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REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH COMMUTING NORMAL JACOBI OPERATOR

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Abstract. We give a complete classification of $\mathfrak{D}^{\perp}$-invariant real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbf{C}^{m+2})$ with commuting normal Jacobi operator $\bar{R}_N$.

0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_n(c)$ Kimura [7] has proved that Hopf real hypersurfaces M in a complex projective space $P_n(\mathbf{C})$ with commuting Ricci tensor are locally congruent to a tube over a totally geodesic $P_k(\mathbf{C})$ (type A), a tube over a complex quadric Q_{n-1}, $\cot^2 2r = n-2$ (type B), a tube over $P_1(\mathbf{C}) \times P_{(n-1)/2}(\mathbf{C})$, $\cot^2 2r = \frac{1}{n-2}$ and n is odd (type C), a tube over a complex two-plane Grassmannian $G_2(\mathbf{C}^5)$, $\cot^2 2r = \frac{3}{5}$ and $n = 9$ (type D), a tube over a Hermitian symmetric space

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$SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ and $n = 15$ (type E). (See also Cecil and Ryan [6].)

The notion of Hopf real hypersurfaces means that the structure vector ξ defined by $\xi = -JN$ satisfies $A\xi = \alpha\xi$, where J denotes a Kaehler structure of $P_n(\mathbf{C})$, N and A a unit normal and the shape operator of M in $P_n(\mathbf{C})$.

In a quaternionic projective space $\mathbf{Q}P^m$, Pérez [8] classified real hypersurfaces in $\mathbf{Q}P^m$ with commuting Ricci tensor $S\phi_i = \phi_i S$, $i = 1, 2, 3$, where S (resp. ϕ_i) denotes the Ricci tensor (resp. the structure tensor) of M in $\mathbf{Q}P^m$. They are locally congruent to of A_1, A_2-type, that is, a tube over $\mathbf{Q}P^k$ with radius $0 < r < \frac{\pi}{2}$, $k \in \{0, \dots, m-1\}$.

The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where J_i, $i = 1, 2, 3$, denote a quaternionic Kähler structure of $\mathbf{Q}P^m$ and N a unit normal field of M in $\mathbf{Q}P^m$. Moreover, the first and third authors [9] considered the notion of $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of a real hypersurface M in $\mathbf{Q}P^m$, and proved that M is locally congruent to a tube of radius $\frac{\pi}{4}$ over $\mathbf{Q}P^k$.

For a commuting problem in quaternionic space forms Berndt [2] introduced the notion of normal Jacobi operator $\bar{R}(X, N)N \in \mathrm{E}ndT_xM$, $x \in M$ for real hypersurfaces M in quaternionic projective space $\mathbf{Q}P^m$ or in quaternionic hyperbolic space $\mathbf{Q}H^m$, where $\bar{R}$ denotes the curvature tensor of a quaternionic projective space $\mathbf{Q}P^m$ and a quaternionic hyperbolic space $\mathbf{Q}H^m$. Berndt [2] also has shown that the curvature adaptedness, that is, the normal Jacobi operator commutes with the shape operator A, and this is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \mathrm{Span}\,\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator A of M, where $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

Now let us consider complex two-plane Grassmannians $G_2(\mathbf{C}^{m+2})$ which consist of all complex 2-dimensional linear subspaces in $\mathbf{C}^{m+2}$. Then the situation for real hypersurfaces in $G_2(\mathbf{C}^{m+1})$ with commuting normal Jacobi operator is not so simple and will be quite different from the cases mentioned above.

So in this paper we consider a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbf{C}^{m+2})$ with commuting normal Jacobi operator, $\bar{R}_N \circ A = A \circ \bar{R}_N$, where $\bar{R}$ and A denote the curvature tensor of the ambient space $G_2(\mathbf{C}^{m+2})$ and the shape operator of M in $G_2(\mathbf{C}^{m+2})$, respectively.

The curvature tensor $\bar{R}(X, Y)Z$ of $G_2(\mathbf{C}^{m+2})$ is explicitly defined in Section 2 and the normal Jacobi operator $\bar{R}_N$ can be derived from the curvature tensor $\bar{R}(X, Y)Z$ by putting $Y = Z = N$ and the geometric structure $JJ_i = J_iJ$, $i = 1, 2, 3$ between the Kaehler structure J and the quaternionic Kaehler structure J_i, $i = 1, 2, 3$ (see Section 3), where N denotes a unit normal vector on M in $G_2(\mathbf{C}^{m+2})$.

