J. S. PAK
KŌDAI MATH. SEM. REP
29 (1977), 22—61

REAL HYPERSURFACES IN QUATERNIONIC KAEHLERIAN MAMIFOLDS WITH CONSTANT Q-SECTIONAL CURVATURE

BY JIN SUK PAK

Recently determinations of some kinds of real hypersurfaces in a complex projective space $CP(m)$ have been done by several authors (Lawson [7], Maeda [9], Okumura [10], [11], [12] and etc.). They have obtained sufficient conditions or necessary and sufficient conditions for a real hypersurface in $CP(m)$ to be one of model hypersurfaces $M^C_{p,q}(a, b)$, where $M^C_{p,q}(a, b)$ are defined in $CP(m)$ by the same way as will be taken in § 7 to define model hypersurfaces $M^Q_{p,q}(a, b)$ in a quaternionic projective space $QP(m)$. Lawson also gave in his paper [7] a sufficient condition for a real minimal hypersurface in $QP(m)$ to be one of model hypersurfaces $M^Q_{p,q}(a, b)$. In the present paper, we shall obtain quaternionic analogies to theorems proved in [7], [9], [10], [11] and [12].

On the other hand Eum and the present author [1] gave a characterization of quaternionic Kaehlerian manifold $QP(m)$ of real dimension $4m$ with constant Q-sectional curvature c by the existence of a real hypersurface, which satisfies the condition

$$(0.1)\quad A(X, Y)=\frac{c}{4}g(X, Y)-\{u(X)u(Y)+v(X)v(Y)+w(X)w(Y)\},$$

passing through an arbitrary point and being tangent to an arbitrary $(4m\text{-}1)$-direction at that point, where A denotes the second fundamental tensor and u, v, w some local 1-forms. So, we shall prove in § 7 that a real hypersurface in $QP(m)$ satisfying the condition (0.1) is necessarily one of model subspaces $M^Q_{p,q}(a, b)$.

Real hypersurfaces in a quaternionic Kaehlerian manifold admit, under certain conditions, what we call an almost contact 3-structure. In § 1, we define almost contact 3-structures and give some formulas for later use. And we prove there Theorem 1 concerning their normality. In § 2, we show that there exist a contact 3-structure on real hypersurface M in a quaternionic Kaehlerian manifold (see Theorem 2). And we give there some necessary and sufficient conditions for the induced contact 3-structure of a real hypersurface M to be normal (see Theorem 3). In § 3, we recall some formulas concerning real hypersurfaces in a quaternionic Kaehlerian manifold with constant Q-sectional curvature for later use and prove Theorem 4. And we characterize there real quaternionic cylinders imbedded in Q^m in terms of the second fundamental tensor (see Theorem 5).

Received May. 10, 1976.

In § 4, using the Laplacian $\Delta\|A\|^2$, we find sufficient conditions for a compact real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature $c\geqq 0$ to satisfy the condition (0.1) (see Theorems 6 and 7). In § 5, using an integral formula, we give a necessary and sufficient condition for a compact real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature to admit a normal almost contact 3-structure (see Theorem 8).

In § 6, we shall recall definitions and some formulas concerning the submersion $\tilde{\pi}: S^{4m+3}\to QP(m)$ and an immersion $\imath: M\to QP(m)$ and prove some lemmas for later use. And we prove there Theorem 9 giving some conditions equivalent to the condition that a real hypersurface in $QP(m)$ admits a normal contact 3-structure. The last § 7 is devoted to give characterizations of the model subspace $M_{p,q}^{Q}(a, b)$ in $QP(m)$ (see Theorems 10~14 and Corollaries 15 and 16). Manifolds, submanifolds, geometric objects and mappings we discuss in this paper will be assumed to be differentiable and of class C^∞. We use in the present paper systems of indices as follows:

$$A, B, C, D=1, 2, \cdots, 4m+4; \qquad \kappa, \lambda, \mu, \nu=1, 2, \cdots, 4m+3,$$

$$\alpha, \beta, \gamma, \delta=1, 2, \cdots, 4m+2; \qquad h, \imath, j, k=1, 2, \cdots, 4m,$$

$$a, b, c, d, e=1, 2, \cdots, 4m-1; \qquad r, s, t, u=1, 2, 3.$$

The summation convention will be used with respect to these systems of indices.

The author wishes to express his appreciation to Professors Shigeru Ishihara and Masafumi Okumura for their encouragement and help in developing this paper.

§ 1. Almost contact 3-structures.

Let M be a differentiable manifold with Riemannian metric g and covered by an open covering $\sigma=\{0, '0, \cdots\}$. Then M is called *a manifold with almost contact 3-structure* if the following conditions (1) and (2) are satisfied:

(1) In each 0 there are given three 1-forms u_1, u_2, u_3 and three tensor fields ϕ_1, ϕ_2, ϕ_3 of type (1, 1) satisfying

$$\begin{aligned}
&\phi_1^2X=-X+u_1(X)U_1,\ u_1(\phi_1X)=0,\ \phi_1U_1=0,\ g(U_1, U_1)=1,\\
&\phi_2^2X=-X+u_2(X)U_2,\ u_2(\phi_2X)=0,\ \phi_2U_2=0,\ g(U_2, U_2)=1,\\
&\phi_3^2X=-X+u_3(X)U_3,\ u_3(\phi_3X)=0,\ \phi_3U_3=0,\ g(U_3, U_3)=1,\\
&\phi_1(\phi_2X)=\phi_3X+u_2(X)U_1,\ \phi_2(\phi_1X)=-\phi_3X+u_1(X)U_2,\\
&\phi_2(\phi_3X)=\phi_1X+u_3(X)U_2,\ \phi_3(\phi_2X)=-\phi_1X+u_2(X)U_3,\\
&\phi_3(\phi_1X)=\phi_2X+u_1(X)U_3,\ \phi_1(\phi_3X)=-\phi_2X+u_3(X)U_1,
\end{aligned}\tag{1.1}$$

$$\phi_1 U_2 = U_3,\ \phi_1 U_3 = -U_2,\ \phi_2 U_3 = U_1,\ \phi_2 U_1 = -U_3,\ \phi_3 U_1 = U_2,\ \phi_3 U_2 = -U_1,$$

$$g(\phi_1 X, Y) = -g(X, \phi_1 Y),\ g(\phi_2 X, Y) = -g(X, \phi_2 Y),\ g(\phi_3 X, Y) = -g(X, \phi_3 Y)$$

for any vector fields X and Y, where U_1, U_2 and U_3 are the vector fields associated respectively to u_1, u_2 and u_3, i.e. $g(U_x, X) = u_x(X)$, $x = 1, 2, 3$.

(2) If $O \cap 'O \neq \phi$, there are differentiable functions S_{xy} in $O \cap 'O$ such that

$$\begin{pmatrix} '\phi_1 \\ '\phi_2 \\ '\phi_3 \end{pmatrix} = (S_{xy}) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad \begin{pmatrix} 'u_1 \\ 'u_2 \\ 'u_3 \end{pmatrix} = (S_{xy}) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad (x, y = 1, 2, 3)$$

the matrix $S = (S_{xy})$ being contained in the orthogonal group $O(3)$. Then the set $\{(0, u_1, u_2, u_3, \phi_1, \phi_2, \phi_3, g) \mid 0 \in \mathcal{A}\}$ is called an *almost contact 3-structure*. In such a case the manifold M is necessarily of dimension $4m$-1.

We define locally in O a tensor field T of type (1, 1) by

$$T = u_1 \otimes U_1 + v_1 \otimes V_1 + w_1 \otimes W_1 .$$

Then, as a consequence of the condition (2), it follows that T determines a global tensor field in M, which will be also denoted by T. The condition (1) shows that T satisfies the equation $T^2 = T$ and hence it is a projection tensor field of rank 3. Therefore there exists in the manifold M a distribution D determined by T, and hence a 3-dimensional vector bundle B over M consisting of all vectors belonging to the distribution D.

We assume that $\{O; z\}$, $O \in \mathcal{A}$ are coordinate neighborhoods in the manifold M. Let there be given a connection ω in the vector bundle B and denote in each coordinate neighborhood $\{O; z\}$ of M by ω_x^y the components of ω with respect to the local frame (U_1, U_2, U_3) in B. Then the condition (2) implies that in $O \cap 'O \neq \phi$ the following relation is valid:

$$'\Omega = S^{-1} \Omega S + S^{-1} dS, \tag{1.3}$$

$\Omega = (\omega_x^y)$ being defined in each neighborhood O and dS the differential of the matrix $S = (S_{xy})$.

Denoting by ∇ the Riemannian connection determined by the Riemannian metric g and putting

$$\begin{pmatrix} \mathring{\nabla}_X \phi_1 \\ \mathring{\nabla}_X \phi_2 \\ \mathring{\nabla}_X \phi_3 \end{pmatrix} = \begin{pmatrix} \nabla_X \phi_1 \\ \nabla_X \phi_2 \\ \nabla_X \phi_3 \end{pmatrix} + (\omega_x^y(X)) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

$$\begin{pmatrix} \mathring{\nabla}_X U_1 \\ \mathring{\nabla}_X U_2 \\ \mathring{\nabla}_X U_3 \end{pmatrix} = \begin{pmatrix} \nabla_X U_1 \\ \nabla_X U_2 \\ \nabla_X U_3 \end{pmatrix} + (\omega_x^y(X)) \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix},$$

$(x, y=1, 2, 3)$ for any vector field X in M, we can easily verify by using (1.3) that in $O\cap 'O$

$$\begin{pmatrix} \mathring{\nabla}'\phi_1 \\ \mathring{\nabla}'\phi_2 \\ \mathring{\nabla}'\phi_3 \end{pmatrix}=(S_{xy})\begin{pmatrix} \mathring{\nabla}\phi_1 \\ \mathring{\nabla}\phi_2 \\ \mathring{\nabla}\phi_3 \end{pmatrix}, \quad \begin{pmatrix} \mathring{\nabla}'U_1 \\ \mathring{\nabla}'U_2 \\ \mathring{\nabla}'U_3 \end{pmatrix}=(S_{xy})\begin{pmatrix} \mathring{\nabla}U_1 \\ \mathring{\nabla}U_2 \\ \mathring{\nabla}U_3 \end{pmatrix}.$$

Now we consider in each neighborhood O local tensor field $\Phi(\phi_x, \phi_y)$, $(x, y=1, 2, 3)$ of type (1, 2) with components

$$\begin{aligned}\Phi(\phi_x, \phi_y)^a_{cb}=&(\phi_x)^e_c\mathring{\nabla}_e(\phi_y)^a_b-(\phi_x)^e_b\mathring{\nabla}_e(\phi_y)^a_c-\{\mathring{\nabla}_c(\phi_y)^e_b-\mathring{\nabla}_b(\phi_y)^e_c\}(\phi_x)^a_e\\ &+(\phi_y)^e_c\mathring{\nabla}_e(\phi_x)^a_b-(\phi_y)^e_b\mathring{\nabla}_e(\phi_x)^a_c-\{\mathring{\nabla}_c(\phi_x)^e_b-\mathring{\nabla}_b(\phi_x)^e_c\}(\phi_y)^a_e\\ &+\{\mathring{\nabla}_c(u_x)_b-\mathring{\nabla}_b(u_x)_c\}(u_y)^a+\{\mathring{\nabla}_c(u_y)_b-\mathring{\nabla}_b(u_y)_c\}(u_x)^a,\end{aligned} \tag{1.5}$$

where $(u_x)_c$, $(u_x)^b$ and $(\phi_x)^b_c$ are components of local tensor fields u_x, U_x and ϕ_x respectively. Then a simple calculation by using (1.2) and (1.4) gives the following relation

$$\begin{pmatrix} \Phi('\phi_1, '\phi_1)\Phi('\phi_1, '\phi_2)\Phi('\phi_1, '\phi_3) \\ \Phi('\phi_2, '\phi_1)\Phi('\phi_2, '\phi_2)\Phi('\phi_2, '\phi_3) \\ \Phi('\phi_3, '\phi_1)\Phi('\phi_3, '\phi_2)\Phi('\phi_3, '\phi_3) \end{pmatrix}=(S_{st})\begin{pmatrix} \Phi(\phi_1, \phi_1)\Phi(\phi_1, \phi_2)\Phi(\phi_1, \phi_3) \\ \Phi(\phi_2, \phi_1)\Phi(\phi_2, \phi_2)\Phi(\phi_2, \phi_3) \\ \Phi(\phi_3, \phi_1)\Phi(\phi_3, \phi_2)\Phi(\phi_3, \phi_3) \end{pmatrix}(S_{st})^{-1}$$

in $O\cap 'O\neq\phi$ because of $\Phi(\phi_x, \phi_y)=\Phi(\phi_y, \phi_x)$. Hence there is a global tensor field Σ_1 on M defined by

$$\Sigma_1=\Phi(\phi_1, \phi_1)+\Phi(\phi_2, \phi_2)+\Phi(\phi_3, \phi_3) \tag{1.6}$$

and a tensor Σ_2 globally defined on M by

$$\begin{aligned}\Sigma_2=&\Phi(\phi_1, \phi_1)\otimes\Phi(\phi_2, \phi_2)+\Phi(\phi_2, \phi_2)\otimes\Phi(\phi_3, \phi_3)+\Phi(\phi_3, \phi_3)\otimes\Phi(\phi_1, \phi_1)\\ &-\Phi(\phi_1, \phi_2)\otimes\Phi(\phi_2, \phi_1)-\Phi(\phi_2, \phi_3)\otimes\Phi(\phi_3, \phi_2)-\Phi(\phi_3, \phi_1)\otimes\Phi(\phi_1, \phi_3)\end{aligned} \tag{1.7}$$

up to sign. We now have

THEOREM 1. *In a* $(4m-1)$*-dimensional differentiable manifold with almost contact* 3*-structure a necessary and sufficient condition for the global tensors* Σ_1 *and* Σ_2 *defined respectivery by* (1.6) *and* (1.7) *to vanish is that*

$$\Phi(\phi_x, \phi_y)=0, (x, y=1, 2, 3).$$

We say that an almost contact 3-structure is *normal* (with respect to a connection ω in the vector bundle B) when $\Sigma_1=0$ and $\Sigma_2=0$. Then by means of Theorem 1 a necessary and sufficient condition for an almost contact 3-structure to be normal is that $\Phi(\phi_x, \phi_y)=0$ are established.

§ 2. Hypersurfaces in a quaternionic Kaehlerian manifold.

We first recall the definition of a quaternionic Kaehlerian structure given by S. Ishihara [3]. Let $\bar{M}$ be a $4m$-dimensional differentiable manifold and assume that there is a 3-dimensional vector bundle V consisting of tensors of type (1.1) over $\bar{M}$ satisfying the following conditions (a), (b) and (c):

(a) In any coordinate neighborhood $\{\bar{U}; y^h\}$, there is a local base $\{F, G, H\}$ of V such that

$$(2.1)\qquad \begin{aligned} &F_h^i F_j^h = -\delta_j^i,\ G_h^i G_j^h = -\delta_j^i,\ H_h^i H_j^h = -\delta_j^i, \\ &F_h^i G_j^h = -G_h^i F_j^h = H_j^i,\ G_h^i H_j^h = -H_h^i G_j^h = F_j^i, \\ &H_h^i F_j^h = -F_h^i H_j^h = G_j^i, \end{aligned}$$

F_j^i, G_j^i and H_j^i denoting components of F, G and H in $\bar{U}$ respectively.

(b) There is a Riemannian metric tensor g_{ji} such that

$$F_{ji} = -F_{ij},\ G_{ji} = -G_{ij},\quad H_{ji} = -H_{ij},$$

where $F_{ji} = g_{hi}F_j^h$, $G_{ji} = g_{hi}G_j^h$ and $H_{ji} = g_{hi}H_j^h$.

(c) For the Riemannian connection D of $(\bar{M}, g)$

$$(2.2)\qquad \begin{aligned} &D_j F_i^h = r_j G_i^h - q_j H_i^h, \\ &D_j G_i^h = -r_j F_i^h + p_j H_i^h, \\ &D_j H_i^h = p_j F_i^h - p_j G_i^h, \end{aligned}$$

where $p = p_i dy^i$, $q = q_i dy^i$ and $r = r_i dy^i$ are certain local 1-forms defined in $\bar{U}$. Such a local base $\{F, G, H\}$ is called a *canonical local base* of the bundle V in $\bar{U}$, and $(\bar{M}, g, V)$ or $\bar{M}$ is called a *quaternionic Kaehlerian manifold* and (g, V) a *quaternionic Kaehlerian structure*.

In a quaternionic Kaehlerian manifold $(\bar{M}, g, V)$ we take intersecting coordinate neighborhoods $\bar{U}$ and $'\bar{U}$. Let $\{F, G, H\}$ and $\{'F, 'G, 'H\}$ be canonical local bases of V in $\bar{U}$ and $'\bar{U}$ respectively. Then it follows that in $\bar{U} \cap {'\bar{U}}$

$$(2.3)\qquad \begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix} = (S_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix}, \qquad (x, y = 1, 2, 3)$$

with differentiable function S_{xy}, where the matrix $S = (S_{xy})$ is contained in the special orthogonal group $SO(3)$ as a consequence of (2.1).
As is well known, a quaternionic Kaehlerian manifold is orientable.

We consider a real hypersurface M in a quaternionic Kaehlerian manifold $\bar{M}$ of dimension $4m$. Let $\bar{M}$ is covered by a system of coordinate neighborhoods $\{\bar{U} : y^h\}$. Then M is covered by a system of coordinate neighborhoods

$\{U: y^a\}$, where $U=\bar{U}\cap M$. Let M be represented by $y^i=y^i(x^a)$ with respect to local coordinates (y^i) in $\bar{U}(\subset\bar{M})$ and (y^a) in $U(\subset M)$. Denoting the vectors $\partial_a y^i(\partial_a=\partial/\partial y^a)$ tangent to M by β_a^i and a unit normal vector field by N^i, we can put in each coordinate neighborhood $U=\bar{U}\cap M$

$$(2.4)\qquad \begin{aligned} &\text{(i)} & F_h^i B_a^h&=\phi_a^b B_b^i+u_a N^i, & F_h^i N^h&=-u^a B_a^i,\\ &\text{(ii)} & G_h^i B_a^h&=\psi_a^b B_b^i+v_a N^i, & G_h^i N^h&=-v^a B_a^i,\\ &\text{(iii)} & H_h^i B_a^h&=\theta_a^b B_b^i+w_a N^i, & H_h^i N^h&=-w^a B_a^i, \end{aligned}$$

$\phi_a^b, \psi_a^b, \theta_a^b$ being local tensor fields of type (1.1) and u_a, v_a, w_a local 1-forms defined in U, where $g_{ba}=g_{ji}B_b^j B_a^i$ are the components of the induced metric tensor in M. We have easily $u^b=g^{ba}u_a$, $v^b=g^{ba}v_a$ and $w^b=g^{ba}w_a$, where $(g^{ba})=(g_{ba})^{-1}$. Applying F_i^j to (2.4), (i) and taking account of (2.1) and (2.4), (i) itself, we find

$$\phi_e^a\phi_b^e=-\delta_b^a+u_b u^a,\ u_e\phi_b^e=0,\ \phi_e^a u^e=0,\ u_e u^e=1\,.$$

Transvecting F_i^j to (2.4), (ii) and using (2.1) give

$$\theta_a^b B_b^j+w_a N^j=\psi_a^e(\phi_e^b B_b^j+u_e N^j)-v_a u^b B_b^j,$$

$$-w^b B_b^j=-v^e(\phi_e^b B_b^j+u_e N^j)$$

because of (2.4), (i) and (ii). Thus we obtain

$$\phi_e^b\psi_a^e=\theta_a^b+v_a u^b,\ u_e\psi_a^e=w_a,\ \phi_e^b v^e=w^b,\ u_e v^e=0\,.$$

Transvecting H_i^j to (2.4), (ii) and using (2.1) imply

$$-\phi_a^b B_b^j-u_a N^j=\psi_a^e(\theta_e^b B_b^j+w_e N^j)-v_a w^b B_b^j,$$

$$-u^b B_b^j=-v^e(\theta_e^b B_b^j+w_e N^j)$$

because of (2.4), (i) and (iii). Thus we have

$$\theta_e^b\psi_a^e=-\phi_a^b+v_a w^b,\ w_e\psi_a^e=-u_a,\ \theta_e^b v^e=u^b,\ w_e v^e=0\,.$$

Similarly, using equations (2.1) and (2.4), we can prove the following formulas (2.5)~(2.13):

$$(2.5)\qquad \phi_e^b\phi_a^e=-\delta_b^a+u_b u^a,\ u_e\phi_a^e=0,\ \phi_e^b u^e=0,\ u_e u^e=1,$$

$$(2.6)\qquad \psi_e^b\psi_a^e=-\delta_b^a+v_b v^a,\ v_e\psi_a^e=0,\ \psi_e^b v^e=0,\ v_e v^e=1,$$

$$(2.7)\qquad \theta_e^b\theta_a^e=-\delta_b^a+w_b w^a,\ w_e\theta_a^e=0,\ \theta_e^b w^e=0,\ w_e w^e=1,$$

$$(2.8)\qquad \phi_e^b\psi_a^e=\theta_a^b+v_a u^b,\ u_e\psi_a^e=w_a,\ \phi_e^b v^e=w^b,\ u_e v^e=0,$$

$$(2.9)\qquad \theta_e^b\psi_a^e=-\phi_a^b+v_a w^b,\ w_e\psi_a^e=-u_a,\ \theta_e^b v^e=-u^b,\ w_e v^e=0,$$

$$(2.10)\qquad \psi_e^b\theta_a^e=\phi_a^b+w_a v^b,\ v_e\theta_a^e=u_a,\ \psi_e^b w^e=u^b,\ v_e w^e=0,$$

$$(2.11)\qquad \phi_e^b\theta_a^e=-\psi_a^b+w_a u^b,\ u_e\theta_a^e=-v_a,\ \phi_e^b w^e=-v^b,\ u_e w^e=0,$$

$$\theta_e^b \phi_a^e = \psi_a^b + u_a w^b,\ w_e \phi_a^e = v_a,\ \theta_e^b u^e = v^b,\ w_e u^e = 0\,, \tag{2.12}$$

$$\psi_e^b \phi_a^e = -\theta_a^b + u_a v^b,\ v_e \phi_a^e = -w_a,\ \psi_e^b u^e = -w^b,\ v_e u^e = 0\,. \tag{2.13}$$

