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REAL HYPERSURFACES IN QUATERNIONIC PROJECTIVE SPACES WITH COMMUTING TANGENT JACOBI OPERATORS

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Abstract. From the classical differential equation of Jacobi fields, one naturally defines the Jacobi operator of a Riemannian manifold with respect to any tangent vector. A straightforward computation shows that any real, complex and quaternionic space forms satisfy that any two Jacobi operators commute. In this way, we classify the real hypersurfaces in quaternionic projective spaces all of whose tangent Jacobi operators commute.

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1. Introduction. Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, \tilde{g})$ satisfy a very well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operator. That is, if $\tilde{R}$ is the curvature operator of $\tilde{M}$, and X is any tangent vector field to $\tilde{M}$, the Jacobi operator (with respect to X) at $p \in M$, $\tilde{R}_X \in \text{End}\,(T_p\tilde{M})$, is defined as $(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p)$ for all $Y \in T_p\tilde{M}$, being a selfadjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Clearly, each tangent vector field X to $\tilde{M}$ provides a Jacobi operator with respect to X. The study of Riemannian manifolds by means of their Jacobi operators has been developed following several ideas (see **[1]** and **[4]** among many others). For instance, given a submanifold in $\tilde{M}$, some authors have studied whether the Jacobi operators with respect to certain vector fields commute with the Weingarten endomorphism. Regarding this, we have been able to find the following in the literature.

1. Let M be a real hypersurface in a non-flat complex space form $\bar{M}^n$, and let ξ be the (local) vector field of the almost contact metric structure of M naturally induced from the complex structure of $\bar{M}^n$. In **[7]**, the authors classified those real hypersurfaces in non-flat complex space forms $\bar{M}^n$, such that the Jacobi operator with respect to ξ commutes with the Weingarten endomorphism, obtaining a characterization of the tubes over totally geodesic complex space forms $\bar{M}^k$ with $k \in \{0, \dots, n-1\}$ and horospheres. See also **[5]**.

2. J. Berndt introduced the definition of curvature-adapted hypersurfaces in a Riemannian manifold $\tilde{M}$ in **[2]**. We will keep the above notations. Indeed, let M be a connected hypersurface in $\tilde{M}$. Given N a unit normal vector of M at $p \in M$, let A be the shape operator associated with N. He considered the *normal* Jacobi operator of M with

respect to N, $K_N := \tilde{R}(\cdot, N)N \in \mathrm{End}(T_pM)$. Thus, curvature-adapted hypersurfaces in $\tilde{M}$ are those satisfying $K_N \circ A = A \circ K_N$ for all unit normal vector fields N or M. For instance, curvature-adapted real hypersurfaces in non-flat complex space forms are those whose structure vector field ξ is principal (on the whole real hypersurface). In the case of a quaternionic projective space $\mathbb{Q}P^m$, J. Berndt obtained the complete classification of curvature-adapted real hypersurfaces in $\mathbb{Q}P^m$ in the following

THEOREM A. *Let M be a connected real hypersurface in $\mathbb{Q}P^m$, $m \geq 2$. Then M is curvature-adapted if and only if M is congruent to an open part of one of the following real hypersurfaces in $\mathbb{Q}P^m$:*

(a) *a tube of radius r, $0 < r < \pi/2$, over a totally geodesic $\mathbb{Q}P^k$, for some $k \in \{1, \ldots, m-1\}$;*

(b) *a tube of radius r, $0 < r < \pi/4$, over a totally geodesic embedded complex projective space $\mathbb{C}P^m$.*

This theorem is a cornerstone of the theory of real hypersurfaces in $\mathbb{Q}P^m$ because, as far as the authors know, most of the results involving real hypersurfaces in $\mathbb{Q}P^m$ make use of it.

3. Furthermore, J. Berndt and L. Vanhecke in [**3**] generalized the definition of curvature-adapted real hypersurfaces to submanifolds in $\mathbb{Q}P^m$. Let $\bar{R}$ be the curvature operator of $\mathbb{Q}P^m$. They called a submanifold P in $\mathbb{Q}P^m$ curvature-adapted if for every normal vector N to P at each point $p \in P$, the normal Jacobi operator $\bar{R}_N$ with respect to N satisfies $\bar{R}_N(T_pP) \subset T_pP$ and $\bar{R}_N$ commutes with the shape operator A_N. They also obtained the complete classification of curvature-adapted submanifolds in $\mathbb{Q}P^m$.

