BULL. AUSTRAL. MATH. SOC. 53C40, 53C15.
VOL. 68 (2003) [379–393]

REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH COMMUTING SHAPE OPERATOR

YOUNG JIN SUH

In this paper we give a non-existence property of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ which have a shape operator A commuting with the structure tensors $\{\phi_1, \phi_2, \phi_3\}$. From this view point we give a characterisation of real hypersurfaces of type B in $G_2(\mathbb{C}^{m+2})$.

0. INTRODUCTION

In the geometry of real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms there have been many characterisations of model hypersurfaces of type A_1, A_2, B, C, D and E in complex projective space $\mathbb{C}P^m$, of type A_0, A_1, A_2 and B in complex hyperbolic space $\mathbb{C}H^m$ or A_1, A_2, B in quaternionic projective space $\mathbb{Q}P^m$, which are completely classified by Cecil and Ryan [**4**], Kimura [**5**], Berndt [**1**], Martinez and Pérez [**7**] respectively. Among them there were only a few characterisations of homogeneous real hypersurfaces of type B in complex projective space $\mathbb{C}P^m$. For example, the condition that the shape operator A and the structure tensor ϕ satisfy $A\phi + \phi A = k\phi$, for some constant k, is a model characterisation of this kind for type B, which is a tube over a real projective space $\mathbb{R}P^m$ in $\mathbb{C}P^m$ (See Yano and Kon [**8**]).

On the other hand, when we consider real hypersurfaces in quaternionic projective space $\mathbb{Q}P^m$, Pak [**6**] has considered a geometric condition that $A\phi_\nu - \phi_\nu A = 0$, $\nu = 1, 2, 3$ that is, the structure tensor ϕ_ν and the shape operator A commute with each other. Moreover, it was known to be a characterisation of type A_1, and A_2 in quaternionic projective space $\mathbb{Q}P^m$, which is a tube of radius r, $0 < r < \pi/2$, over a totally geodesic $\mathbb{Q}P^k$ in $\mathbb{Q}P^m$.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold, being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing

Received 18th March, 2003
This work was supported by grant Proj. No. R05-2002-000-00047-0 and Proj. No. R14-2002-003-01001-0 from the Korea Science & Engineering Foundation.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/03 $A2.00+0.00.

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces M; that $[\xi] = \mathrm{Span}\{\xi\}$ or $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M. The almost contact structure vector field ξ mentioned above is defined by $\xi = -JN$, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ and the almost contact 3-structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$, where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$.

The first result in this direction is the classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying both conditions. Berndt and the present author [**2**] have proved the following theorem.

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In Theorem A the vector ξ contained in the one-dimensional distribution $[\xi]$ is said to be a *Hopf* vector when it becomes a principal vector for the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. Moreover in such a situation M is said to be a *Hopf* hypersurface. Besides of this, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ also admits the 3-dimensional distribution $\mathfrak{D}^\perp$, which are spanned by almost contact 3-structure vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$.

Real hypersurfaces of type B in Theorem A are just the case that the one dimensional distribution $[\xi]$ is contained in $\mathfrak{D}$. It can be easily proved in Section 3 that the tube of type B satisfies the following formula on the orthogonal complement of the one-dimensional distribution $[\xi]$

$$(*) \qquad A\phi_\nu - \phi_\nu A = 0, \quad \nu = 1, 2, 3.$$

Also in the paper [**3**] Berndt and the present author have given a characterisation of real hypersurfaces of type A when the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, which is equivalent to the condition that the Reeb flow on M is isometric.

Now in this paper we consider another condition that the almost contact 3-structure tensors $\{\phi_1, \phi_2, \phi_3\}$ and the shape operator A of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ commute with each other. Then we are able to assert the following.

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

THEOREM 1. *There do not exist any real hypersurfaces of M in $G_2(\mathbb{C}^{m+2})$ satisfying the condition that the shape operator A and the structure tensors $\{\phi_1, \phi_2, \phi_3\}$ commute with each other.*

From this view point, Theorem 1 and the argument mentioned above give a characterisation of real hypersurfaces of type B in $G_2(\mathbb{C}^{m+2})$. Namely, we give a characterisation of type B in Theorem A as follows.

