Real hypersurfaces in non-flat complex space form with structure Jacobi operator of Lie-Codazzi type

George Kaimakamis, Konstantina Panagiotidou and Juan de Dios Pérez

Abstract: In this paper the notion of Lie-Codazzi type of a tensor field T of type (1,1) on real hypersurfaces, which generalizes the notion of Lie-parallel, is introduced. Real hypersurfaces in non-flat complex space forms, whose structure Jacobi operator is of Lie-Codazzi type are studied. More precisely, the non-existence of such real hypersurfaces is proved.

Keywords: Real hypersurface, Structure Jacobi operator, Lie-Codazzi type, Complex projective space, Complex hyperbolic space.

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1 Introduction

An n-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called *complex space form* and is denoted by $M_n(c)$. Additionally, a complete and simply connected complex space form is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n$, if $c > 0$,
- a complex Euclidean space $\mathbb{C}^n$, if $c = 0$,
- or a complex hyperbolic space $\mathbb{C}H^n$, if $c < 0$.

Let *M* be a real hypersurface in a non-flat complex space form $M_n(c)$, $c \neq 0$. Then an almost contact metric structure (φ, ξ, η, g) can be defined on *M* induced from the Kaehler metric *G* and the complex structure *J* on $M_n(c)$. The *structure vector field* ξ is called *principal* if $A\xi = \alpha\xi$, where *A* is the shape operator of *M* and $\alpha = \eta(A\xi)$ is a smooth function. A real hypersurface is said to be a *Hopf hypersurface*, if ξ is principal.

The problem of classifying real hypersurfaces in $M_n(c)$, $c \neq 0$, is a classical one in the area of Differential Geometry. It was initiated by Takagi, who classified homogeneous real hypersurfaces in $\mathbb{C}P^n$. It was shown that they could be divided into six types, namely (A_1), (A_2), (B), (C), (D) and (E) ([12], [13]). In case of $\mathbb{C}H^n$, the study of real hypersurfaces with constant principal curvatures, was started by Montiel [6] and completed by Berndt in [1] for the Hopf case. They are divided into two types, namely (A) and (B), depending on the number of constant principal curvatures. The real hypersurfaces found by them are homogeneous. More information on the problem of classification of real hypersurfaces with constant principal curvatures in complex space forms can be found in [3] by Díaz-Ramos and Domínguez-Vázquez. Recently, Berndt and Tamaru in [2] have given a complete classification of homogeneous real hypersurfaces in $\mathbb{C}H^n$, $n \geq 2$.

Many researchers have been studying real hypersurfaces in non-flat complex space forms, when they satisfy certain geometric conditions such as parallelism, Lie paralellism etc. The structure Jacobi operator plays an important role in this direction. Generally, on a manifold M the Jacobi operator with respect to a vector field X is defined by $R(\cdot, X)X$, where R is the Riemmanian curvature of M. In case of real hypersurfaces, for $X = \xi$ the Jacobi operator is called *structure Jacobi operator* and is denoted by $l = R(\cdot, \xi)\xi$.

The Lie derivative of the structure Jacobi operator is an issue, which has been extensively studied. More precisely, in [8] Perez and Santos proved the non-existence of real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose structure Jacobi operator is *Lie parallel* , i.e. $\mathcal{L}_X l = 0$, for any $X \in TM$. Another type of parallelness of structure Jacobi operator, that has been studied is that of Lie ξ-parallel, i.e. $\mathcal{L}_\xi l = 0$. More precisely, in [9] real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, equipped with *Lie ξ-parallel* structure Jacobi operator are classified. In [4] Ivey and Ryan extend some of the above results in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. More analytically, it is proved that in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ there exist no real hypersurfaces, whose structure Jacobi operator is Lie parallel, but real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, whose structure Jacobi operator is Lie ξ-parallel exist and a classification of them is given. Furthermore, they proved that no real hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$, $n \geq 3$, equipped with Lie parallel structure Jacobi operator exist. Recently, in [11] the third author and Suh studied the condition of *Lie $\mathbb{D}$-parallel* structure Jacobi operator, i.e. $\mathcal{L}_X l = 0$, where X is orthogonal to ξ. They proved that no Hopf real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, satisfying the previous condition exist. In [7] the previous work is extended for the case of three dimensional real hypersurfaces in non-flat complex space forms. More precisely, it is proved the non-existence of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ equipped with $\mathbb{D}$-parallel structure Jacobi operator.

