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Real Hypersurfaces in Complex Two-plane Grassmannians with $\mathfrak{F}$-parallel Normal Jacobi Operator

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Abstract. In this paper we give a non-existence theorem for Hopf hypersurfaces M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ whose normal Jacobi operator $\bar{R}_N$ is parallel on the distribution $\mathfrak{F}$ defined by $\mathfrak{F} = [\xi] \cup \mathfrak{D}^{\perp}$, where $[\xi] = \text{Span}\{\xi\}$, $\mathfrak{D}^{\perp} = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ and $T_xM = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$, $x \in M$.

1. Introduction

The geometry of real hypersurfaces in complex space forms or in quaternionic space forms is one of interesting parts in the field of differential geometry. Until now there have been many characterizations for homogeneous hypersurfaces of type (A_1), (A_2), (B), (C), (D) and (E) in complex projective space $\mathbb{C}P^m$, of type (A_1), (A_2) and (B) in quaternionic projective space $\mathbb{H}P^m$, or of type (A) and (B) in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. Each corresponding geometric features are classified and investigated by Berndt [2], Pérez, Santos and Suh [13], Ki and Suh [9], Kimura [10], Berndt and Suh [3] and [4], respectively.

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold. A vector field U along a geodesic γ in a Riemannian manifold $\bar{M}$ is said to be a *Jacobi field* if it satisfies a differential equation

$$\bar{\nabla}^2_{\dot{\gamma}} U + \bar{R}(U(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0,$$

where $\bar{\nabla}_{\dot{\gamma}} U$ and $\bar{R}$ respectively denotes the covariant derivative of the vector field U along the curve γ in $\bar{M}$ and the curvature tensor of the Riemannian manifold $(\bar{M}, \bar{g})$. Then this equation is called the *Jacobi* equation.

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The Jacobi operator $\bar{R}_X$ for any tangent vector field X at $x \in \bar{M}$, is defined by

$$(\bar{R}_X Y)(x) = (\bar{R}(Y, X)X)(x)$$

for any $Y \in T_x\bar{M}$, becomes a self adjoint endomorphism of the tangent bundle $T\bar{M}$ of $\bar{M}$. That is, the Jacobi operator satisfies $\bar{R}_X \in \mathrm{End}(T_x\bar{M})$ and is symmetric in the sense of $\bar{g}(\bar{R}_X Y, Z) = \bar{g}(\bar{R}_X Z, Y)$ for any vector fields Y and Z on $\bar{M}$.

The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$ of $\mathbb{H}P^m$ and N a unit normal field of M in $\mathbb{H}P^m$. In a quaternionic projective space $\mathbb{H}P^m$, Pérez and Suh [11] have classified real hypersurfaces in $\mathbb{H}P^m$ with $\mathfrak{D}^\perp$-parallel curvature tensor, that is, $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of M in $\mathbb{H}P^m$ and $\mathfrak{D}^\perp$ a distribution defined by $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$. In such a case they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic $\mathbb{H}P^k$ in $\mathbb{H}P^m$, $0 \leq k \leq m-1$.

Kimura [10] proved that any tube M around a complex submanifold in complex projective space $\mathbb{C}P^m$ are characterized by the invariant of $A\xi = \alpha\xi$, where the Reeb vector ξ is defined by $\xi = -JN$ for a Kähler structure J and a unit normal N to hypersurfaces M in $\mathbb{C}P^m$. Moreover, the corresponding geometrical feature for hypersurfaces in $\mathbb{H}P^m$ is the invariant of the distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ by the shape operator, where $\xi_i = -J_i N$, $J_i \in \mathfrak{J}$. In fact, every tube around a quaternionic submanifold $\mathbb{H}P^m$ satisfies such kind of geometrical feature (See Alekseevskii [1]).

The complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ which consists of all complex two dimensional linear subspaces in $\mathbb{C}^{m+2}$ has a remarkable geometric structure. It is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ (See Berndt and Suh [3]). From such a view point, Berndt and Suh [3] considered two natural geometric conditions for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ that $[\xi] = \mathrm{Span}\{\xi\}$ and $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator. By using such conditions and the result in Alekseevskii [1], Berndt and Suh [3] proved the following

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The structure vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a *Reeb* vector field. If the *Reeb* vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, M is said to be a *Hopf hypersurface*. In such a case the integral curves of the *Reeb* vector field ξ are geodesics (See Berndt and Suh [4]). Moreover, the flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a *geodesic Reeb flow.* And the corresponding principal curvature α is non-vanishing we say that M is with non-vanishing *geodesic Reeb flow.*

Now by putting a unit normal vector N into the curvature tensor $\bar{R}$ of the ambient space $G_2(\mathbb{C}^{m+2})$, we calculate the normal Jacobi operator $\bar{R}_N$ in such a way that

$$\begin{aligned}\bar{R}_N X &= \bar{R}(X,N)N \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\}\end{aligned}$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$ (See [5]).