The ambient space $G_2(\mathbf{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see Berndt [3]). So, in $G_2(\mathbf{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces that $[\xi] = \mathrm{Span}\,\{\xi\}$ or $\mathfrak{D}^{\perp} = \mathrm{Span}\,\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such kind of geometric conditions Berndt and the third author [4] have proved the following:

THEOREM A. *Let M be a connected real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m \geqq 3$. Then both $[\xi]$ and $\mathfrak{D}^{\perp}$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbf{Q}P^n$ in $G_2(\mathbf{C}^{m+2})$.*

In Theorem A the vector ξ contained in the one-dimensional distribution $[\xi]$ is said to be a Hopf vector when it becomes a principal vector for the shape operator A of M in $G_2(\mathbf{C}^{m+2})$. Moreover in such a situation M is said to be a Hopf hypersurface. Besides, a real hypersurface M in $G_2(\mathbf{C}^{m+2})$ admits the 3-dimensional distribution $\mathfrak{D}^{\perp}$, which is spanned by *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$.

On the other hand, Berndt and the third author [5] considered the geometric condition that the shape operator A of real hypersurfaces M in $G_2(\mathbf{C}^{m+2})$ commutes with the structure tensor, that is, $A\phi = \phi A$. This condition also has the geometric meaning that the flow of the Reeb vector field is isometric. Moreover, Berndt and the third author [3] proved that a real hypersurface in $G_2(\mathbf{C}^{m+2})$ with isometric flow is a tube over a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$.

By putting a unit normal vector N to M in $G_2(\mathbf{C}^{m+2})$ into the curvature tensor $\bar{R}$ of the ambient space $G_2(\mathbf{C}^{m+2})$, we introduce the so called normal Jacobi operator $\bar{R}_N$ defined by

$$(*) \qquad \bar{R}_N(X) = \bar{R}(X,N)N = X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu$$
$$- \sum_{\nu=1}^{3}\big\{\eta_\nu(\xi)J_\nu(\phi X + \eta(X)N) - \eta_\nu(\phi X)\big\{\phi_\nu\xi + \eta_\nu(\xi)N\big\}\big\}$$
$$= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu$$

$$-\sum_{\nu=1}^{3}\left\{\eta_\nu(\xi)\left(\phi_\nu\phi X-\eta(X)\xi_\nu\right)-\eta_\nu(\phi X)\phi_\nu\xi\right\}$$

for any tangent vector fields X on M in $G_2(\mathbf{C}^{m+2})$.

Now in this paper we want to give a complete classification of real hypersurfaces M in $G_2(\mathbf{C}^{m+2})$ concerned with the commuting normal Jacobi operator as follows:

THEOREM 1. *Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbf{C}^{m+2})$. Then M is Hopf if and only if the normal Jacobi operator $\bar{R}_N$ commutes with the shape operator A provided with $\xi\in\mathfrak{D}$ or $\xi\in\mathfrak{D}^\perp$.*

By virtue of Theorem 1 and Theorem A, we assert our main theorem as follows:

THEOREM 2. *Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbf{C}^{m+2})$. If the normal Jacobi operator $\bar{R}_N$ commutes with the structure tensor and the shape operator, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$.*

1. Riemannian geometry of $G_2(\mathbf{C}^{m+2})$

In this section we summarize the basic material about $G_2(\mathbf{C}^{m+2})$, for details we refer to [4] and [5]. By $G_2(\mathbf{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbf{C}^{m+2}$. The special unitary group $G=SU(m+2)$ acts transitively on $G_2(\mathbf{C}^{m+2})$ with stabilizer isomorphic to $K=S\big(U(2)\times U(m)\big)\subset G$. Then $G_2(\mathbf{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbf{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ is an $\mathrm{Ad}\,(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o=eK$ and identify $T_oG_2(\mathbf{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m}\times\mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $\mathrm{Ad}\,(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbf{C}^{m+2})$. In this way $G_2(\mathbf{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbf{C}^{m+2}),g)$ is eight. Since $G_2(\mathbf{C}^3)$ is isometric to the three-dimensional complex projective space $\mathbf{C}P^3$ with constant holomorphic sectional curvature eight, we will assume $m\geqq 2$

from now on. Note that the isomorphism Spin (6) $\simeq SU(4)$ yields an isometry between $G_2(\mathbf{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbf{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbf{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbf{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbf{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\operatorname{tr}(JJ_1) = 0$.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar\nabla$ of $\big(G_2(\mathbf{C}^{m+2}), g\big)$, for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ there exist three local one-forms q_1, q_2, q_3 such that

$$\bar\nabla_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbf{C}^{m+2})$.