Generally, a tensor field T of type (1,1) on M is of Codazzi type, when the following relation is satisfied

$$(\nabla_X T)Y = (\nabla_Y T)X, \quad \text{where } X, Y \in TM.$$

In case of real hypersurfaces in [10] is proved that in complex projective space there exist no real hypersurfaces equipped with structure Jacobi operator of Codazzi type, i.e. $(\nabla_X l)Y = (\nabla_Y l)X$. In [14] and [15] Theofanidis and Xenos extended the previous result also for the case of three dimensional real hypersurfaces in non-flat complex space forms and in the case of the ambient space being the complex hyperbolic space.

Motivated by the work that is done so far the following question raises naturally

Question: Do real hypersurfaces in non-flat complex space forms, whose Lie derivative is of Codazzi type, exist?

First of all, a tensor field T of type (1,1) defined on a real hypersurface, will be called of *Lie-Codazzi type*, when the following relation is satisfied

$$(\mathcal{L}_X T)Y = (\mathcal{L}_Y T)X \quad X, Y \in TM.$$

In the present paper, real hypersurfaces in non-flat complex space forms equipped with structure Jacobi operator of Lie-Codazzi type are studied, i.e.

$$(\mathcal{L}_X l)Y = (\mathcal{L}_Y l)X \quad X, Y \ \in \ TM. \tag{1.1}$$

The following Theorem is proved

Theorem 1.1 *There exist no real hypersurfaces in $M_n(c)$, $n \geq 2$ and $c \neq 0$, whose structure Jacobi operator is of Lie-Codazzi type.*

It would be interesting to study the condition of Lie-Codazzi type also for other tensor fields of type (1,1), which are defined on real hypersurfaces such as the shape operator A, the Ricci operator S and the structure tensor φ. Furthermore, the above condition can be studied also for tensor fields of type (1,1) of real hypersurfaces in complex two-plane Grassmannians.

This paper is organized as follows: In Section 2 basic relations and definitions for real hypersurfaces in non-flat complex space forms are given. In Section 3 we present some basic Lemmas and the proof of 1.1.

2 Preliminaries

Throughout this paper all manifolds, vector fields etc are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces are supposed to be oriented and without boundary. Let M be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with almost complex structure J of constant holomorphic sectional curvature c. Let N be a unit normal vector field on M and $\xi = -JN$. For a vector field X tangent to M we can write $JX = \varphi X + \eta(X)N$, where φX and $\eta(X)N$ are the tangential and the normal component of JX respectively. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M

$$\overline{\nabla}_Y X = \nabla_Y X + g(AY, X)N,$$

$$\overline{\nabla}_X N = -AX,$$

where g is the Riemannian metric induced from the metric G and A is the shape operator of M in $M_n(c)$ with respect to N. M has an almost contact metric structure (φ, ξ, η, g) induced from J on $M_n(c)$, where φ is a (1,1) tensor field and η a 1-form on M such that

$$g(\varphi X, Y) = G(JX, Y), \quad \eta(X) = g(X, \xi) = G(JX, N).$$

Then we have

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \;\; g(X, \varphi Y) = -g(\varphi X, Y),$$

$$\nabla_X \xi = \varphi AX, \quad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX,Y)\xi. \tag{2.1}$$

Since the ambient space is of constant holomorphic sectional curvature c, the Gauss and Codazzi equations are respectively given by

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X \tag{2.2}$$

$$-g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z] + g(AY,Z)AX - g(AX,Z)AY,$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X,Y)\xi], \tag{2.3}$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M.

Relation (2.2) implies that the structure Jacobi operator l is given by

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi, \tag{2.4}$$

where $\alpha = \eta(A\xi)$ and for any X tangent vector to M.

For every point $P \in M$, the tangent space $T_P M$ can be decomposed as

$$T_P M = span\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \{X \in T_P M : \eta(X) = 0\}$. Due to the above decomposition, the vector field $A\xi$ can be written:

$$A\xi = \alpha\xi + \beta U, \tag{2.5}$$

where $\beta = |\varphi\nabla_\xi \xi|$ and $U = -(1/\beta)\varphi\nabla_\xi \xi \in \ker(\eta)$, provided that $\beta \neq 0$.

3 Proof of Theorem 1.1

In this section the symbol $M_n(c)$ is used to denote $\mathbb{C}P^n$ and $\mathbb{C}H^n$, $n \geq 2$. Let M be a real hypersurface in $M_n(c)$, whose structure Jacobi operator satisfies (1.1), which analytically is written

$$\nabla_X(lY) + \nabla_{lX}Y - \nabla_{lY}X - \nabla_Y(lX) = 2l\nabla_X Y - 2l\nabla_Y X, \tag{3.1}$$

where $X, Y \in TM$.