THEOREM 2. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying (*) on the orthogonal complement of the one-dimensional disribution $[\xi]$. Then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

The present author would like to express his sincere gratitude to the referee for his valuable comments and suggestions to develop the first version of the manuscript.

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarise basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [**2**] and [**3**]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabiliser isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalise g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the three-dimensional complex projective space $\mathbb{C}P^3$ with constant holomorphic sectional curvature eight we shall assume $m \geqslant 2$ from now on. Note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = su(m) \oplus su(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the centre of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the centre $\mathfrak{R}$ induces a Kähler structure J and the $su(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

$G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\overline{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\overline{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$. This fact will be used frequently.

Let $p \in G_2(\mathbb{C}^{m+2})$ and W a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (respectively totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (respectively totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$.

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression for $\overline{R}$, the Codazzi equation becomes

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\}\end{aligned}$$

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

$$+\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\}$$
$$+\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\}\xi_\nu\,.$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}&\phi_{\nu+1}\xi_\nu=-\xi_{\nu+2},\quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\ &\phi\xi_\nu=\phi_\nu\xi,\quad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\ &\phi_\nu\phi_{\nu+1}X=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\ &\phi_{\nu+1}\phi_\nu X=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}.\end{aligned}\tag{2.1}$$

Now let us put

$$JX=\phi X+\eta(X)N,\quad J_\nu X=\phi_\nu X+\eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi,\quad \nabla_X\xi=\phi AX,\tag{2.2}$$

$$\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX,\tag{2.3}$$

$$\begin{aligned}(\nabla_X\phi_\nu)Y=&-q_{\nu+1}(X)\phi_{\nu+2}Y+q_{\nu+2}(X)\phi_{\nu+1}Y+\eta_\nu(Y)AX\\ &-g(AX,Y)\xi_\nu.\end{aligned}\tag{2.4}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi)&=\nabla_X(\phi\xi_\nu)\\ &=(\nabla_X\phi)\xi_\nu+\phi(\nabla_X\xi_\nu)\\ &=q_{\nu+2}(X)\phi_{\nu+1}\xi-q_{\nu+1}(X)\phi_{\nu+2}\xi+\phi_\nu\phi AX\\ &\quad-g(AX,\xi)\xi_\nu+\eta(\xi_\nu)AX.\end{aligned}\tag{2.5}$$

Moreover, from $JJ_\nu=J_\nu J$, $\nu=1,2,3$, it follows that

$$\phi\phi_\nu X=\phi_\nu\phi X+\eta_\nu(X)\xi-\eta(X)\xi_\nu.\tag{2.6}$$

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

3. Proof of Theorem 1

Before giving the proof of Theorem 1 let us ask what kind of model hypersurfaces given in Theorem A satisfy the formula (*). In other words, it will be interesting to know whether there exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the condition (*). In this section we shall show that only a tube over a quaternionic projective space $\mathbb{Q}P^m$ in $G_2(\mathbb{C}^{m+2})$ satisfies the formula (*) on the orthogonal complement of the one-dimensional distribution $[\xi]$.

To solve such a problem let us recall some propositions given by Berndt and the present author ([2]). For a tube of type A in Theorem A we have the following.

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three(if $r = \pi/2$) or four(otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot\left(\sqrt{8}r\right)\ ,\ \beta = \sqrt{2}\cot\left(\sqrt{2}r\right)\ ,\ \lambda = -\sqrt{2}\tan\left(\sqrt{2}r\right),\ \mu = 0$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 2,\ m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN, \\ T_\beta &= \mathfrak{C}^{\perp}\xi = \mathfrak{C}^{\perp}N, \\ T_\lambda &= \{X \mid X \perp \mathbb{H}\xi, JX = J_1X\}, \\ T_\mu &= \{X \mid X \perp \mathbb{H}\xi, JX = -J_1X\}. \end{aligned}$$

Moreover, for a tube of type B in Theorem A we have the following.