In papers [5] and [6] due to Jeong, Pérez and Suh we have introduced a notion of normal Jacobi operator $\bar{R}_N$ for hypersurfaces in $G_2(\mathbb{C}^{m+2})$ in such a way that

$$\bar{R}_N = \bar{R}(X,N)N \in \mathrm{End}(T_xM), \quad x \in M,$$

where $\bar{R}$ denotes the curvature tensor of $G_2(\mathbb{C}^{m+2})$. They [5] have also classified real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting normal Jacobi operator, that is, $\bar{R}_N \circ \phi = \phi \circ \bar{R}_N$ or otherwise $\bar{R}_N \circ A = A \circ \bar{R}_N$ and proved that the commuting normal Jacobi operator $\bar{R}_N$ with the shape operator A is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$ are invariant by the shape operator A for Hopf hypersurfaces M in $G_2(\mathbb{C}^{m+2})$, where $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$. Moreover, a normal Jacobi operator for hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be parallel if $\nabla_X \bar{R}_N = 0$ for any X in T_xM, $x \in M$ (See [7]). And in paper [7], the present authors and Kim proved a non-existence theorem for Hopf hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator as follows:

Theorem B. *There do not exist any connected Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator.*

On the other hand, Suh [15] obtained a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator on the distribution $\mathfrak{F}$ defined by $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$, where $[\xi] = \mathrm{Span}\{\xi\}$, $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ and $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

As a generalization of parallel normal Jacobi operator, let us introduce the notion of parallel normal Jacobi operator along the distribution $\mathfrak{F}$, that is, $\nabla_X \bar{R}_N = 0$ for any X in $\mathfrak{F}$, defined by $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$ on M in $G_2(\mathbb{C}^{m+2})$.

A $\mathfrak{F}$-parallel normal Jacobi operator means that the normal Jacobi operator $\bar{R}_N$ is *parallel* along the distribution $\mathfrak{F}$ of M in $G_2(\mathbb{C}^{m+2})$, that is, the eigenspaces of the normal Jacobi operator $\bar{R}_N$ is *parallel* along the distribution $\mathfrak{F}$ of M. Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be *parallel* along the distribution $\mathfrak{F}$ if they are *invariant* with respect to any *parallel displacement* along the distribution $\mathfrak{F}$.

As a generalization of Theorem B, we prove a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{F}$-parallel normal Jacobi operator as follows:

Main Theorem. *There do not exist any connected Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{F}$-parallel normal Jacobi operator, where $\mathfrak{F} = [\xi] \cup \mathfrak{D}^{\perp}$.*

2. Riemannian geometry of Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [3], [4], [16] and [17].

By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$.

We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric $\bar{g}$ on $G_2(\mathbb{C}^{m+2})$.

In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize $\bar{g}$ such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), \bar{g})$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight.

When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$.

Let J_ν, $\nu = 1, 2, 3$, be a canonical local basis, which becomes an almost Hermitian structure in $\mathfrak{J}$. Then $JJ_\nu = J_\nu J$ and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{Tr}(JJ_\nu) = 0$, $\nu = 1, 2, 3$.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar\nabla$ of $(G_2(\mathbb{C}^{m+2}), \bar g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar\nabla_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{2.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar R$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned} \bar R(X,Y)Z = {} & \bar g(Y,Z)X - \bar g(X,Z)Y + \bar g(JY,Z)JX \\ & - \bar g(JX,Z)JY - 2\bar g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\big\{\bar g(J_\nu Y,Z)J_\nu X - \bar g(J_\nu X,Z)J_\nu Y - 2\bar g(J_\nu X,Y)J_\nu Z\big\} \\ & + \sum_{\nu=1}^{3}\big\{\bar g(J_\nu JY,Z)J_\nu JX - \bar g(J_\nu JX,Z)J_\nu JY\big\}, \end{aligned} \tag{2.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$ (See [3]).