Let $p \in G_2(\mathbf{C}^{m+2})$ and W a subspace of $T_pG_2(\mathbf{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbf{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbf{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbf{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbf{C}^{m+2})$.

The Riemannian curvature tensor $\bar R$ of $G_2(\mathbf{C}^{m+2})$ is locally given by

$$\begin{aligned} \bar R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ &- 2g(JX,Y)JZ + \sum_{\nu=1}^{3}\big\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\big\} \\ &+ \sum_{\nu=1}^{3}\big\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\big\}, \end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbf{C}^{m+2})$

In this section we want to derive the normal Jacobi operator from the curvature tensor of the complex two-plane Grassmannian $G_2(\mathbf{C}^{m+2})$ given in (1.2) and the equation of Gauss. Moreover, we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbf{C}^{m+2})$ (see [4], [5], [10], [11], [14] and [15]).

Let M be a real hypersurface of $G_2(\mathbf{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbf{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbf{C}^{m+2})$ induces an almost contact metric structure (ϕ,ξ,η,g) on M. Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M. Using the above expression for $\bar{R}$, the Codazzi equation becomes

$$\begin{aligned}(\nabla_X A)Y-(\nabla_Y A)X&=\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\\&+\sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu Y-\eta_\nu(Y)\phi_\nu X-2g(\phi_\nu X,Y)\xi_\nu\big\}\\&+\sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\big\}\\&+\sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu.\end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

(2.1)
$$\begin{cases}\phi_{\nu+1}\xi_\nu=-\xi_{\nu+2},\quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\quad \phi\xi_\nu=\phi_\nu\xi,\quad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\ \phi_\nu\phi_{\nu+1}X=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\quad \phi_{\nu+1}\phi_\nu X=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}.\end{cases}$$

Now let us put

$$JX=\phi X+\eta(X)N,\quad J_\nu X=\phi_\nu X+\eta_\nu(X)N \tag{2.2}$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbf{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbf{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi,\quad \nabla_X\xi=\phi AX, \tag{2.3}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.4}$$

$$(\nabla_X \phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu. \tag{2.5}$$

Summing up these formulas, we find

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) = (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{2.6}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.7}$$

3. The normal Jacobi operator and the shape operator

Now by putting $Y = Z = N$ in (1.2) for a unit normal vector N to M in $G_2(\mathbf{C}^{m+2})$ and using (2.2), we define a normal Jacobi operator $\bar{R}_N$ by $(*)$ for any vector field X on M, where we have used the following

$$g(J_\nu JN, N) = -g(JN, J_\nu N) = -g(\xi,\xi_\nu) = -\eta_\nu(\xi),$$

$$g(J_\nu JX, N) = g(X, JJ_\nu N) = -g(X, J\xi_\nu)$$

$$= -g\big(X, \phi\xi_\nu + \eta(\xi_\nu)N\big) = -g(X, \phi\xi_\nu),$$

and

$$J_\nu JN = -J_\nu\xi = -\phi_\nu\xi - \eta_\nu(\xi)N.$$

Then by (2.7) we know that the normal Jacobi operator $\bar{R}_N$ could be a symmetric endomorphism of T_xM, $x \in M$.

Now consider the cases $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$. On such distributions, first of all, we calculate the normal Jacobi operator given in $(*)$ as follows:

Case I: $\xi \in \mathfrak{D}$. For any $X \in \mathfrak{D}^\perp$ from the definition of normal Jacobi operator $(*)$ we know that

$$\bar{R}_N X = \bar{R}(X,N)N = X + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu.$$

Hence the structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are eigenvectors of the normal Jacobi opeartor $\bar{R}_N$ as follows:

$$\bar{R}(\xi_1, N)N = 4\xi_1, \quad \bar{R}(\xi_2, N)N = 4\xi_2, \quad \bar{R}(\xi_3, N)N = 4\xi_3. \tag{3.1}$$

Moreover, for any $X \in \mathfrak{D}$ we have

$$\bar{R}(X, N)N = X + 3\eta(X)\xi + \sum_{\nu=1}^{3} \eta_\nu(\phi X)\phi_\nu \xi. \tag{3.2}$$

Thus together with the fact that $\xi, \phi\xi_i \in \mathfrak{D}$, $i = 1, 2, 3$ we have

$$\bar{R}(\xi, N)N = 4\xi, \quad \bar{R}(\phi_i\xi, N)N = 0, \quad i = 1, 2, 3. \tag{3.3}$$