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 3 = m(\gamma),\ m(\lambda) = 4n - 4 = m(\mu)$$

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi,\ T_\beta = \mathfrak{J}J\xi,\ T_\gamma = \mathfrak{J}\xi,\ T_\lambda,\ T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp,\ \mathfrak{J}T_\lambda = T_\lambda,\ \mathfrak{J}T_\mu = T_\mu,\ JT_\lambda = T_\mu\,.$$

Of course we have proved that all of the principal curvatures and its eigenspaces of the tube A (respectively, the tube B) in Theorem A satisfies all of the properties in Proposition A (respectively, Proposition B).

Now by using this Proposition B let us check whether a tube of type B in Theorem A, that is, a tube over a totally geodesic QP^m in $G_2(\mathbb{C}^{m+2})$, satisfies the formula (*) on the orthogonal complement of the one-dimensional distribution $[\xi]$.

For any $X \in T_\lambda \oplus T_\mu$ by the invariance of the eigenspaces with respect to $\mathfrak{J}$ we have $A\phi_\nu = \phi_\nu A$.

To check for ξ_ν, we note it follows that

$$A\phi_\nu\xi_{\nu+1} - \phi_\nu A\xi_{\nu+1} = A\xi_{\nu+2} - \alpha_{\nu+1}\xi_{\nu+2} = 0.$$

Finally, we check for any $\phi_\nu\xi$ in an eigenspace T_β in Proposition B whether the formula (*) holds or not. Then (*) follows

$$A\phi_\nu\phi_{\nu+1}\xi - \phi_\nu A\phi_{\nu+1}\xi = A\phi_{\nu+2}\xi - \phi_\nu A\phi_{\nu+1}\xi = 0.$$

Summing up all of these facts, we know that the formula (*) holds on the distribution $\xi^\perp$, where $\xi^\perp$ denotes the orthogonal complement of the distribution $[\xi]$.

But without the restriction of formula (*) on the distrbution $\xi^\perp$, the situation could be quite different from the above. We are going to prove our Theorem 1 in the introduction. From the condition that $A\phi_\nu = \phi_\nu A$ it follows that

$$0 = A\phi_\nu\xi_\nu = \phi_\nu A\xi_\nu.$$

Naturally this implies $A\xi_\nu = \beta_\nu\xi_\nu$, $\nu = 1,2,3$, which gives $g(A\mathfrak{D},\mathfrak{D}^\perp) = 0$. Moreover we know that

$$A\xi_3 = A\phi_1\xi_2 = \phi_1 A\xi_2 = \beta_2\phi_1\xi_2 = \beta_2\xi_3.$$

From this we know that all of principal curvatures are equal to each other, that is,

$$\beta_1 = \beta_2 = \beta_3. \tag{3.1}$$

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

Now let us take an inner product ξ_1 with the Codazzi equation. Then using (2.1) it follows that

$$\begin{aligned}(3.2)\qquad & 2\eta(X)\eta_1(\phi Y) - 2\eta(Y)\eta_1(\phi X) - 2g(\phi X, Y)\eta(\xi_1)\\ &\quad + 2\eta_2(X)\eta_3(Y) - 2\eta_3(X)\eta_2(Y) - 2g(\phi_1 X, Y)\\ &\quad + 2\eta_2(\phi X)\eta_3(\phi Y) - 2\eta_2(\phi Y)\eta_3(\phi X)\\ &= g\big((\nabla_X A)\xi_1, Y\big) - g\big((\nabla_Y A)\xi_1, X\big).\end{aligned}$$

Now differentiating $A\xi_1 = \beta_1\xi_1$, and using (2.3) and the formulas $A\xi_2 = \beta_2\xi_2$ and $A\xi_3 = \beta_3\xi_3$, we have

$$\begin{aligned}(3.3)\qquad g\big((\nabla_X A)\xi_1, Y\big) - g\big((\nabla_Y A)\xi_1, X\big) &= (X\beta_1)\eta_1(Y) - (Y\beta_1)\eta_1(X)\\ &\quad + (\beta_1 - \beta_2)\big\{q_3(X)\eta_2(Y) - q_3(Y)\eta_2(X)\big\}\\ &\quad - (\beta_1 - \beta_3)\big\{q_2(X)\eta_3(Y) - q_2(Y)\eta_3(X)\big\}\\ &\quad + \beta_1 g\big((\phi_1 A + A\phi_1)X, Y\big) - 2g(A\phi_1 AX, Y).\end{aligned}$$

Then we are going to prove our theorem as follows.

