion of the g-Tanaka-Webster connection, many geometers have studied some characterizations of real hypersurfaces in complex space form $\widetilde{M}_n(c)$ with constant holomorphic sectional curvature c. For instance, when $c > 0$, that is, $\widetilde{M}_n(c)$ is a complex projective space $\mathbb{C}P^n$, Cho [5] proved that if the shape operator A of M in $\mathbb{C}P^n$ is $\widehat{\nabla}^{(k)}$*-parallel* (it means that the shape operator A satisfies $\widehat{\nabla}^{(k)} A = 0$), then ξ is a principal curvature vector field and M is locally congruent to a real hypersurface of Type (A_2) and Type (B). (In fact, he also gave the classification of real hypersurfaces in a complex hyperbolic space $(c < 0)$ and complex Euclidean space $(c = 0)$ under the assumption $\widehat{\nabla}^{(k)}$-parallel shape operator [5]). Moreover in [6] he gave the classification theorem of Levi-parallel Hopf hypersurface in $\widetilde{M}_n(c)$, $c \neq 0$. Here, a real hypersurface of $\widetilde{M}_n(c)$ is called *Levi-parallel* if its Levi form is parallel with respect to the g-Tanaka-Webster connection. In [9], Kon gave a characterization for real hypersurfaces of Type (A_1) in complex projective space $\mathbb{C}P^n$ under the assumption that the Ricci tensor related to the g-Tanaka-Webster connection identically vanishes.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometric conditions for real hypersurfaces M that the 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M (see section 2).

Here the almost contact structure vector field ξ is defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

By using two invariant conditions mentioned above and the result in Alekseevskii [1], Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Furthermore, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in Section 2 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

Moreover, we say that the Reeb vector field ξ on M is Killing, when the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*. In [4], Berndt and Suh gave some equivalent conditions of this property as follows:

Theorem B. *Let M be a connected orientable real hypersurface in a Kähler manifold $\widetilde{M}$. The following statements are equivalent:*

(1) *The Reeb flow on M is isometric,*

(2) *The shape operator A and the structure tensor field ϕ commute with each other,*

(3) *The Reeb vector field ξ is a principal curvature vector of M everywhere and the principal curvature spaces contained in the maximal complex subbundle $\mathcal{D}$ of TM are complex subspaces.*

Also in [4], a characterization of real hypersurfaces of Type (A) in Theorem A was given in terms of the Reeb flow on M as follows:

Theorem C. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Recently, Lee and Suh [10] gave a new characterization of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

Theorem D. *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

In particular, if the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ satisfies $(\nabla_X A)Y = 0$ for any vector fields X, Y on M, we say that the shape operator A is *parallel with* *respect to the Levi-Civita connection.* Using this notion, Suh [12] proved the non-existence theorem of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator. Moreover, in [13], he also considered a generalized condition weaker than $\nabla A = 0$, which is said to be $\mathfrak{F}$-parallel, and proved that there does not exist any real hypersurface with $\mathfrak{F}$-parallel shape operator. Here, a shape operator A of M in $G_2(\mathbb{C}^{m+2})$ is said to be $\mathfrak{F}$*-parallel* if the shape operator A satisfies $(\nabla_X A)Y = 0$ for any tangent vector fields $X \in \mathfrak{F}$ and $Y \in T_xM$, where the subdistribution $\mathfrak{F}$ is defined by $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$ (see [13]).

Now in this paper we consider a new parallel shape operator for real hypersurface M in $G_2(\mathbb{C}^{m+2})$. Here the shape operator A is called *generalized Tanaka-Webster parallel* (in short, *g-Tanaka-Webster parallel*) if the shape operator A is parallel with respect to the g-Tanaka-Webster connection $\widehat{\nabla}^{(k)}$, that is, $(\widehat{\nabla}^{(k)}_X A)Y = 0$ for any vector fields X, Y on M. If we consider such a notion in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, its situation is quite different from the case in complex space forms $\widetilde{M}_n(c)$.