Then (3.1), (3.2) and (3.3) give that there exists an orthogonal matrix$P$ such that

$$P = \left[\xi_1, \xi_2, \xi_3, \xi, \phi_1\xi, \phi_2\xi, \phi_3\xi, e_1, \ldots, e_{4(m-2)}\right]$$

and for any vector $e_i \in \mathfrak{D}$, $i = 1, \ldots, 4(m-2)$ in T_xM, $x \in M$, the normal Jacobi operator $\bar{R}_N$ of a real hypersurface M in $G_2(\mathbf{C}^{m+2})$ can be diagonalized as follows:

$${}^tP\bar{R}_N P = \begin{bmatrix} B & & \\ & C & 0 \\ 0 & & I \end{bmatrix},$$

where I denotes the $4(m-2) \times 4(m-2)$ identity matrix and the matrices B and C are

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Case II: $\xi \in \mathfrak{D}^\perp$. Without loss of generality we can put $\xi = \xi_1$. For any $X \in \mathfrak{D}$ the normal Jacobi operator $\bar{R}_N$ is given by

$$\bar{R}_N(X) = \bar{R}(X, N)N = X - \phi_1\phi X$$

and for any $X \in \mathfrak{D}^\perp$ the nomal Jacobi operator is

$$\bar{R}_N(X) = X + 4\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu - \phi_1\phi X + \sum_{\nu=1}^{3}\eta_\nu(\phi X)\phi_\nu\xi. \tag{3.4}$$

Then the normal Jacobi operator for $\mathfrak{D}^{\perp} = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$ can be calculated as

$$\bar{R}_N(\xi_1) = 8\xi_1, \quad \bar{R}_N(\xi_2) = 2\xi_2, \quad \bar{R}_N(\xi_3) = 2\xi_3.$$

On the other hand, we calculate the normal Jacobi operator $\bar{R}_N$ for $\mathfrak{D} = \{X \in T_xM \mid X \perp \xi_i,\ i = 1,2,3\}$. Then for any $X \in \mathfrak{D}$ such that $\phi X = \phi_1 X$ or $\phi X = -\phi_1 X$, the normal Jacobi operator $\bar{R}_N$ is given by

$$\bar{R}_N(X) = 2X \quad \text{or} \quad \bar{R}_N(X) = 0, \tag{3.5}$$

respectively. Here the dimension of the eigenspace corresponding to an eigenvalue 2 (resp. 0) is equal to $2(m-1)$ (resp. $2(m-1)$) (see Berndt [3]). Then the normal Jacobi operator $\bar{R}_N$ for the orthogonal matrix P can be given by

$${}^tP\bar{R}_NP = \begin{bmatrix} 8 & & & & \text{\Large 0} \\ & 2 & & & \\ & & 2 & & \\ & & & B & \\ \text{\Large 0} & & & & C \end{bmatrix},$$

where (3.5) gives that the matrices B and C are $2(m-1) \times 2(m-1)$-matrices respectively given by

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now recall a proposition from [2] concerned with a tube of type (A) as follows:

Proposition A. *Let M be a connected real hypersurface in $G_2(\mathbf{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot\big(\sqrt{8}r\big), \quad \beta = \sqrt{2}\cot\big(\sqrt{2}r\big), \quad \lambda = -\sqrt{2}\tan\big(\sqrt{2}r\big), \quad \mu = 0 \tag{3.6}$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m-2 = m(\mu),$$

and for the corresponding eigenspaces we have

$$T_\alpha = \mathbf{R}\xi = \mathbf{R}JN, \quad T_\beta = \mathbf{C}^{\perp}\xi = \mathbf{C}^{\perp}N = \{\xi_2, \xi_3\},$$

$$T_\lambda = \{X \mid X \perp \mathbf{H}\xi,\ JX = J_1 X\}, \quad T_\mu = \{X \mid X \perp \mathbf{H}\xi,\ JX = -J_1 X\}.$$

Now let us check for a tube over a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$ whether it satisfies the commuting normal Jacobi operator, that is $\bar{R}_N \circ A = A \circ \bar{R}_N$ or not as follows:

Case I: $\xi = \xi_1 \in T_\alpha$. Then by (3.4) we have

$$(\bar{R}_N \circ A)\xi_1 = \alpha \bar{R}_N \xi_1 = 8\alpha \xi_1.$$

On the other hand, the left side becomes $A \circ (\bar{R}_N \xi_1) = 8A\xi_1 = 8\alpha\xi_1$. So in this case $\bar{R}_N \circ A = A \circ \bar{R}_N$ holds.