We consider the open subset $\mathcal{N}$ of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

Furthermore, we consider $\mathcal{V}$, Ω open subsets of $\mathcal{N}$ such that

$$\mathcal{V} = \{P \in \mathcal{N} : \alpha = 0 \text{ in a neighborhood of } P\},$$

$$\Omega = \{P \in \mathcal{N} : \alpha \neq 0 \text{ in a neighborhood of } P\},$$

where $\mathcal{V} \cup \Omega$ is open and dense in the closure of $\mathcal{N}$.

Lemma 3.1 *Let M be a real hypersurface in $M_n(c)$, whose structure Jacobi operator is of Lie-Codazzi type. Then $\mathcal{V}$ is empty.*

Proof: In $\mathcal{V}$ relation (2.5) becomes $A\xi = \beta U$. From (2.4) for $X = \varphi U$ and $X = \xi$ we obtain $l\varphi U = (c/4)\varphi U$ and $l\xi = 0$. Furthermore, the first of (2.1) implies $\nabla_\xi \xi = \beta\varphi U$.

Relation (3.1) for $X = \xi$ and $Y = \varphi U$, due to the first (2.1) yields

$$\frac{c}{4}\nabla_\xi \varphi U - \frac{c}{4}\varphi A\varphi U = 2l\nabla_\xi \varphi U - 2l\varphi A\varphi U.$$

The inner product of the above relation with ξ, due to $l\xi = 0$, $\nabla_\xi \xi = \beta\varphi U$ and $\beta \neq 0$, results in $c = 0$, which is a contradiction and this completes the proof the present Lemma. □

Lemma 3.2 *Let M be a real hypersurface in $M_n(c)$, whose structure Jacobi operator is of Lie-Codazzi type. Then on Ω the following relations hold*

$$AU = (\frac{\beta^2}{\alpha} - \frac{c}{4\alpha})U + \beta\xi, \quad A\varphi U = -\frac{c}{4\alpha}\varphi U. \tag{3.2}$$

Proof: In Ω relation (2.5) holds and the first of (2.1), because of the latter, implies $\nabla_\xi \xi = \beta\varphi U$. The inner product of relation (3.1) with ξ, due to the first of (2.1) and $l\xi = 0$ implies

$$g(l\varphi AX + \varphi AlX + lA\varphi X + A\varphi lX, Y) + lY[g(X,\xi)] - lX[g(Y,\xi)] = 0, \quad \text{where } X, Y \in TM. \tag{3.3}$$

Relation (3.3) for $X = \xi$, because of (2.5) and $\beta \neq 0$, yields $g(l\varphi U, Y) = 0$, for any $Y \in TM$, which leads to $l\varphi U = 0$. Therefore, relation (2.4) for $X = \varphi U$, owing to the latter results in

$$A\varphi U = -\frac{c}{4\alpha}\varphi U. \tag{3.4}$$

Relation (3.3) for $X = \varphi U$, taking into account $l\varphi U = 0$ and (3.4) yields $g((c/4\alpha)lU - lAU, Y) = 0$, for any $Y \in TM$ and this leads to

$$lAU = \frac{c}{4\alpha}lU. \tag{3.5}$$

The inner product of relation (3.1) for $Y = \xi$ with φU, due to the first of (2.1) and $l\varphi U = l\xi = 0$ yields $g(lAU + l\nabla_\xi \varphi U, X) = 0$, for any $X \in TM$, which results in

$$lAU = -l\nabla_\xi \varphi U. \tag{3.6}$$

Furthermore, relation (3.1) for $X = \xi$ and $Y = \varphi U$, because of the first of (2.1), $l\varphi U = l\xi = 0$ and (3.4) implies

$$\frac{c}{4\alpha}lU = l\nabla_\xi \varphi U. \tag{3.7}$$

The combination of relations (3.6) and (3.7), taking into consideration (3.5) implies $lU = 0$. Then relation (2.4) for $X = U$ implies $AU = (\beta^2/\alpha - c/4\alpha)U + \beta\xi$ and this completes the proof of the present Lemma. □

Lemma 3.3 *Let M be a real hypersurface in $M_n(c)$, whose structure Jacobi operator is of Lie-Codazzi type. Then Ω is empty.*