From such a point of view, in this paper we give a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator in the generalized Tanaka-Webster connection as follows:

Main Theorem. *There does not exist any Hopf hypersurface, $\alpha \neq 2k$, in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel shape operator in the generalized Tanaka-Webster connection.*

2 Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2], [3] and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(C^3)$ is isometric to the two-dimensional complex projective space CP^2 with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(C^4)$ and the real Grassmann manifold $G_2^+(R^6)$ of oriented two-dimensional linear subspaces in R^6. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\operatorname{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\widetilde{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\widetilde{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \tag{2.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned} \widetilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3} \Big\{ g(J_\nu Y, Z) J_\nu X - g(J_\nu X, Z) J_\nu Y - 2g(J_\nu X, Y) J_\nu Z \Big\} \\ & + \sum_{\nu=1}^{3} \Big\{ g(J_\nu JY, Z) J_\nu JX - g(J_\nu JX, Z) J_\nu JY \Big\}, \end{aligned} \tag{2.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

3 Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [3], [4], [10], [11], [12] and [13]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.1}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in section 1, induced an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned} &\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0, \\ &\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1} \end{aligned} \tag{3.2}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$ in section 1 and (3.1), the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{aligned} &\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\ &\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi. \end{aligned} \tag{3.3}$$

On the other hand, from the Kähler structure J, that is, $\widetilde{\nabla} J = 0$ and the quaternionic Kähler structure J_ν, together with Gauss and Weingarten equations it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{3.4}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{3.5}$$

$$\begin{aligned} (\nabla_X\phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &- g(AX, Y)\xi_\nu. \end{aligned} \tag{3.6}$$

Summing up these formulas, we find the following:

$$\begin{aligned} \nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\ &\quad - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX. \end{aligned} \tag{3.7}$$

Using the above expression (2.2) for the curvature tensor $\widetilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi becomes:

$$(3.8)\quad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi + \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\Big\} + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\Big\} + \sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\Big\}\xi_\nu\,.$$

4 The g-Tanaka-Webster connection for real hypersurfaces

In this section, we introduce the notion of generalized Tanaka-Webster connection (see [5], [6], [7] and [9]).

As mentioned above, the Tanaka-Webster connection is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold (see [14], [16]). In [15], Tanno defined the g-Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR-structure is integrable.

From now on, we introduce the g-Tanaka-Webster connection due to Tanno [15] for real hypersurfaces in Kähler manifolds by natural extending of the canonical affine connection on a non-degenerate, pseudo-Hermitian CR manifold.

Now let us recall the g-Tanaka-Webster connection $\widehat{\nabla}$ define by Tanno [15] for contact metric manifolds as follows:

$$\widehat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X\xi - \eta(X)\phi Y$$

for all vector fields X and Y (see [15]).
By taking (3.4) into account, the g-Tanaka-Webster connection $\widehat{\nabla}^{(k)}$ for real hypersurfaces of Kähler manifolds is defined by

$$(4.1)\qquad \widehat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$$

for a non-zero real number k (see [5], [6] and [7]) (Note that $\widehat{\nabla}^{(k)}$ is invariant under the choice of the orientation. Namely, we may take $-k$ instead of k in (4.1) for the opposite orientation $-N$).
Let us put

$$(4.2)\qquad F_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y.$$

Then the torsion tensor $\widehat{T}^{(k)}$ is given by $\widehat{T}^{(k)}(X,Y) = F_X Y - F_Y X$. Also, by using (3.4) and (4.1) we can see that

$$(4.3)\qquad \widehat{\nabla}^{(k)}\eta = 0,\quad \widehat{\nabla}^{(k)}\xi = 0,\quad \widehat{\nabla}^{(k)}g = 0,\quad \widehat{\nabla}^{(k)}\phi = 0.$$

Next the g-Tanaka-Webster curvature tensor $\widehat{R}^{(k)}$ with respect to $\widehat{\nabla}^{(k)}$ can be defined by

$$\widehat{R}^{(k)}(X,Y)Z = \widehat{\nabla}^{(k)}_X(\widehat{\nabla}^{(k)}_Y Z) - \widehat{\nabla}^{(k)}_Y(\widehat{\nabla}^{(k)}_X Z) - \widehat{\nabla}^{(k)}_{[X,Y]}Z \tag{4.4}$$

for all vector fields X, Y, Z on M. Then we have the following identities

$$\widehat{R}^{(k)}(X,Y)Z = -\widehat{R}^{(k)}(Y,X)Z,$$
$$g(\widehat{R}^{(k)}(X,Y)Z, W) = -g(\widehat{R}^{(k)}(X,Y)W, Z).$$

Here we remark that the identities of type Jacobi and of type Bianchi do not hold in general, because the g-Tanaka-Webster connection is not torsion-free. Moreover, the g-Tanaka-Webster Ricci tensor $\widehat{S}$ is defined by

$$\widehat{S}(Y,Z) = \text{trace of } \{X \mapsto \widehat{R}(X,Y)Z\}. \tag{4.5}$$

5 Key Lemmas

Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with g-Tanaka-Webster parallel shape operator. First of all, we find the fundamental equation for the condition that the shape operator A is parallel with respect to $\widehat{\nabla}^{(k)}$, that is, $(\widehat{\nabla}^{(k)}_X A)Y = 0$ for any tangent vector fields X and Y.