These ideas have made us think of another point of view to study Riemannian manifolds by means of the behaviour of the Jacobi operators. Thus, we consider the following problem:

Problem 1: To classify the Riemannian manifolds all of whose Jacobi operators commute.

A straightforward computation shows that all real, complex and quaternionic space forms satisfy this property. We would like to make an approach to the solution of Problem 1 by studying a certain family of Riemannian manifolds, namely, real hypersurfaces in the quaternionic projective space $\mathbb{Q}P^m$ of quaternionic dimension $m \geq 2$, endowed with the metric g of constant quaternionic sectional curvature 4. Since we are going to use both the normal Jacobi operator and the (usual) Jacobi operator, we will introduce the following notation. If R is the curvature operator of a real hypersurface M in $\mathbb{Q}P^m$, given a tangent vector X to M at $p \in M$, we will call the *tangent* Jacobi operator (with respect to X) of M the endomorphism of T_pM given by $R_X = R(\cdot, X)X$. Thus, this paper is devoted to classifying the (connected) real hypersurfaces in $\mathbb{Q}P^m$, $m \geq 3$, all of whose tangent Jacobi operators commute in the following

THEOREM 1. *Let M be a connected real hypersurface in $\mathbb{Q}P^m$, $m \geq 3$. All tangent Jacobi operators of M commute if and only if M is locally congruent to one of the following real hypersurfaces:*

1. *a tube of radius r, $0 < r < \pi/2$, over a totally geodesic $\mathbb{Q}P^k$, for some $k \in \{1, \ldots, m-1\}$;*
2. *a tube of radius r, $0 < r < \pi/4$, over a totally geodesic embedded complex projective space $\mathbb{C}P^m$.*

2. Preliminaries. Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{Q}P^m$ without boundary. The restriction of g to M will also be called g. Let N be a locally defined unit normal vector field of M. Given a local basis $\{J_1, J_2, J_3\}$ of the quaternionic structure of $\mathbb{Q}P^m$, we put $U_k = -J_k N$, $k = 1, 2, 3$. Let $\mathbb{D}$ be the maximal quaternionic distribution of M. We will denote the orthogonal complement of $\mathbb{D}$ in TM by $\mathbb{D}^\perp$, which is locally spanned by $\{U_1, U_2, U_3\}$. Also, let A be the Weingarten endomorphism associated with N. Let X be a tangent vector field to M. We put $J_i X = \phi_i X + f_i(X)N$, $i = 1, 2, 3$, where $\phi_i X$ is the tangent component of $J_i X$, and $f_i(X) = g(U_i, X)$, $i = 1, 2, 3$. As $J_i^2 = -Id$, $i = 1, 2, 3$, where Id denotes the identity endomorphism on $T\mathbb{Q}P^m$, we get

$$\phi_i^2 X = -X + f_i(X)U_i, \quad f_i(\phi_i X) = 0, \quad \phi_i U_i = 0, \quad i = 1, 2, 3, \tag{2.1}$$

for any tangent vector X to M. As $J_i J_j = -J_j J_i = J_k$, where (i, j, k) is a cyclic permutation of (1, 2, 3), we obtain

$$\begin{aligned} \phi_i X = \phi_j \phi_k X - f_k(X)U_k = -\phi_k \phi_j X + f_j(X)U_k, \quad i = 1, 2, 3 \\ f_i(X) = f_j(\phi_k X) = -f_k(\phi_j X), \end{aligned} \tag{2.2}$$

for any tangent vector X to M, where (i, j, k) is a cyclic permutation of (1, 2, 3). It is also easy to check that for any tangent vectors X, Y to M and $i = 1, 2, 3$,

$$g(\phi_i X, Y) + g(X, \phi_i Y) = 0, \quad g(\phi_i X, \phi_i Y) = g(X, Y) - f_i(X)f_i(Y), \tag{2.3}$$

and

$$\phi_i U_j = -\phi_j U_i = U_k, \tag{2.4}$$

(i, j, k) being a cyclic permutation of (1, 2, 3). Given a tangent vector $X \in \mathbb{D}$, we denote $Q(X) = \mathrm{Span}\{X, \phi_1 X, \phi_2 X, \phi_3 X\}$.