Putting $\phi_{ba} = g_{ae}\phi_b^e$, $\psi_{ba} = g_{ae}\psi_b^e$ and $\theta_{ba} = g_{ae}\theta_b^e$, we have from (2.4)

$$\phi_{ba} = F_{ji} B_b^j B_a^i,\ \psi_{ba} = G_{ji} B_b^j B_a^i,\ \theta_{ba} = H_{ji} B_b^j B_a^i\,,$$

from which and the condition (b)

$$\phi_{ba} = -\phi_{ab},\ \psi_{ba} = -\psi_{ab},\ \theta_{ba} = -\theta_{ab}\,. \tag{2.14}$$

We now consider intersections of coordinate neighborhoods $U = \bar{U} \cap M$ and $'U = '\bar{U} \cap M$. Then, taking account of (2.3) and of (2.4) established in $\bar{U} \cap '\bar{U}$, we can prove that

$$\begin{pmatrix} '\phi \\ '\psi \\ '\theta \end{pmatrix} = (S_{xy}) \begin{pmatrix} \phi \\ \psi \\ \theta \end{pmatrix},\quad \begin{pmatrix} 'u \\ 'v \\ 'w \end{pmatrix} = (S_{xy}) \begin{pmatrix} u \\ v \\ w \end{pmatrix},\qquad (x, y = 1, 2, 3)\,. \tag{2.15}$$

hold in $U \cap 'U$, where the restriction of functions S_{xy} defined in $\bar{U} \cap '\bar{U}$ to $U \cap 'U$ is denoted also by the same letter S_{xy}. Thus we have proved

THEOREM 2. *A real hypersurface of a $4m$-dimensional quaternionic Kaehlerian manifold admits an almost contact 3-structure.*

We denote by ∇ the Riemannian connection induced on M from the Riemannian connection D of $\bar{M}$. Then equations of Gauss and Weingarten are given by

$$\nabla_b B_a^i = A_{ba} N^i,\ \nabla_b N^i = -A_b^a B_a^i \tag{2.16}$$

respectively, A_{ba} being the components of the second fundamental tensor with respect to the unit normal vector N^i and A_b^a being defined by $A_b^a = g^{ae} A_{be}$, where

$$\nabla_b B_a^i = \partial_b B_a^i + \left\{ {i \atop jh} \right\} B_b^j B_a^h - \left\{ {c \atop ba} \right\} B_c^i\,,$$

$$\nabla_b N^i = \partial_b N^i + \left\{ {i \atop jh} \right\} B_b^j N^h\,,$$

and $\left\{ {i \atop jh} \right\}$, $\left\{ {c \atop ba} \right\}$ are christoffel symbols formed respectively with g_{ji} and g_{ba}.

Applying the operator $\nabla_c = B_c^j D_j$ to the first equation of (2.4), (i), we obtain

$$B_c^j (D_j F_h^i) B_a^h + F_h^i \nabla_c B_a^h = (\nabla_c \phi_a^b) B_b^i + \phi_a^b \nabla_c B_b^i + (\nabla_c u_a) N^i + u_a \nabla_c N^i,$$

from which, substituting (2.2) and (2.16) and using (2.4),

$$(r_j B_c^j)(\psi_a^b B_b^i + v_a N^i) - (q_j B_c^j)(\theta_a^b B_b^i + w_a N^i) - A_{ca} u^b B_b^i$$

$$= (\nabla_c \phi_a^b) B_b^i + (A_{ce} \phi_a^e) N^i + (\nabla_c u_a) N^i - A_c^b u_a B_b^i\,.$$

Consequently, putting $p_c = p_j B_c^j$, $q_c = q_j B_c^j$ and $r_c = r_j B_c^j$, we have

$$\nabla_c\phi_a^b=r_c\psi_a^b-q_c\theta_a^b-A_{ca}u^b+A_c^bu_a,\ \nabla_cu_a=r_cv_a-q_cw_a-A_{ce}\phi_a^e .$$

Similarly, using (2.2), (2.4) and (2.16), we can find

$$\begin{cases}\nabla_c\phi_a^b=r_c\psi_a^b-q_c\theta_a^b-A_{ca}u^b+A_c^bu_a\ ,\\ \nabla_cu_a=r_cv_a-q_cw_a-A_{ce}\phi_a^e\ ,\end{cases} \tag{2.17}$$

$$\begin{cases}\nabla_c\psi_a^b=-r_c\phi_a^b+p_c\theta_a^b-A_{ca}v^b+A_c^bv_a\ ,\\ \nabla_cv_a=-r_cu_a+p_cw_a-A_{ce}\psi_a^e\ ,\end{cases} \tag{2.18}$$

$$\begin{cases}\nabla_c\theta_a^b=q_c\phi_a^b-p_c\psi_a^b-A_{ca}w^b+A_c^bw_a\ ,\\ \nabla_cw_a=q_cu_a-p_cv_a-A_{ce}\theta_a^e\ .\end{cases} \tag{2.19}$$

We now define a matrix ω consisting of local 1-forms $p=p_bdy^b$, $q=q_bdy^b$ and $r=r_bdy^b$ in M by

$$\Omega=\begin{pmatrix}0 & r & -q\\ -r & 0 & p\\ q & -p & 0\end{pmatrix}$$

in each coordinate neighborhood U, which is really the connection form of a linear connection ω induced in the vector bundle B determined by the projec tion tensor field $T=u\otimes U+v\otimes V+w\otimes W$ of rank 3. Obviously, we have

$$'\Omega=S^{-1}\Omega S+S^{-1}(dS)$$

in $U\cap 'U$, where $'\Omega$ is the connection form of ω in $'U$. If we now put

$$\mathring{\nabla}_c\phi_b^a=\nabla_c\phi_b^a-r_c\psi_b^a+q_c\theta_b^a,\ \mathring{\nabla}_cu^a=\nabla_cu^a-r_cv^a+q_cw^a\ ,$$

$$\mathring{\nabla}_c\psi_b^a=\nabla_c\psi_b^a+r_c\phi_b^a-p_c\theta_b^a,\ \mathring{\nabla}_cv^a=\nabla_cv^a+r_cu^a-p_cw^a\ ,$$

$$\mathring{\nabla}_c\theta_b^a=\nabla_c\theta_b^a-q_c\phi_b^a+p_c\psi_b^a,\ \mathring{\nabla}_cw^a=\nabla_cw^a-q_cu^a+p_cv^a\ ,$$

then we have from (2.15)

$$\begin{pmatrix}\mathring{\nabla}'\phi\\ \mathring{\nabla}'\psi\\ \mathring{\nabla}'\theta\end{pmatrix}=(S_{st})\begin{pmatrix}\mathring{\nabla}\phi\\ \mathring{\nabla}\psi\\ \mathring{\nabla}\theta\end{pmatrix},\quad \begin{pmatrix}\mathring{\nabla}'u\\ \mathring{\nabla}'v\\ \mathring{\nabla}'w\end{pmatrix}=(S_{st})\begin{pmatrix}\mathring{\nabla}u\\ \mathring{\nabla}v\\ \mathring{\nabla}w\end{pmatrix}$$

in $U\cap 'U$. On the other hand, (2.17), (2.18) and (2.19) give respectively

$$\mathring{\nabla}_c\phi_b^a=-A_{cb}u^a+A_c^au_b,\ \mathring{\nabla}_cu_b=-A_{ce}\phi_b^e\ , \tag{2.20}$$

$$\mathring{\nabla}_c\psi_b^a=-A_{cb}v^a+A_c^av_b,\ \mathring{\nabla}_cv_b=-A_{ce}\psi_b^e\ , \tag{2.21}$$

$$\mathring{\nabla}_c\theta_b^a=-A_{cb}w^a+A_c^aw_b,\ \mathring{\nabla}_cw_b=-A_{ce}\theta_b^e\ . \tag{2.22}$$

We compute components of local tensor fields $\Phi(\phi, \phi), \Phi(\psi, \psi), \Phi(\theta, \theta)$, $\Phi(\phi, \psi), \Phi(\psi, \theta)$ and $\Phi(\theta, \phi)$ define by (1.5). Denoting by $\Psi(\phi, \phi)_{cba}=g_{ae}\Phi(\phi, \phi)_{cb}{}^{e}$, we have from (2.20)

$$\begin{aligned}\Phi(\phi, \phi)_{cba}=&\phi_c^e(-A_{eb}u_a+A_{ea}u_b)-\phi_b^e(-A_{ec}u_a+A_{ea}u_c)\\&+(A_{ce}u_b-A_{be}u_c)\phi_a^e-(A_{ce}\phi_b^e-A_{be}\phi_c^e)u_a\,,\end{aligned}$$

this is,

$$\Phi(\phi, \phi)_{cba}=(A_{ce}\phi_a^e+A_{ae}\phi_c^e)u_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)u_c\,.$$

Similarly we have by using (2.20), (2.21) and (2.22)

$$\begin{aligned}&\Phi(\phi, \phi)_{cba}=(A_{ce}\phi_a^e+A_{ae}\phi_c^e)u_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)u_c\,,\\ (2.23)\qquad&\Phi(\psi, \psi)_{cba}=(A_{ce}\psi_a^e+A_{ae}\psi_c^e)v_b-(A_{be}\psi_a^e+A_{ae}\psi_b^e)v_c\,,\\&\Phi(\theta, \theta)_{cba}=(A_{ce}\theta_a^e+A_{ae}\theta_c^e)w_b-(A_{be}\theta_a^e+A_{ae}\theta_b^e)w_c\,.\end{aligned}$$

On the other hand, denoting by $\Phi(\phi, \psi)_{cba}=g_{ae}\Phi(\phi, \psi)_{cb}{}^{e}$, we have from (2.20) and (2.21)

$$\begin{aligned}\Phi(\phi, \psi)_{cba}=&\phi_c^e(-A_{eb}v_a+A_{ea}v_b)-\phi_b^e(-A_{ec}v_a+A_{ea}v_c)\\&+(A_{ce}v_b-A_{be}v_c)\phi_a^e+\psi_c^e(-A_{eb}u_a+A_{ea}u_b)\\&-\psi_b^e(-A_{ec}u_a+A_{ea}u_c)+(A_{ce}u_b-A_{be}u_c)\psi_a^e\\&-(A_{ce}\phi_b^e-A_{be}\phi_c^e)v_a-(A_{ce}\psi_b^e-A_{be}\psi_c^e)u_a\,,\end{aligned}$$

and consequently

$$\begin{aligned}\Phi(\phi, \psi)_{cba}=&(A_{ce}\phi_a^e+A_{ae}\phi_c^e)v_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)v_c\\&+(A_{ce}\psi_a^e+A_{ae}\psi_c^e)u_b-(A_{be}\psi_a^e+A_{ae}\psi_b^e)u_c\,.\end{aligned}$$

Similarly we have from (2.20), (2.21) and (2.22)

$$\begin{aligned}&\Phi(\phi, \psi)_{cba}=(A_{ce}\phi_a^e+A_{ae}\phi_c^e)v_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)v_c\\&\qquad+(A_{ce}\psi_a^e+A_{ae}\psi_c^e)u_b-(A_{be}\psi_a^e+A_{ae}\psi_b^e)u_c\,,\\ (2.24)\qquad&\Phi(\psi, \theta)_{cba}=(A_{ce}\psi_a^e+A_{ae}\psi_c^e)w_b-(A_{be}\psi_a^e+A_{ae}\psi_b^e)w_c\\&\qquad+(A_{ce}\theta_a^e+A_{ae}\theta_c^e)v_b-(A_{be}\theta_a^e+A_{ae}\theta_b^e)v_c\,,\\&\Phi(\theta, \phi)_{cba}=(A_{ce}\theta_a^e+A_{ae}\theta_c^e)u_b-(A_{be}\theta_a^e+A_{ae}\theta_b^e)u_c\\&\qquad+(A_{ce}\phi_a^e+A_{ae}\phi_c^b)w_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)w_c\,.\end{aligned}$$

We now assume the global tensor Σ_1 defined by (1.6) vanishes. Then substituting (2.23) into (1.6) gives

(2.25)
$$(A_{ce}\phi_a^e+A_{ae}\phi_c^e)u_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)u_c+(A_{ce}\psi_a^e+A_{ae}\psi_c^e)v_b$$
$$-(A_{be}\psi_a^e+A_{ae}\psi_b^e)v_c+(A_{ce}\theta_a^e+A_{ae}\theta_c^e)w_b-(A_{be}\theta_a^e+A_{ae}\theta_b^e)w_c=0\,.$$

Transvecting (2.25) with u^b and using (2.5), (2.8) and (2.11), we have

(2.26)
$$A_{ce}\phi_a^e+A_{ae}\phi_c^e-(u^bA_{be})\phi_a^e u_c-(u^bA_{be}\psi_a^e-A_{ae}w^e)v_c$$
$$-(u^bA_{be}\theta_a^e+A_{ae}v^e)w_c=0\,,$$

from which, transvecting with u^a,

(2.27)
$$(u^aA_{ae})\phi_c^e+2A(U,\,W)v_c-2A(U,\,V)w_c=0\,,$$

where and in the sequel the function $A_{ba}X^bY^a$ is denoted by $A(X, Y)$ for arbitrary vector fields $X=X^a\partial/\partial y^a$ and $Y=Y^a\partial/\partial y^a$ in M. Therefore, transvecting (2.27) with v^c and w^c respectively gives $A(U, V)=0$ and $A(U, W)=0$. Consequently (2.27) becomes

$$(u^aA_{ae})\phi_c^e=0\,.$$

Transvecting the equation above with ϕ_b^c and using (2.5) imply

$$A_{ba}u^a=A(U,\,U)u_b\,.$$

Similarly, using (2.5)~(2.13) and (2.25), we have

(2.28)
$$A_{ba}u^a=A(U,\,U)u_b,\ A_{ba}v^a=A(V,\,V)v_b,\ A_{ba}w^a=A(W,\,W)w_b\,.$$

Substituting (2.28) into (2.26) and taking account of (2.5), (2.12) and (2.13), we obtain

(2.29)
$$A_{ce}\phi_a^e+A_{ae}\phi_c^e=(A(U,\,U)-A(W,\,W))v_cw_a-(A(V,\,V)-A(U,\,U))w_cv_a\,,$$

from which, taking the skew-symmetric part,

$$(A(V,\,V)-A(W,\,W))(v_cw_a-w_cv_a)=0\,,$$

which implies $A(V, V)=A(W, W)$. On the other hand, transvecting (2.29) with v^cw^a and using (2.12), (2.13) and (2.28) give $A(U, U)=A(W, W)$. Consequently we have from (2.29)

$$A_{ce}\phi_a^e+A_{ae}\phi_c^e=0\,.$$

By the same way as above we can find

(2.30)
$$A_{ce}\phi_a^e+A_{ae}\phi_c^e=0,\ A_{ce}\psi_a^e+A_{ae}\psi_c^e=0,\ A_{ce}\theta_a^e+A_{ae}\theta_c^e=0\,.$$

Therefore, comparing (2.23) and (2.24) with (2.30) and taking account of (1.7), we see that the global tensor field Σ_2 also vanishes. Thus $\Sigma_1=0$ implies $\Sigma_2=0$ for real hypersurfaces. Hence, combining Theorm 1, we have

THERREM 3. *In a real hypersurface, of a quaternionic Kaehlerian manifold the following conditions* (1)~(3) *are equivalent to each other*:

(1) *The induced almost contact* 3-*structure in the hypersurface is normal.*

(2) *The induced almost contact* 3-*structure tensors* $\{\phi, \psi, \theta\}$ *commute with the second fundamental tensor.*

(3) $\Sigma_1=0$.

§ 3. Hypersurfaces in a quaternionic Kaehlerian manifold of constant Q-sectional curvature.

Let $\bar{M}$ be a $4m$-dimensional quaternionic Kaehlerian manifold with constant Q-sectional curvature c. It is well known that its curvature tensor has components of the form

$$(3.1)\quad \begin{aligned} K_{kji}{}^{h}=\frac{c}{4}(&\delta_k^h g_{ji}-\delta_j^h g_{ki}+F_k^h F_{ji}-F_j^h F_{ki}-2F_{kj}F_i^h+G_k^h G_{ji} \\ &-G_j^h G_{ki}-2G_{kj}G_i^h+H_k^h H_{ji}-H_j^h H_{ki}-2H_{kj}H_i^h), \end{aligned}$$

where c is necessary a constant, provided $m\geqq 2$ (See Ishihara [3]). On the other hand, as a characterization of quaternionic Kaehlerian manifold with constant Q-sectional curvature c, Eum and the present author [1] proved

THEOREM A. *A necessary and sufficient condition that a* $4m$-*dimensional Kaehlerian manifold* ($m\geqq 2$) *is of constant* Q-*sectional curvature* c *is there exists a hypersurface with the second fundamental tensor* A_{ba} *of the form*

$$A_{ba}=\frac{c}{4}g_{ba}-(u_b u_a+v_b v_a+w_b w_a),$$

u, v *and* w *being appeared in* (2.4), *through every point with every* ($4m$-1)-*direction at the point.*

So, it seems interesting to study real hypersurfaces with second fundamental tensor of the form

$$(3.2)\quad A_{ba}=\mu g_{ba}-\lambda(u_b u_a+v_b v_a+w_b w_a),$$

μ, λ being assumed to be functions, in a quaternionic Kaehlerian manifold with constant Q-sectional curvature.

Let M be a real hypersurface in the manifold $\bar{M}$. Then the structure equations of Gauss and Codazzi

$$K_{kjih}B_d^k B_c^j B_b^i B_a^h=K_{dcba}-A_{da}A_{cb}+A_{ca}A_{db},$$

$$K_{kjih}B_c^k B_b^j B_a^i N^h=\nabla_c A_{ba}-\nabla_b A_{ca}$$

are established, where $K_{kjih}=g_{hl}K_{kji}{}^{l}$ and $K_{dcba}=g_{ae}K_{dcb}{}^{e}$, $K_{dcb}{}^{e}$ being components of the curvature tensor determined by the induced metric g_{cb} in M. Substituting (2.4) and (3.1) into the equations above give respectively

$$K_{dcba}=\frac{c}{4}(g_{da}g_{cb}-g_{ca}g_{db}+\phi_{da}\phi_{cb}-\phi_{ca}\phi_{db}-2\phi_{dc}\phi_{ba}+\psi_{da}\psi_{cb}-\psi_{ca}\psi_{db}-2\psi_{dc}\psi_{ba}+\theta_{da}\theta_{cb}-\theta_{ca}\theta_{db}-2\theta_{dc}\theta_{ba})+A_{da}A_{cb}-A_{ca}A_{db}, \tag{3.3}$$

$$\nabla_c A_{ba}-\nabla_b A_{ca}=\frac{c}{4}(u_c\phi_{ba}-\phi_{ca}u_b-2\phi_{cb}u_a+v_c\psi_{ba}-\psi_{ca}v_b-2\psi_{cb}v_a+w_c\theta_{ba}-\theta_{ca}w_b-2\theta_{cb}w_a). \tag{3.4}$$

We now denote by K_{cb} components of the Ricci tensor in M. Transvecting (3.3) with g^{da}, we have from (2.5), (2.6) and (2.7)

$$K_{cb}=\frac{c}{4}\{(4m+7)g_{cb}-3(u_cu_b+v_cv_b+w_cw_b)\}+BA_{cb}-A_{ce}A_b^e, \tag{3.5}$$

where and in the sequel the mean curvature $A_b^b=g^{cb}A_{cb}$ will be denoted by B.