From (4.1), we have

$$\begin{aligned}(\widehat{\nabla}^{(k)}_X A)Y &= \widehat{\nabla}^{(k)}_X(AY) - A(\widehat{\nabla}^{(k)}_X Y)\\ &= \nabla_X(AY) + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY\\ &\quad - A\Big(\nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y\Big)\\ &= (\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY\\ &\quad - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y.\end{aligned} \tag{5.1}$$

Under our conditions, $(\widehat{\nabla}^{(k)}_X A)Y = 0$ and $A\xi = \alpha\xi$, it follows that

$$\begin{aligned}&(\nabla_X A)Y + g(\phi AX, AY)\xi - \alpha\eta(Y)\phi AX - k\eta(X)\phi AY\\ &\quad - \alpha g(\phi AX, Y)\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y = 0\end{aligned} \tag{5.2}$$

for any tangent vector fields X and Y on M.

From the equation (5.2), we can assert following:

Lemma 5.1. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M has the generalized Tanaka-Webster parallel shape operator, then the smooth function $\alpha = g(A\xi, \xi)$ is constant.*

On the other hand, under the assumption of $A\xi = \alpha\xi$, the Codazzi equation (3.8) becomes

$$(\nabla_\xi A)Y - (\nabla_Y A)\xi = \phi Y + \sum_{\nu=1}^{3} \Big\{ \eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi - 3g(\phi_\nu\xi, Y)\xi_\nu \Big\}$$

for any tangent vector field Y on M.
From this, taking an inner product with ξ, it gives that

$$g\big((\nabla_\xi A)Y, \xi\big) - g\big((\nabla_Y A)\xi, \xi\big) = 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y)\,.$$

On the other hand, taking the covariant derivative for $A\xi = \alpha\xi$ along any direction X, we get

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX.$$

Using this equation, we obtain

$$\begin{aligned} g\big((\nabla_\xi A)Y, \xi\big) - g\big((\nabla_Y A)\xi, \xi\big) &= g\big(Y, (\nabla_\xi A)\xi\big) - g\big(\xi, (\nabla_Y A)\xi\big) \\ &= g\big(Y, (\xi\alpha)\xi\big) - g\big(\xi, (Y\alpha)\xi + \alpha\phi AY - A\phi AY\big) \\ &= (\xi\alpha)\eta(Y) - (Y\alpha), \end{aligned}$$

where we have used two formulas that $(\nabla_\xi A)\xi = (\xi\alpha)\xi$ and $(\nabla_Y A)\xi = (Y\alpha)\xi + \alpha\phi AY - A\phi AY$.
Consequently, we have the following

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y) \tag{5.3}$$

for any tangent vector field Y on M (see [4]).

Now we give one of Key Lemmas as follows:

Lemma 5.2. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M has the parallel shape operator with respect to the generalized Tanaka-Webster connection, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Before giving the proof of our Main Theorem in the introduction, let us check whether the shape operator A of real hypersurfaces of Type (A) or Type (B) in Theorem A is parallel with respect to the g-Tanaka-Webster connection.

In order to do this, we recall the following propositions due to Berndt and Suh [3] as follows:

Proposition E. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\},$$
$$T_\beta = \mathbb{C}^{\perp}\xi = \mathbb{C}^{\perp}N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2,\ \xi_3\},$$
$$T_\lambda = \{X|\, X \perp \mathbb{H}\xi,\ JX = J_1X\,\},$$
$$T_\mu = \{X|\, X \perp \mathbb{H}\xi,\ JX = -J_1X\,\}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector field ξ and $\mathbb{C}^{\perp}\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Proposition F. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \in \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi = \mathrm{Span}\{\xi\},$$
$$T_\beta = \mathfrak{J}J\xi = \mathrm{Span}\{\xi_\nu \mid \nu = 1, 2, 3\,\},$$
$$T_\gamma = \mathfrak{J}\xi = \mathrm{Span}\{\phi_\nu\xi \mid \nu = 1, 2, 3\,\},$$
$$T_\lambda, \quad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

Using these propositions, we conclude remarks as follows:

Remark 5.3. *The shape operator A of real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ is not parallel with respect to the generalized Tanaka-Webster connection.*

Remark 5.4. *The shape operator A of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ is not parallel with respect to the generalized Tanaka-Webster connection.*

6 The proof of Main Theorem

In this section, let M be a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with the g-Tanaka-Webster parallel shape operator. Then by Lemma 5.2 we consider the following two cases:

- **Case I:** the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$,
- **Case II:** the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$.