From the expression of the curvature tensor of $\mathbb{Q}P^m$, $m \geq 2$, we obtain the equation of Gauss and Codazzi respectively:

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + \sum_{k=1}^{3}\{g(\phi_k Y, Z)\phi_k X - g(\phi_k X, Z)\phi_k Y \\ - 2g(\phi_k X, Y)\phi_k Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \sum_{k=1}^{3}\{f_k(X)\phi_k Y - f_k(Y)\phi_k X - 2g(\phi_k X, Y)U_k\}, \tag{2.6}$$

for any tangent vectors X, Y, Z to M, where ∇ denotes the covariant derivative on M.

From the expressions of the covariant derivatives of J_i, $i = 1, 2, 3$, it is easy to see

$$\nabla_X U_i = -p_j(X)U_k + p_k(X)U_j + \phi_i AX, \tag{2.7}$$

for any tangent vector X to M, (i, j, k) being a cyclic permutation of (1, 2, 3) and p_i, $i = 1, 2, 3$, local 1-forms on $\mathbb{Q}P^m$.

A tangent vector X to M is said to be principal if it is an eigenvector of A everywhere, and its associated eigenfunction will be called a principal curvature. Sometimes, we may call a locally defined tangent vector X to M principal if there is an open subset of M where it is defined and principal.

J. Berndt proved in **[2]** that a real hypersurface is curvature-adapted if and only if $A\mathbb{D} \subset \mathbb{D}$, equivalently, $A\mathbb{D}^{\perp} \subset \mathbb{D}^{\perp}$. Given a subset Ω of M, we will say that a real hypersurface is curvature-adapted on Ω if $A_p\mathbb{D}_p \subset \mathbb{D}_p$, equivalently, $A_p\mathbb{D}_p^{\perp} \subset \mathbb{D}_p^{\perp}$ for all $p \in \Omega$. Furthermore, we may say that M is (not) curvature-adapted at a point $p \in M$ if it is (not) curvature-adapted on $\{p\}$. Moreover, all real hypersurfaces appearing in Theorem A have constant principal curvatures. In the case (a), for $k \in \{1, \ldots, m-1\}$ and $r \in (0, \pi/2)$, the principal curvatures are $\cot(r)$ with multiplicity $4(m-k-1)$, $-\tan(r)$ with multiplicity $4k$, whose eigenspaces are contained in $\mathbb{D}$, and $2\cot(2r)$ with multiplicity 3, whose eigenspace is $\mathbb{D}^{\perp}$. In the case (b), for $r \in (0, \pi/4)$, the principal curvatures are $\cot(r)$, $-\tan(r)$ with multiplicity $2m-2$ respectively, whose eigenspaces are contained in $\mathbb{D}$, and $2\cot(2r)$ with multiplicity 1 and $-2\tan(2r)$ with multiplicity 2, whose eigenspaces are contained in $\mathbb{D}^{\perp}$.

3. Proof of Theorem 1.

Before starting the proof of Theorem 1, we need a lemma.

LEMMA 3.1. *There are no real hypersurfaces in $\mathbb{Q}P^m$, $m \geq 2$, satisfying both of the following:*

(a) *there exists a unit tangent vector $Z \in \mathbb{D}^{\perp}$ and a smooth function μ defined on M such that $AZ = \mu Z$;*

(b) *there exists a distribution $\Pi \subset \mathbb{D}$ such that $\phi_i\Pi \subset \Pi$, $i = 1, 2, 3$, and $A\Pi = \{0\}$.*

Proof. Suppose that there is a real hypersurface in $\mathbb{Q}P^m$, $m \geq 2$, satisfying statements (a) and (b). We can assume that M is connected. If $m = 2$, then $\Pi = \mathbb{D}$, so that M is curvature-adapted. Then M is one of the real hypersurfaces of Theorem A. But none of them has 0 as a principal curvature, which is a contradiction. Thus, we have to assume $m \geq 3$. Choose a point $p \in M$. As it is shown in [6], there is a connected open neighbourhood $\tilde{G}$ of p in $\mathbb{Q}P^m$, and a basis $\{J_1, J_2, J_3\}$ defined on $\tilde{G}$ of the quaternionic structure of $\mathbb{Q}P^m$ such that the corresponding vectors U_1, U_2, U_3 are defined on $G = \tilde{G} \cap M$, and $U_1 = Z$. Take a unit $X \in \Pi$ defined on G. Then $AX = A\phi_1 X = 0$. Putting $Y = \phi_1 X$ and inserting X and Y in (2.6), by (2.4) and (2.7),