Now, we assume that the second fundamental tensor A_{ba} of M has the form (3.2), μ, λ being differentiable functions. Then substituting (3.2) into the second equation of (2.17) and using (2.5), (2.12) and (2.13), we have

$$\nabla_c u_a=(r_c+\lambda w_c)v_a-(q_c+\lambda v_c)w_a+\mu\phi_{ca}. \tag{3.6}$$

Similarly from those of (2.18) and those of (2.19) the equations

$$\begin{aligned}\nabla_c v_a&=-(r_c+\lambda w_c)u_a+(p_c+\lambda u_c)w_a+\mu\psi_{ca},\\ \nabla_c w_a&=(q_c+\lambda v_c)u_a-(p_c+\lambda u_c)v_a+\mu\theta_{ca}\end{aligned} \tag{3.7}$$

will be obtained. Differentiating (3.2) covariantly along M and taking account of (3.6) and (3.7), we find

$$\nabla_c A_{ba}=(\nabla_c\mu)g_{ba}-\nabla_c\lambda(u_bu_a+v_bv_a+w_bw_a)-\lambda\mu(u_a\phi_{cb}+u_b\phi_{ca}+v_a\psi_{cb}+v_b\psi_{ca}+w_a\theta_{cb}+w_b\theta_{ca}),$$

from which, taking the skew-symmetric part with respect to c and b and using (3.4), we have

$$(\nabla_c\mu)g_{ba}-(\nabla_b\mu)g_{ca}-\nabla_c\lambda(u_bu_a+v_bv_a+w_bw_a)+\nabla_b\lambda(u_cu_a+v_cv_a+w_cw_a)=\left(\frac{c}{4}-\lambda\mu\right)(u_c\phi_{ba}-\phi_{ca}u_b-2\phi_{cb}u_a+v_c\psi_{ba}-\psi_{ca}v_b-2\psi_{cb}v_c+w_c\theta_{ba}-\theta_{ca}w_b-2\theta_{cb}w_a).$$

Transvecting the above equation with g^{ba} and $u^bu^a+v^bv^a+w^bw^a$, we find respectively

$$(4m-2)\nabla_c\mu-3\nabla_c\lambda+(u^a\nabla_a\lambda)u_c+(v^a\nabla_a\lambda)v_c+(w^a\nabla_a\lambda)w_c=0 \tag{3.8}$$

and

$$3\nabla_c\mu-(u^a\nabla_a\mu)u_c-(v^a\nabla_a\mu)v_c-(w^a\nabla_a\mu)w_c$$
$$=3\nabla_c\lambda-(u^a\nabla_a\lambda)u_c-(v^a\nabla_a\lambda)v_c-(w^a\nabla_a\lambda)w_c\,.$$

Combining the last two equations, we get

$$(4m-5)\nabla_c\mu=-(u^a\nabla_a\mu)u_c-(v^a\nabla_a\mu)v_c-(w^a\nabla_a\mu)w_c\,,$$

which implies that

$$u^c\nabla_c\mu=v^c\nabla_c\mu=w^c\nabla_c\mu=0$$

and consequently that $\nabla_c\mu=0$. Substituting $\nabla_c\mu=0$ into (3.8), we obtain

$$3\nabla_c\lambda=(u^a\nabla_a\lambda)u_c+(v^a\nabla_a\lambda)v_c+(w^a\nabla_a\lambda)w_c\,,$$

from which

$$u^a\nabla_a\lambda=v^a\nabla_a\lambda=w^a\nabla_a\lambda=0\,.$$

Hence $\nabla_c\lambda=0$. Thus μ and λ are both constants and $\lambda\mu=c/4$. Thus we have

THEOREM 4. *Let M be a real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature c. If the second fundamental tensor A_{ba} has the form*

$$A_{ba}=\mu g_{ba}-\lambda(u_bu_a+v_bv_a+w_bw_a)\,,$$

μ, λ being differentiable functions, then μ and λ are both constants and $\lambda\mu=c/4$.

We now consider the case where the ambient manifold is of zero Q-sectional curvature. Identifying the quaternionic Q^m naturally with $\boldsymbol{R}^{4m}$, Q^m can be considered as a quaternionic Kaehlerian manifold of zero Q-sectional curvature with the natural quaternionic Kaehlerian structure $\{F, G, H\}$ having numerical components of the form

$$(3.9)\quad F:\begin{pmatrix}0 & -E & 0 & 0\\ E & 0 & 0 & 0\\ 0 & 0 & 0 & -E\\ 0 & 0 & E & 0\end{pmatrix},\quad G:\begin{pmatrix}0 & 0 & -E & 0\\ 0 & 0 & 0 & E\\ E & 0 & 0 & 0\\ 0 & -E & 0 & 0\end{pmatrix},\quad H:\begin{pmatrix}0 & 0 & 0 & -E\\ 0 & 0 & -E & 0\\ 0 & E & 0 & 0\\ E & 0 & 0 & 0\end{pmatrix},$$

where E denotes the identity (m, m)-matrix. We assume that there exists a real hypersurface in Q^m with the second fundamental tensor A_{ba} of the form (3.2). Then by means of Theorem 4 μ and λ are constants and $\lambda\mu=0$. Therefore A_{ba} is one of the following forms

$$(3.10)\quad A_{ba}=0\,;\qquad A_{ba}=\mu g_{ba}\,;\qquad A_{ba}=-\lambda(u_bu_a+v_bv_a+w_bw_a)\,.$$

Now let the second fundamental tensor A_{ba} of a real hypersurface M in Q^m

be of the form (3.10). Since in this case

$$D_jF_i^h=0, \quad D_jG_i^h=0, \quad D_jH_i^h=0,$$

the local 1-forms p, q and r in M are all vanish. Therefore taking account of our assumption (3.10) implies

$$\nabla_c u_a = w_c v_a - w_a v_c, \quad \nabla_c v_a = u_c w_a - u_a w_c, \quad \nabla_c w_a = v_c u_a - v_a u_c.$$

Applying the operator ∇_c to (3.10) and substituting the equations above, we can easily verify $\nabla_c A_{ba}=0$. On the other hand the condition (3.10) implies that the second fundamental tensor A_b^a has exactly two eigenvalues $-\lambda$ and 0 whose multiplicities are 3 and $4(m-1)$ respectively. Hence, using $\nabla_c A_b^a=0$, we see that the eigenspaces corresponding to $-\lambda$ and 0 define respectively 3-and $4(m-1)$-dimensional distribution D_{-1} and D_0 over M which are integrable and parallel. Denoting maximal integral manifolds of D_{-1} and D_0 by $M_{-\lambda}$ and M_0 respectively, M_{-1} and M_0 and both totally geodesic in M. Taking account of (3.10) and using (2.5)~(2.13), we have by a simple calculation

$$A_{be}\phi_a^e + A_{ae}\phi_b^e = 0, \quad A_{be}\psi_a^e + A_{ae}\psi_b^e = 0, \quad A_{be}\theta_a^e + A_{ae}\theta_b^e = 0.$$

Thus, for an arbitrary eigenvector X^a of A_b^a corresponding to an eigenvalue ρ, $\phi_b^a X^b$, $\psi_b^a X^b$ and $\theta_b^a X^b$ are also eigenvectors corresponding to the same eigenvalue ρ. Putting $q^j = q^b B_b^j$ for an eigenvector q^b of A_b^a and taking account of (2.4), we see that the subspaces $\{q^j \mid q^b \in D_{-1}\} \oplus \{N^j\}^*$ and $\{q^j \mid q^b \in D_0\}$ are both invariant under the actions of F, G and H, where $\{N^j\}^*$ is the linear closure of the set $\{N^j\}$. Consequently M_0 can be regarded as quaternionic submanifolds of Q^m. Let M_{-1} be represented by $y^a = y^a(z^\alpha)$ in M. Then the local expression of M_{-1} in Q^m can be written by $y^j = y^j(y^a(z^\alpha))$. Denoting the tangent vectors $\partial_\alpha y^j$ to M_{-1} by B_α^j, we have $B_\alpha^j = B_\alpha^b B_b^j$. Since M_{-1} is totally geodesic in M and B_α^b are eigenvectors of A_b^a corresponding to eigenvalue -1, we obtain $\nabla_\beta B_\alpha^j = -g_{\beta\alpha} N^j$, which means that M_{-1} is totally umbilical in Q^m. Similarly we can prove that M_0 is totally geodesic in Q^m and hence identified with Q^{m-1}. Therefore, since $M_{-1} \times M_0 = S^3 \times Q^{m-1}$ is complete, we have

THEOREM 5. *Let M be a complete real hypersurface of Q^m with the second fundamental tensor A_{ba} of the form*

$$A_{ba} = \mu g_{ba} - \lambda(u_b u_a + v_b v_a + w_b w_a),$$

μ and λ being differentiable functions. Then M is a Euclidean plane R^{4m-1}, $S^{4m-1}(1/\sqrt{\mu})$ or $S^3\ (1/\sqrt{\lambda}) \times Q^{m-1}$.

§ 4. The Laplacian $\Delta\|A\|^2$

Let M be a real hypersurface in a quaternionic Kaehlerian manifold of constant Q-sectional curvature c. In this section we compute the Laplacian $\Delta\|A\|^2$ of the function $\|A\|^2 = A_{ba}A^{ba}$, which is globally defined in M, where $\Delta =$

$g^{cb}\nabla_c\nabla_b$. We thus have

$$\frac{1}{2}\Delta\|A\|^2=g^{dc}(\nabla_d\nabla_c A_{ba})A^{ba}+\|\nabla_c A_{ba}\|^2,$$

where $\|\nabla_c A_{ba}\|^2=(\nabla_c A_{ba})(\nabla^c A^{ba})$. By using Ricci identity and the equation (3.4) of Codazzi we find

$$\begin{aligned}(4.1)\qquad \frac{1}{2}\Delta\|A\|^2=&(\nabla_b\nabla_a B)A^{ba}+K_c^b A_b^a A_a^c-K_{dcba}A^{da}A^{cb}+\frac{3}{4}c\{B(A(U,U)\\&+A(V,V)+A(W,W))-(\|A_{cb}u^b\|^2+\|A_{cb}v^b\|^2+\|A_{cb}w^b\|^2)\\&-(A_{ce}\phi_b^e)(\phi_d^c)(\phi_a^c A^{db})-(A_{ce}\psi_b^e)(\psi_a^c A^{db})-(A_{ce}\theta_b^e)(\theta_a^c A^{db})\}\\&+\|\nabla_c A_{ba}\|^2.\end{aligned}$$

On the other hand a straight forward calculation by using (2.5), (2.6) and (2.7) gives

$$\begin{aligned}&\|A_{ce}\phi_b^e+A_{be}\phi_c^e\|^2+\|A_{ce}\psi_b^e+A_{be}\psi_c^e\|^2+\|A_{ce}\theta_b^e+A_{be}\theta_c^e\|^2\\&=6A_{cb}A^{cb}-2\{(A_{ce}\phi_b^e)(\phi_a^c A^{db})+(A_{ce}\psi_b^e)(\psi_a^c A^{db})+(A_{ce}\theta_b^e)(\theta_a^c A^{db})\}\\&\quad-2(\|A_{ce}u^e\|^2+\|A_{ce}v^e\|^2+\|A_{ce}w^e\|^2),\end{aligned}$$

from which, using the equation (3.3) of Gauss and (3.5), we can easily see that

$$\begin{aligned}K_{dcba}A^{da}A^{cb}=\frac{c}{4}[&-\frac{3}{2}\{\|A_{ce}\phi_b^e+A_{be}\phi_c^e\|^2+\|A_{ce}\psi_b^e+A_{be}\psi_c^e\|^2\\&+\|A_{ce}\theta_b^e+A_{be}\theta_c^e\|^2\}-3(\|A_{ce}u^e\|^2+\|A_{ce}v^e\|^2+\|A_{ce}w^e\|^2)\\&+B^2+8(A_{cb}A^{cb})+(A_{cb}A^{cb})^2-\|A_{ce}A_b^e\|^2]\end{aligned}$$

and

$$\begin{aligned}K_{cb}A_a^b A^{ca}=\frac{c}{4}\{&(4m+7)A_{cb}A^{cb}-3(\|A_{ce}u^e\|^2+\|A_{ce}v^e\|^2+\|A_{ce}w^e\|^2)\\&+B(A_c^b A_b^a A_a^c)-\|A_{ce}A_b^e\|^2\}.\end{aligned}$$

Therefore (4.1) becomes

$$\begin{aligned}(4.2)\qquad \frac{1}{2}\Delta\|A\|^2=&(\nabla_b\nabla_a B)A^{ba}+\frac{c}{4}[3\{\|A_{ce}\phi_b^e+A_{be}\phi_c^e\|^2\\&+\|A_{ce}\psi_b^e+A_{be}\psi_c^e\|^2+\|A_{ce}\theta_b^e+A_{be}\theta_c^e\|^2\}\\&+3B\{A(U,U)+A(V,V)+A(W,W)\}+B(A_c^b A_b^a A_a^c)\\&-B^2+(4m-10)A_{cb}A^{cb}-(A_{cb}A^{cb})^2]+\|\nabla_c A_{ba}\|^2.\end{aligned}$$

In order to get further results, we shall prove some lemmas.

LEMMA 4.1. *On a real hypersurface in a quaternionic Kaehlerian manifold the following inequality holds*:

$$B^2 \leqq 4(m-1)A_{cb}A^{cb}+2B\{A(U, U)+A(V, V)+A(W, W)\}\ . \tag{4.3}$$

Proof. We define a symmetric tensor P_{ba} by

$$P_{ba}=A_{ba}+\frac{1}{4(m-1)}B(u_b u_a+v_b v_a+w_b w_a)\ .$$

Putting $P^{ba}=g^{be}g^{ad}P_{ed}$ and $P=g^{ba}P_{ba}$ gives

$$\|P_{ba}-(P/4m-1)g_{ba}\|^2=P_{ba}P^{ba}-\frac{1}{4m-1}P^2 \geqq 0\ ,$$

which implies (4.3).

LEMMA 4.2. *On a real hypersurface in a quaternionic Kaehlerian manifold of constant Q-sectional curvature c*

$$\|\nabla_c A_{ba}\|^2 \geqq \frac{3}{2}(m-1)c^2$$

holds and that equality holds if and only if

$$\nabla_c A_{ba}+\frac{c}{4}(\phi_{ca}u_b+\phi_{cb}u_a+\psi_{ca}v_b+\psi_{cb}v_a+\theta_{ca}w_b+\theta_{cb}w_a)=0\ .$$

Proof. Putting

$$\overset{*}{\nabla}_c A_{ba}=\nabla_c A_{ba}+\frac{c}{4}(\phi_{ca}u_b+\phi_{cb}u_a+\psi_{ca}v_b+\psi_{cb}v_a+\theta_{ca}w_b+\theta_{cb}w_a) \tag{4.4}$$

and using the equation (3.4) of Codazzi, we can easily check that

$$\|\overset{*}{\nabla}_c A_{ba}\|^2=\|\nabla_c A_{ba}\|^2-\frac{3}{2}(m-1)c^2,$$

which implies our assertion.

By means of (4.2), Lemmas 4.1 and 4.2 we have the following inequality

$$\begin{aligned}\frac{1}{2}\Delta\|A\|^2 \geqq (\nabla_c\nabla_b B)A^{cb}+\frac{c}{4}[3\{\|A_{ce}\phi^e_b+A_{be}\phi^e_c\|^2 \\ +\|A_{ce}\psi^e_b+A_{be}\psi^e_c\|^2+\|A_{ce}\theta^e_b+A_{be}\theta^e_c\|^2\} \\ +3B\{A(U, U)+A(V, V)+A(W, W)\}+B(A^b_c A^a_b A^c_a)-(A_{cb}A^{cb})^2 \\ +6\{(m-1)c-A_{ba}A^{ba}\}]+\|\overset{*}{\nabla}_c A_{ba}\|^2,\end{aligned} \tag{4.5}$$

where $\overset{*}{\nabla}_c A_{ba}$ is defined by (4.4). Thus we have

THEOREM 6. *Let M be a compact real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature* $c \geqq 0$. *If the second* *fundamental tensor A_{ba} is semi-definite and the mean curvature B constant and if $A_{ba}A^{ba} \leqq (m-1)c$, then A_{ba} has the form*

$$A_{ba} = \frac{\sqrt{c}}{2}\{g_{ba} - (u_b u_a + v_b v_a + w_b w_a)\}.$$

Proof. When $c=0$, the lemma is trivially established. When $c>0$, (4.5) and our assumptions imply

$$A_{ce}\phi^e_b + A_{be}\phi^e_c = 0,\ A_{ce}\psi^e_b + A_{be}\psi^e_c = 0,\ A_{ce}\theta^e_b + A_{be}\theta^e_c = 0, \tag{4.6}$$

$$B(A^b_c A^a_b A^c_a) = (A_{cb}A^{cb})^2, \tag{4.7}$$

$$A_{cb}A^{cb} = (m-1)c, \tag{4.8}$$

$$A(U, U) = A(V, V) = A(W, W) = 0. \tag{4.9}$$

As already show in section 2, (4.6) and (4.9) imply

$$A_{ba}u^a = 0, \quad A_{ba}v^a = 0, \quad A_{ba}w^a = 0. \tag{4.10}$$

Applying the operator ∇_c to the first equation of (4.10) and taking the skew-symmetric part with respect to the indices c and b, we find

$$(\nabla_c A_{ba} - \nabla_b A_{ca})u^a + A_{ba}\nabla_c u^a - A_{ca}\nabla_b u^a = 0.$$

Substituting (2.17) and (3.4) in the equation above and using (2.5), (2.8), (2.12) and (2.13) give

$$\frac{c}{4}(v_c w_b - v_b w_c - \phi_{cb}) + A_{ed}A^e_c\phi^d_b = 0$$

because of (4.6) and (4.10). Transvecting the equation above with ϕ^c_a and making use of (2.5), (2.12), (2.13) and (4.10), we can easily verify that

$$A_{be}A^e_a = \frac{c}{4}\{g_{ba} - (u_b u_a + v_b v_a + w_b w_a)\}$$

Combining (4.7) and (4.8), we see that the second fundamental tensor A^a_b has the components on M

$$(A^a_b) = \begin{pmatrix} 0 & & & & & & 0 \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & \frac{\sqrt{c}}{2} & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & \frac{\sqrt{c}}{2} \end{pmatrix} \quad \text{or} \quad (A^a_b) = \begin{pmatrix} 0 & & & & & & 0 \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & -\frac{\sqrt{c}}{2} & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & -\frac{\sqrt{c}}{2} \end{pmatrix}$$

with respect to the adapted orthonormal frame $\{U, V, W, X_1, \cdots, X_{4(m-1)}\}$. Thus we may consider only one case, for example, the first case. In this case we can write the matrix (A_b^a) in the form

$$(A_b^a)=\frac{\sqrt{c}}{2}\begin{pmatrix}1 & & & & 0\\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & 1\end{pmatrix}-\frac{\sqrt{c}}{2}\begin{pmatrix}1 & & & & 0\\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & 0\end{pmatrix}$$

$$-\frac{\sqrt{c}}{2}\begin{pmatrix}0 & & & & 0\\ & 1 & & & \\ & & 0 & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & 0\end{pmatrix}-\frac{\sqrt{c}}{2}\begin{pmatrix}0 & & & & & 0\\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \cdot & \\ 0 & & & & & 0\end{pmatrix},$$

this is,

$$A_b^a=\frac{\sqrt{c}}{2}\{\delta_b^a-(u_b u^a+v_b v^a+w_b w^a)\}.$$

which is a tensor equation and so holds for any frame, especially for natural frame. Thus the theorem is completely proved.

THEOREM 7. *Let M be a compact real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature $c\geqq 0$. If the second fundamental tensor A_{ba} is semi-definite and the mean curvature B constant and if $B^2\leqq 4(m-1)^2c$, then A_{ba} is of the form*

$$A_{ba}=\frac{\sqrt{c}}{2}\{g_{ba}-(u_b u_a+v_b v_a+w_b w_a)\}.$$

Proof. The equation (4.2), Lemmas 4.1 and 4.2 also give the following inequality:

$$\frac{1}{2}\Delta\|A\|^2\geqq(\nabla_b\nabla_a B)A^{ba}+\frac{c}{4}[3\{\|A_{ce}\phi_b^e+A_{be}\phi_c^e\|^2+\|A_{ce}\psi_b^e+A_{be}\psi_c^e\|^2$$
$$+\|A_{ce}\theta_b^e+A_{be}\theta_c^e\|^2\}+\frac{m+2}{m-1}B\{A(U, U)+A(V, V)+A(W, W)\}$$
$$+\frac{3}{2(m-1)}\{4(m-1)^2c-B^2\}+B(A_c^b A_b^a A_a^c)-(A_{ba}A^{ba})^2]+\|\overset{*}{\nabla}_c A_{ba}\|^2.$$

Consequently our assumptions give (4.6), (4.7), (4.9) and $B^2=4(m-1)^2c$. Thus the theorem is proved by the same method as in the proof of Theorem 6.

§ 5. An integral formula.