Case 1. ξ is principal.

In such a case we are able to apply Theorem A. Then M is congruent to real hypersurfaces of type A or type B. When M is congruent to of type A, by Proposition A its principal curvatures β_ν, $\nu = 1, 2, 3$ are not equal to each other. Then by (3.1) we have a contradiction.

When M is congruent to of type B, by Proposition B its principal curvatures β_ν are equal to each other. As mentioned in Proposition B of this section, all eigenvectors in T_λ and T_μ satisfy $A\phi_\nu = \phi_\nu A$. But we know $\{\phi_1\xi, \phi_2\xi, \phi_3\xi\}\in T_\gamma$ for the structure vector $\xi\in T_\alpha$. Moreover, its principal curvature is $\gamma = 0$. This implies that for any $\nu = 1, 2, 3$ we have

$$0 = A\phi_\nu\xi - \phi_\nu A\xi = -\phi_\nu A\xi = 2\tan(2r)\phi_\nu\xi,$$

which also gives a contradiction. So the Case 1 can not occur.

Case II. ξ is not principal.

Let us put

$$A\xi = \alpha\xi + \nu U,$$

where U is orthogonal to ξ. Then we are able to consider an open set $\mathfrak{U} = \{p\in M \mid \nu(p) \neq 0\}$ in M. On this open subset we continue our proof of Theorem 1 by proving the following Lemmas.

LEMMA 3.1. *Under the above assumptions we have*

$$\xi \in \mathfrak{D}, \quad A\xi \in \mathfrak{D} \quad \text{and} \quad U \in \mathfrak{D}.$$

PROOF: Let

$$\xi = \eta(X)X + \eta(Z)Z$$

for any $X \in \mathfrak{D}$ and $Z \in \mathfrak{D}^\perp$. Then without loss of generality we are able to choose a vector Z so that

$$Z = \xi_3 \in \mathfrak{D}^\perp = \operatorname{Span}\{\xi_1, \xi_2, \xi_3\}.$$

Then $\xi = \eta(X)X + \eta(\xi_3)\xi_3$ implies

$$\eta(\xi_1) = 0 = \eta(\xi_2).$$

This means $\xi \perp \xi_1, \xi_2$, that is, $\xi_1, \xi_2 \in \xi^\perp$, where $\xi^\perp$ denotes an orthogonal complement of the structure vector ξ in T_xM. Now from (3.2) and (3.3) for any $X \in \mathfrak{D}$ such that $AX = \lambda X$ we have

$$\begin{aligned}(2\lambda - \beta_1)A\phi_1 X - (\lambda\beta_1 + 2)\phi X - (X\beta_1)\xi_1 \\ - (\beta_1 - \beta_2)q_3(X)\xi_2 + \beta_1 q_2(X)\xi_3 - q_2(X)A\xi_3 \\ - 2\eta(X)\phi\xi_1 - 2\eta_1(\phi X)\xi - 2\eta(\xi_1)\phi X \\ - 2\phi_1 X - 2\eta_2(\phi X)\phi\xi_3 + 2\eta_3(\phi X)\phi\xi_2 = 0.\end{aligned}$$

From this, take an inner product with ξ_2 and use the fact that $X \in \mathfrak{D}$ and (3.1) to get

$$(\beta_1 - \beta_2)q_3(X) - 2\eta(X)\eta(\xi_3) = 0.$$

Then by the fact $\beta_1 = \beta_2$ we have $\eta(X)\eta(\xi_3) = 0$. This means that

$$\xi \in \mathfrak{D} \quad \text{or} \quad \xi \in \mathfrak{D}^\perp.$$

Now let us show $\xi \in \mathfrak{D}$. By the above result we are able to suppose $\xi \in \mathfrak{D}^\perp$. From this, the structure vector ξ is equal to one of ξ_ν, $\nu = 1, 2, 3$ and could be regarded as a principal vector field. Together with Case 1 we have a contradiction. Accordingly, we must have that $\xi \in \mathfrak{D}$. From this fact we are able to assert the second and the third formulae, because