Case II: $\xi_2, \xi_3 \in T_\beta$. Also by (3.4) we have $(\bar{R}_N \circ A)\xi_2 = \beta \bar{R}_N \xi_1 = 2\beta\xi_2$. Moreover, the left side becomes $A \circ (\bar{R}_N \xi_2) = 2A\xi_2 = 2\beta\xi_2$.

Case III: $X_i \in T_\lambda$, $i = 1, \dots, 2(m-1)$. Then by (3.5) we assert

$$(\bar{R}_N \circ A)X_i = \bar{R}_N(AX_i) = \lambda \bar{R}_N X_i = 2\lambda X_i$$

and

$$(A \circ \bar{R}_N)X_i = A(\bar{R}_N X_i) = 2AX_i = 2\lambda X_i$$

for $i = 1, \dots, 2(m-1)$.

Case IV: $Y_i \in T_\mu$, $i = 1, \dots, 2(m-1)$. Then also by (3.5) we have $(\bar{R}_N \circ A)Y_i = \bar{R}_N Y_i = 0$ and $A\bar{R}_N Y_i = 0$.

Summarizing these cases, we know that a real hypersurface of type (A) in Theorem A satisfies $\bar{R}_N \circ A = A \circ \bar{R}_N$.

Next we consider M congruent to a tube of type (B) mentioned in Theorem A, that is, a tube of radius r over $\mathbf{H}P^m$, $m = 2n$ in $G_2(\mathbf{C}^{m+2})$. That is, for a tube of type B in Theorem A we introduce the following

PROPOSITION B. *Let M be a connected real hypersurface of $G_2(\mathbf{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbf{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbf{R}\xi, \quad T_\beta = \mathfrak{J}J\xi, \quad T_\gamma = \mathfrak{J}\xi, \quad T_\lambda, \quad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbf{HC}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

Now let us check for a real hypersurface of type (B) whether it satisfies the property of commuting normal Jacobi operator as follows:

Case I: $\xi \in T_\alpha$. Then by (3.3) we know

$$A \circ \bar{R}_N\xi = 4A\xi_1 = 4\alpha\xi \quad \text{and} \quad \bar{R}_N A\xi = \alpha\bar{R}_N\xi = 4\alpha\xi.$$

Case II: $\xi_1, \xi_2, \xi_3 \in T_\beta$. Then also by (3.2) we have

$$A \circ \bar{R}_N\xi_i = 4A\xi_i = 4\beta\xi_1 \quad \text{and} \quad \bar{R}_N A\xi_i = \beta\bar{R}_N\xi_i = 4\beta\xi_i$$

for $i = 1, 2, 3$.

Case III: $\phi_i\xi \in T_\gamma$, $i = 1, 2, 3$. Then by (3.3) we have

$$A \circ \bar{R}_N\phi\xi_i = 0 \quad \text{and} \quad \bar{R}_N A\phi\xi_i = 0.$$

Case IV: $X_i \in T_\lambda$, $i = 1, \dots, 2(m-2)$. Then by (3.2) we have

$$A \circ \bar{R}_N X_i = AX_i = \lambda X_i \quad \text{and} \quad \bar{R}_N AX_i = \lambda\bar{R}_N X_i = \lambda X_i.$$

Case V: $Y_i \in T_\mu$, $i = 1, \dots, 2(m-2)$. Then also by (3.2) we have

$$A \circ \bar{R}_N Y_i = AX_i = \mu X_i \quad \text{and} \quad \bar{R}_N AX_i = \mu\bar{R}_N X_i = \mu X_i.$$

Summarizing the cases mentioned above, we know that real hypersurfaces of type (B) satisfy the property of commuting normal Jacobi operator, that is, $\bar{R}_N \circ A = A \circ \bar{R}_N$.

4. Commuting normal Jacobi operator

In this section we consider a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbf{C}^{m+2})$ with commuting normal Jacobi operator.

Consider a normal Jacobi operator

$$\bar{R}_N(X) = \bar{R}(X, N)N \in T_xM, \quad x \in M$$

defined in $(*)$. Then the commuting Jacobi operator $A \circ \bar{R}_N = \bar{R}_N \circ A$ gives that

$$(4.1) \qquad 3\sum_{\nu=1}^{3}\eta_\nu(X)A\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\xi)A\phi_\nu\phi X + \sum_{\nu=1}^{3}\eta(X)\eta_\nu(\xi)A\xi_\nu$$

$$+\sum_{\nu=1}^{3}\eta_\nu(\phi X)A\phi_\nu\xi+3\eta(X)A\xi$$

$$=3\sum_{\nu=1}^{3}\eta_\nu(AX)\xi_\nu-\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\phi AX+\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta(AX)\xi_\nu$$

$$+\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\phi_\nu\xi+3\eta(AX)\xi.$$

Now we consider the following two cases.