It is well known (Ishihara [3]) that for a $4m$-dimensional quaternionic Kaehlerian manifold with constant Q-sectional curvature c, when $m\geqq 2$, the followings are valid:

$$D_j p_i - D_i p_j + q_j r_i - r_j q_i = -cF_{ji},$$
$$D_j q_i - D_i q_j + r_j p_i - p_j r_i = -cG_{ji},$$
$$D_j r_i - D_i r_j + p_j q_i - q_j p_i = -cH_{ji}.$$

Therefore, in a real hypersurface M the local 1-forms p, q, r defined by

$$p_b = p_i B_b{}^i, \qquad q_b = q_i B_b{}^i, \qquad r_b = r_i B_b{}^i$$

satisfy

$$
\begin{aligned}
&\nabla_b p_a - \nabla_a p_b + q_b r_a - r_b q_a = -c\phi_{ba}, \\
(5.1)\qquad &\nabla_b q_a - \nabla_a q_b + r_b p_a - p_b r_a = -c\psi_{ba}, \\
&\nabla_b r_a - \nabla_a r_b + p_b q_a - q_b p_a = -c\theta_{ba}.
\end{aligned}
$$

On the other hand, taking account of arguments developed in section 2, we see easily that there are two global vector fields S_1 and S_2 on M with components

$$u^e(\mathring{\nabla}_e u^b) + v^e(\mathring{\nabla}_e v^b) + w^e(\mathring{\nabla}_e w^b), \qquad (\mathring{\nabla}_e u^e)u^b + (\mathring{\nabla}_e v^e)v^b + (\mathring{\nabla}_e w^e)w^b$$

respectively. In this section by using these global vector fields S_1 and S_2 we shall find an integral formula which corresponds to an integral formula given by K. Yano (Theorem 1.9 in [13]). Putting

$$\mathring{\nabla}_c \mathring{\nabla}_b u_a = \nabla_c \mathring{\nabla}_b u_a - r_c \mathring{\nabla}_b v_a + q_c \mathring{\nabla}_b w_a,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b v_a = \nabla_c \mathring{\nabla}_b v_a + r_c \mathring{\nabla}_b u_a - p_c \mathring{\nabla}_b w_a,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b w_a = \nabla_c \mathring{\nabla}_b w_a - q_c \mathring{\nabla}_b u_a + p_c \mathring{\nabla}_b v_a$$

and taking account of (5.1), we can verify

$$\mathring{\nabla}_c \mathring{\nabla}_b u_a - \mathring{\nabla}_b \mathring{\nabla}_c u_a = -K_{abc}{}^e u_e + c\theta_{cb} v_a - c\psi_{cb} w_a,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b v_a - \mathring{\nabla}_b \mathring{\nabla}_c v_a = -K_{cba}{}^e v_e - c\theta_{cb} u_a + c\phi_{cb} w_a,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b w_a - \mathring{\nabla}_b \mathring{\nabla}_c w_a = -K_{cba}{}^e w_e + c\psi_{cb} u_a - c\phi_{cb} v_a,$$

which implies

$$
\begin{aligned}
\mathring{\nabla}_b S_1{}^b - \mathring{\nabla}_b S_2{}^b = {} & K_{ba}(u^b u^a + v^b v^a + w^b w^a) - 6c + (\mathring{\nabla}_b u^a)(\mathring{\nabla}_a u^b) \\
& + (\mathring{\nabla}_b v^a)(\mathring{\nabla}_a v^b) + (\mathring{\nabla}_b w^a)(\mathring{\nabla}_a w^b) - (\|\mathring{\nabla}_b u_a\|^2 + \|\mathring{\nabla}_b v_a\|^2 + \|\mathring{\nabla}_b w_a\|^2),
\end{aligned}
$$

or equivalently

$$(5.2)\qquad \mathring{\nabla}_b S_1{}^b - \mathring{\nabla}_b S_2{}^b = K_{ba}(u^b u^a + v^b v^a + w^b w^a) - 6c - \{(\mathring{\mathrm{div}}\, u)^2 + (\mathring{\mathrm{div}}\, v)^2 + (\mathring{\mathrm{div}}\, w)^2\} + \frac{1}{2}\{\|\mathring{\mathcal{L}}_u g\|^2 + \|\mathring{\mathcal{L}}_v g\|^2 + \|\mathring{\mathcal{L}}_w g\|^2\} - (\|\mathring{\nabla}_b u_a\|^2 + \|\mathring{\nabla}_b v_a\|^2 + \|\mathring{\nabla}_b w_a\|^2),$$

where $\mathring{\mathcal{L}}_u g = \mathring{\nabla}_b u_a + \mathring{\nabla}_a u_b$ and $\mathring{\mathrm{div}}\, u = \mathring{\nabla}_a u^a$. On the other side, (2.20), (2.21) and (2.22) imply

$$\|\mathring{\nabla}_b u_a\|^2 + \|\mathring{\nabla}_b v_a\|^2 + \|\mathring{\nabla}_b w_a\|^2 = 3A_{ba}A^{ba} - (\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2),$$
$$\mathring{\mathrm{div}}\, u = \mathring{\mathrm{div}}\, v = \mathring{\mathrm{div}}\, w = 0.$$

And (3.4) gives

$$K_{ba}(u^b u^a + v^b v^a + w^b w^a) = \frac{c}{4}\{12(m-1) + B(A(U,U) + A(V,V) + A(W,W)) - (\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2)\}.$$

Substituting these equalities in (5.2), we obtain

$$(5.3)\qquad \mathring{\nabla}_b S_1{}^b - \mathring{\nabla}_b S_2{}^b = \frac{1}{2}\{\|\mathring{\mathcal{L}}_u g\|^2 + \|\mathring{\mathcal{L}}_v g\|^2 + \|\mathring{\mathcal{L}}_w g\|^2 + \frac{c}{4}\{12(m-1) + B(A(U,U) + A(V,V) + A(W,W))\} - 3A_{ba}A^{ba} + \left(1 - \frac{c}{4}\right)\{\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2\}.$$

We can now prove

THEOREM 8. *For a compact and orientable real hypersurface M of a $4m$-dimensional quaternionic Kaehlerian manifold $(m \geqq 2)$ with constant Q-sectional curvature c, any one of the three conditions* (1), (2) *and* (3) *stated in Theorem* 3 *is equivalent to the following conditions*:

$$\int_M \Big[\frac{c}{4}\{12(m-1) + B(A(U,U) + A(V,V) + A(W,W))\} - 3A_{ba}A^{ba} + \left(1 - \frac{c}{4}\right)\{\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2\}\Big] * 1 \geqq 0.$$

Proof. From (5.3) we find

$$-\int_M \frac{1}{2}\{\|\mathring{\mathcal{L}}_u g\|^2 + \|\mathring{\mathcal{L}}_v g\|^2 + \|\mathring{\mathcal{L}}_w g\|^2\} * 1$$
$$= \int_M \Big[\frac{c}{4}\{12(m-1) + B(A(U,U) + A(V,V) + A(W,W))\} - 3A_{ba}A^{ba} + \left(1 - \frac{c}{4}\right)\{\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2\}\Big] * 1.$$

Thus taking account of $\overset{\circ}{\mathcal{L}}_u g=\overset{\circ}{\nabla}_b u_a+\overset{\circ}{\nabla}_a u_b=A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e$, $\overset{\circ}{\mathcal{L}}_v g=\overset{\circ}{\nabla}_b v_a+\overset{\circ}{\nabla}_a v_b=A_{be}\psi_a{}^e+A_{ae}\psi_b{}^e$ and $\overset{\circ}{\mathcal{L}}_w g=\overset{\circ}{\nabla}_b w_a+\overset{\circ}{\nabla}_a w_b=A_{be}\theta_a{}^e+A_{ae}\theta_b{}^e$, we have our theorem.

§ 6. Submersion $\tilde{\pi}$: $S^{4m+3}\to QP(m)$ and immersion $\imath$: $M\to QP(m)$

Let $S^{4m+3}(1)$ be the hypersphere $\{(q^1,\cdots,q^{m+1})\,|\,|q^1|^2+\cdots+|q^{m+1}|^2=1\}$ of radius 1 in a $(m+1)$-dimensional space Q^{m+1} of quaternions, which will be identified naturally with $\boldsymbol{R}^{4(m+1)}$. The sphere $S^{4m+3}(1)$ will be simply denoted by S^{4m+1}. Let $\tilde{\pi}$: $S^{4m+3}\to QP(m)$ be the natural projection of S^{4m+3} onto a quaternionic projective space $QP(m)$ which is defined by the Hopf fibration. As is well known S^{4m+3} admits a Sasakian 3-structure $\{\tilde{\xi},\tilde{\eta},\tilde{\zeta}\}$ (See Ishihara and Konishi [4]) and any fibre $\tilde{\pi}^{-1}(P)$, $P\in QP(m)$, is a maximal integral manifold of the distribution spanned by $\tilde{\xi},\tilde{\eta}$ and $\tilde{\zeta}$. Therefore, the base space $QP(m)$ of a fibred Riemannian space with Sasakian 3-structure admits the induced a quaternionic Kaehlerian structure, and moreover, is of constant Q-sectional curvature 4 (See Ishihara [2], [3]). We consider a Riemannian submersion π: $\bar{M}\to M$ compatible with the Hopf fibration $\tilde{\pi}$: $S^{4m+3}\to QP(m)$, where M is a real hypersurface in $QP(m)$ and $\bar{M}=\tilde{\pi}^{-1}(M)$ a hypersurface of S^{4m+3}. More precisely speaking, π: $\bar{M}\to M$ is a Riemannian submersion with totally geodesic fibres such that the following diagram is commutative:

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\;\tilde{\imath}\;} & S^{4m+3} \\ \pi\downarrow & & \downarrow\tilde{\pi} \\ M & \xrightarrow[\imath]{} & QP(m) \end{array}$$

where $\tilde{\imath}$: $M\to S^{4m+3}$ and $\imath$: $M\to QP(m)$ are certain isometric immersions.

We take coordinate neighborhoods $\{\bar{U};\,X^\alpha\}$ of $\bar{M}$ such that $\pi(\bar{U})=U$ are coordinate neighborhoods of M with local coordinate (y^a). Then the projection π: $\bar{M}\to M$ may be expressed by

$$y^a=y^a(x^\alpha)\,, \tag{6.1}$$

where $y^a(x^\alpha)$ are differentiable functions of variables x^α with Jacobian $(\partial y^a/\partial x^\alpha)$ of the maximum rank $4m-1$. We take a fibre $\mathcal{F}$ such that $\mathcal{F}\cap\bar{U}\neq\phi$. Then we can introduce local coordinates (z^s) in $\mathcal{F}\cap\bar{U}$ in such a way that (y^a, z^s) is a system of local coordinate in $\bar{U}$, (y^a) being coordinates of $\pi(\mathcal{F})$ in U. Differentiating (6.1) with respect to x^α, we put $E_\alpha{}^a=\partial_\alpha y^a(\partial_\alpha=\partial/\partial x^\alpha)$ and denote by E^a local covector fields with components $E_\alpha{}^a$ in $\bar{U}$. On the other side, $C_s=\partial/\partial z^s$ form a natural frame tangent to each fibre $\mathcal{F}$ in $\mathcal{F}\cap\bar{U}$. Denoting by $C^\alpha{}_s$ components of C_s in $\bar{U}$, we put $C_\alpha{}^s=g_{\alpha\beta}g^{st}C^\beta{}_t$, where $g_{\alpha\beta}$ are components of the induced metric of $\bar{M}$ from that of S^{4m+3} in $\bar{U}$, $g_{st}=g_{\alpha\beta}C^\alpha{}_sC^\beta{}_t$ and $(g^{st})=(g_{st})^{-1}$. We now denote by C^s local covector fields with components $C_\alpha{}^s$ in

$\bar{U}$. We next define $E^{\alpha}{}_{a}$ by $(E^{\alpha}{}_{a}, C^{\alpha}{}_{s})=(E_{\alpha}{}^{a}, C_{\alpha}{}^{s})^{-1}$ and denote by E_a local vector fields with components $E^{\alpha}{}_{a}$ in $\bar{U}$. Then $\{E_b, C_s\}$ is a local frame in $\bar{U}$ and $\{E^b, C^s\}$ the coframe dual to $\{E_b, C_s\}$ in $\bar{U}$.

We now take coordinate neighborhoods $\{\tilde{U}; x^{\kappa}\}$ of S^{4m+3} such that $\tilde{\pi}(\tilde{U})=\hat{U}$ are coordinate neighborhoods of $QP(m)$ with local coordinates (y^j). Then we can also define similarly a local frame $\{\tilde{E}_j, \tilde{C}_s\}$ and the coframe $\{\tilde{E}^j, \tilde{C}^s\}$ dual to $\{\tilde{E}_j, \tilde{C}_s\}$ in $\tilde{U}$ (See Ishihara [2], [3], [4], [5] and Konishi [4], [5]). We denote by $\{\tilde{E}^{\kappa}{}_{j}, \tilde{C}^{\kappa}{}_{s}\}$ and $\{\tilde{E}_{\kappa}{}^{j}, \tilde{C}_{\kappa}{}^{s}\}$ components of $\{\tilde{E}_j, \tilde{C}_s\}$ and $\{\tilde{E}^j, \tilde{C}^s\}$ respectively in $\tilde{U}$.

Let the isometric immersions $\tilde{i}$ and i be locally expressed by $x^{\kappa}=x^{\kappa}(x^{\alpha})$ and $y^j=y^j(y^a)$ respectively. Then the commutativity $\tilde{\pi}\circ\tilde{i}=i\circ\pi$ of the diagram implies

$$y^j(y^a(x^{\alpha}))=y^j(x^{\kappa}(x^{\alpha})),$$

and hence

$$B_a{}^j E_{\alpha}{}^a=\tilde{E}_{\kappa}{}^j B_{\alpha}{}^{\kappa}, \tag{6.2}$$

where $B_a{}^j=\partial_a y^j$ and $B_{\alpha}{}^{\kappa}=\partial_{\alpha}x^{\kappa}$.

For an arbitrary point $P\in M$ we choose a unit normal vector field N^j to M defined in a neighborhood U of P in such a way that $\{B_a{}^j, N^j\}$ span the tangent space of $QP(m)$ at $i(P)$. Let $\bar{P}$ be an arbitrary point of the fibre $\mathcal{F}$ over P, then the lift $N^{\kappa}=N^j E^{\kappa}{}_{j}$ of N^j is a unit normal vector to $\bar{M}$ defined in the tubular neighborhood over U because of (6.2).

Let's denote by $\tilde{\xi}^{\kappa}, \tilde{\eta}^{\kappa}$ and $\tilde{\zeta}^{\kappa}$ components of $\tilde{\xi}, \tilde{\eta}$ and $\tilde{\zeta}$ of the induced Sasakian 3-structure $\{\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}\}$ in S^{4m+3} respectively. Since any fibre $\tilde{\pi}^{-1}(\hat{P})$, $\hat{P}\in QP(m)$, is a maximal integral manifold of the distribution spanned by $\tilde{\xi}, \tilde{\eta}$ and $\tilde{\zeta}$, $\tilde{\xi}^{\kappa}, \tilde{\eta}^{\kappa}$ and $\tilde{\zeta}^{\kappa}$ can be represented by

$$\tilde{\xi}^{\kappa}=\xi^{\alpha}B_{\alpha}{}^{\kappa}, \qquad \tilde{\eta}^{\kappa}=\eta^{\alpha}B_{\alpha}{}^{\kappa}, \qquad \tilde{\zeta}^{\kappa}=\zeta^{\alpha}B_{\alpha}{}^{\kappa}, \tag{6.3}$$

where ξ^{α}, η^{α} and ζ^{α} are unit vector fields in $\bar{M}$ which are vertical and span the tangent space to the fibre $\mathcal{F}$ at each point of $\bar{M}$ because of (6.2). We now put in $\tilde{U}$

$$\tilde{\xi}=a^s\tilde{C}_s, \qquad \tilde{\eta}=b^s\tilde{C}_s, \qquad \tilde{\zeta}=c^s\tilde{C}_s,$$

$$a_s=a^t\tilde{g}_{ts}, \qquad b_s=b^t\tilde{g}_{ts}, \qquad c_s=c^t\tilde{g}_{ts},$$

where $\tilde{g}_{ts}=\tilde{g}_{\lambda\mu}\tilde{C}^{\lambda}{}_{t}\tilde{C}^{\mu}{}_{s}$ and $\tilde{g}_{\lambda\mu}$ components of the induced metric in $S^{4m+3}(\subset Q^{m+1})$. Then it follows that

$$\tilde{C}_s=a_s\tilde{\xi}+b_s\tilde{\eta}+c_s\tilde{\zeta}, \tag{6.4}$$

$$a_s a^t+b_s b^t+c_s c^t=\delta_s^t \tag{6.5}$$

Transvecting (6.2) with $\tilde{E}^{\mu}{}_{j}$ and substituting (6.4) imply

$$(\tilde{E}^{\mu}{}_{j}B_a{}^j)E_{\alpha}{}^a=B_{\alpha}{}^{\mu}-(a^s\xi_{\alpha}+b^s\eta_{\alpha}+c^s\zeta_{\alpha})\tilde{C}^{\mu}{}_{s},$$

where $\xi_\alpha=\xi^\beta g_{\beta\alpha}$, $\eta_\alpha=\eta^\beta g_{\beta\alpha}$ and $\zeta_\alpha=\zeta^\beta g_{\beta\alpha}$. Thus, transvecting the equation above with $E^\alpha{}_b$ and using the fact ξ_α, η_α and ζ_α being vertical, we have

$$\tilde{E}^\mu{}_j B_b{}^j = B_\alpha{}^\mu E^\alpha{}_b . \tag{6.6}$$

Hence the vertical vectors C_s can be written as

$$C_s = a_s \xi + b_s \eta + c_s \zeta \tag{6.7}$$

in such a way that the functions a_s, b_s and c_s satisfy (6.5), where a_s, b_s and c_s are respectively the restrictions of a_s, b_s and c_s appearing in (6.4) and in the sequel these restrictions will be denoted by the corresponding letters respectively.

Denoting by $\{{}^{\lambda}_{\mu\nu}\}$, $\{{}^{i}_{jh}\}$, $\{{}^{\alpha}_{\beta\gamma}\}$ and $\{{}^{a}_{bc}\}$ the Christoffel symbols formed with the Riemannian metrics $\tilde{g}_{\lambda\mu}, g_{ji}, g_{\alpha\beta}$ and g_{ba} respectively, we put

$$\tilde{D}_\mu \tilde{E}_\lambda{}^i = \partial_\mu \tilde{E}_\lambda{}^i - \{{}^{\kappa}_{\mu\lambda}\} \tilde{E}_\kappa{}^i + \{{}^{i}_{jh}\} \tilde{E}_\mu{}^j \tilde{E}_\lambda{}^h ,$$
$$\tilde{D}_\mu \tilde{E}^\lambda{}_i = \partial_\mu \tilde{E}^\lambda{}_i + \{{}^{\lambda}_{\mu\kappa}\} \tilde{E}^\kappa{}_i - \{{}^{h}_{ij}\} \tilde{E}_\mu{}^j \tilde{E}^\lambda{}_h ,$$

and

$$\bar{\nabla}_\beta E_\alpha{}^a = \partial_\beta E_\alpha{}^a - \{{}^{\gamma}_{\beta\alpha}\} E_\gamma{}^a + \{{}^{a}_{bc}\} E_\beta{}^b E_\alpha{}^c ,$$
$$\bar{\nabla}_\beta E^\alpha{}_a = \partial_\beta E^\alpha{}_a + \{{}^{\alpha}_{\beta\gamma}\} E^\gamma{}_a - \{{}^{c}_{ba}\} E_\beta{}^b E^\alpha{}_c .$$

Since the metrics $\tilde{g}_{\lambda\mu}$ and $g_{\alpha\beta}$ are invariant with respect to the submersions $\tilde{\pi}$ and π respectively the van der Waerden-Bortolotti covariant derivatives of $\tilde{E}_\lambda{}^i$, $\tilde{E}^\lambda{}_i$ and $E_\alpha{}^a, E^\alpha{}_a$ are given by

$$\begin{cases} \bar{D}_\mu \tilde{E}_\lambda{}^i = h_j{}^i{}_s (\tilde{E}_\mu{}^j \tilde{C}_\lambda{}^s + \tilde{C}_\mu{}^s E_\lambda{}^j) , \\ \tilde{D}_\mu \tilde{E}^\lambda{}_i = h_{ji}{}^s \tilde{E}_\mu{}^j \tilde{C}^\lambda{}_s - h_i{}^j{}_s \tilde{C}_\mu{}^s \tilde{E}^\lambda{}_j , \end{cases} \tag{6.8}$$

$$\begin{cases} \bar{\nabla}_\beta E_\alpha{}^a = h_b{}^a{}_s (E_\beta{}^b C_\alpha{}^s + C_\beta{}^s E_\alpha{}^b) , \\ \bar{\nabla}_\beta E^\alpha{}_a = h_{ba}{}^s E_\beta{}^b C^\alpha{}_s - h_a{}^b{}_s C_\beta{}^s E^\alpha{}_b \end{cases} \tag{6.9}$$

respectively, where $h_j{}^i{}_s = \tilde{g}^{ih} \tilde{g}_{st} h_{jh}{}^t$, $h_b{}^a{}_s = g^{ac} g_{st} h_{bc}{}^t$, $h_{ji}{}^s$ being $h_{ba}{}^s$ are the structure tensors induced from the submersions $\tilde{\pi}$ and π respectively (See Ishihara and Konishi [5]).