$$g(A\xi, \xi_\nu) = g(\xi, A\xi_\nu) = \beta_\nu g(\xi, \xi_\nu) = 0$$

and because $A\xi = \alpha\xi + \nu U$. ∎

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

Lemma 3.2. *Under the above assumptions we have that the functions* $2\alpha = \beta_1 = \beta_2 = \beta_3$ *are constant or* $g(\phi_\nu U, \xi) = 0$ *for any* $\nu = 1, 2, 3$.

Proof: From (3.1), (3.2) and (3.3) it follows that

$$\begin{aligned} 2\eta(X)\eta_1(\phi Y) - 2\eta(Y)\eta_1(\phi X) - 2g(\phi X, Y)\eta(\xi_1) \\ + 2\eta_2(X)\eta_3(Y) - 2\eta_3(X)\eta_2(Y) - 2g(\phi_1 X, Y) \\ + 2\eta_2(\phi X)\eta_3(\phi Y) - 2\eta_2(\phi Y)\eta_3(\phi X) \\ = (X\beta_1)\eta_1(Y) - (Y\beta_1)\eta_1(X) + \beta_1 g\big((\phi_1 A + A\phi_1)X, Y\big) - 2g(A\phi_1 AX, Y). \end{aligned} \tag{3.4}$$

Then, by putting $Y = \xi_1$ and using Lemma 3.1 and the assumptions, we have

$$X\beta_1 = (\xi_1\beta_1)\eta_1(X)$$

for any X on the open subset $\mathfrak{U}$ of M. Similarly we are able to calculate

$$X\beta_2 = (\xi_2\beta_2)\eta_2(X) \text{ and } X\beta_3 = (\xi_3\beta_3)\eta_3(X).$$

From this $\xi_1\beta_2 = (\xi_2\beta_2)\eta_2(\xi_1) = 0$. Then by (3.1) we know that $X\beta_2 = X\beta_1 = 0$ for any X. This means $\beta_1 = \beta_2 = \beta_3$ are all constant. Then (3.4) can be written as

(3.5)
$$\begin{aligned} \beta_1(\phi_1 A + A\phi_1)X - 2A\phi_1 AX = -2\eta(X)\phi\xi_1 - 2\eta_1(\phi X)\xi - 2\eta(\xi_1)\phi X + 2\eta_2(X)\xi_3 \\ - 2\eta_3(X)\xi_2 - 2\phi_1 X - 2\eta_2(\phi X)\phi\xi_3 + 2\eta_3(\phi X)\phi\xi_2. \end{aligned}$$

Now by putting $X = \xi$ in (3.5) and using $\xi \in \mathfrak{D}$ and $A\xi \in \mathfrak{D}$ in Lemma 3.1 we have

$$\begin{aligned} \beta_1(\phi_1 A + A\phi_1)\xi &= 2A\phi_1 A\xi - 4\phi_1\xi \\ &= 2\phi_1 A^2\xi - 4\phi_1\xi. \end{aligned} \tag{3.6}$$

Since in this case the structure vector ξ is not principal, we can put

$$A\xi = \alpha\xi + \nu U$$

on the open set $\mathfrak{U}$. Then $\phi_1 A\xi = \alpha\phi_1\xi + \nu\phi_1 U$. Substituting this into (3.6), we have

$$\begin{aligned} (2\alpha - \beta_1)A\phi_1\xi &= (\alpha\beta_1 + 4)\phi_1\xi + \nu\beta_1\phi_1 U - 2\nu A\phi_1 U \\ &= (\alpha\beta_1 + 4)\phi_1\xi + \nu\beta_1\phi_1 U - 2\nu\phi_1 AU. \end{aligned}$$

On the other hand, the left side reduces to the following

$$\begin{aligned} (2\alpha - \beta_1)A\phi_1\xi &= (2\alpha - \beta_1)\phi_1 A\xi \\ &= \alpha(2\alpha - \beta_1)\phi_1\xi + \nu(2\alpha - \beta_1)\phi_1 U. \end{aligned}$$