3. Some fundamental formulas

In this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (See [3], [4], [14], [15], [16] and [17]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression (2.2) for the curvature tensor $\bar R$, the Gauss and Codazzi equations are respectively given by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y \\ & + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ & + \sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X-g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\}\\
&-\sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X-\eta(X)\eta_\nu(Z)\phi_\nu\phi Y\}\\
&-\sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y,Z)-\eta(Y)g(\phi_\nu\phi X,Z)\}\xi_\nu\\
&+g(AY,Z)AX-g(AX,Z)AY
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla_X A)Y-(\nabla_Y A)X=&\ \eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\\
&+\sum_{\nu=1}^{3}\left\{\eta_\nu(X)\phi_\nu Y-\eta_\nu(Y)\phi_\nu X-2g(\phi_\nu X,Y)\xi_\nu\right\}\\
&+\sum_{\nu=1}^{3}\left\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\right\}\\
&+\sum_{\nu=1}^{3}\left\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\right\}\xi_\nu,
\end{aligned}
$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$
\begin{aligned}
&\phi_{\nu+1}\xi_\nu=-\xi_{\nu+2},\quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\
&\phi\xi_\nu=\phi_\nu\xi,\quad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\
&\phi_\nu\phi_{\nu+1}X=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}.
\end{aligned}
\tag{3.1}
$$

Now let us put

$$
JX=\phi X+\eta(X)N,\quad J_\nu X=\phi_\nu X+\eta_\nu(X)N \tag{3.2}
$$

for any tangent vector X of M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (2.1) and (3.1) we have that

$$
(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi,\quad \nabla_X\xi=\phi AX, \tag{3.3}
$$

$$
\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX, \tag{3.4}
$$

$$
\begin{aligned}
(\nabla_X\phi_\nu)Y=&-q_{\nu+1}(X)\phi_{\nu+2}Y+q_{\nu+2}(X)\phi_{\nu+1}Y+\eta_\nu(Y)AX\\
&-g(AX,Y)\xi_\nu.
\end{aligned}
\tag{3.5}
$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, it follows that

$$\phi\phi_\nu X = \phi_\nu \phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{3.6}$$

4. Key lemmas

Now let us consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator $\bar{R}_N$, that is, $\nabla_X \bar{R}_N = 0$ for any vector field X on M. Then first of all, we want to derive the normal Jacobi operator from the curvature tensor $\bar{R}(X, Y)Z$ of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ given in (2.2). So the normal Jacobi operator $\bar{R}_N$ is given by

$$\begin{aligned}
\bar{R}_N(X) &= \bar{R}(X, N)N \\
&= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&\quad - \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)J_\nu(\phi X + \eta(X)N) - \eta_\nu(\phi X)(\phi_\nu\xi + \eta_\nu(\xi)N)\Big\} \\
&= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&\quad - \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\Big\},
\end{aligned}$$

where we have used the following

$$\begin{aligned}
g(J_\nu JN, N) &= -g(JN, J_\nu N) = -g(\xi, \xi_\nu) = -\eta_\nu(\xi), \\
g(J_\nu JX, N) &= g(X, JJ_\nu N) = -g(X, J\xi_\nu) \\
&= -g(X, \phi\xi_\nu + \eta(\xi_\nu)N) = -g(X, \phi\xi_\nu)
\end{aligned}$$

and

$$J_\nu JN = -J_\nu\xi = -\phi_\nu\xi - \eta_\nu(\xi)N.$$

Of course, by (3.6) the normal Jacobi operator $\bar{R}_N$ could be symmetric endomorphism of T_xM, $x \in M$ (See [5]).

Now let us consider a covariant derivative of the normal Jacobi operator $\bar{R}_N$ along any direction X of T_xM, $x \in M$ (See [6], [7]). Then it is given by

$$\begin{aligned}
(\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N Y) - \bar{R}_N(\nabla_X Y) \\
&= 3(\nabla_X\eta)(Y)\xi + 3\eta(Y)\nabla_X\xi + 3\sum_{\nu=1}^{3}(\nabla_X\eta_\nu)(Y)\xi_\nu \\
&\quad + 3\sum_{\nu=1}^{3}\eta_\nu(Y)\nabla_X\xi_\nu - \sum_{\nu=1}^{3}\Big[X(\eta_\nu(\xi))(\phi_\nu\phi Y - \eta(Y)\xi_\nu) \\
&\quad + \eta_\nu(\xi)\big\{(\nabla_X\phi_\nu\phi)Y - (\nabla_X\eta)(Y)\xi_\nu - \eta(Y)\nabla_X\xi_\nu\big\}
\end{aligned}$$