On the other side the equations of Gauss and Weingarten for the immersion $\bar{i}: \bar{M} \to S^{4m+3}$ are given by

$$\begin{aligned} &\bar{\nabla}_\beta B_\alpha{}^\kappa = \partial_\beta B^{\alpha\kappa} + \{{}^{\kappa}_{\mu\lambda}\} B_\beta{}^\mu B_\alpha{}^\lambda - \{{}^{\gamma}_{\beta\alpha}\} B_\gamma{}^\kappa = A_{\beta\alpha} N^\kappa , \\ &\bar{\nabla}_\beta N^\kappa = \partial_\beta N^\kappa + \{{}^{\kappa}_{\mu\lambda}\} B_\beta{}^\mu N^\lambda = -A_\beta{}^\alpha B_\alpha{}^\kappa , \end{aligned} \tag{6.10}$$

and those for the immersion $i: M \to QP(m)$ by

$$\begin{aligned} &\nabla_b B_a{}^i = \partial_b B_a{}^i + \{{}^{i}_{jh}\} B_b{}^j B_a{}^h - \{{}^{c}_{ba}\} B_c{}^i = A_{ba} N^i , \\ &\nabla_b N^i = \partial_b N^i + \{{}^{i}_{jh}\} B_b{}^j N^h = -A_b{}^a B_a{}^i , \end{aligned} \tag{6.11}$$

where $A_{\beta}{}^{\alpha}=A_{\beta\gamma}g^{\gamma\alpha}$, $A_b{}^a=A_{be}g^{ea}$, $A_{\beta\alpha}$ being A_{ba} are the second fundamental tensors of $\bar{M}$ and M with respect to the unit normals N^{κ} and N^j respectively. Moreover in this case (6.2) and (6.6) imply

$$\nabla_b=E^{\alpha}{}_b\bar{\nabla}_{\alpha}.$$

Putting $\tilde{\phi}_{\mu}{}^{\lambda}=\bar{D}_{\mu}\tilde{\xi}^{\lambda}$, $\tilde{\psi}_{\mu}{}^{\lambda}=\bar{D}_{\mu}\tilde{\eta}^{\lambda}$ and $\tilde{\theta}_{\mu}{}^{\lambda}=\bar{D}_{\mu}\tilde{\zeta}^{\lambda}$, we have by definition of Sasakian 3-structure

$$\begin{aligned}
&\tilde{\phi}_{\mu}{}^{\lambda}\tilde{\phi}_{\kappa}{}^{\mu}=-\delta_{\kappa}^{\lambda}+\tilde{\xi}_{\kappa}\tilde{\xi}^{\lambda}, \quad \tilde{\phi}_{\mu}{}^{\lambda}\tilde{\xi}^{\mu}=0, \quad \tilde{\xi}_{\lambda}\tilde{\phi}_{\mu}{}^{\lambda}=0, \quad \tilde{\xi}_{\lambda}\tilde{\xi}^{\lambda}=1,\\
&\tilde{\psi}_{\mu}{}^{\lambda}\tilde{\psi}_{\kappa}{}^{\lambda}=-\delta_{\kappa}^{\lambda}+\tilde{\eta}_{\kappa}\tilde{\eta}^{\lambda}, \quad \tilde{\psi}_{\mu}{}^{\lambda}\tilde{\eta}^{\mu}=0, \quad \tilde{\eta}_{\lambda}\tilde{\psi}_{\mu}{}^{\lambda}=0, \quad \tilde{\eta}_{\lambda}\tilde{\eta}^{\lambda}=1,\\
&\tilde{\theta}_{\mu}{}^{\lambda}\tilde{\theta}_{\kappa}{}^{\mu}=-\delta_{\kappa}^{\lambda}+\tilde{\zeta}_{\kappa}\tilde{\zeta}^{\lambda}, \quad \tilde{\theta}_{\mu}{}^{\lambda}\tilde{\zeta}^{\mu}=0, \quad \tilde{\zeta}_{\lambda}\tilde{\theta}_{\mu}{}^{\lambda}=0, \quad \tilde{\zeta}_{\lambda}\tilde{\zeta}^{\lambda}=1,\\
(6.12)\quad &\tilde{\theta}_{\mu}{}^{\lambda}\tilde{\eta}^{\mu}=-\tilde{\psi}_{\mu}{}^{\lambda}\tilde{\zeta}^{\mu}=\tilde{\xi}^{\lambda}, \quad \tilde{\phi}_{\mu}{}^{\lambda}\tilde{\zeta}^{\mu}=-\tilde{\theta}_{\mu}{}^{\lambda}\tilde{\xi}^{\mu}=\tilde{\eta}^{\lambda}, \quad \tilde{\psi}_{\mu}{}^{\lambda}\tilde{\xi}^{\mu}=-\tilde{\phi}_{\mu}{}^{\lambda}\tilde{\eta}^{\mu}=\tilde{\zeta}^{\lambda},\\
&\tilde{\phi}_{\mu}{}^{\lambda}\tilde{\psi}_{\kappa}{}^{\mu}=-\tilde{\theta}_{\kappa}{}^{\lambda}+\tilde{\eta}_{\kappa}\tilde{\xi}^{\lambda}, \quad \tilde{\psi}_{\mu}{}^{\lambda}\tilde{\theta}_{\kappa}{}^{\mu}=-\phi_{\kappa}{}^{\lambda}+\tilde{\zeta}_{\kappa}\tilde{\eta}^{\lambda}, \quad \tilde{\theta}_{\mu}{}^{\lambda}\tilde{\phi}_{\kappa}{}^{\mu}=\tilde{\psi}_{\kappa}{}^{\lambda}+\tilde{\xi}_{\kappa}\tilde{\zeta}^{\lambda},\\
&\tilde{\psi}_{\mu}{}^{\lambda}\tilde{\phi}_{\kappa}{}^{\mu}=\tilde{\theta}_{\kappa}{}^{\lambda}+\tilde{\xi}_{\kappa}\tilde{\eta}^{\lambda}, \quad \tilde{\theta}_{\mu}{}^{\lambda}\tilde{\psi}_{\kappa}{}^{\mu}=\tilde{\phi}_{\kappa}{}^{\lambda}+\tilde{\eta}_{\kappa}\tilde{\zeta}^{\lambda}, \quad \tilde{\phi}_{\mu}{}^{\lambda}\tilde{\theta}_{\kappa}{}^{\mu}=\tilde{\psi}_{\kappa}{}^{\lambda}+\tilde{\zeta}_{\kappa}\tilde{\xi}^{\lambda},\\
&\tilde{\phi}_{\mu\lambda}+\tilde{\phi}_{\lambda\mu}=0, \quad \tilde{\psi}_{\mu\lambda}+\tilde{\psi}_{\lambda\mu}=0, \quad \tilde{\theta}_{\mu\lambda}+\tilde{\theta}_{\lambda\mu}=0,
\end{aligned}$$

and

$$(6.13)\qquad \bar{D}_{\mu}\tilde{\phi}_{\lambda}{}^{\kappa}=\tilde{\xi}_{\lambda}\delta_{\mu}^{\kappa}-\tilde{\xi}^{\kappa}\tilde{g}_{\mu\lambda}, \quad \bar{D}_{\mu}\tilde{\psi}_{\lambda}{}^{\kappa}=\tilde{\eta}_{\lambda}\delta_{\mu}^{\kappa}-\tilde{\eta}^{\kappa}\tilde{g}_{\mu\lambda}, \quad \bar{D}_{\mu}\tilde{\theta}_{\lambda}{}^{\kappa}=\tilde{\zeta}_{\lambda}\delta_{\mu}^{\kappa}-\tilde{\zeta}^{\kappa}\tilde{g}_{\mu\lambda},$$

where we have put $\tilde{\xi}_{\kappa}=\tilde{\xi}^{\lambda}\tilde{g}_{\lambda\kappa}$, $\tilde{\eta}_{\kappa}=\tilde{\eta}^{\lambda}\tilde{g}_{\lambda\kappa}$, $\tilde{\zeta}_{\kappa}=\tilde{\zeta}^{\lambda}\tilde{g}_{\lambda\kappa}$, $\tilde{\phi}_{\mu\lambda}=\tilde{\phi}_{\mu}{}^{\nu}\tilde{g}_{\nu\lambda}$, $\tilde{\psi}_{\mu\lambda}=\tilde{\psi}_{\mu}{}^{\nu}\tilde{g}_{\nu\lambda}$ and $\tilde{\theta}_{\mu\lambda}=\tilde{\theta}_{\mu}{}^{\nu}\tilde{g}_{\nu\lambda}$ (See Kuo [6]).

We now put in $\tilde{U}$

$$\phi_j{}^i=\tilde{\phi}_{\mu}{}^{\lambda}\tilde{E}^{\mu}{}_j\tilde{E}_{\lambda}{}^i, \quad \psi_j{}^i=\tilde{\psi}_{\mu}{}^{\lambda}\tilde{E}^{\mu}{}_j\tilde{E}_{\lambda}{}^i, \quad \theta_j{}^i=\tilde{\theta}_{\mu}{}^{\lambda}\tilde{E}^{\mu}{}_j\tilde{E}_{\lambda}{}^i.$$

Then we have from (6.12)

$$(6.14)\quad \begin{aligned}
&\phi_h{}^i\phi_j{}^h=-\delta_j^i, \quad \psi_h{}^i\psi_j{}^h=-\delta_j{}^i, \quad \theta_h{}^i\theta_j{}^h=-\delta_j^i,\\
&\phi_h{}^i\psi_j{}^h=-\psi_h{}^i\phi_j{}^h=\theta_j{}^i, \psi_h{}^i\theta_j{}^h=-\theta_h{}^i\psi_j{}^h=\phi_j^i, \quad \theta_h{}^i\phi_j{}^h=-\phi_h{}^i\theta_j{}^h=\psi_j{}^i.
\end{aligned}$$

We also have by using (6.8), (6.12) and (6.13)

$$(6.15)\quad \begin{aligned}
&\mathcal{L}_{\tilde{\xi}}\phi_j{}^i=0, \quad \mathcal{L}_{\tilde{\eta}}\phi_j{}^i=-2\theta_j{}^i, \quad \mathcal{L}_{\tilde{\zeta}}\phi_j{}^i=2\psi_j{}^i,\\
&\mathcal{L}_{\tilde{\xi}}\psi_j{}^i=2\theta_j{}^i, \quad \mathcal{L}_{\tilde{\eta}}\psi_j{}^i=0, \quad \mathcal{L}_{\tilde{\zeta}}\psi_j{}^i=-2\phi_j{}^i,\\
&\mathcal{L}_{\tilde{\xi}}\theta_j{}^i=-2\psi_j{}^i, \quad \mathcal{L}_{\tilde{\eta}}\theta_j{}^i=2\phi_j{}^i, \quad \mathcal{L}_{\tilde{\zeta}}\vartheta_j{}^i=0,
\end{aligned}$$

$\mathcal{L}_{\tilde{\xi}}$ denoting the Lie derivation with respect to $\tilde{\xi}$, and

$$(6.16)\qquad h_{ji}{}^s=-(a^s\phi_{ji}+b^s\psi_{ji}+c^s\theta_{ji}),$$

where $\phi_{ji}=\phi_j{}^hg_{hi}$, $\psi_{ji}=\psi_j{}^hg_{hj}$ and $\theta_{ji}=\theta_j{}^hg_{hi}$.

Consider a point $\hat{P}$ of $QP(m)$ and a point $\tilde{P}$ of S^{4m+3} such that $\tilde{\pi}(\tilde{P})=\hat{P}$. Denoting by $\tilde{\phi}_{\tilde{P}}, \tilde{\psi}_{\tilde{P}}$ and $\tilde{\theta}_{\tilde{P}}$ respectively the values of $\tilde{\phi}, \tilde{\psi}$ and $\tilde{\theta}$ at $\tilde{P}$, we can define tensors $\hat{F}_{\tilde{p}}, \hat{G}_{\tilde{p}}$ and $\hat{H}_{\tilde{p}}$ of type (1.1) at $\hat{P}\in QP(m)$ respectively by

$$\hat{F}_{\tilde{p}}A=d\tilde{\pi}(\tilde{\phi}_{\tilde{p}}A^L), \quad \hat{G}_{\tilde{p}}A=d\tilde{\pi}(\tilde{\psi}_{\tilde{p}}A^L), \quad \hat{H}_{\tilde{p}}A=d\tilde{\pi}(\tilde{\theta}_{\tilde{p}}A^L) \tag{6.17}$$

for any vector A tangent to $QP(m)$ at $\hat{P}$, where $d\tilde{\pi}$ means the differential of $\tilde{\pi}$ and A^L denote the horizontal lift of A. We now denote by $V_{\hat{p}}^*$ the linear closure of the set

$$(\bigcup_{\tilde{p}\in\tilde{\pi}^{-1}(\hat{p})}\hat{F}_{\tilde{p}})\cup(\bigcup_{\tilde{p}\in\tilde{\pi}^{-1}(\hat{p})}\hat{G}_{\tilde{p}})\cup(\bigcup_{\tilde{p}\in\tilde{\pi}^{-1}(\hat{p})}\hat{H}_{\tilde{p}})$$

of tensors of type (1.1) at $\hat{P}\in QP(m)$ and put $V_{\hat{p}}^*=\cup_{\hat{p}\in QP(m)}V_{\hat{p}}^*$, which is a linear subbundle of the tensor bundle of type (1, 1) over $QP(m)$.

Take a coordinate neighborhood $\hat{U}\ni\hat{P}$ of $QP(m)$ and consider a local cross-section τ of S^{4m+3} over $\hat{U}$. If we put

$$F_{\hat{p}}=\hat{F}_{\tau(\hat{p})}, \quad G_{\hat{p}}=\hat{G}_{\tau(\hat{p})}, \quad H_{\hat{p}}=\hat{H}_{\tau(\hat{p})}, \quad \hat{P}\in\hat{U}, \tag{6.18}$$

then the correspondence $\hat{P}\to F_{\hat{p}}, \hat{P}\to G_{\hat{p}}$ and $\hat{P}\to H_{\hat{p}}$ define respectively local tensor fields F, G and H of type (1, 1) on $\hat{U}$. Thus, taking account of (6.14), (6.17) and (6.18), we find

$$\begin{gathered} F_h{}^iF_j{}^h=-\delta_j^i, \quad G_h{}^iG_j{}^h=-\delta_j^i, \quad H_h{}^iH_j{}^h=-\delta_j^i, \\ F_h{}^iG_j{}^h=-G_h{}^iF_j{}^h=H_j{}^i, \quad G_h{}^iH_j{}^h=-H_h{}^iG_j{}^h=F_j{}^i, \quad H_h{}^iF_j{}^h=-F_h{}^iH_j{}^h=G_j{}^i, \\ F_{ji}=-F_{ij}, \quad G_{ji}=-G_{ij}, \quad H_{ji}=-H_{ij} \end{gathered} \tag{6.19}$$

where $F_{ji}=F_j{}^hg_{hi}, G_{ji}=G_j{}^hg_{hi}, H_{ji}=H_j{}^hg_{hi}, F_j{}^i, G_j{}^i$ and $H_j{}^i$ being respectively local components of F, G and H in $\hat{U}$.

We take another local cross-section τ' of $QP(m)$ in $'\hat{U}$. Then we can construct a triple $\{'F, 'G, 'H\}$ in $'\hat{U}$ by the same way as above and $\{'F, 'G, 'H\}$ also satisfy (6.19). Thus, taking account of (6.15) implies in $\hat{U}\cap{}'\hat{U}\neq\phi$

$$\begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix}=S_{(xy)}\begin{pmatrix} F \\ G \\ H \end{pmatrix}, \quad (x, y=1, 2, 3) \tag{6.20}$$

with functions S_{xy} in $\hat{U}\cap{}'\hat{U}$, where the matrix (S_{xy}) is contained in the special orthogonal group $SO(3)$.

Next, denoting by $(\tau^\kappa(y))$ coordinates of the point $\tau(\hat{P})$, we have from (6.18)

$$F_j{}^i(y)=\phi_j{}^i(\tau^\kappa(y)), \quad G_j{}^i(y)=\psi_j{}^i(\tau^\kappa(y)), \quad H_j{}^i(y)=\theta_j{}^i(\tau^\kappa(y)).$$

Differentiating the first equation above with respect to y^h and using $(\partial_h\tau^\kappa)\tilde{E}_\kappa{}^i=\delta_h^i$ imply

$$\partial_hF_j{}^i=\partial_h\phi_j{}^i+(\partial_h\tau^\kappa)\tilde{C}_\kappa{}^s\partial_s\phi_j{}^i.$$

Thus, taking account of (6.16), we obtain $D_hF_j{}^i=r_hG_j{}^i-q_hH_j{}^i$, where we have put $q_h=-b_s\tilde{C}_\kappa{}^s\partial_h\tau^\kappa$ and $r_h=-c_s\tilde{C}_\kappa{}^s\partial_h\tau^\kappa$. Similarly, using (6.16), we obtain in $\hat{U}$

$$\begin{aligned} &D_h F_j{}^i = r_h G_j{}^i - q_h H_j{}^i, \\ (6.21)\qquad &D_h G_j{}^i = -r_h F_j{}^i + p_h H_j{}^i, \\ &D_h H_j{}^i = q_h F_j{}^i - p_h G_j{}^i. \end{aligned}$$

for certain local 1-forms p, q, r defined in $\hat{U}$. By means of (6.19), (6.20) and (6.21) the quaternionic projective space $QP(m)$ admits a quaternionic Kaehlerian structure (See Ishihara [2], [3], [5] and Konishi [5]).

Let's denote by $K_{\kappa\mu\nu}{}^{\lambda}$ and $K_{kji}{}^h$ components of the curvature tensors of $(S^{4m+3}, g_{\lambda\mu})$ and $(QP(m), g_{ji})$ respectively. Since the unit sphere S^{4m+3} is a space of constant curvature 1, using the euqation of co-Gauss (See Ishihara and Konishi [5])

$$K_{kji}{}^h = K_{\kappa\mu\nu}{}^{\lambda} \tilde{E}^{\kappa}{}_k \tilde{E}^{\mu}{}_j \tilde{E}^{\nu}{}_i \tilde{E}_{\lambda}{}^h + h_k{}^h{}_s h_{ji}{}^s - h_j{}^h{}_s h_{ki}{}^s - 2h_{kj}{}^s h_i{}^h{}_s$$

and (6.16) implies

$$\begin{aligned} K_{kji}{}^h = {} & \delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^h F_{ji} - F_j{}^h F_{ki} - 2F_{kj} F_i{}^h + G_k{}^h G_{ji} - G_j{}^h G_{ki} \\ & - 2G_{kj} G_i{}^h + H_k{}^h H_{ji} - H_j{}^h H_{ki} - 2H_{kj} H_i{}^h. \end{aligned}$$

Hence $QR(m)$ is a quaternionic Kaehlerian manifold with constant Q-sectional curvature 4 (See Ishihara [2], [3], [5] and Kanishi [5]), and consequently the real hypersurface M of $QP(m)$ can be regarded as a manifold with almost contact 3-structure as already shown in section 2.

We are now going to prove that the structure of M induced by the immersion $\tilde{\imath}: \bar{M} \to S^{4m+3}$ and the submersion $\pi: \bar{M} \to M$ is the same as the structure induced by the submersion $\tilde{\pi}: S^{4m+3} \to QP(m)$ and the immersion $\imath: M \to QP(m)$.

Applying the operator $\bar{\nabla}_{\beta} = B_{\beta}{}^{\mu} \bar{D}_{\mu}$ to (6.3) and using the euqations (6.10) of Gauss and Weingarten, we find

$$\begin{aligned} \tilde{\phi}_{\mu}{}^{\kappa} B_{\beta}{}^{\mu} &= (\bar{\nabla}_{\beta} \xi^{\alpha}) B_{\alpha}{}^{\kappa} + A_{\beta\alpha} \xi^{\alpha} N^{\kappa}, \\ \tilde{\psi}_{\mu}{}^{\kappa} B_{\beta}{}^{\mu} &= (\bar{\nabla}_{\beta} \eta^{\alpha}) B_{\alpha}{}^{\kappa} + A_{\alpha\beta} \eta^{\alpha} N^{\kappa}, \\ \tilde{\theta}_{\mu}{}^{\kappa} B_{\beta}{}^{\mu} &= (\bar{\nabla}_{\beta} \zeta^{\alpha}) B_{\alpha}{}^{\kappa} + A_{\beta\alpha} \zeta^{\alpha} N^{\kappa}, \end{aligned}$$

from which, putting

$$(6.22)\qquad \phi_{\beta}{}^{\alpha} = \bar{\nabla}_{\beta} \xi^{\alpha}, \qquad \psi_{\beta}{}^{\alpha} = \bar{\nabla}_{\beta} \eta^{\alpha}, \qquad \theta_{\beta}{}^{\alpha} = \bar{\nabla}_{\beta} \zeta^{\alpha},$$

$$\begin{aligned} (6.23)\qquad & u_{\beta} = A_{\beta\alpha} \xi^{\alpha}, \qquad v_{\beta} = A_{\beta\alpha} \eta^{\alpha}, \qquad w_{\beta} = A_{\beta\alpha} \zeta^{\alpha}, \\ & u^{\alpha} = g^{\beta\alpha} u_{\beta}, \qquad v^{\alpha} = g^{\beta\alpha} v_{\beta}, \qquad w^{\alpha} = g^{\beta\alpha} w_{\beta}, \end{aligned}$$

we also have

$$\begin{aligned} & \tilde{\phi}_{\mu}{}^{\kappa} B_{\beta}{}^{\mu} = \phi_{\beta}{}^{\alpha} B_{\alpha}{}^{\kappa} + u_{\beta} N^{\kappa}, \qquad \tilde{\phi}_{\mu}{}^{\kappa} N^{\mu} = -u^{\beta} B_{\beta}{}^{\kappa}, \\ (6.24)\qquad & \tilde{\psi}_{\mu}{}^{\kappa} B_{\beta}{}^{\mu} = \psi_{\beta}{}^{\alpha} B_{\alpha}{}^{\kappa} + v_{\beta} N^{\kappa}, \qquad \tilde{\psi}_{\mu}{}^{\kappa} N^{\mu} = -v^{\beta} B_{\beta}{}^{\kappa}, \\ & \tilde{\theta}_{\mu}{}^{\kappa} B_{\beta}{}^{\mu} = \theta_{\beta}{}^{\alpha} B_{\alpha}{}^{\kappa} + w_{\beta} N^{\kappa}, \qquad \tilde{\theta}_{\mu}{}^{\kappa} N^{\mu} = -w^{\beta} B_{\beta}{}^{\kappa}. \end{aligned}$$