$$\begin{aligned} g((\nabla_X A)Y - (\nabla_Y A)X, U_1) &= \sum_{k=1}^{3}\{-2g(\phi_k X, Y)g(U_k, U_1)\} = -2 \\ &= -g(A\nabla_X Y, U_1) + g(A\nabla_Y X, U_1) \\ &= \mu\{-g(\nabla_X Y, U_1) + g(\nabla_Y X, U_1)\} \\ &= \mu\{g(Y, \phi_1 AX) - g(X, \phi_1 AY)\} = 0, \end{aligned}$$

which is a contradiction. □

Proof of Theorem 1: Given a point $p \in M$, let G be a connected neighbourhood of $p \in M$ where the local vector fields N, U_1, U_2, U_3, etc. are defined. As is it shown in **[10]**, shrinking G if necessary, we can assume

$$g(AU_i, U_j) = 0, \quad \text{for any } i, j \in \{1, 2, 3\}, \quad i \neq j \text{ on } G.$$

We will use this assumption as well as equations (2.1), (2.3), (2.2) and (2.4) very often, although we may not explicitly say it. From now on, all the computations will be made on G unless otherwise stated.

From (2.5) and our hypothesis, as $R_{U_1}(R_{U_2}(U_1)) = 0$, we have

$$-g(AU_2, U_2)g(AU_1, U_1)U_1 + g(AU_2, U_2)(1 - g(AU_1, AU_1))AU_1 + g(AU_1, U_1)g(AU_2, U_2)A^2U_1 = 0. \tag{3.1}$$

This means that at each point $p \in G$, either $g(AU_2, U_2) = 0$ or $-g(AU_1, U_1)U_1 + (1 - g(AU_1, AU_1))AU_1 + g(AU_1, U_1)A^2U_1 = 0$.

From $R_{U_1}(R_{U_2}(U_3)) = R_{U_2}(R_{U_1}(U_3))$ we obtain

$$3g(AU_1, U_1)AU_3 + 3g(AU_2, U_2)g(AU_3, U_3)U_3 - g(AU_2, U_2)g(AU_3, AU_1)AU_1 = 3g(AU_2, U_2)AU_3 + 3g(AU_1, U_1)g(AU_3, U_3)U_3 - g(AU_1, U_1), g(AU_3, AU_2)AU_2. \tag{3.2}$$

If we take the scalar product of (3.2) with U_1 we get

$$g(AU_1, U_1)g(AU_2, U_2)g(AU_3, AU_1) = 0.$$

Similarly, we obtain

$$g(AU_i, U_i)g(AU_j, U_j)g(AU_k, AU_i) = 0, \quad \text{for distinct} i, j, k \in \{1, 2, 3\}. \tag{3.3}$$

Thus, if at a point $q \in G$, $g(AU_3, AU_1)$ and $g(AU_3, AU_2)$ are both nonzero, we get $g(AU_1, U_1)g(AU_2, U_2) = 0$, so that either $g(AU_1, U_1) = 0$ or $g(AU_2, U_2) = 0$ at q. If $g(AU_1, U_1) = 0$, from (3.2), then $3g(AU_2, U_2)\{g(AU_3, U_3)U_3 - g(AU_3, AU_1)AU_1\} = 3g(AU_3, U_3)U_3 - g(AU_3, AU_1)AU_1 = 3AU_3$. Taking the scalar product with AU_1 we get $(3 + \|AU_1\|^2)g(AU_3, AU_1) = 0$, which is a contradiction. Similarly, we get a contradiction if $g(AU_2, U_2) = 0$.

If at a point $q \in G$, $g(AU_3, AU_1)$ is zero and $g(AU_3, AU_2)$ is not zero, then $g(AU_1, U_1)g(AU_2, U_2) = 0$. Therefore, either $g(AU_1, U_1) = 0$ or $g(AU_2, U_2) = 0$ at q. If $g(AU_1, U_1) = 0$, from (3.2), $g(AU_2, U_2)g(AU_3, U_3)U_3 = g(AU_2, U_2)AU_3$. If $g(AU_2, U_2) \neq 0$, $AU_3 = g(AU_3, U_3)U_3$, that is to say, U_3 is principal at q. A similar result is obtained if we suppose $g(AU_3, AU_1) \neq 0$ and $g(AU_3, AU_2) = 0$ at $q \in G$.