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

Then from these two equations, by taking an inner product with ξ we have

$$\begin{aligned}\nu(2\alpha-\beta_1)g(\phi_1U,\xi) &= \nu\beta_1 g(\phi_1U,\xi)-2\nu g(\phi_1AU,\xi)\\ &= \nu(\beta_1-2\alpha)g(\phi_1U,\xi).\end{aligned}$$

This completes the proof of our Lemma 3.2. □

LEMMA 3.3. *If the functions on* $\mathfrak{U}$ *satisfy* $2\alpha=\beta_1=\beta_2=\beta_3$, *then* $g(\phi_\nu U,\xi)=0$ *for any* $\nu=1,2,3$.

PROOF: By Lemma 3.1 we know that $2\alpha=\beta_\nu$, $\nu=1,2,3$, are constant or $g(\phi_\nu U,\xi)=0$. By differentiating $\alpha=g(A\xi,\xi)$, we have

$$\begin{aligned}0 &= g\big((\nabla_XA)\xi+A\nabla_X\xi,\xi\big)+g(A\xi,\nabla_X\xi)\\ &= g\big((\nabla_XA)\xi,\xi\big)+\nu g(\phi AX,U).\end{aligned}\tag{3.7}$$

Now differentiating $A\xi=\alpha\xi+\nu U$ implies

$$(\nabla_XA)\xi+A\nabla_X\xi=\alpha\nabla_X\xi+\nu\nabla_XU+(X\nu)U,$$

where we have used that the function α is constant. Then, by taking an inner product with ξ, we have

$$g\big((\nabla_XA)\xi,\xi\big)=-g(A\phi AX,\xi)-\nu g(U,\phi AX).$$

From this, together with (3.7) we assert that

$$0=g(A\phi AX,\xi)=g(\phi AX,\alpha\xi+\nu U)=\nu g(\phi AX,U).$$

Then on the open set $\mathfrak{U}$ we have $A\phi U=0$, which gives that

$$0=g(A\phi U,\xi_\nu)=\beta_i g(\phi U,\xi_\nu).$$

From this, together with Lemma 3.2 we know that $2\alpha=\beta_\nu=0$ or $g(\phi U,\xi_\nu)=0$, $\nu=1,2,3$.

Let us consider for a case where $2\alpha=\beta_\nu=0$, $\nu=1,2,3$. Then substituting into (3.5), we have

$$\begin{aligned}-2A\phi_1AX = &-2\eta(X)\phi\xi_1-2\eta_1(\phi X)\xi-2\eta(\xi_1)\phi X+2\eta_2(X)\xi_3\\ &-2\eta_3(X)\xi_2-2\phi_1X-2\eta_2(\phi X)\phi\xi_3+2\eta_3(\phi X)\phi\xi_2.\end{aligned}\tag{3.8}$$

Now by putting $X=\xi$ into the above equation and using $\xi\in\mathfrak{D}$ and $A\xi=\nu U$ we have

$$\nu A\phi_1U=2\phi\xi_1.\tag{3.9}$$

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

Also putting $X = U$ in (3.8) and using $U \in \mathfrak{D}$ and $\xi \in \mathfrak{D}$ in Lemma 3.1, we have

$$-2A\phi_1 AU = -2\eta_1(\phi U)\xi - 2\phi_1 U - 2\eta_2(\phi U)\phi\xi_3 + 2\eta_3(\phi U)\phi\xi_2.$$

From this, taking an inner product with ξ, we have

$$g(A\phi_1 AU, \xi) = 2\eta_1(\phi U) = -2g(\phi\xi_1, U).$$

On the other hand, the left side reduces to the following

$$g(A\phi_1 AU, \xi) = \nu g(\phi_1 AU, U) = \nu g(A\phi_1 U, U) = 2g(\phi\xi_1, U),$$

where in the second equality we have used the assumption (*) and in the third equality we have used the formula (3.9). The above two equations complete the proof of Lemma 3.3. $\square$

Lemma 3.4. *There does not exist any open subset $\mathfrak{U}$ in M satisfying $g(\phi_\nu U, \xi) = 0$ for any $\nu = 1, 2, 3$.*