Case I: $\xi\in\mathfrak{D}$. Then (4.1) gives that

$$3\sum_{\nu=1}^{3}\eta_\nu(X)A\xi_\nu+\sum_{\nu=1}^{3}\eta_\nu(\phi X)A\phi_\nu\xi+3\eta(X)A\xi$$

$$=3\sum_{\nu=1}^{3}\eta_\nu(AX)\xi_\nu+3\eta(AX)\xi+\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi)\phi_\nu\xi.$$

Putting $X=\xi$ into this equation,

$$3A\xi=3\sum_{\nu=1}^{3}\eta_\nu(A\xi)\xi_\nu+3\eta(A\xi)\xi+\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi)\phi_\nu\xi. \tag{4.2}$$

On the other hand, from the assumption we know that the shape operator A is invariant, that is $g(A\mathfrak{D},\mathfrak{D}^\perp)=0$. This means $\eta_\nu(A\xi)=0$ for $\xi\in\mathfrak{D}$. Moreover, differentiating $g(\xi,\xi_\nu)=0$ along the direction ξ gives

$$g(\nabla_\xi\xi,\xi_\nu)+g(\xi,\nabla_\xi\xi_\nu)=0.$$

Hence, together with the formulas in Section 2 it follows that

$$\eta_\nu(\phi A\xi)=-g(\xi,q_{\nu+2}(\xi)\xi_{\nu+1}-q_{\nu+1}(\xi)\xi_{\nu+2}+\phi_\nu A\xi)$$

$$=-g(\xi,\phi_\nu A\xi)=-\eta_\nu(\phi A\xi).$$

Then $\eta_\nu(\phi A\xi)=0$, and together with (4.2), we assert that M is Hopf, that is, the structure vector ξ is principal.

Case II: $\xi\in\mathfrak{D}^\perp$. Without loss of generality we may put $\xi=\xi_1$. Now putting $X=\xi_1$ in (4.1), we have

$$7A\xi=2\sum_{\nu=1}^{3}\eta_\nu(A\xi_1)\xi_\nu-\phi_1\phi A\xi+5\eta(A\xi)\xi. \tag{4.3}$$

Hence, by applying ϕ to both sides and using the formulas in Section 2 we have

$$7\phi A\xi = 2\sum_{\nu=1}^{3}\eta_\nu(A\xi_1)\phi\xi_\nu - \phi\phi_1\phi A\xi = 2\sum_{\nu=1}^{3}\eta_\nu(A\xi_1)\phi\xi_\nu + \phi_1 A\xi.$$

Substituting this into (4.3), we have

$$49A\xi = 14\sum_{\nu=1}^{3}\eta_\nu(A\xi_1)\xi_\nu - \phi_1\left\{2\sum_{\nu=1}^{3}\eta_\nu(A\xi_1)\phi_1\phi\xi_\nu + \phi_1 A\xi\right\} + 35\eta(A\xi)\xi.$$

This implies

$$49A\xi = 14\sum_{\nu=1}^{3}\eta_\nu(A\xi_1)\xi_\nu - 2\big\{\eta_2(A\xi_1)\xi_2 + \eta_3(A\xi_1)\xi_3\big\} + A\xi + 34\eta(A\xi)\xi,$$

which yields

$$48A\xi = 12\sum_{\nu=1}^{3}\eta_\nu(A\xi_1)\xi_\nu + 36\eta(A\xi)\xi. \tag{4.4}$$

Hence, taking an inner product with ξ_2 and ξ_3 respectively, we have

$$\eta_2(A\xi) = 0 \quad \text{and} \quad \eta_3(A\xi) = 0.$$

Substituting this into (4.4) finally gives $A\xi = \eta(A\xi)\xi$. This means that a real hypersurface M satisfying the commuting normal Jacobi operator is also a Hopf hypersurface in this case.