$$- (\nabla_X \eta_\nu)(\phi Y)\phi_\nu \xi - \eta_\nu((\nabla_X \phi)Y)\phi_\nu \xi - \eta_\nu(\phi Y)\nabla_X(\phi_\nu \xi)\Big],$$

where the formula $X(\eta_\nu(\xi))$ in the right side is given by

$$\begin{aligned} X(\eta_\nu(\xi)) &= g(\nabla_X \xi_\nu, \xi) + g(\xi_\nu, \nabla_X \xi) \\ &= q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) + 2g(\phi_\nu AX, \xi). \end{aligned}$$

From this, together with the formulas given in section 3, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator, that is, $\nabla_X \bar{R}_N = 0$ for any vector field X on M, satisfies the following

$$\begin{aligned} 0 = {} & 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX \\ & + 3\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+1}(X)\eta_{\nu+2}(Y) \\ & + g(\phi_\nu AX, Y)\Big\}\xi_\nu \\ & + 3\sum_{\nu=1}^{3}\eta_\nu(Y)\Big\{q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX\Big\} \\ & - \sum_{\nu=1}^{3}\Big[\Big\{q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) \\ & + 2\eta_\nu(\phi AX)\Big\}(\phi_\nu \phi Y - \eta(Y)\xi_\nu) \\ & + \eta_\nu(\xi)\Big\{ - q_{\nu+1}(X)\phi_{\nu+2}\phi Y + q_{\nu+2}(X)\phi_{\nu+1}\phi Y \\ & + \eta_\nu(\phi Y)AX - g(AX, \phi Y)\xi_\nu \\ & + \eta(Y)\phi_\nu AX - g(AX, Y)\phi_\nu \xi - g(\phi AX, Y)\xi_\nu \\ & - \eta(Y)(q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX)\Big\} \\ & - \Big\{q_{\nu+2}(X)\eta_{\nu+1}(\phi Y) - q_{\nu+1}(X)\eta_{\nu+2}(\phi Y) \\ & + g(\phi_\nu AX, \phi Y)\Big\}\phi_\nu \xi \\ & - \Big\{\eta(Y)\eta_\nu(AX) - g(AX, Y)\eta_\nu(\xi)\Big\}\phi_\nu \xi \\ & - \eta_\nu(\phi Y)\Big\{q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi \\ & + \phi_\nu \phi AX - g(AX, \xi)\xi_\nu + \eta(\xi_\nu)AX\Big\}\Big]. \end{aligned} \tag{4.1}$$

On the other hand, we know that the following formulas vanish.

$$\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(X)\eta_{\nu+1}(Y)\xi_\nu - q_{\nu+1}(X)\eta_\nu(Y)\xi_{\nu+2}\Big\} = 0,$$
$$\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(X)\eta_\nu(Y)\xi_{\nu+1} - q_{\nu+1}(X)\eta_{\nu+2}(Y)\xi_\nu\Big\} = 0,$$
$$\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(X)\eta_{\nu+1}(\xi)\phi_\nu\phi Y - q_{\nu+1}(X)\eta_\nu(\xi)\phi_{\nu+2}\phi Y\Big\} = 0,$$
$$\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(X)\eta_\nu(\xi)\phi_{\nu+1}\phi Y - q_{\nu+1}(X)\eta_{\nu+2}(\xi)\phi_\nu\phi Y\Big\} = 0,$$
$$\sum_{\nu=1}^{3}\Big\{q_{\nu+1}(X)\eta(Y)\eta_\nu(\xi)\xi_{\nu+2} - q_{\nu+2}(X)\eta(Y)\eta_{\nu+1}(\xi)\xi_\nu\Big\} = 0,$$
$$\sum_{\nu=1}^{3}\Big\{q_{\nu+1}(X)\eta(Y)\eta_{\nu+2}(\xi)\xi_\nu - q_{\nu+2}(X)\eta(Y)\eta_\nu(\xi)\xi_{\nu+1}\Big\} = 0,$$
$$\sum_{\nu=1}^{3}\Big\{q_{\nu+1}(X)\eta_{\nu+2}(\phi Y)\phi_\nu\xi - q_{\nu+2}(X)\eta_\nu(\phi Y)\phi_{\nu+1}\xi\Big\} = 0$$

and

$$\sum_{\nu=1}^{3}\Big\{q_{\nu+1}(X)\eta_\nu(\phi Y)\phi_{\nu+2}\xi - q_{\nu+2}(X)\eta_{\nu+1}(\phi Y)\phi_\nu\xi\Big\} = 0.$$