Proof: If $n = 2$ suppose that $\nabla_\xi U = \kappa_3\varphi U$. If $n \geq 3$ then let $\nabla_\xi U = \kappa_3\varphi U + \lambda_1 Z_1$, for a unit vector Z_1 taken in the orthogonal complement of the distribution spanned by $\{U, \varphi U, \xi\}$. Then from (3.1) for $X = U$ and $Y = \xi$, due to the latter and (2.4) for $X = Z_1$ and $lU = l\varphi U = l\xi = 0$ we obtain

$$\lambda_1(\frac{c}{4}Z_1 + \alpha A Z_1) = 0.$$

Suppose that $\lambda_1 \neq 0$, then due to the latter we have $AZ_1 = -(c/4\alpha)Z_1$. The inner product of the Codazzi equation, because of $\nabla_\xi U = \kappa_3\varphi U + \lambda_1 Z_1$ and (3.2) implies

$$Z_1\alpha = \beta\lambda_1, \text{ for } X = Z_1 \text{ and } Y = \xi \text{ with } \xi$$
$$g(\nabla_U U, Z_1) = \frac{\beta\lambda_1}{\alpha}, \text{ for } X = U \text{ and } Y = \xi \text{ with } Z_1$$
$$Z_1\beta = \frac{\beta^2\lambda_1}{\alpha}, \text{ for } X = Z_1 \text{ and } Y = U \text{ with } \xi \text{ owing to the previous one.}$$

Moreover, the inner product of the Codazzi equation for $X = Z_1$ and $Y = U$ with U, due to (3.2) and all the above relations results in $c = 0$, which is a contradiction.

Therefore, in Ω $\lambda_1 = 0$ and $\nabla_\xi U = \kappa_3\varphi U$. In the following the method is the same when $n = 2$ or $n \geq 3$. The inner product of Codazzi equation , because of (3.2) yields

$$\frac{\beta^2\kappa_3}{\alpha} = \beta\kappa_1 + \frac{c}{4\alpha}(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}), \text{ for } X = U \text{ and } Y = \xi \text{ with } \varphi U, \tag{3.8}$$
$$(\varphi U)\beta = \beta^2 + \beta\kappa_1 + \frac{c}{2\alpha}(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}), \text{ for } X = \varphi U \text{ and } Y = \xi \text{ with } U \text{ due to (3.8)} \tag{3.9}$$
$$(\varphi U)\alpha = \beta(\alpha + \kappa_3 + \frac{3c}{4\alpha}), \text{ for } X = \varphi U \text{ and } Y = \xi \text{ with } \xi, \tag{3.10}$$
$$\xi\alpha = \frac{4\alpha^2\beta\kappa_2}{c}, \text{ for } X = \varphi U \text{ and } Y = \xi \text{ with } \varphi U \tag{3.11}$$
$$(\varphi U)(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}) = \beta(\frac{\beta^2}{\alpha} + \frac{\beta\kappa_1}{\alpha} - \frac{3c}{4\alpha}), \text{ for } X = U \text{ and } Y = \varphi U \text{ with } U, \tag{3.12}$$
$$U\alpha = \frac{4\alpha\beta^2\kappa_2}{c}, \text{ for } X = U \text{ and } Y = \varphi U \text{ with } \varphi U \tag{3.13}$$
$$U\alpha = \xi\beta = \frac{4\alpha\beta^2\kappa_2}{c}, \text{ for } X = U \text{ and } Y = \xi \text{ with } \xi \text{ due to (3.13)} \tag{3.14}$$
$$U\beta = \beta\kappa_2(\frac{4\beta^2}{c} + 1), \text{ for } X = U \text{ and } Y = \xi \text{ with } U \text{ due to (3.11) and (3.14),} \tag{3.15}$$

where$\kappa_1 = g(\nabla_U U, \varphi U)$, $\kappa_2 = g(\nabla_{\varphi U} U, \varphi U)$ and $\kappa_3 = g(\nabla_\xi U, \varphi U)$.