Proof: Differentiating the formula $A\phi_\nu - \phi_\nu A = 0$ and using (2.4) and $A\xi_\nu = \beta_\nu \xi_\nu$, $\nu = 1, 2, 3$ we have

$$(\nabla_X A)Y = \eta_\nu((\nabla_X A)Y)\xi_\nu - \phi_\nu(\nabla_X A)\phi_\nu Y - \eta_\nu(Y)\phi_\nu A^2 X + \eta_\nu(AY)\phi_\nu AX. \tag{3.10}$$

Now putting $Y = \xi_\nu$ into (3.10) and using the fact that $\beta_\nu = g(A\xi_\nu, \xi_\nu)$, $\nu = 1, 2, 3$ are constants, we have

$$(\nabla_X A)\xi_\nu = -\phi_\nu A^2 X + \beta_\nu \phi_\nu AX. \tag{3.11}$$

Then by taking an inner product (3.10) with $\xi_{\nu+1}$ and using (2.1) and the assumption (*) on $\mathfrak{U}$ we have

$$\begin{aligned} g\big((\nabla_X A)Y, \xi_{\nu+1}\big) &= g\big((\nabla_X A)\phi_\nu Y, \xi_{\nu+2}\big) + \eta_\nu(Y)g\big(A^2 X, \xi_{\nu+2}\big) \\ &\quad - \eta_\nu(AY)g(AX, \xi_{\nu+2}) \\ &= g\big(\phi_\nu Y, -\phi_{\nu+2}A^2 X + \beta_{\nu+2}\phi_{\nu+2}AX\big) \\ &= -g\big(\phi_{\nu+1}Y, A^2 X\big) + \beta_{\nu+2}g(\phi_{\nu+1}Y, AX), \end{aligned} \tag{3.12}$$

where we have used Lemma 3.2, (3.11) and $\phi_{\nu+2}\phi_\nu Y = -\phi_{\nu+1}Y + \eta_\nu(Y)\xi_{\nu+2}$ in (2.1).

By applying (3.11) to the left side of (3.12) we have

$$\begin{aligned} g\big((\nabla_X A)Y, \xi_{\nu+1}\big) &= g(Y, (\nabla_X A)\xi_{\nu+1}) \\ &= -g\big(Y, \phi_{\nu+1}A^2 X\big) + \beta_{\nu+1}g(\phi_{\nu+1}AX, Y). \end{aligned} \tag{3.13}$$

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

Then (3.12) and (3.13) imply for any vector field X

$$A^2\phi_{\nu+1}X = \beta_{\nu+2}A\phi_{\nu+1}X.$$

From this, together with Lemma 3.2, we have

$$A^2X = \beta_{\nu+2}AX. \tag{3.14}$$

On the other hand, by putting $X = U$ into (3.5) and using Lemma 3.3 and the assumption $g(\phi_\nu U, \xi) = 0$ we have

$$\beta_1 A\phi_1 U = A\phi_1 AU - \phi_1 U.$$

Then by using our assumption (*) and Lemma 3.1 we have

$$A^2U - \beta_1 AU - U = 0. \tag{3.15}$$

From this, together with (3.14), we have a contradiction. So we have proved our Lemma. □

Summing up these four Lemmas 3.1, 3.2, 3.3 and 3.4 we have completed the proof of our Theorem 1. That is, we know that there do not exist any real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ satisfying the condition that the structure tensors $\{\phi_1, \phi_2, \phi_3\}$ commute with the shape operator A of M.

4. Another characterisation

In paper [**3**] of Berndt and the present author we give a characterisation of real hypersurfaces of type A in $G_2(\mathbb{C}^{m+2})$ when the Reeb flow is isometric. Moreover, in this paper we have proved that the Reeb flow on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if the structure tensor ϕ and the shape operator A commute with each other.

But in Section 3 we have proved that there do not exist any real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ if the structure tensors $\{\phi_1, \phi_2, \phi_3\}$ commute with its shape operator A of M. Moreover, we have known that if we restrict the formula (*) on the orthogonal complement $\xi^\perp$, a tube over a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$, satisfies such a condition.