From these, the derivative of the normal Jacobi operator (4.1) becomes

$$\begin{aligned}
0 &= (\nabla_X \bar{R}_N)Y \\
&= 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX \\
&\quad + 3\sum_{\nu=1}^{3}\Big\{g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\Big\} \\
&\quad - \sum_{\nu=1}^{3}\Big[2\eta_\nu(\phi AX)(\phi_\nu\phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu AX, \phi Y)\phi_\nu\xi \\
&\quad - \eta(Y)\eta_\nu(AX)\phi_\nu\xi - \eta_\nu(\phi Y)(\phi_\nu\phi AX - g(AX,\xi)\xi_\nu)\Big]
\end{aligned} \tag{4.2}$$

for any tangent vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$.

By putting $X=\xi_\mu$ and replacing Y by X in (4.2) we have

$$\begin{aligned}0&=(\nabla_{\xi_\mu}\bar{R}_N)X\\&=3g(\phi A\xi_\mu,X)\xi+3\eta(X)\phi A\xi_\mu\\&\quad+3\sum_{\nu=1}^{3}\Big\{g(\phi_\nu A\xi_\mu,X)\xi_\nu+\eta_\nu(X)\phi_\nu A\xi_\mu\Big\}\\&\quad-\sum_{\nu=1}^{3}\Big[2\eta_\nu(\phi A\xi_\mu)(\phi_\nu\phi X-\eta(X)\xi_\nu)-g(\phi_\nu A\xi_\mu,\phi X)\phi_\nu\xi\\&\quad-\eta(X)\eta_\nu(A\xi_\mu)\phi_\nu\xi-\eta_\nu(\phi X)(\phi_\nu\phi A\xi_\mu-g(A\xi_\mu,\xi)\xi_\nu)\Big].\end{aligned}\tag{4.3}$$

From this, by putting $X=\xi$ in (4.3), it follows that

$$\begin{aligned}0&=(\nabla_{\xi_\mu}\bar{R}_N)\xi\\&=3\phi A\xi_\mu+5\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\xi_\nu+3\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu A\xi_\mu+\sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\phi_\nu\xi\end{aligned}\tag{4.4}$$

for any $\mu=1,2,3$.

We consider the notion of parallel normal Jacobi operator along the direction of the Reeb vector ξ, that is, $\nabla_\xi\bar{R}_N=0$ for a hypersurface M in $G_2(\mathbb{C}^{m+2})$. We assert the following

Lemma 4.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m\geq 3$. If the normal Jacobi operator $\bar{R}_N$ is parallel in the direction of the structure vector ξ, then the Reeb vector ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. Putting $X=Y=\xi$ in (4.2), we have

$$0=4\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi,\tag{4.5}$$

where we have used $A\xi=\alpha\xi$. From this, we have $\alpha=0$ or $\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi=0$.

In the case where M is with vanishing geodesic Reeb flow, it can be verified directly by Pérez and Suh [12].

Now let us consider the other case that M is with non-vanishing geodesic Reeb flow, that is, $\alpha\neq 0$. Then we assume that $\xi=\eta(X_0)X_0+\eta(\xi_1)\xi_1$ for some unit $X_0\in\mathfrak{D}$ and the non-zero functions $\eta(X_0)$ and $\eta(\xi_1)$. From this, together with (4.5), it follows that

$$\begin{aligned}0&=\eta_1(\xi)\phi_1\xi\\&=\eta_1(\xi)\phi_1(\eta(X_0)X_0+\eta(\xi_1)\xi_1)\\&=\eta_1(\xi)\eta(X_0)\phi_1X_0.\end{aligned}$$

Then two non-zero functions $\eta_1(\xi)$ and $\eta(X_0)$ give

$$\phi_1 X_0 = 0,$$

which makes a contradiction. This means $\eta(X_0) = 0$ or $\eta(\xi_1) = 0$, that is, the Reeb vector ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$. □