Relation (3.12), because of (3.8), (3.9) and (3.10), yields

$$\kappa_3 = -4\alpha, \tag{3.16}$$

and so relation (3.8) becomes

$$\beta\kappa_1 = \frac{c}{4\alpha}(\frac{c}{4\alpha} - \frac{\beta^2}{\alpha}) - 4\beta^2. \tag{3.17}$$

The Riemannian curvature on M satisfies relation (2.2) and on the other hand is given by the relation $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. The combination and the inner product of these two relations for $X = Z = U$, $Y = \xi$ with φU and $X = \xi$, $Y = \varphi U$, $Z = U$ with φU, owing to $\nabla_\xi(\varphi U) = (\nabla_\xi\varphi)U + \varphi\nabla_\xi U$ and the second of (2.1) implies respectively:

$$U\kappa_3 - \xi\kappa_1 = \kappa_2(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha} - \kappa_3), \tag{3.18}$$

$$(\varphi U)\kappa_3 - \xi\kappa_2 = \kappa_1(\kappa_3 + \frac{c}{4\alpha}) + \beta(\kappa_3 - \frac{c}{2\alpha}). \tag{3.19}$$

Differentiating the relations (3.16) and (3.17) with respect to U and ξ respectively and substituting in (3.18) and taking into account (3.11), (3.14) and (3.15) we obtain

$$\kappa_2(c - 2\beta^2 - 4\alpha^2) = 0.$$

Suppose that $\kappa_2 \neq 0$, then because of the above relation we obtain $2\beta^2 + 4\alpha^2 = c$ holds. Differentiation of the last relation with respect to ξ, because of (3.11), (3.14) and $2\beta^2 + 4\alpha^2 = c$ yields $\kappa_2 = 0$, which is a contradiction.

Therefore, in Ω, $\kappa_2 = 0$ and relations (3.11), (3.14) and (3.15) become

$$U\alpha = U\beta = \xi\alpha = \xi\beta = 0.$$

Using the above relations and (3.16) we obtain

$$[U,\xi]\alpha = U(\xi\alpha) - \xi(U\alpha) = 0,$$

$$[U,\xi]\alpha = (\nabla_U\xi - \nabla_\xi U)\alpha = \frac{1}{4\alpha}(4\beta^2 + 16\alpha^2 - c)(\varphi U)\alpha.$$

Combining the last two relations we have

$$(4\beta^2 + 16\alpha^2 - c)(\varphi U)\alpha = 0.$$

Suppose that $(\varphi U)\alpha \neq 0$ then due to the above relation we have $16\alpha^2 + 4\beta^2 = c$. Differentiating the last relation with respect to φU and taking into account (3.9), (3.10), (3.16), (3.17) and $c = 16\alpha^2 + 4\beta^2$, implies $\alpha^2 = 0$, which is impossible.

Hence, on Ω relation $(\varphi U)\alpha = 0$ holds and relations (3.10) and (3.16) imply $c = 4\alpha^2$ and (3.17), due to the latter, yields $\beta\kappa_1 = \alpha^2 - 5\beta^2$. On the other hand relation (3.19), because of (3.16) and $c = 4\alpha^2$, leads to $\kappa_1 = -2\beta$. Substitution of κ_1 in $\beta\kappa_1 = \alpha^2 - 5\beta^2$ yields $3\beta^2 = \alpha^2$. The covariant derivative of $3\beta^2 = \alpha^2$ with respect to φU, because of (3.9), $\kappa_1 = -2\beta$, $c = 4\alpha^2$ and $3\beta^2 = \alpha^2$, results in $\beta = 0$, which is a contradiction and this completes the proof of the present Lemma. □

From Lemmas 3.1 and 3.3, the following proposition holds

Proposition 3.4 *Every real hypersurface in $M_n(c)$, $n \geq 2$, whose structure Jacobi operator is of Lie-Codazzi type, is a Hopf hypersurface.*

Since M is a Hopf hypersurface, $A\xi = \alpha\xi$ and due Theorem 2.1 [5] α is constant. We consider a unit vector $W \in \mathbb{D}$, such that $AW = \lambda W$, then $(\lambda - \alpha/2)A\varphi W = (\lambda\alpha/2 + c/4)\varphi W$ at some point $P \in M$, (Corollary 2.3 [5]). We have the following cases

- Case I: $\alpha^2 + c \neq 0$.

In this case we have that $\lambda \neq \alpha/2$, so $A\varphi W = \nu\varphi W$. On M the following relation holds ,(Corollary 2.3 [5]):

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{3.20}$$

The first of relation (2.1) and relation (2.4), for $X = W$ and $X = \varphi W$, due to $AW = \lambda W$ and $A\varphi W = \nu\varphi W$, imply respectively

$$\nabla_W\xi = \lambda\varphi W \quad \text{and} \quad \nabla_{\varphi W}\xi = -\nu W, \tag{3.21}$$

$$lW = (\frac{c}{4} + \alpha\lambda)W \quad \text{and} \quad l\varphi W = (\frac{c}{4} + \alpha\nu)\varphi W. \tag{3.22}$$