Transvecting $\tilde{\phi}_{\prime s}$ to (6.24) and using (6.12) and (6.24) itself in the usual way, we can easily obtain that

$$
(6.25)\quad
\begin{aligned}
&\phi_\gamma{}^\alpha\phi_\beta{}^\gamma=-\delta_\beta^\alpha+u_\beta u^\alpha+\xi_\alpha\xi^\beta, \quad \phi_\beta{}^\alpha u^\beta=\phi_\beta{}^\alpha\xi^\beta=0, \quad u_\beta u^\beta=1, \quad \xi_\beta\xi^\beta=1,\\
&\psi_\gamma{}^\alpha\psi_\beta{}^\gamma=-\delta_\beta^\alpha+v_\beta v^\alpha+\eta_\beta\eta^\alpha, \quad \psi_\beta{}^\alpha v^\beta=\psi_\beta{}^\alpha\eta^\beta=0, \quad v_\beta v^\beta=1, \quad \eta_\beta\eta^\beta=1,\\
&\theta_\gamma{}^\alpha\theta_\beta{}^\gamma=-\delta_\beta^\alpha+w_\beta w^\alpha+\zeta_\beta\zeta^\alpha, \quad \theta_\beta{}^\alpha w^\beta=\theta_\beta{}^\alpha\zeta^\beta=0, \quad w_\beta w^\beta=1, \quad \zeta_\beta\zeta^\beta=1,\\
&\phi_\gamma{}^\alpha\psi_\beta{}^\gamma=-\theta_\beta{}^\alpha+v_\beta u^\alpha+\eta_\beta\xi^\alpha, \quad \psi_\beta{}^\alpha u_\alpha=w_\beta, \quad u_\beta\xi^\beta=0,\\
&\psi_\gamma{}^\alpha\phi_\beta{}^\gamma=\theta_\beta{}^\alpha+u_\beta v^\alpha+\xi_\beta\eta^\alpha, \quad \phi_\beta{}^\alpha v_\alpha=-w_\beta, \quad v_\beta\eta^\beta=0,\\
&\psi_\gamma{}^\alpha\theta_\beta{}^\gamma=-\phi_\beta{}^\alpha+w_\beta v^\alpha+\zeta_\beta\eta^\alpha, \quad \theta_\beta{}^\alpha v_\alpha=u_\beta, \quad w_\beta\zeta^\beta=0,\\
&\theta_\gamma{}^\alpha\psi_\beta{}^\gamma=\phi_\beta{}^\alpha+v_\beta w^\alpha+\eta_\beta\zeta^\alpha, \quad \psi_\beta{}^\alpha w_\alpha=-u_\beta, \quad \xi_\beta u^\beta=0,\\
&\theta_\gamma{}^\alpha\phi_\beta{}^\gamma=-\psi_\beta{}^\alpha+u_\beta w^\alpha+\xi_\beta\zeta^\alpha, \quad \phi_\beta{}^\alpha w_\alpha=v_\beta, \quad \eta_\beta v^\beta=0,\\
&\phi_\gamma{}^\alpha\theta_\beta{}^\gamma=\psi_\beta{}^\gamma+w_\beta u^\alpha+\zeta_\beta\xi^\alpha, \quad \theta_\beta{}^\alpha u_\alpha=-v_\beta, \quad \zeta_\beta w^\beta=0,\\
&\theta_\beta{}^\alpha\eta^\beta=-\psi_\beta{}^\alpha\zeta^\beta=\xi^\alpha, \quad w_\beta\eta^\beta=0, \quad v_\beta\zeta^\beta=0,\\
&\phi_\beta{}^\alpha\zeta^\beta=-\theta_\beta{}^\alpha\xi^\beta=\eta^\alpha, \quad u_\beta\zeta^\beta=0, \quad w_\beta\xi^\beta=0,\\
&\psi_\beta{}^\alpha\xi^\beta=-\phi_\beta{}^\alpha\eta^\beta=\zeta^\alpha, \quad u_\beta\eta^\beta=0, \quad v_\beta\xi^\beta=0.
\end{aligned}
$$

Applying the operator $\bar{\nabla}_\gamma=B_\gamma{}^\kappa\bar{D}_\kappa$ to (6.24) and using (6.11), (6.13) and (6.24) itself, we also have

$$
(6.26)\quad
\begin{aligned}
&\bar{\nabla}_\gamma\phi_\beta{}^\alpha=\xi_\beta\delta_\gamma^\alpha-\xi^\alpha g_{\gamma\beta}+u_\beta A_\gamma{}^\alpha-u^\alpha A_{\gamma\beta}, \quad \bar{\nabla}_\gamma u_\beta=-A_{\gamma\alpha}\phi_\beta{}^\alpha,\\
&\bar{\nabla}_\gamma\psi_\beta{}^\alpha=\eta_\beta\delta_\gamma^\alpha-\eta^\alpha g_{\gamma\beta}+v_\beta A_\gamma{}^\alpha-v^\alpha A_{\gamma\beta}, \quad \bar{\nabla}_\gamma v_\beta=-A_{\gamma\alpha}\psi_\beta{}^\alpha,\\
&\bar{\nabla}_\gamma\theta_\beta{}^\alpha=\zeta_\beta\delta_\gamma^\alpha-\zeta^\alpha g_{\gamma\beta}+w_\beta A_\gamma{}^\alpha-w^\alpha A_{\gamma\beta}, \quad \bar{\nabla}_\gamma w_\beta=-A_{\gamma\alpha}\theta_\beta{}^\alpha,
\end{aligned}
$$

which and (6.25) imply

$$
(6.27)\quad
\begin{aligned}
&\mathcal{L}_\xi\phi_\beta{}^\alpha=0, \quad \mathcal{L}_\eta\phi_\beta{}^\alpha=-2\theta_\beta{}^\alpha, \quad \mathcal{L}_\zeta\phi_\beta{}^\alpha=2\psi_\beta{}^\alpha,\\
&\mathcal{L}_\xi\psi_\beta{}^\alpha=2\theta_\beta{}^\alpha, \quad \mathcal{L}_\eta\psi_\beta{}^\alpha=0, \quad \mathcal{L}_\zeta\psi_\beta{}^\alpha=-2\phi_\beta{}^\alpha,\\
&\mathcal{L}_\xi\theta_\beta{}^\alpha=-2\psi_\beta{}^\alpha, \quad \mathcal{L}_\eta\theta_\beta{}^\alpha=2\phi_\beta{}^\alpha, \quad \mathcal{L}_\zeta\theta_\beta{}^\alpha=0.
\end{aligned}
$$

and

$$
(6.28)\quad
\begin{aligned}
&\mathcal{L}_\xi u^\alpha=0, \quad \mathcal{L}_\eta u^\alpha=-2w^\alpha, \quad \mathcal{L}_\zeta u^\alpha=2v^\alpha,\\
&\mathcal{L}_\xi v^\alpha=2w^\alpha, \quad \mathcal{L}_\eta v^\alpha=0, \quad \mathcal{L}_\zeta v^\alpha=-2u^\alpha,\\
&\mathcal{L}_\xi w^\alpha=-2v^\alpha, \quad \mathcal{L}_\eta w^\alpha=2u^\alpha, \quad \mathcal{L}_\zeta w^\alpha=0.
\end{aligned}
$$

If we put in a neighborhood $\bar{U}$ of $\bar{M}$

$$
\phi_a{}^b=\phi_\alpha{}^\beta E^\alpha{}_a E_\beta{}^b, \quad \psi_a{}^b=\psi_\alpha{}^\beta E^\alpha{}_a E_\beta{}^b, \quad \theta_a{}^b=\theta_\alpha{}^\beta E^\alpha{}_a E_\beta{}^b,
$$

$$u^a=u^\alpha E_\alpha{}^a,\qquad v^a=v^\alpha E_\alpha{}^a,\qquad w^a=w^\alpha E_\alpha{}^a,$$

then, taking account of (6.7), we find from (6.25)

$$\phi_\alpha{}^\beta=\phi_a{}^b E_\alpha{}^a E^\beta{}_b+(c_s b^t-b_s c^t)C_\alpha{}^s C^\beta{}_t\,,$$
$$\psi_\alpha{}^\beta=\psi_a{}^b E_\alpha{}^a E^\beta{}_b+(a_s c^t-c_s a^t)C_\alpha{}^s C^\beta{}_t\,,$$
$$\theta_\alpha{}^\beta=\theta_a{}^b E_\alpha{}^a E^\beta{}_b+(b_s a^t-a_s b^t)C_\alpha{}^s C^\beta{}_t\,.$$

and

$$u^\alpha=u^a E^\alpha{}_a\,,\qquad v^\alpha=v^a E^\alpha{}_a\,,\qquad w^\alpha=w^a E^\alpha{}_a\,,$$

which imply the following formulas

$$\phi_c{}^a\phi_b{}^c=-\delta_b^a+u_b u^a,\qquad \phi_a{}^b u^a=0\,,\qquad u_b\phi_a{}^b=0\,,\qquad u_b u^b=1\,,$$
$$\psi_c{}^a\psi_b{}^c=-\delta_b^a+v_b v^a,\qquad \psi_a{}^b v^a=0\,,\qquad v_b\psi_a{}^b=0\,,\qquad v_b v^b=1\,,$$
$$\theta_c{}^a\theta_b{}^c=-\delta_b^a+w_b w^a,\qquad \theta_a{}^b w^a=0\,,\qquad w_b\theta_a{}^b=0\,,\qquad w_b w^b=1\,,$$

which are already given by (2.5), (2.6) and (2.7) respectively, where $u_b=u^a g_{ab}$, $v_b=v^a g_{ab}$ and $w_b=w^a g_{ab}$. Therefore we can construct a triple $\{\bar\phi, \bar\psi, \bar\theta\}$ of almost contact metric structures defined in each coordinate neighborhood $\{U;\ y^a\}$ of the hypersurface M by the same method as in the construction of the quaternionic Kaehlerian structure $\{F, G, H\}$, and moreover prove that they satisfy the other algebraic conditions given by (2.8)~(2.13). Since $\tilde\pi\circ\tilde\imath=i\circ\pi$, choosing suitably local coordinates in M and in $QP(m)$, we can find in $U\cap'U\neq\phi$ the relations

$$\begin{pmatrix}'\bar\phi\\ '\bar\psi\\ '\bar\theta\end{pmatrix}=(S_{xy})\begin{pmatrix}\bar\phi\\ \bar\psi\\ \bar\theta\end{pmatrix},\qquad \begin{pmatrix}'\bar u\\ '\bar v\\ '\bar w\end{pmatrix}=(S_{xy})\begin{pmatrix}\bar u\\ \bar v\\ \bar w\end{pmatrix},\qquad (x, y=1, 2, 3)$$

with functions S_{xy} defined in $U\cap'U$ which coincide with those appearing in (6.20). By denoting by $\{\bar\phi_a{}^b, \bar\psi_a{}^b, \bar\theta_a{}^b\}$ and $\{\bar u^a, \bar v^a, \bar w^a\}$ respectively the components of $\{\bar\phi, \bar\psi, \bar\theta\}$ and $\{\bar u, \bar v, \bar w\}$ with respect to coordinate neighborhood $\{U;\ y^a\}$, the commutativity of the diagram gives in U

$$F_j{}^i B_a{}^j=\bar\phi_a{}^b B_b{}^i+\bar u_a N^i,\qquad F_j{}^i N^j=-\bar u^a B_a{}^i,$$
$$G_j{}^i B_a{}^j=\bar\psi_a{}^b B_b{}^i+\bar v_a N^i,\qquad G_j{}^i N^j=-\bar v^a B_a{}^i,$$
$$H_j{}^i B_a{}^j=\bar\theta_a{}^b B_b{}^i+\bar w_a N^i,\qquad H_j{}^i N^j=-\bar w^a B_a{}^i,$$

where $\bar u_a=\bar u^b g_{ba}$, $\bar v_a=\bar v^b g_{ba}$ and $\bar w_a=\bar w^b g_{ba}$.

Here and in the sequel we use the notations $\{\phi_a{}^b, \psi_a{}^b, \theta_a{}^b\}$ and $\{u^b, v^b, w^b\}$ instead of $\{\bar\phi_a{}^b, \bar\psi_a{}^b, \bar\theta_a{}^b\}$ and $\{\bar u^b, \bar v^b, \bar w^b\}$ respectively. In the followings the algebraic relations (2.5)~(2.13) and the structure equations (2.17)~(2.19) will be very useful.

First we apply the operator $\nabla_b = E^\alpha{}_b \bar{\nabla}_\alpha = B_b{}^j D_j$ to (6.2). Then we have

$$(\nabla_b B_a{}^j)E_\alpha{}^a + B_a{}^j E^\beta{}_b \bar{\nabla}_\beta E_\alpha{}^a = B_b{}^i \tilde{E}^\mu{}_i (\bar{D}_\mu \tilde{E}_\kappa{}^j) B_\alpha{}^\kappa + \tilde{E}_\kappa{}^j E^\beta{}_b \bar{\nabla}_\beta B_\alpha{}^\kappa,$$

from which, substituting (6.8), (6.9), (6.10) and (6.11),

$$A_{ba} E_\alpha{}^a N^j + h_b{}^a{}_s C_\alpha{}^s B_a{}^j = h_i{}^j{}_s \tilde{C}_\kappa{}^s B_\alpha{}^\kappa B_b{}^i + A_{\beta\alpha} E^\beta{}_b N^j,$$

and consequently

$$A_{ba} = A_{\beta\alpha} E^\beta{}_b E^\alpha{}_a, \tag{6.29}$$

$$h_b{}^a{}_s C_\alpha{}^s B_a{}^j = h_i{}^j{}_s \tilde{C}_\kappa{}^s B_\alpha{}^\kappa B_b{}^i \tag{6.30}$$

because of (6.7), (6.23) and (6.25). Transvecting $E_\gamma{}^b E_\delta{}^A$ to (6.29) and replacing the indices γ and δ with β and α respectively, we get

$$A_{ba} E_\beta{}^b E_\alpha{}^a = A_{\beta\alpha} - A_{\beta\gamma}(a_s \xi^\gamma + b_s \eta^\gamma + c_s \zeta^\gamma) C_\alpha{}^s - A_{\alpha\gamma}(a_s \xi^\gamma + b_s \eta^\gamma + c_s \zeta^\gamma) C_\beta{}^s,$$

or equivalently

$$A_{\beta\alpha} = A_{ba} E_\beta{}^b E_\alpha{}^a + (u_\beta \xi_\alpha + v_\beta \eta_\alpha + w_\beta \zeta_\alpha) + (u_\alpha \xi_\beta + v_\alpha \eta_\beta + w_\alpha \zeta_\beta). \tag{6.31}$$

Then, transvecting (6.31) with $g^{\beta\alpha}$ and using (6.25), we find

$$A_a{}^a = A_\alpha{}^\alpha.$$

And also transvecting (6.31) with $A^{\beta\alpha}$ and using (6.25) and (6.29) give

$$A_{ba} A^{ba} = A_{\beta\alpha} A^{\beta\alpha} - 6.$$

Thus we have

LEMMA 6.1. (See also Lawson [6])

$$A_a{}^a = A_\alpha{}^\alpha \quad \textit{and} \quad A_{ba} A^{ba} = A_{\beta\alpha} A^{\beta\alpha} - 6.$$

On the other hand, as a consequence of (6.16) and (6.18), we have

$$F_j{}^i = -h_j{}^i{}_s a^s, \qquad G_j{}^i = -h_j{}^i{}_s b^s, \qquad H_j{}^i = -h_j{}^i{}_s c^s.$$

Thus substituting these equations into (6.30) and taking account of (2.4) imply

$$\phi_a{}^b = -h_a{}^b{}_s a^s, \qquad \psi_a{}^b = -h_a{}^b{}_s b^s, \qquad \theta_a{}^b = -h_a{}^b{}_s c^s. \tag{6.32}$$

Applying $\nabla_c = E^\gamma{}_c \bar{\nabla}_\gamma$ to (6.31), we can easily obtain

$$\begin{aligned} E^\gamma{}_c \bar{\nabla}_\gamma A_{\beta\alpha} = {} & (\nabla_c A_{ba}) E_\beta{}^b E_\alpha{}^a + A_{ba} E^\gamma{}_c (\bar{\nabla}_\gamma E_\beta{}^b) E_\alpha{}^a + A_{ba} E_\beta{}^b E^\gamma{}_c \bar{\nabla}_\gamma E_\alpha{}^a \\ & + E^\gamma{}_c \{(\bar{\nabla}_\gamma u_\beta)\xi_\alpha + (\bar{\nabla}_\gamma u_\alpha)\xi_\beta + (\bar{\nabla}_\gamma v_\beta)\eta_\alpha + (\bar{\nabla}_\gamma v_\alpha)\eta_\beta + (\bar{\nabla}_\gamma w_\beta)\zeta_\alpha \\ & + (\bar{\nabla}_\gamma w_\alpha)\zeta_\beta + u_\beta \bar{\nabla}_\gamma \xi_\alpha + u_\alpha \bar{\nabla}_\gamma \xi_\beta + v_\beta \bar{\nabla}_\gamma \eta_\alpha + v_\alpha \bar{\nabla}_\gamma \eta_\beta + w_\beta \bar{\nabla}_\gamma \zeta_\alpha + w_\alpha \bar{\nabla}_\gamma \zeta_\beta\}, \end{aligned}$$

from which, substituting (6.9), (6.22) and (6.26) and using (6.32),

$$E^{\gamma}{}_{c}\bar{\nabla}_{\gamma}A_{\alpha\beta}=(\nabla_{c}A_{ba}+\phi_{ca}u_{b}+\psi_{ca}v_{b}+\theta_{ca}w_{b}+\phi_{cb}u_{a}+\psi_{cb}v_{a}+\theta_{cb}w_{a})E_{\beta}{}^{b}E_{\alpha}{}^{a}$$
$$-(A_{be}\phi_{c}{}^{e}+A_{ce}\phi_{b}{}^{e})(E_{\beta}{}^{b}\xi_{\alpha}+E_{\alpha}{}^{b}\xi_{\beta})-(A_{be}\psi_{c}{}^{e}+A_{ce}\psi_{b}{}^{e})(E_{\beta}{}^{b}\eta_{\alpha}+E_{\alpha}{}^{b}\eta_{\beta})$$
$$-(A_{be}\theta_{c}{}^{e}+A_{ce}\theta_{b}{}^{e})(E_{\beta}{}^{b}\zeta_{\alpha}+E_{\alpha}{}^{b}\zeta_{\beta}).$$

Thus, using Lemma 4.2, we have

LEMMA 6.2. *If the second fundamental tensor $A_{\beta\alpha}$ of $\bar{M}=\tilde{\pi}^{-1}(M)$ is parallel, then the following two conditions* (1) *and* (2) *are valid in M:*

(1) $$\|\nabla_{c}A_{ba}\|^{2}=24(m-1),$$

(2) $$A_{ce}\phi_{b}{}^{e}+A_{be}\phi_{c}{}^{e}=0,\qquad A_{ce}\psi_{b}{}^{e}+A_{be}\psi_{c}{}^{e}=0,\qquad A_{ce}\theta_{b}{}^{e}+A_{be}\theta_{c}{}^{e}=0.$$

Next we prove

LEMMA 6.3. *If the second fundamental tensor A_{ba} of M satisfies*

(6.33) $$A_{be}\phi_{a}{}^{e}+A_{ae}\phi_{b}{}^{e}=0,\qquad A_{be}\psi_{a}{}^{e}+A_{ae}\psi_{b}{}^{e}=0,\qquad A_{be}\theta_{a}{}^{e}+A_{ae}\theta_{b}{}^{e}=0,$$

then the following two conditions (1) *and* (2) *are valid in $\bar{M}=\tilde{\pi}^{-1}(M)$.*

(1) $$\bar{\nabla}_{\gamma}A_{\beta}{}^{\alpha}=0,$$

(2) $$A_{\beta\gamma}A_{\alpha}{}^{\gamma}=\lambda A_{\beta\alpha}+g_{\beta\alpha},$$

where λ is a function defined by $\lambda=A_{ba}u^{b}u^{a}$.