If at a point $q \in G$, $g(AU_3, AU_1) = g(AU_3, AU_2) = 0$, then from (3.2) we obtain $g(AU_1, U_1)AU_3 + g(AU_2, U_2)g(AU_3, U_3)U_3 = g(AU_2, U_2)AU_3 + g(AU_1, U_1)g(AU_3, U_3)U_3$. This yields $\{g(AU_1, U_1) - g(AU_2, U_2)\}AU_3 = \{g(AU_1, U_1) - g(AU_2, U_2)\}g(AU_3, U_3)U_3$. If at $q \in G$, $g(AU_1, U_1) \neq g(AU_2, U_2)$, then $AU_3 = g(AU_3, U_3)U_3$, that is to say, U_3 is principal at $q \in G$.

Summing up, on a connected neighbourhood G of any point $p \in M$, there are two possibilities. Either

$$\begin{gathered} g(AU_1, AU_3) = g(AU_2, AU_3) = 0. \text{ In this case,} \\ \text{either } g(AU_1, U_1) = g(AU_2, U_2) \text{ or } U_3 \text{ is principal.} \end{gathered} \tag{3.4}$$

or

$$\begin{gathered} g(AU_1, U_1)g(AU_2, U_2) = 0. \text{ In this case, either} \\ g(AU_1, U_1) = g(AU_2, U_2) = 0, \text{ or there exists } i, j \in \{1, 2\}, i \neq j, \\ \text{such that } g(AU_i, U_i) = 0, g(AU_j, U_j) \neq 0, \text{ and } U_3 \text{ is principal.} \end{gathered} \tag{3.5}$$

Similar results to (3.4) and (3.5) hold for a cyclic permutation of (1, 2, 3).

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Now, we consider $R_{U_1}(R_{U_2}(X)) = R_{U_2}(R_{U_1}(X))$ for any $X \in \mathbb{D}$,

$$\begin{aligned}
-g(AU_2, U_2)g(AX, U_1)U_1 + 3g(AU_2, U_2)g(AX, U_3)U_3 - g(AU_1, U_1)g(AX, U_2)A^2U_2 \\
- g(AU_2, U_2)g(AX, AU_1)AU_1 + g(AX, U_2)g(AU_2, AU_1)AU_1 \\
= -g(AU_1, U_1)g(AX, U_2)U_2 + 3g(AU_1, U_1)g(AX, U_3)U_3 \\
- g(AU_2, U_2)g(AX, U_1)A^2U_1 - g(AU_1, U_1)g(AX, AU_2)AU_2 \\
+ g(AX, U_1)g(AU_1, AU_2)AU_2.
\end{aligned} \tag{3.6}$$

Developing $R_X(R_{U_1}(U_2)) = R_{U_1}(R_X(U_2))$ for any unit $X \in \mathbb{D}$,

$$\begin{aligned}
-g(AU_1, U_1)g(AU_2, X)X + 3g(AU_1, U_1)\sum_{i=1}^{3} g(AU_2, \phi_i X)\phi_i X + 3g(AX, X)AU_2 \\
-3g(U_2, AX)AX - g(AU_1, U_1)g(AU_2, AX)AX \\
= g(AU_2, X)g(AU_1, X)U_1 - 3g(AU_2, X)g(AU_3, X)U_3 + 3g(AX, X)g(AU_2, U_2)U_2 \\
-3g(AU_2, X)^2U_2 - g(AU_1, U_1)g(AU_2, X)A^2X - g(AX, X)g(AU_2, AU_1)AU_1.
\end{aligned} \tag{3.7}$$

In order to prove the theorem, and bearing in mind (3.4) and (3.5), we discuss the following cases. All the computations will be made on G, shrinking it if necessary.