First, let us consider the Case I, that is, $\xi \in \mathfrak{D}$. By Theorem D, we see that M is locally congruent to a real hypersurface of Type (B) under our assumption. But in section 4 we have checked that the shape operator A of real hypersurface of Type (B) is not g-Tanaka-Webster parallel (see Remark 5.4). From these facts, first we assert the following:

Theorem 6.1. *There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with generalized Tanaka-Webster parallel shape operator if the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.*

Next we consider for the case $\xi \in \mathfrak{D}^\perp$. Accordingly, we may put $\xi = \xi_1$. Then we have the following:

Lemma 6.2. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $\xi \in \mathfrak{D}^\perp$. If M has the parallel shape operator in the generalized Tanaka-Webster connection and $\alpha \neq 2k$, then the structure tensor ϕ commutes with the shape operator A of M.*

Therefore from Theorems B and C in the introduction, we assert the following:

Lemma 6.3. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M satisfies the assumptions in Lemma 6.2, M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

As mentioned in Remark 5.3, the shape operator A for real hypersurfaces of Type (A) can not parallel with respect to the g-Tanaka-Webster connection. From this we assert the following:

Theorem 6.4. *There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator with respect to the generalized Tanaka-Webster connection if $\xi \in \mathfrak{D}^{\perp}$ and $\alpha \neq 2k$.*

Summing up Theorems 6.1 and 6.4, we give a complete proof of our Main Theorem in the introduction. □

Acknowledgement. This article is a brief survey on the purpose of 2010 NIMS Hot Topic Workshop " The 14th International Workshop on Differential Geometry". The detailed proof of this article was given in a paper [8] due to Jeong, Lee and Suh.

References

[1] D.V. Alekseevskii, *Compact quaternion spaces*, Funct. Anal. Appl. **2** (1968), 106–114.

[2] J. Berndt, *Riemannian geometry of complex two-plane Grassmannian*, Rend. Sem. Mat. Univ. Politec. Torino **55** (1997), 19–83.

[3] J. Berndt and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians*, Monatshefte für Math. **127** (1999), 1–14.

[4] J. Berndt and Y.J. Suh, *Isometric flows on real hypersurfaces in complex two-plane Grassmannians*, Monatshefte für Math. **137** (2002), 87–98.

[5] J.T. Cho, *CR structures on real hypersurfaces of a complex space form*, Publ. Math. Debrecen **54** no. 3-4 (1999), 473–487.

[6] J.T. Cho, *Levi-parallel hypersurfaces in a complex space form*, Tsukuba J. Math. **30** no. 2 (2006), 329–344.

[7] J.T. Cho and M. Kimura, *Pseudo-holomorphic sectional curvatures of real hypersrufaces in a complex space form*, Kyushu J. Math. **62** (2008), 75–87.

[8] I. Jeong, H. Lee and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster parallel shape operator*, Submitted.

[9] M. Kon, *Real hypersurfaces in complex space forms and the generalized-Tanaka-Webster connection*, Proc. of the 13th International Workshop on Differential Geometry anad Related Fields, Edited by Y.J. Suh, J. Berndt and Y.S. Choi (NIMS, 2009), 145–159.

[10] H. Lee and Y.J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector*, Bull. Korean Math. Soc. **47** no. 3 (2010), 551–561.

[11] J.D. Pérez and Y.J. Suh, *The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians*, J. Korean Math. Soc. **44** (2007), 211–235.

[12] Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator*, Bull. of Austral. Math. Soc. **68** (2003), 493–502.

[13] Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator II*, J. Korean Math. Soc. **41** (2004), 535–565.

[14] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan J. Math. **20** (1976), 131–190.

[15] S. Tanno, *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc. **314** (1989), 349–379.

[16] S.M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, J. Diff. Geom. **13** (1978), 25–41.