Proof. We have already seen in section 2 that the condition (6.33) implies

$$A_{ba}u^{a}=A(U,U)u_{b},\qquad A_{ba}v^{a}=A(V,V)v_{b},\qquad A_{ba}w^{a}=A(W,W)w_{b}.$$

On the other hand transvecting the first equation of (6.33) with v^{a} and making use of (2.13) give $A_{be}w^{e}+v^{a}A_{ae}\phi_{b}{}^{e}=0$, and consequently $A(V,V)=A(W,W)$. Similarly we can also obtain $A(U,U)=A(V,V)=A(W,W)$. If we put $\lambda=A(U,U)=A(V,V)=A(W,W)$, then we get

(6.34) $$A_{ba}u^{a}=\lambda u_{b},\qquad A_{ba}v^{a}=\lambda v_{b},\qquad A_{ba}w^{a}=\lambda w_{b}.$$

Substituting (2.17) and (6.34) itself in the equation obtained by applying the operator ∇_{c} to the first equation of (6.34), we have

$$(\nabla_{c}A_{ba})u^{a}+A_{ba}A_{c}{}^{e}\phi_{e}{}^{a}=(\nabla_{c}\lambda)u_{b}-\lambda A_{ce}\phi_{b}{}^{e},$$

from which, taking the skew-symmetric part and using the euqation (3.3) of Codazzi and (6.33),

(6.35) $$v_{c}w_{b}-w_{c}v_{b}-\phi_{cb}-A_{ce}A_{b}{}^{d}\phi_{d}{}^{e}=\frac{1}{2}\{(\nabla_{c}\lambda)u_{b}-(\nabla_{b}\lambda)u_{c}\}-\lambda A_{ce}\phi_{b}{}^{e},$$

and consequently $\nabla_{c}\lambda=(u^{e}\nabla_{e}\lambda)u_{c}$. By the similar way as above the second equation of (6.34) implies $\nabla_{c}\lambda=(v^{e}\nabla_{e}\lambda)v_{c}$. Accordingly, since u and v are mutually orthogonal unit vectors, $u^e \nabla_e \lambda = v^e \nabla_e \lambda = 0$ and hence λ=const. If we substitute $\nabla_c \lambda = 0$ into (6.35) and take account of (6.33), then we have

$$v_c w_b - w_c v_b - \phi_{cb} - A_c{}^e A_{ed} \phi_b{}^d = -\lambda A_{ce} \phi_b{}^e,$$

from which, transvecting with $\phi_a{}^b$ and using (2.5), (2.12) and (2.13), because of (6.34)

$$A_{ce} A_a{}^e = \lambda A_{ca} + g_{ca} - (u_c u_a + v_c v_a + w_c w_a). \tag{6.36}$$

On the other side, if we transvect (6.31) with $A_\gamma{}^\alpha$ and use (6.25), (6.29), (6.34) and (6.36) itself, then we get

$$A_{\beta\alpha} A_\gamma{}^\alpha = (A_{ce} A_a{}^e) E_\beta{}^c E_\gamma{}^a + (\lambda \xi_\gamma + u_\gamma) u_\beta + (\lambda \eta_\gamma + v_\gamma) v_\beta + (\lambda \zeta_\gamma + w_\gamma) w_\beta$$
$$+ (\lambda \mu_\gamma + \xi_\gamma) \xi_\beta + (\lambda v_\gamma + \eta_\gamma) \eta_\beta + (\lambda w_\gamma + \zeta_\gamma) \zeta_\beta,$$

from which, substituting (6.36),

$$A_{\beta\alpha} A_\gamma{}^\alpha = \lambda A_{\beta\gamma} + g_{\beta\gamma}. \tag{6.37}$$

If we now apply the operator $\bar{\nabla}_\delta$ to (6.37), then using λ=const. implies

$$(\bar{\nabla}_\delta A_{\beta\alpha}) A_\gamma{}^\alpha + A_{\beta\alpha} \bar{\nabla}_\delta A_\gamma{}^\alpha = \lambda \bar{\nabla}_\delta A_{\beta\gamma}.$$

Thus, taking account of $\bar{\nabla}_\delta A_{\beta\alpha} - \bar{\nabla}_\beta A_{\delta\alpha} = 0$, we get

$$A_{\beta\alpha} \bar{\nabla}_\delta A_\gamma{}^\alpha = A_{\delta\alpha} \bar{\nabla}_\beta A_\gamma{}^\alpha,$$

and consequently $A_{\beta\alpha} \bar{\nabla}_\delta A_\gamma{}^\alpha = A_{\gamma\alpha} \bar{\nabla}_\delta A_\beta{}^\alpha$. Therefore we find

$$2 A_{\beta\alpha} \bar{\nabla}_\delta A_\gamma{}^\alpha = \lambda \bar{\nabla}_\delta A_{\beta\gamma}.$$

from which, transvecting $A_\sigma{}^\beta$ and using (6.37),

$$2\lambda A_{\sigma\alpha} \bar{\nabla}_\delta A_\gamma{}^\alpha + 2 \bar{\nabla}_\delta A_{\gamma\sigma} = \lambda A_\sigma{}^\beta \bar{\nabla}_\delta A_{\beta\gamma},$$

and consequently

$$\bar{\nabla}_\delta A_{\gamma\sigma} = -\frac{1}{2} \lambda A_\sigma{}^\alpha \bar{\nabla}_\delta A_{\gamma\alpha}.$$

Hence $\{2 + (\lambda^2/2)\} A_{\beta\alpha} \bar{\nabla}_\delta A_\gamma{}^\alpha = 0$, which implies $\bar{\nabla}_\delta A_{\beta\gamma} = 0$. Therefore the lemma is completely proved.

LEMMA 6.4. *If* $\|\nabla_c A_{ba}\|^2 = 24(m-1)$, *then*

$$A_{ce} \phi_b{}^e + A_{be} \phi_c{}^e = 0, \qquad A_{ce} \psi_b{}^e + A_{be} \psi_c{}^e = 0, \qquad A_{ce} \theta_b{}^e + A_{be} \theta_c{}^e = 0.$$

Proof. By means of Lemma 4.2 the assumption $\|\nabla_c A_{ba}\|^2 = 24(m-1)$ implies

$$\nabla_c A_{ba} + \phi_{ca} u_b + \phi_{cb} u_a + \psi_{ca} v_b + \psi_{cb} v_a + \theta_{ca} w_b + \theta_{cb} w_a = 0. \tag{6.38}$$

Differentiating (6.38) covariantly along M and applying Ricci identity to the equation thus obtained, we can easily find from (2.17), (2.18) and (2.19)

$$\begin{aligned}
&-K_{dcb}{}^{e}A_{ea}-K_{dca}{}^{e}A_{be}\\
&-(A_{de}\phi_{b}{}^{e})\phi_{ca}+(A_{ce}\phi_{b}{}^{e})\phi_{da}-\phi_{cb}(A_{de}\phi_{a}{}^{e})+\phi_{db}(A_{ce}\phi_{a}{}^{e})\\
&-(A_{de}\psi_{b}{}^{e})\psi_{ca}+(A_{ce}\psi_{b}{}^{e})\psi_{da}-\psi_{cb}(A_{de}\psi_{a}{}^{e})+\psi_{db}(A_{ce}\psi_{a}{}^{e})\\
&-(A_{de}\theta_{b}{}^{e})\theta_{ca}+(A_{ce}\theta_{b}{}^{e})\theta_{da}-\theta_{cb}(A_{de}\theta_{a}{}^{e})+\theta_{db}(A_{ce}\theta_{a}{}^{e})\\
&+A_{da}(u_{b}u_{c}+v_{b}v_{c}+w_{b}w_{c})+A_{db}(u_{c}u_{a}+v_{c}v_{a}+w_{c}w_{a})\\
&-A_{ca}(u_{b}u_{d}+v_{b}v_{d}+w_{b}w_{d})-A_{cb}(u_{d}u_{a}+v_{d}v_{a}+w_{d}w_{a})=0\,.
\end{aligned}$$

On the other hand, by using the equation (3.3) of Gauss and $c=4$ a direct simple calculation gives

$$\begin{aligned}
K_{dcb}{}^{e}A_{ea}=A_{da}g_{cb}-g_{db}A_{ca}&\\
&+(\phi_{d}{}^{e}A_{ea})\phi_{cb}-\phi_{db}(\phi_{c}{}^{e}A_{ea})-2\phi_{dc}(\phi_{b}{}^{e}A_{ea})+(\psi_{d}{}^{e}A_{ea})\psi_{cb}\\
&-\psi_{db}(\psi_{c}{}^{e}A_{ea})-2\psi_{dc}(\psi_{b}{}^{e}A_{ea})+(\theta_{d}{}^{e}A_{ea})\theta_{cb}-\theta_{db}(\theta_{c}{}^{e}A_{ea})\\
&-2\theta_{dc}(\theta_{b}{}^{e}A_{ea})+(A_{d}{}^{e}A_{ea})A_{cb}-A_{db}(A_{c}{}^{e}A_{ea})\,.
\end{aligned}$$

Consequently the equation above reduces to

$$\begin{aligned}
(6.39)\qquad & A_{da}(A_{c}{}^{e}A_{eb}-g_{cb}+u_{c}u_{b}+v_{c}v_{b}+w_{c}w_{b})-A_{ca}(A_{d}{}^{e}A_{eb}-g_{db}\\
&+u_{d}u_{b}+v_{d}v_{b}+w_{d}w_{b})+A_{db}(A_{c}{}^{e}A_{ea}-g_{ca}+u_{c}u_{a}+v_{c}v_{a}+w_{c}w_{a})\\
&-A_{cb}(A_{d}{}^{e}A_{ea}-g_{da}+u_{d}u_{a}+v_{d}v_{a}+w_{d}w_{a})+\phi_{da}(A_{ce}\phi_{b}{}^{e}+A_{bd}\phi_{c}{}^{e})\\
&-\phi_{ca}(A_{de}\phi_{b}{}^{e}+A_{be}\phi_{d}{}^{e})+\phi_{db}(A_{ce}\phi_{a}{}^{e}+A_{ae}\phi_{c}{}^{e})\\
&-\phi_{cb}(A_{de}\phi_{a}{}^{e}+A_{ae}\phi_{d}{}^{e})+2\phi_{dc}(\phi_{b}{}^{e}A_{ea}+\phi_{a}{}^{e}A_{eb})\\
&+\psi_{da}(A_{ce}\psi_{b}{}^{e}+A_{be}\psi_{c}{}^{e})-\psi_{ca}(A_{de}\psi_{b}{}^{e}+A_{be}\psi_{d}{}^{e})\\
&+\psi_{db}(A_{ce}\psi_{a}{}^{e}+A_{ae}\psi_{c}{}^{e})-\psi_{cb}(A_{de}\psi_{a}{}^{e}+A_{ae}\psi_{d}{}^{e})\\
&+2\psi_{dc}(\psi_{b}{}^{e}A_{ea}+\psi_{a}{}^{e}A_{eb})+\theta_{da}(A_{ce}\theta_{b}{}^{e}+A_{be}\theta_{c}{}^{e})\\
&-\theta_{ca}(A_{de}\theta_{b}{}^{e}+A_{be}\theta_{d}{}^{e})+\theta_{db}(A_{ce}\theta_{a}{}^{e}+A_{ae}\theta_{c}{}^{e})\\
&-\theta_{cb}(A_{de}\theta_{a}{}^{e}+A_{ae}\theta_{d}{}^{e})+2\theta_{dc}(\theta_{b}{}^{e}A_{ea}+\theta_{a}{}^{e}A_{eb})=0\,.
\end{aligned}$$

Transvecting (6.39) with $u^{c}u^{b}$ and using (2.5), (2.8), (2.11), (2.12) and (2.13), we can easily verify that

$$\begin{aligned}
&A(U,U)(A_{de}A_{a}{}^{e}-g_{da}+u_{d}u_{a}+v_{d}v_{a}+w_{d}w_{a})-\|A_{ce}u^{e}\|^{2}A_{da}\\
&\quad+(A_{ae}u^{e})(A_{dc}A_{b}{}^{c}u^{b})-(A_{de}u^{e})(A_{ac}A_{b}{}^{c}u^{b})+2A(U,W)\psi_{da}-2A(U,V)\theta_{da}
\end{aligned}$$

$$+v_a\{A_{de}v^e+(A_{ce}u^c)\theta_d{}^e\}+3v_d\{A_{ae}v^e+(A_{ce}u^c)\theta_a{}^e\}$$

$$+w_a\{A_{de}w^e-(A_{ce}u^c)\phi_d{}^e\}+3w_d\{A_{ae}w^e-(A_{ce}u^c)\phi_a{}^e\}=0\,,$$

from which, taking its symmetric and skew-symmetric part, we get respectively

$$\begin{aligned}(6.40)\qquad A(U,U)A_d{}^eA_{ea}&=A(U,U)(g_{da}-u_du_a-v_dv_a-w_dw_a)+\|A_{ce}u^e\|^2A_{da}\\&-2v_a\{A_{de}v^e+(A_{ce}u^c)\theta_d{}^e\}-2v_d\{A_{ae}v^e+(A_{ce}u^e)\theta_a{}^e\}\\&-2w_a\{A_{de}w^e-(A_{ce}u^c)\phi_d{}^e\}-2w_d\{A_{ae}w^e-(A_{ce}u^c)\phi_a{}^e\}\end{aligned}$$

and

$$\begin{aligned}(6.41)\qquad &v_d\{A_{ae}v^e+(A_{ce}u^c)\theta_a{}^e\}-v_a\{A_{de}v^e+(A_{ce}u^c)\theta_d{}^e\}+2A(u,w)\phi_{da}\\&+w_d\{A_{ae}w^e-(A_{ce}u^c)\phi_a{}^e\}-w_a\{A_{de}w^e-(A_{ce}u^c)\phi_d{}^e\}-2A(u,v)\theta_{da}\\&-(A_{de}u^e)(A_{ac}A_b{}^cu^b)+(A_{ae}u^e)(A_{dc}A_b{}^cu^b)=0\,.\end{aligned}$$

If we transvect (6.40) with u^a and use (2.5), (2.12) and (2.13), then we have

$$A(U,U)A_{de}A_b{}^eu^b=\|A_{ce}u^e\|^2A_{db}u^b-4A(U,V)v_d-4A(U,W)w_d\,.$$

Similarly using (2.5)~(2.13) and (6.39) implies

$$\begin{aligned}&A(U,U)A_{de}A_b{}^eu^b=\|A_{ce}u^e\|^2A_{db}u^b-4A(U,V)v_d-4A(U,W)w_d\,,\\(6.24)\qquad&A(V,V)A_{de}A_b{}^ev^b=\|A_{ce}v^e\|^2A_{db}v^b-4A(U,V)u_d-4A(V,W)w_d\,,\\&A(W,W)A_{de}A_b{}^ew^b=\|A_{ce}w^e\|^2A_{db}w^b-4A(U,W)u_d-4A(V,W)v_d\,.\end{aligned}$$

Multiplying $A(U,U)$ to (6.41) and substituting the first equation of (6.42), we have

$$\begin{aligned}&A(U,U)\{v_d(A_{ae}v^e+A_{ce}u^c\theta_a{}^e)-v_a(A_{de}v^e+A_{ce}u^c\theta_d{}^e)\}+2A(U,U)A(U,W)\phi_{da}\\&+A(U,U)\{w_d(A_{ae}w^e-A_{ce}u^c\phi_a{}^e)-w_a(A_{de}w^e-A_{ce}u^c\phi_d{}^e)\}\\&-2A(U,U)A(U,V)\theta_{da}+4(A_{de}u^e)\{A(U,V)v_a+A(U,W)w_a\}\\&-4(A_{ae}u^e)\{A(U,V)v_d+A(U,W)w_d\}=0\,,\end{aligned}$$

from which, transvecting ϕ^{da} and θ^{da} respectively and using (2.5)~(2.13),

$$A(U,U)A(U,V)=0\,,\qquad A(U,U)A(U,W)=0\,,$$

and consequently

$$\begin{aligned}(6.43)\qquad &A(U,U)\{(A_{ae}v^e+A_{ce}u^c\theta_a{}^e)-(A(V,V)-A(U,U))v_a-A(V,W)w_a\}=0\,,\\&A(U,U)\{(A_{ae}w^e-A_{ce}u^c\phi_a{}^e)-A(V,W)v_a+(A(U,U)-A(W,W))w_a\}=0\,.\end{aligned}$$

Therefore, (6.40) and (6.43) imply

$$(6.44)\qquad A(U,U)A_{de}A_a{}^e=A(U,U)(g_{da}-u_du_a-v_dv_a-w_dw_a)+\|A_{ce}u^e\|^2A_{da}$$

$$-4(A(V,V)-A(U,U))v_dv_a+4(A(U,U)-A(W,W))w_dw_a$$
$$-4A(V,W)(v_dw_a+w_dv_a)\,.$$

On the other side, transvecting (6.39) with ϕ^{dc} and taking account of (2.5)~(2.13), we obtain

$$(6.45)\qquad (4m-1)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=-\phi^{dc}\{A_{da}(A_{ce}A_b{}^e)+A_{db}(A_{ce}A_a{}^e)\}$$
$$-v_a\{A_{be}w^e-(A_{ce}u^c)\phi_b{}^e\}-v_b\{A_{ae}w^e-(A_{ce}u^c)\phi_a{}^e\}$$
$$+w_a\{A_{be}v^e+(A_{ce}u^c)\theta_b{}^e\}+w_b\{A_{ae}v^e+(A_{ce}u^c)\theta_a{}^e\}\,,$$

from which, multiplying $A(U,U)$ and substituting (6.44),

$$(6.46)\quad 2(2m-1)A(U,U)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)$$
$$=A(U,U)\{(A_{ae}v^e)w_b-(A_{ae}w^e)v_b+(A_{be}v^e)w_a-(A_{be}w^e)v_a\}$$
$$-4\{A(U,U)-A(W,W)\}\{(A_{ae}v^e)w_b+(A_{be}v^e)w_a\}$$
$$-4\{A(V,V)-A(U,U)\}\{(A_{ae}w^e)v_b+(A_{be}w^e)v_a\}$$
$$+A(U,U)\{u_a(A_{ce}u^c)\phi_b{}^e+u_b(A_{ce}u^c)\phi_a{}^e\}-2A(U,U)A(V,W)(v_av_b-w_aw_b)$$
$$+A(U,U)\{A(V,V)-A(W,W)\}(v_aw_b+w_av_b)\,.$$

Transvecting v^av^b to (6.46) and using (2.5)~(2.13) imply

$$A(U,U)A(V,W)=0\,.$$

Thus transvecting u^a to (6.46) gives

$$(4m-3)A(U,U)(A_{ae}u^a)\phi_b{}^e=0\,,$$

and hence $A(U,U)\{A_{be}u^e-A(U,U)u_b\}=0$. Moreover, substituting this equation into (6.43), we also find

$$A(U,U)\{A_{be}v^e-A(V,V)v_b\}=0\,,\qquad A(U,U)\{A_{be}w^e-A(W,W)w_b\}=0\,.$$

Accordingly (6.46) becomes

$$(2m-1)A(U,U)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=A(U,U)\{A(W,W)-A(V,V)\}(v_bw_a+w_bv_a)\,,$$

from which, transvecting v^bw^a, we find

$$A(U,U)\{A(W,W)-A(V,V)\}=0\,,$$

and consequently

$$A(U,U)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=0\,.$$

Similarly, using (6.39) and (6.42), we can derive

$$A(U,U)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=0\,,\qquad A(U,U)\{A(V,V)-A(W,W)\}=0\,,$$

$$(6.47)\qquad A(V,V)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=0\,,\qquad A(V,V)\{A(U,U)-A(W,W)\}=0\,,$$
$$A(W,W)(A_{be}\theta_a{}^e+A_{ae}\theta_b{}^e)=0\,,\qquad A(W,W)\{A(U,U)-A(V,V)\}=0\,.$$

Now, we consider the following three cases. Let P be an arbitrarily fixed point of M.
Case I. $A(U,U)_p=0$ and $A(V,V)_p\neq 0$.
Case II. $A(U,U)_p=0$, $A(V,V)_p=0$ and $A(W,W)_p\neq 0$.
Case III. $A(U,U)_p=A(V,V)_p=A(W,W)_p=0$.
In Case I, we have from (6.47)

$$(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)_P=0\,,$$

from which, transvecting $u^b u^a$ and $\phi_c{}^b v^a$ respectively,

$$A(U,W)_P=0 \quad\text{and}\quad A(U,V)_P=0\,.$$

Hence, from (6.42) we obtain $(A_{be}u^e)_P=0$.

In Case II, we can similarly prove that $(A_{be}u^e)_P=0$ by using (6.24) and (6.47).

In case III, using (6.42), at the point P

$$\|A_{ce}u^e\|^2 A_{ba}u^a=4A(U,V)v_b+4A(U,W)w_b\,,$$
$$\|A_{ce}v^e\|^2 A_{ba}v^a=4A(U,V)u_b+4A(V,W)w_b\,,$$
$$\|A_{ce}w^e\|^2 A_{ba}w^a=4A(U,W)u_b+4A(V,W)v_b\,.$$

Suppose that $A(U,V)_P\neq 0$. Then we have $\|A_{ce}u^e\|^2{}_P=4$ and $\|A_{ce}v^e\|^2{}_P=4$, and consequently

$$A_{be}u^e=A(U,V)v_b+A(U,W)w_b\,,\qquad A_{be}v^e=A(U,V)u_b+A(V,W)w_b\,,$$
$$\|A_{ce}w^e\|^2 A_{ba}w^a=4A(U,W)u_b+4A(V,W)v_b$$

at that point P. Substituting these relations into (6.41) and transvecting θ^{da}, we can easily see that $4(m-1)A(U,V)_P=0$ because of $\|A_{ce}u^e\|^2{}_P=4$. In contradicts the assumption $A(U,V)_P\neq 0$. Hence $A(U,V)_P=0$ and similarly $A(U,W)_P=0$ will be obtained.