Summarizing the above cases, we assert the following

THEOREM 4.1. *Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbf{C}^{m+2})$ with commuting Jacobi operator. Then M is Hopf provided with $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.*

Hence, together with a theorem due to Berndt and Suh [4], we have the following

THEOREM 4.2. *Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m \geqq 3$, with commuting normal Jacobi operator. Then M is congruent to one of the following provided $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$:*

(A) *an open part of a tube around a totally geodesic $G_2(\mathbf{C}^{m+1})$ in $G_2(\mathbf{C}^{m+2})$, or*

(B) *an open part of a tube around a totally geodesic and totally real $\mathbf{Q}P^n$, $m = 2n$, in $G_2(\mathbf{C}^{m+2})$.*

Now recall a lemma due to Suh [12] as follows:

Lemma 4.3. *Let M be a real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m \geqq 3$. If the shape operator is parallel along the distribution $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$, then $\xi \in \mathfrak{D}$.*

By Lemma 4.3 and Theorem 4.2 we assert the following

Theorem 4.4. *Let M be a real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m \geqq 3$. If the shape operator is parallel along the distribution $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$, then M is locally congruent to a tube over a totally geodesic, totally real $\mathbf{Q}P^m$, $m = 2n$, in $G_2(\mathbf{C}^{m+2})$.*

Next we consider M congruent to a tube of type (B) mentioned in Theorem 4.4, that is, a tube of radius r over $\mathbf{Q}P^m$, $m = 2n$ in $G_2(\mathbf{C}^{m+2})$. Then for any $Y \in T_\lambda$ due to Proposition B in Section 3 we put

$$AY = \cot rY, \quad A\phi Y = -\tan r\phi Y, \quad \phi AY = \cot r\phi Y,$$

$$A\phi AY = \cot rA\phi Y = -\phi Y.$$

So by the equation of Codazzi we have for any $Y \in T_\lambda$

$$0 = (\nabla_\xi A)Y = (\nabla_Y A)\xi + \eta(\xi)\phi Y = (\alpha I - A)\phi AY + \phi Y$$

$$= \alpha\phi AY - A\phi AY + \phi Y = \{-2\tan 2r \cdot \cot r + 2\}\phi Y,$$

where in the second equality we have used $\mathfrak{J}T_\lambda = T_\lambda$, $\mathfrak{J}T_\mu = T_\mu$. This gives $\tan^2 r + 1 = 0$, which is a contradiction. Then by Theorem 4.4 and Proposition B in Section 3 we assert the following

Theorem 4.5. *There do not exist any real hypersurfaces M in $G_2(\mathbf{C}^{m+2})$ with parallel second fundamental tensor on $\mathfrak{F}$ when M has a commuting Jacobi operator.*

5. Proof of the main theorem

In this section we consider a real hypersurface M in $G_2(\mathbf{C}^{m+2})$ whose normal Jacobi operator $\bar{R}_N$ commutes with the structure tensor ϕ. Then the commuting condition $\bar{R}_N \circ \phi = \phi \circ \bar{R}_N$ gives

$$(5.1) \quad 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu - \sum_{\nu=1}^{3}\big\{\eta_\nu(\xi)\big(\phi\phi_\nu\phi X - \eta(X)\phi\xi_\nu\big) - \eta_\nu(\phi X)\phi\phi_\nu\xi\big\}$$

$$= 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu - \sum_{\nu=1}^{3}\big\{\eta_\nu(\xi)\big(\phi_\nu\phi^2 X - \eta(\phi X)\xi_\nu\big) - \eta_\nu(\phi^2 X)\phi_\nu\xi\big\}.$$

This can be written as follows:

$$4\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu-\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta(\phi_\nu X)\xi=4\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu. \tag{5.2}$$

Hence, taking an inner product with ξ, we have

$$\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta(\phi_\nu X)=0.$$

Then (5.2) becomes

$$\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu=\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu. \tag{5.3}$$

Putting $X=\xi$ in (5.3), we have

$$\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu=0. \tag{5.4}$$

Now we put $\xi=X_1+X_2$ for some $X_1\in\mathfrak{D}^\perp$ and $X_2\in\mathfrak{D}$. Then (5.4) gives

$$0=\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi=\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu X_1+\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu X_2.$$

From this it follows that

$$\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu X_1=0 \quad\text{and}\quad \sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu X_2=0, \tag{5.5}$$

because $\phi_\nu X_1\in\mathfrak{D}^\perp$ and $\phi_\nu X_2\in\mathfrak{D}$ for any $\nu=1,2,3$. The term in the second formula of (5.5) is included in a distribution $\mathfrak{D}$. If X_2 vanishes, then $\xi\in\mathfrak{D}^\perp$. If not, then $\{\phi_1X_2,\phi_2X_2,\phi_3X_3\}\in\mathfrak{D}$ are linearly independent. So naturally we have $\eta_\nu(\xi)=0$ for any $\nu=1,2,3$. This means that $\xi\in\mathfrak{D}$.