Next, we consider the notion of parallel normal Jacobi operator along the distribution $\mathfrak{D}^\perp$, that is, $\nabla_{\xi_\mu} \bar{R}_N = 0$, $\mu = 1, 2, 3$, for a hypersurface M in $G_2(\mathbb{C}^{m+2})$. Then we obtain the following

Lemma 4.2. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}^\perp$-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. Assume that ξ is tangent to $\mathfrak{D}^\perp$. Then the unit normal N is a singular tangent vector of $G_2(\mathbb{C}^{m+2})$ of type $JN \in \mathfrak{J}N$. So there exists an almost Hermitian structure $J_1 \in \mathfrak{J}$ such that $JN = J_1 N$. Then we have

$$\xi = \xi_1, \quad \phi\xi_2 = -\xi_3, \quad \phi\xi_3 = \xi_2, \quad \phi\mathfrak{D} \subset \mathfrak{D}.$$

Putting $\xi = \xi_1$ into (4.4), we have

$$0 = 3\phi A\xi_\mu + 5\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)\xi_\nu + 3\phi_1 A\xi_\mu + \sum_{\nu=1}^{3}\eta_\nu(A\xi_\mu)\phi_\nu\xi.$$

From this, taking an inner product with any $X \in \mathfrak{D}$ and using $g(\phi_\nu\xi, X) = 0$, we have

$$0 = 3g(\phi A\xi_\mu, X) + 3g(\phi_1 A\xi_\mu, X). \tag{4.6}$$

On the other hand, by using (3.4) we know that

$$\begin{aligned} \phi A\xi_\mu &= \nabla_{\xi_\mu}\xi \\ &= \nabla_{\xi_\mu}\xi_1 \\ &= q_3(\xi_\mu)\xi_2 - q_2(\xi_\mu)\xi_3 + \phi_1 A\xi_\mu. \end{aligned}$$

From this, taking an inner product with any $X \in \mathfrak{D}$, we have

$$g(\phi A\xi_\mu, X) = g(\phi_1 A\xi_\mu, X).$$

Substituting this formula into (4.6) gives

$$0 = g(\phi A\xi_\mu, X). \tag{4.7}$$

From this, let us replace X by ϕX in (4.8). Then it follows that

$$\begin{aligned} 0 &= g(\phi A\xi_\mu, \phi X) \\ &= -g(A\xi_\mu, \phi^2 X) \\ &= g(AX, \xi_\mu) \end{aligned}$$

for any $X \in \mathfrak{D}$, $\mu = 1, 2, 3$. This gives a complete proof of our Lemma. □

Moreover, in order to prove our theorem, we need the following

Lemma 4.3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}^\perp$-parallel normal Jacobi operator and $\xi \in \mathfrak{D}$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. Put the subset $\mathfrak{D}_0$ of $\mathfrak{D}$ as $\mathfrak{D}_0 = \{X \in \mathfrak{D} \mid X \perp \xi, \phi_1\xi, \phi_2\xi, \phi_3\xi\}$. Then the tangent vector space T_xM for any point $x \in M$ is denoted by

$$\begin{aligned} T_xM &= \mathfrak{D} \oplus \mathfrak{D}^\perp \\ &= [\xi] \oplus [\phi_1\xi, \phi_2\xi, \phi_3\xi] \oplus \mathfrak{D}_0 \oplus \mathfrak{D}^\perp, \end{aligned}$$

where $[\phi_1\xi, \phi_2\xi, \phi_3\xi]$ denotes a subspace of the distribution $\mathfrak{D}$ spanned by the vectors $\{\phi_1\xi, \phi_2\xi, \phi_3\xi\}$.

In order to show that $g(AX, \xi_\mu) = 0$ for any $X \in \mathfrak{D}$ and $\mu = 1, 2, 3$, first we consider for $X = \xi$. Then we have $g(A\xi, \xi_\mu) = \alpha g(\xi, \xi_\mu) = 0$ for any $\mu = 1, 2, 3$.