Summing up the results obtained in these Cases I, II and III, we can say that if there exists a point $P\in M$ such that $A(U,V)_P=0$, then $(A_{be}u^e)_P=0$. On the other hand, (6.47) implies that at the point P satisfying $A(U,U)_P=0$ at least one of $A(V,V)$ and $A(W,W)$, say $A(V,V)$, is zero. Then $(A_{be}v^e)_P=0$. Transvecting $w^a v^b$ to (6.45) and taking account of $(A_{be}u^e)_P=0$ and $(A_{be}v^e)_P=0$, we have $A(W,W)_P=0$ and consequently $(A_{be}w^e)_P=0$. Summing up, if we put $S=\{P\in M|(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)_P\neq 0\}$, then we have

$$(6.48)\qquad A_{ae}u^e=0\,,\qquad A_{ae}v^e=0\,,\qquad A_{ae}w^e=0 \quad\text{on}\quad S\,,$$

since (6.47) implies $A(U,U)=0$ on S. As was proved in section 6, (6.34) with $\lambda=0$ implies (6.35) with $\lambda=0$. Thus, (6.48) implies (6.35) with $\lambda=0$, that is,

$$v_b w_a - w_b v_a - \phi_{ba} - A_{be} A_a{}^d \phi_d{}^e = 0 \quad \text{on} \quad S,$$

from which, transvecting $A_c{}^a$, we have

$$-\phi^{de} A_{be} A_{ad} A_c{}^a = A_{ce} \phi_b{}^e \quad \text{on} \quad S.$$

Hence, from (6.45) we have $A_{be}\phi_a{}^e + A_{ae}\phi_b{}^e = 0$ on S, and consequently the set S should be void. Therefore, the equation $A_{be}\phi_a{}^e + A_{ae}\phi_b{}^e = 0$ holds identically in M. Similarly, using (6.42) and (6.47), we obtain

$$A_{be}\phi_a{}^e + A_{ae}\phi_b{}^e = 0, \qquad A_{be}\psi_a{}^e + A_{ae}\psi_b{}^e = 0, \qquad A_{be}\theta_a{}^e + A_{ae}\theta_b{}^e = 0,$$

which completes the proof of Lemma 6.4.

Thus, joining Theorem 2, Lemmas 6.2, 6.3 and 6.4, we have

THEOREM 9. *Let M be a real hypersurface of $QP(m)$ and $\pi : \bar{M} \to M$ the submersion which is compatible with the Hopf fibration $S^{4m+3} \to QP(m)$. Then the following conditions* (1)$\sim$(5) *are equivalent to each other*:

(1) *The second fundamental tensor of $\bar{M}$ is parallel.*

(2) *The induced almost contact* 3*-structure in M is normal.*

(3) *The induced almost contact* 3*-structure tensors $\{\phi, \psi, \theta\}$ in M commute with its second fundamental tensor.*

(4) *The square of the length of the derivative of the second fundamental tensor in M is equal to a constant* $24(m-1)$.

(5) *The global tensor field Σ_1 defined by* (1.6) *vanishes.*

§ 7. Characterizations of hypersurfaces $M^Q_{p,q}(a, b)$ in $QP(m)$

Before we state our main results we should explain model subspaces which will appear in our theorems. We denote by $S^{4p+3}(a)$ the hypersphere of radius a centered at the origin in Q^{p+1}. If we identify Q^{p+q+2} with the product space $Q^{p+1} \times Q^{q+1}$, then, taking spheres $S^{4p+3}(a)$ in Q^{p+1} and $S^{4q+3}(b)$ in Q^{q+1}, we consider the product space $\bar{M}^Q_{p,q}(a, b) = S^{4p+3}(a) \times S^{4q+3}(b)$, which is naturally considered as a submanifold in Q^{p+q+2}. When $a^2 + b^2 = 1$, $\bar{M}^Q_{p,q}(a, b)$ is a hypersurface in $S^{4(p+q+1)+3}(1) \subset Q^{p+q+2}$. Thus, if $a^2 + b^2 = 1$, for any portion (p, q) of an integer $m-1$ such that $p+q = m-1$, $p \geqq 0$, $q \geqq 0$, $\bar{M}^Q_{p,q}(a, b)$ may be considered as a real hypersurface of $S^{4m+3}(1) \subset Q^{m+1}$. Considering the Hopf fibering $\tilde{\pi}: S^{4m+3}(1) \to QP(m)$, we put $M^Q_{p,q}(a, b) = \tilde{\pi}(\bar{M}^Q_{p,q}(a, b))$, which gives an example for submanifolds satisfying the commutative diagram shown in the previous section. We are now going to prove

THEOREM 10. *Let M be a complete real hypersurface of $QP(m)$. Suppose one of the following conditions* (1), (2) *and* (3) *which are equivalent to each other is valid*:

(1) *The induced almost contact* 3*-structure in M is normal.*

(2) *The derivative of the second fundamental tensor in M has constant norm* $24(m-1)$.

(3) *The global tensor field* Σ_1 *defined by* (1.6) *vanishes.*
Then $M=M^Q_{p,q}(a, b)$ *for some portion* (p, q) *of* $m-1$ *and some* a, b *such that* $a^2+b^2=1$.

However in order to prove this theorem we need the following Lemmas 7.1 and 7.2.

LEMMA 7.1. *Assume the relations*

$$A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e=0, \qquad A_{be}\psi_a{}^e+A_{ae}\psi_b{}^e=0, \qquad A_{be}\theta_a{}^e+A_{ae}\theta_b{}^e=0$$

are valid. Then the second fundamental tensor $A_\alpha{}^\beta$ *of* $\bar{M}$ *has exactly two eigenvalues whose multiplicities are* $4p+3$ *and* $4q+3$ *respectively, where* $p+q=m-1, p\geqq 0, q\geqq 0$.

Proof. As shown in Lemma 6.3 the assumption implies

$$A_{\beta\gamma}A_\alpha{}^\gamma=\lambda A_{\beta\alpha}+g_{\beta\alpha}, \tag{7.1}$$

where λ is constant defined by $\lambda=A_{ba}u^b u^a$. Denoting by ρ the eigenvalue corresponding to an eigenvector of $A_\alpha{}^\beta$, the equation (7.1) implies $\rho^2-\lambda\rho-1=0$. Consequently $A_\alpha{}^\beta$ has exactly two eigenvalues $\rho_1=(\lambda+\sqrt{\lambda^2+4})/2$ and $\rho_2=(\lambda-\sqrt{\lambda^2+4})/2$. On the other hand, transvecting (7.1) with ξ^α, η^α and ζ^α and using (6.23), we have respectively

$$A_{\beta\gamma}u^\gamma=\lambda u_\beta+\xi_\beta, \qquad A_{\beta\gamma}v^\gamma=\lambda v_\beta+\eta_\beta, \qquad A_{\beta\gamma}w^\gamma=\lambda w_\beta+\zeta_\beta,$$

from which, taking account of $\rho_1{}^2=\lambda\rho_1+1$,

$$A_\alpha{}^\beta(\rho_1 u^\alpha+\xi^\alpha)=\rho_1(\rho_1 u^\beta+\xi^\beta), \qquad A_\alpha{}^\beta(\rho_1 v^\alpha+\eta^\alpha)=\rho_1(\rho_1 v^\beta+\eta^\beta),$$
$$A_\alpha{}^\beta(\rho_1 w^\alpha+\zeta^\alpha)=\rho_1(\rho_1 w^\beta+\zeta^\beta).$$

Therefore $\rho_1 u^\alpha+\xi^\alpha, \rho_1 v^\alpha+\eta^\alpha$ and $\rho_1 w^\alpha+\zeta^\alpha$, which will be denoted by $e_1{}^\alpha, e_2{}^\alpha$ and $e_3{}^\alpha$ respectively, are eigenvectors of $A_\alpha{}^\beta$ corresponding to ρ_1, where $e_1{}^\alpha$, $e_2{}^\alpha$ and $e_3{}^\alpha$ are mutually orthogonal because of (6.25). Assume there exists another eigenvector $e_4{}^\alpha$ of $A_\alpha{}^\beta$ corresponding to ρ_1. Suppose $e_4{}^\alpha$ is orthogonal to $e_1{}^\alpha, e_2{}^\alpha$ and $e_3{}^\alpha$. Then we find

$$\rho_1(u_\alpha e_4{}^\alpha)+(\xi_\alpha e_4{}^\alpha)=0, \qquad \rho_1(v_\alpha e_4{}^\alpha)+(\eta_\alpha e_4{}^\alpha)=0, \qquad \rho_1(w_\alpha e_4{}^\alpha)+(\zeta_\alpha e_4{}^\alpha)=0. \tag{7.2}$$

On the other side, taking account of (6.23) and $A_\alpha{}^\beta e_4{}^\alpha=\rho_1 e_4{}^\alpha$, we get

$$(u_\alpha e_4{}^\alpha)-\rho_1(\xi_\alpha e_4{}^\alpha)=0, \qquad (v_\alpha e_4{}^\alpha)-\rho_1(\eta_\alpha e_4{}^\alpha)=0, \qquad (w_\alpha e_4{}^\alpha)-\rho_1(\zeta_\alpha e_4{}^\alpha)=0. \tag{7.3}$$

Since $\rho_1{}^2+1\neq 0$, (7.2) and (7.3) give

$$u_\alpha e_4{}^\alpha=v_\alpha e_4{}^\alpha=w_\alpha e_4{}^\alpha=0, \qquad \xi_\alpha e_4{}^\alpha=\eta_\alpha e_4{}^\alpha=\zeta_\alpha e_4{}^\alpha=0. \tag{7.4}$$

Moreover, by means of (6.25), (6.31) and

$$\phi_\alpha{}^\beta E^\alpha{}_b=\phi_b{}^a E^\beta{}_a, \qquad \psi_\alpha{}^\beta E^\alpha{}_b=\psi_b{}^a E^\beta{}_a, \qquad \theta_\alpha{}^\beta E^\alpha{}_b=\theta_b{}^a E^\beta{}_a,$$

our assumption implies

$$A_{\beta\gamma}\phi_\alpha{}^\gamma+A_{\alpha\gamma}\phi_\beta{}^\gamma=0\,,\qquad A_{\beta\gamma}\psi_\alpha{}^\gamma+A_{\alpha\gamma}\psi_\beta{}^\gamma=0\,,\qquad A_{\beta\gamma}\theta_\alpha{}^\gamma+A_{\alpha\gamma}\theta_\beta{}^\gamma=0\,,$$

from which, taking account of skew-symmetry of $\phi_{\beta\alpha}$, $\psi_{\beta\alpha}$ and $\theta_{\beta\alpha}$, we find

$$A_\gamma{}^\beta(\phi_\alpha{}^\gamma e_4{}^\alpha)=\rho_1(\phi_\alpha{}^\beta e_4{}^\alpha)\,,\qquad A_\gamma{}^\beta(\psi_\alpha{}^\gamma e_4{}^\alpha)=\rho_1(\psi_\alpha{}^\beta e_4{}^\alpha)\,,\qquad A_\gamma{}^\beta(\theta_\alpha{}^\gamma e_4{}^\alpha)=\rho_1(\theta_\alpha{}^\beta e_4{}^\alpha)\,.$$

Thus $\phi_\alpha{}^\beta e_4{}^\alpha$, $\psi_\alpha{}^\beta e_4{}^\alpha$ and $\theta_\alpha{}^\beta e_4{}^\alpha$ are also eigenvectors of $A_\alpha{}^\beta$ corresponding to ρ_1, which are mutually orthogonal and also orthogonal to $e_1{}^\alpha, e_2{}^\alpha, e_3{}^\alpha$ and $e_4{}^\alpha$ because of (7.4). Hence multiplicity of the eigenvalue ρ_1 is necessarily $4p+3$ for some integer p. Similarly we can prove that multiplicity of ρ_2 is $4q+3$, where $q=m+1-p$.

By means of Lemma 7.1 and $\bar{\nabla}_\gamma A_\alpha{}^\beta=0$ the eigenspaces corresponding to ρ_1 and ρ_2 define respectively $(4p+3)$- and $(4q+3)$-dimensional distributions D_{ρ_1} and D_{ρ_2} over $\bar{M}$ which are both integrable and parallel. Moreover each integral manifold of D_{ρ_1} is totally geodesic in $\bar{M}$ and so is each integral manifold of D_{ρ_2}.

Let $\{\tilde{F},\tilde{G},\tilde{H}\}$ be the natural quaternionic Kaehlerian structure of Q^{m+1} whose numerical components $\{\tilde{F}_A{}^B,\tilde{G}_A{}^B,\tilde{H}_A{}^B\}$ are given by (3.9). Denoting by $\tilde{B}_\alpha{}^A$ and $B_\kappa{}^A$ the differentials of the isometric immersions $\imath_1:\bar{M}(\subset S^{4m+3})\subset Q^{m+1}$ and $i_2: S^{4m+3}\subset Q^{m+1}$ in terms of local coordinates respectively, we can see that $\tilde{B}_\alpha{}^A=B_\alpha{}^\kappa B_\kappa{}^A$. Accordingly the vector $\tilde{N}^A=N^\kappa B_\kappa{}^A$ and the position vector N^A of S^{4m+3} can be chosen as unit normals for the immersion $\imath_1$ and then (6.21) implies

$$(7.5)\qquad \begin{aligned}\tilde{F}_A{}^B\tilde{B}_\alpha{}^A&=\phi_\alpha{}^\beta\tilde{B}_\beta{}^\beta+u_\alpha\tilde{N}^B+\xi_\alpha N^B,\\ \tilde{G}_A{}^B\tilde{B}_\alpha{}^A&=\psi_\alpha{}^\beta\tilde{B}_\beta{}^B+v_\alpha\tilde{N}^B+\eta_\alpha N^B,\\ \tilde{H}_A{}^B\tilde{B}_\alpha{}^A&=\theta_\alpha{}^\beta\tilde{B}_\beta{}^B+w_\alpha\tilde{N}^B+\zeta_\alpha N^B\,.\end{aligned}$$

and

$$(7.6)\qquad \begin{aligned}\tilde{F}_B{}^A(\rho_1\tilde{N}^B+N^B)&=-(\rho_1u^\alpha+\xi^\alpha)\tilde{B}_\alpha{}^A, & \tilde{F}_B{}^A(\rho_2\tilde{N}^B+N^B)&=-(\rho_2u^\alpha+\xi^\alpha)\tilde{B}_\alpha{}^A,\\ \tilde{G}_B{}^A(\rho_1\tilde{N}^B+N^B)&=-(\rho_1v^\alpha+\eta^\alpha)\tilde{B}_\alpha{}^A, & \tilde{G}_B{}^A(\rho_2\tilde{N}^B+N^B)&=-(\rho_2v^\alpha+\eta^\alpha)\tilde{B}_\alpha{}^A,\\ \tilde{H}_B{}^A(\rho_1\tilde{N}^B+N^B)&=-(\rho_1w^\alpha+\zeta^\alpha)\tilde{B}_\alpha{}^A, & \tilde{H}_B{}^A(\rho_2\tilde{N}^B+N^B)&=-(\rho_2w^\alpha+\zeta^\alpha)\tilde{B}_\alpha{}^A\,.\end{aligned}$$

In this case the equations of Gauss and Weingarten are given by

$$(7.7)\qquad \begin{aligned}&\bar{\nabla}_\beta\tilde{B}_\alpha{}^A=A_{\beta\alpha}\tilde{N}^A+g_{\beta\alpha}N^A,\\ &\bar{\nabla}_\beta\tilde{N}^A=-A_\beta{}^\alpha\tilde{B}_\alpha{}^A,\qquad \bar{\nabla}_\beta N^A=-\tilde{B}_\beta{}^A.\end{aligned}$$

If we put $q^A=q^\alpha\tilde{B}_\alpha{}^A$ for an eigenvector q^α of $A_\alpha{}^\beta$, then the direct sums $\{\tilde{q}^A\,|\,q^\alpha\in D_{\rho_1}\}\oplus\{\rho_1\tilde{N}^A+N^A\}^*$ and $\{\tilde{q}^A\,|\,q^\alpha\in D_{\rho_2}\}\oplus\{\rho_2\tilde{N}^A+N^A\}^*$ are both invariant under the actions of $\tilde{F},\tilde{G}$ and $\tilde{H}$ because of (7.5) and (7.6), where $\{\rho_1\tilde{N}^A+N^A\}^*$ is the linear closure of the set $\{\rho_1\tilde{N}^A+N^A\}$. Moreover we can verify from

(7.7) that $q^\alpha\bar{\nabla}_\alpha(\rho_1\tilde{N}^A+N^A)=0, q^\alpha\in D_{\rho_2}$ and $p^\alpha\bar{\nabla}_\alpha(\rho_2\tilde{N}^A+N^A)=0, p^\alpha\in D_{\rho_1}$ because $\rho_1\rho_2=-1$. Therefore the maximal integral manifolds M_{ρ_1} of D_{ρ_1} and M_{ρ_2} of D_{ρ_2} can be considered as real hypersurfaces in Q^{p+1} and in Q^{q+1} respectively. Now we can easily prove

LEMMA 7.2. *The M_{ρ_1} and M_{ρ_2} are both totally umbilical in Q^{m+1}.*

Proof of Theorem 10. Combining Theorem 9, Lemma 7.1 and Lemma 7.2 implies immediately the theorem.

We shall next prove

THEOREM 11. *Let M be a complete real hypersurface in $QP(m)$ whose second fundamental tensor A_{ba} is of the form*

$$A_{ba}=\mu g_{ba}-(u_b u_a+v_b v_a+w_b w_a)\,, \tag{7.8}$$

μ being a differentiable function. Then $M=M^Q_{m-1,0}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Proof. First by using (2.3)~(2.13) we can easily verify that (7.8) gives

$$A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e=0\,, \qquad A_{be}\psi_a{}^e+A_{ae}\psi_b{}^e=0\,, \qquad A_{be}\theta_a{}^e+A_{ae}\theta_b{}^e=0\,.$$

Since $\mu=1$ which is a consequence of Theorem 4 and $c=4$, $A_\alpha{}^\beta$ has exactly two eigenvalues 1 and -1 whose multiplicities are $4m-1$ and 3 respectively because of Lemma 6.1. Thus by the same way as in the proof of Theorem 10 we can complete the proof.

Combining Theorem 6 and Theorem 11, we have

THEOREM 12. *Let M be a compact real hypersurface in $QP(m)$. If the second fundamental tensor A_{ba} is semi-definite and the mean curvature B constant and if $A_{ba}A^{ba}\leqq 4(m-1)$, then $M=M^Q_{m-1,0}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.*

Combining Theorem 7 and Theorem 11, we have

THEROEM 13. *Let M be a compact real hypersurface in $QP(m)$. If the second fundamental tensor A_{ba} is semi-definite. the mean curvature B constant and $B^2\leqq(4m-4)^2$, then $M=M^Q_{m-1,0}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.*

Combining Theorem 8 and Theorem 10, we have

THEOREM 14. *Let M be a compact and orientable real hypersurface in $QP(m)$. If*

$$\int_M \{12(m-1)+B(A(U,U)+A(V,V)+A(W,W))-3A_{ba}A^{ba}\}*1\geqq 0\,,$$

then $M=M^Q_{p,q}(a,b)$ for some portion (p,q) of $m-1$ and some a,b such that $a^2+b^2=1$.

COROLLARY 15. *Let M be a compact and orientable real hypersurface in $QP(m)$. If*

$$12(m-1)+B(A(U, U)+A(V, V)+A(W, W))-3A_{ba}A^{ba}\geqq 0$$

at each point of M. then $M=M^{Q}_{p,q}(a, b)$, $p+q=m-1$, $p\geqq 0$, $q\geqq 0$ and $a^2+b^2=1$.

COROLLARY 16. (See also Lawson [6]). *Let M be a compact and orientable minimal real hypersurface in $QP(m)$. If $A_{ba}A^{ba}\leqq 4(m-1)$ at each point of M, then $M=M^{Q}_{p,q}(a, b)$, $p+q=m-1$, $p\geqq 0$, $q\geqq 0$ and $a^2+b^2=1$.*

BIBLIOGRAPHY

[1] EUM, S.S. AND J.S. PAK, Some characterizations of quaternionic Kaehlerian manifold with constant Q-sectional curvature, to appear.

[2] ISHIHARA, S., quaternion Kaehlerian manifolds and fibred Riemannian spaces with Sasakian 3-structure, Kōdai Math. Sem. Rep. vol. 25 (1973), 321-329.

[3] ISHIHARA, S., quaternion Kaehlerian manifolds, J. Differential Geometry. vol. 9, No. 4 (1974), 483-500.

[4] ISHIHARA, S. AND M. KONISHI, Fibred Riemannian spaces with Sasakian 3-structure, Differential Geometry in Honor of K. Yano, Kinokuniya, Tokyo, 1972, 179-194.

[5] ISHIHARA, S. AND M. KONISHI, Differential geometry of fibred spaces, Publication of the study group of geometry. vol. 8, Tokyo, 1973.

[6] KUO, Y.Y., On almost contact 3-structure, Tōhoku Math. J. vol. 22 (1970), 325-332.

[7] LAWSON, H.B., Jr., Rigidity theorems in rank 1 symmetric spaces, J. Differential Geometry. vol. 4 (1970), 349-357.

[8] NOMIZU, K. AND B. SMYTH, A formula of Simon's type and hypersurfaces with constant mean curvature, J. Differential Geometry. Vol. 3 (1969), 367-377.

[9] MAEDA, Y., On real hypersurfaces of a complex projective space, to appear.

[10] OKUMURA, M., On some real hypersurfaces of complex projective space, Transactions of AMS. Vol. 212 (1975), 355-364.

[11] OKUMURA, M., Compact real hypersurface with constant mean curvature of a complex projective space, to appear.

[12] OKUMURA, M., Compact real hypersurfaces of a complex projective space, to appear.

[13] YANO, K., Differential geometry on complex and almost complex spaces, Pergamon press, 1965.

[14] YANO, K. AND S. ISHIHARA, Submanifolds with parallel mean curvature vector, J. Differential Geometry. Vol. 6 (1971), 95-118.

DEPARTMENT OF MATHEMATICS
TEACHERS COLLEGE
KYUNGPOOK NATIONAL UNIVERSITY
TAEGU, KOREA