From this view point we give a characterisation of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ by a condition weaker than the commutative condition

$$A\phi_\nu = \phi_\nu A, \ \nu = 1, 2, 3. \tag{4.1}$$

Now we prove the following:

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

THEOREM 4.1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying the formula (4.1) on the orthogonal complement of the one-dimensional distibution $[\xi]$. Then M is locally congruent to an open part of a tube around a totally geodesic QP^n in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

PROOF: Let us put

$$\xi = \eta(X)X + \eta(Z)Z$$

for any $X \in \mathfrak{D}$ and $Z \in \mathfrak{D}^\perp$. Then without loss of generality we are able to choose a vector Z in such a way that

$$Z = \xi_3 \in \mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}.$$

Then $\xi = \eta(X)X + \eta(\xi_3)\xi_3$ implies

$$\eta(\xi_1) = 0 = \eta(\xi_2).$$

This means $\xi \perp \xi_1, \xi_2$, that is, $\xi_1, \xi_2 \in \xi^\perp$. From the condition that $A\phi_\nu = \phi_\nu A$, $\nu = 1, 2$ on the the orthogonal complement $\xi^\perp$ it follows that

$$A\phi_\nu\xi_\nu = \phi_\nu A\xi_\nu, \ \nu = 1, 2.$$

Naturally this implies $A\xi_\nu = \beta_\nu\xi_\nu$, $\nu = 1, 2$. Moreover we know that

$$A\xi_3 = A\phi_1\xi_2 = \phi_1 A\xi_2 = \beta_2\phi_1\xi_2 = \beta_2\xi_3.$$

This means that all structure vector fields ξ_ν, $\nu = 1, 2, 3$ are principal vectors with the same principal curvatures, that is,

$$\beta_1 = \beta_2 = \beta_3. \tag{4.2}$$

This implies $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. From this together with Theorem A we know that a real hypersurface M satisfying (4.1) on the othogonal complement $\xi^\perp$ is congruent to a real hypersurface of type A or type B. But it can be easily checked that real hypersurfaces of type A can not satisfy the condition (4.1), because $\xi = \xi_3$ is in an eigenspace T_α and ξ_2, ξ_3 is in an eigenspace T_β, and the constants α and β are different from each other, which contradicts to (4.2). Accordingly, a real hypersurface M satisfying (4.1) on the orthogonal complement $\xi^\perp$ is congruent to one of real hypersurfaces of type B. From this we completed the proof of Theorem 2. $\square$

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press

References

[1] J. Berndt, 'Real hypersurfaces in quaternionic space forms', *J. Reine Angew. Math.* **419** (1991), 9–26.

[2] J. Berndt and Y.J. Suh, 'Real hypersurfaces in complex two-plane Grassmannians', *Monatsh. Math.* **127** (1999), 1–14.

[3] J. Berndt and Y.J. Suh, 'Isometric flows on real hypersurfaces in complex two-plane Grassmannians', *Monatsh. Math.* **137** (2002), 87–98.

[4] T.E. Cecil and P.J. Ryan, 'Focal sets and real hypersurfaces in complex projective space', *Trans. Amer. Math. Soc.* **269** (1982), 481-499.

[5] M. Kimura, 'Real hypersurfaces and complex submanifolds in complex projective space', *Trans. Amer. Math. Soc.* **296** (1986), 137–149.

[6] J.S. Pak, 'Real hypersurfaces in quaternionic Kaehlerian manifolds with constant Q-sectional curvature', *Kodai Math. Sem. Rep.* **29** (1977), 22–61.

[7] A. Martinez and J.D. Pérez, 'Real hypersurfaces in quaternionic projective space', *Ann. Mat. Pura Appl.* **145** (1986), 355–384.

[8] K. Yano and M. Kon, *CR-submanifolds of Kaehlerian and Sasakian manifolds* (Birkhäuser, Boston, Basel, Strutgart, 1983).

Kyungpook National University
Department of Mathematics
Taegu 702-701
Korea
e-mail: yjsuh@wmail.knu.ac.kr

https://doi.org/10.1017/S0004972700037795 Published online by Cambridge University Press