Next, we consider the case that any $X \in [\phi_1\xi, \phi_2\xi, \phi_3\xi]$. Put $X = \phi_\nu\xi$, $\nu = 1, 2, 3$. Since $\eta(\xi_\nu) = 0$ for any $\nu = 1, 2, 3$, we see that $g(\nabla_{\xi_\mu}\xi, \xi_\nu) = -g(\xi, \nabla_{\xi_\mu}\xi_\nu)$ for any $\mu = 1, 2, 3$. Thus we have

$$\begin{aligned} g(A\phi_\nu\xi, \xi_\mu) &= g(\phi\xi_\nu, A\xi_\mu) \\ &= -g(\xi_\nu, \phi A\xi_\mu) \\ &= -g(\xi_\nu, \nabla_{\xi_\mu}\xi) \\ &= g(\nabla_{\xi_\mu}\xi_\nu, \xi) \\ &= g(q_{\nu+2}(\xi_\mu)\xi_{\nu+1} - q_{\nu+1}(\xi_\mu)\xi_{\nu+2} + \phi_\nu A\xi_\mu, \xi) \\ &= g(\phi_\nu A\xi_\mu, \xi) \\ &= -g(A\phi_\nu\xi, \xi_\mu). \end{aligned}$$

Consequently we have

$$g(A\phi_\nu\xi, \xi_\mu) = 0$$

for any μ, $\nu = 1, 2, 3$.

Finally, we consider the case that any $X \in \mathfrak{D}_0$. By putting $X = \xi_1$ in (4.3), we have

$$\tag{4.8} \begin{aligned} 0 = 3g(\phi A\xi_\mu, \xi_1)\xi &+ 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi_\mu, \xi_1)\xi_\nu + 3\phi_1 A\xi_\mu \\ &- \sum_{\nu=1}^{3} \left\{2\eta_\nu(\phi A\xi_\mu)\phi_\nu\phi\xi_1 - g(\phi_\nu A\xi_\mu, \phi\xi_1)\phi_\nu\xi\right\}. \end{aligned}$$

To avoid confusion, we put $X = X_0 \in \mathfrak{D}_0$, where the distribution $\mathfrak{D}_0$ is defined by

$$\mathfrak{D}_0 = \{X \in \mathfrak{D} \mid X \perp \xi, \phi_1\xi, \phi_2\xi, \phi_3\xi\}.$$

Now let us take an inner product (4.8) with any $X_0 \in \mathfrak{D}_0$, we have

$$0 = 3g(\phi_1 A\xi_\mu, X_0) - 2\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_\mu)g(\phi_\nu\phi\xi_1, X_0). \tag{4.9}$$

On the other hand, by using (3.6) we know that

$$\begin{aligned} g(\phi_\nu\phi\xi_1, X_0) &= g(\phi\phi_\nu\xi_1 - \eta_\nu(\xi_1)\xi + \eta(\xi_1)\xi_\nu, X_0) \\ &= g(\phi\phi_\nu\xi_1, X_0) \\ &= 0 \end{aligned}$$

for any $X_0 \in \mathfrak{D}_0$ and any $\nu = 1, 2, 3$. Substituting this formula into (4.9), we have

$$0 = g(\phi_1 A\xi_\mu, X_0) \tag{4.10}$$

for any $X_0 \in \mathfrak{D}_0$. Here let us replace X_0 by $\phi_1 X_0$ in (4.10). Then we have

$$\begin{aligned} 0 &= g(\phi_1 A\xi_\mu, \phi_1 X_0) \\ &= -g(A\xi_\mu, {\phi_1}^2 X_0) \\ &= g(AX_0, \xi_\mu) \end{aligned}$$

for any $X_0 \in \mathfrak{D}_0$ and $\mu = 1, 2, 3$. From these facts we assert that $g(AX, \xi_\mu) = 0$ for any $X \in \mathfrak{D}$ and any $\mu = 1, 2, 3$. This gives a complete proof of our Lemma. □

5. $\mathfrak{F}$-parallel normal Jacobi operator

Now in this section we consider the weaker condition than having parallel normal Jacobi operator. So as a generalization of the notion of parallel normal Jacobi operator, we consider a notion of parallel normal Jacobi operator along the distribution $\mathfrak{F}$ defined by $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$ on M in $G_2(\mathbb{C}^{m+2})$, where $[\xi] = \text{Span}\{\xi\}$ and $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$. Then, on such a distribution $\mathfrak{F}$ we can use Lemmas 4.1, 4.2 and 4.3. Then we assert the following

Proposition 5.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{F}$-parallel normal Jacobi operator. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

By Proposition 5.1 and Theorem A in the introduction, we have

Proposition 5.2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{F}$-parallel normal Jacobi operator. Then M is congruent to an open part of one of the following real hypersurfaces in $G_2(\mathbb{C}^{m+2})$:*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Before giving the proof of our main theorem, we will check whether any kind of hypersurfaces in Theorem A satisfy $\mathfrak{F}$-parallel normal Jacobi operator.