Summarizing, we assert the following

LEMMA 5.1. *If M is a real hypersurface in $G_2(\mathbf{C}^{m+2})$, $m\geqq 3$, satisfying $\bar{R}_N\circ\phi=\phi\circ\bar{R}_N$, then $\xi\in\mathfrak{D}$ or $\xi\in\mathfrak{D}^\perp$.*

Hence, together with Theorem 3.1, we have the following

THEOREM 5.2. *Let M be a $\mathfrak{D}^{\perp}$-invariant real hypersurface in $G_2(\mathbf{C}^{m+2})$. Then M is Hopf if the Jacobi operator $\bar{R}_N$ commutes both with the shape operator A and the structure tensor ϕ.*

Hence, together with Theorem A due to Berndt and Suh [4], we know that M is congruent to an open part of a tube around a totally geodesic $G_2(\mathbf{C}^{m+1})$ or a tube around a totally geodesic quaternionic projective space $\mathbf{Q}P^n$, $m = 2n$, in $G_2(\mathbf{C}^{m+2})$.

Related to our assumption of commuting normal Jacobi operator with the structure tensor we give the following two remarks:

REMARK 5.1. By Proposition A let us check whether real hypersurfaces of type (A) satisfy $\bar{R}_N \circ \phi = \phi \circ \bar{R}_N$ or not as follows:

Case I: $\xi = \xi_1 \in T_\alpha$. Then by (3.4) we know that $\bar{R}_N \circ \phi\xi_1 = 0$ and $\phi \circ \bar{R}_N \xi_1 = 8\phi\xi_1 = 0$.

Case II: $\xi_2, \xi_3 \in T_\beta$. Then also by (3.4) we have $\bar{R}_N \circ \phi\xi_2 = -\bar{R}_N\xi_3 = -2\xi_3$ and $\phi \circ \bar{R}_N A\xi_2 = 2\phi\xi_2 = -2\xi_3$.

Case III: $X_i \in T_\lambda$, $i = 1, \dots, 2(m-1)$. Then by Proposition A, the eigen space T_λ has the property that $\phi X = \phi_1 X$ for any $X \in T_\lambda$. Moreover, it is invariant by the structure tensor ϕ, that is $\phi T_\lambda \subset T_\lambda$, because for any $X_i \in \mathfrak{D}$ such that $\phi X_i = \phi_1 X_i$ we have $\phi\phi X_i = -X_i$ and $\phi_1\phi X_i = \phi_1^2 X_i = -X_i$. Thus $\phi\phi X_i = \phi_1\phi X_i$. So it follows that $\phi X_i \in T_\lambda$. Hence, together with (3.5), we have $(\bar{R}_N \circ \phi)X_i = 2\phi X_i$ and $(\phi \circ \bar{R}_N)X_i = 2\phi X_i$.

Case IV: $Y_i \in T_\mu$, $i = 1, \dots, 2(m-1)$. The eigen space T_μ has the property that $\phi Y = -\phi_1 Y$ for any $Y \in T_\mu$. Moreover, such an eigen space T_μ is ϕ-invariant, that is, $\phi T_\mu \subset T_\mu$. In fact, suppose $Y_i \in \mathfrak{D}$ such that $\phi Y_i = -\phi_1 Y_i$. Then $\phi Y_i \in \mathfrak{D}$, $\phi\phi Y_i = -Y_i$ and $\phi_1\phi Y_i = -\phi_1^2 Y_i = Y_i$. So it follows that $\phi Y_i \in \mathfrak{D}$. By using (3.5), $\bar{R}_N \circ \phi Y_i = 0$ and $\phi \circ \bar{R}_N Y_i = 0$. Then by Cases I, II, III and IV the normal Jacobi operator $\bar{R}_N$ commutes with the structure tensor ϕ for real hypersurfaces of type (A) in Theorem A.

REMARK 5.2. By Proposition B we want to check whether real hypersurfaces of type (B) in Theorem A satisfy $\bar{R}_N \circ \phi = \phi \circ \bar{R}_N$ or not as follows.

In fact, from (3.3) for $\xi \in \mathfrak{D}$ it follows that $\bar{R}_N \circ \phi\xi_2 = 0$, but from (3.1) we know that $\phi \circ \bar{R}_N\xi_2 = 4\phi\xi_2$. So naturally the normal Jacobi operator $\bar{R}_N$ does not commute with the structure tensor ϕ for $\xi_2 \in T_\beta$ in Proposition B.

By Theorem 5.2 and Theorem A, together with Remarks 5.1 and 5.2 mentioned above we complete the proof of our Theorem 2.

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