The inner product of relation (3.1) for $X = W$ and $Y = \varphi W$ with ξ, taking into account (3.21) and (3.22) implies

$$(\lambda + \nu)[(\frac{c}{2} + \alpha(\lambda + \nu)] = 0.$$

Let M_1 be the open subset of M such that

$$M_1 = \{P \in M : \lambda \neq -\nu, \text{ in a neighborhood of } P\}.$$

Then on M_1 relation $\alpha(\lambda+\nu) = -c/2$ holds. Relation (3.20), due to the latter, implies $\lambda\nu = 0$. Suppose that $\nu \neq 0$. So $\lambda = 0$ and relation $\alpha(\lambda + \nu) = -c/2$ implies $\alpha\nu = -c/2$.

The inner product of relation (3.1) for $X = W$ and $Y = \xi$ with φW, due to the first of (3.21) and (3.22), $\lambda = 0$ and $\alpha\nu = -c/2$, yields $g(\nabla_\xi W, \varphi W) = 0$. On the other hand, the inner product of (3.1) for $X = \varphi W$ and $Y = \xi$ with W, taking into account the second of (3.21) and (3.22), $\alpha\nu = -c/2$ and $g(\nabla_\xi W, \varphi W) = 0$ implies $c\nu = 0$, which is a contradiction, since $c \neq 0$ and $\nu \neq 0$. Therefore, on M_1 relation $\nu = 0$ holds. Following similar steps as in the previous case, we lead to the conclusion that M_1 is empty.

Therefore on M relation $\lambda = -\nu$ holds. Substitution of the last relation in (3.20) implies $c = -4\lambda^2$. So we conclude that $c < 0$ and that λ, ν are constant. The Hopf real hypersurface, which satisfies the previous conditions is that of type (B) in $\mathbb{C}H^n$, since the distribution $\mathbb{D}$ is not φ-invariant. Substituting the eigenvalues of it in relation $\lambda = -\nu$ leads to a contradiction (for the eigenvalues see [1]).

- Case II: $\alpha^2 + c = 0$.

In this case $\alpha \neq 0$, because if $\alpha = 0$, then $c = 0$, which is impossible. Suppose that $\lambda \neq \alpha/2$. Relation (3.20), owing to $\alpha^2 + c = 0$, results in $\nu = \alpha/2$, where ν is defined to be $A\varphi W = \nu\varphi W$. Also in this case relations (3.21) and (3.22) hold. The inner product of relation (3.1) for $X = W$ and $Y = \varphi W$ with ξ, taking into account (3.21), (3.22) and $\alpha^2 + c = 0$ implies

$$\lambda(\lambda + \frac{\alpha}{2}) = 0.$$

Suppose that $\lambda \neq 0$, then we have that $\lambda = -\alpha/2$. So we conclude that λ, ν are constant and the real hypersurface has three distinct eigenvalues. This implies that we have one of type (B). Substituting the eigenvalues of type (B) in the previous relation leads to a contradiction (for the eigenvalues see [1]).

Therefore, we have $\lambda = 0$. The inner product of relation (3.1) for $X = W$ and $Y = \xi$ with φW, due to the first of (3.21) and (3.22) yields $g(\nabla_\xi W, \varphi W) = 0$. On the other hand, the inner product of (3.1) for $X = \varphi W$ and $Y = \xi$ with W, taking into account the second of (3.21) and (3.22) and $g(\nabla_\xi W, \varphi W) = 0$ results in $\alpha = 0$, which is a contradiction.

So the remaining case is that of $\lambda = \alpha/2$, which will be the only eigenvalue for all vectors in $\mathbb{D}$. In this case the real hypersurface is a horosphere. The inner product of relation (3.1) for $X = W$ and $Y = \varphi W$ with ξ, taking into account (3.21) and (3.22) and that the only eigenvalue is $\alpha/2$ implies $\alpha = 0$, which is impossible.

Therefore, we have proved that there exist no real hypersurfaces in non-flat complex space forms $M_n(c)$, $n \geq 2$, whose structure Jacobi operator is of Lie-Codazzi type and this completes the proof of Theorem 1.1.

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G. Kaimakamis, Faculty of Mathematics and Engineering Sciences, Hellenic Military Academy, Vari, Attiki, Greece
e-mail:gmiamis@gmail.com

K. Panagiotidou, Faculty of Engineering, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece
e-mail: kapanagi@gen.auth.gr

J. de Dios Pérez, Departmento de Geometria y Topologia, Universidad de Granada, 18071, Granada Spain
e-mail: jdperez@ugr.es