By Lemma 4.2 and Theorem A we know that M is locally congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Then it naturally rises to a problem that the normal Jacobi operator $\bar{R}_N$ of hypersurfaces of type A in Theorem A satisfies $\mathfrak{F}$-parallel or not? Corresponding to such a problem, we introduce the following due to Berndt and Suh [3]

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r),\ \beta = \sqrt{2}\cot(\sqrt{2}r),\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2,\ m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\ T_\beta &= \mathbb{C}^{\perp}\xi = \mathbb{C}^{\perp}N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\ T_\lambda &= \{X \mid X \perp \mathbb{H}\xi, JX = J_1 X\}, \\ T_\mu &= \{X \mid X \perp \mathbb{H}\xi, JX = -J_1 X\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denotes real, complex and quaternionic span of the structure vector ξ and $\mathbb{C}^{\perp}\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Let us consider for $\mu = 2$ in (4.4). Then we have

$$\begin{aligned} 0 &= (\nabla_{\xi_2}\bar{R}_N)\xi \\ &= 3\phi A\xi_2 + 5\sum_{\nu=1}^{3}\eta_\nu(\phi A\xi_2)\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu A\xi_2 + \sum_{\nu=1}^{3}\eta_\nu(A\xi_2)\phi_\nu\xi. \end{aligned} \tag{5.1}$$

Now let us suppose that M is of type (A) with $\mathfrak{F}$-parallel normal Jacobi operator and its Reeb vector $\xi \in \mathfrak{D}^{\perp}$. Then, by using Proposition A and $\phi\xi_2 = -\xi_3$, $\phi\xi_3 = \xi_2$, the equation (5.1) can be written

$$\begin{aligned} 0 &= 3\beta\phi\xi_2 + 5\beta\sum_{\nu=1}^{3}\eta_\nu(\phi\xi_2)\xi_\nu + 3\beta\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi_2 + \beta\sum_{\nu=1}^{3}\eta_\nu(\xi_2)\phi_\nu\xi \\ &= -3\beta\xi_3 - 5\beta\xi_3 + 3\beta\phi_1\xi_2 + \beta\phi_2\xi \\ &= -6\beta\xi_3. \end{aligned} \tag{5.2}$$

Thus, we have $\beta = 0$ and this case also can not occur for some $r \in (0, \pi/\sqrt{8})$. This makes a contradiction.

On the other hand, by Lemma 4.3 and Theorem A we know that M is locally congruent to a tube over a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$. Now it remains only to check that whether real hypersurfaces of type (B) in Theorem A satisfy $\mathfrak{F}$-parallel normal Jacobi operator or not? Then in order to do this we introduce the following due to Berndt and Suh [3]

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 3 = m(\gamma),\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi,\ T_\beta = \mathfrak{J}J\xi,\ T_\gamma = \mathfrak{J}\xi,\ T_\lambda,\ T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp,\ \mathfrak{J}T_\lambda = T_\lambda,\ \mathfrak{J}T_\mu = T_\mu,\ JT_\lambda = T_\mu.$$

Let us consider for $\mu = 2$ in (4.4) and $\xi \in \mathfrak{D}$. Then, by using Proposition B and (3.1), (4.4) can be written

$$\begin{aligned} 0 &= 3\beta\phi\xi_2 + 5\beta\sum_{\nu=1}^{3}\eta_\nu(\phi\xi_2)\xi_\nu + 3\beta\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi_2 + \beta\sum_{\nu=1}^{3}\eta_\nu(\xi_2)\phi_\nu\xi \\ &= 3\beta\phi\xi_2 + \beta\phi_2\xi \\ &= 4\beta\phi_2\xi. \end{aligned}$$

Thus, we have $\beta = 0$ and this case also can not occur for some $r \in (0, \pi/4)$. This also makes a contradiction. Accordingly, we know that the normal Jacobi operator $\bar{R}_N$ for hypersurfaces of type (A) or of type (B) in Theorem A can not be $\mathfrak{F}$-parallel, respectively.

From this, we complete the proof of our main Theorem in the introduction.

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