Case 1. Suppose $g(AU_i, U_i) = 0$, $i = 1, 2, 3$, on G. This implies $AU_i \in \mathbb{D}$, $i = 1, 2, 3$. From (3.7), for all unit $X \in \mathbb{D}$,

$$\begin{aligned}
3g(AX, X)AU_2 - 3g(U_2, AX)AX \\
= g(AU_2, X)g(AU_1, X)U_1 - 3g(AU_2, X)g(AU_3, X)U_3 \\
-3g(AU_2, X)^2U_2 - g(AX, X)g(AU_2, AU_1)AU_1.
\end{aligned} \tag{3.8}$$

Taking the scalar product of (3.8) and U_1 we get

$$-3g(U_2, AX)g(AX, U_1) = g(U_2, AX)g(AX, U_1).$$

By similar reasonings taking cyclic permutations of (1, 2, 3), we have

$$g(AU_i, X)g(AU_j, X) = 0, \tag{3.9}$$

for any $X \in \mathbb{D}$ on G, and any $i, j \in \{1, 2, 3\}$, $i \neq j$. This implies $g(AU_i, AU_j) = 0$ for any $i, j \in \{1, 2, 3\}$, $i \neq j$. By (3.8) and (3.9), and similar expressions we obtain

$$g(U_i, AX)AX = g(AU_i, X)^2U_i + g(AX, X)AU_i, \tag{3.10}$$

for any unit $X \in \mathbb{D}$, $i \in \{1, 2, 3\}$ on G. Now, as $AU_i \in \mathbb{D}$, given $i \neq j$, we put $X = AU_i + AU_j$ and we insert it in (3.9), so that $\|AU_j\|^2\|AU_i\|^2 = 0$. Therefore, at most one of the vectors $AU_i \neq 0$ at a certain point $q \in G$. Suppose that there is a point $q \in G$ such that $AU_1(q) \neq 0$. Then, there is an open neighbourhood $V \subset G$ of p on which $AU_1 = \delta X_1 \in \mathbb{D}$, where $\delta \in C^\infty(V)$, X_1 is a unit vector lying in $\mathbb{D}$ on V, and $AU_2 = AU_3 = 0$ on V. If we take $X \in \mathbb{D}$ orthogonal to X_1, and we insert it in (3.10), $g(AX, X) = 0$, that is to say, $g(AX, Y) = 0$ for any $X, Y \in \mathbb{D}$ orthogonal to X_1. Moreover, if we insert X_1 in (3.10), then $AX_1 = \delta U_1 + g(AX_1, X_1)X_1$. As $m \geq 3$, $Q(X_1)^\perp \cap \mathbb{D} \neq \{0\}$. From all this, if $X \in Q(X_1)^\perp \cap \mathbb{D}$, then $AX = 0$, and $AU_2 = 0$. But this is impossible due to

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Lemma 3.1. Therefore, $AU_i = 0$ for all $i \in \{1, 2, 3\}$ at any point of G, that is to say, G is a curvature-adapted real hypersurface in $\mathbb{Q}P^m$.

Case 2. Suppose $g(AU_1, U_1) = g(AU_2, U_2) = 0$ and $g(AU_3, U_3) \neq 0$ on G.

From (3.5) we have $AU_1 = AU_2 = 0$. Shrinking G if necessary, we can write $AU_3 = g(AU_3, U_3)U_3 + \delta X_3$, where $X_3 \in \mathbb{D}$ is a unit tangent vector to M on G and $\delta \in C^\infty$. It is easy to obtain a similar equation to (3.7) by changing U_2 by U_3, so that $\delta g(AX, X)X_3 = 0$ for any $X \in \mathbb{D} \cap \mathrm{Span}\{X_3\}^\perp$ on G. Thus, if there is an open subset $V \subset G$ where $\delta \neq 0$, we get a contradiction by a similar reasoning as in the case 1, using Lemma 3.1. As δ is continuous, δ vanishes on the whole G and then G is a curvature-adapted real hypersurface in $\mathbb{Q}P^m$.

Case 3. Suppose $g(AU_1, U_1) = 0$ and $g(AU_2, U_2)g(AU_3, U_3) \neq 0$ on G.

From (3.5), U_2 and U_3 are principal and by (3.3), $g(AU_1, AU_2) = g(AU_1, AU_3) = 0$. This and (3.4) imply that either U_1 is also principal or $g(AU_2, U_2) = g(AU_3, U_3)$. Suppose that there is a point $q \in G$ where U_1 is not principal. Then, there is a connected open subset $V \subset G$ where $0 \neq AU_1 \in \mathbb{D}$. On it, $g(AU_2, U_2) = g(AU_3, U_3)$. There is a non-vanishing C^∞ function γ defined on V such that $AU_2 = \gamma U_2$, $AU_3 = \gamma U_3$. By (3.6), given $X \in \mathbb{D}$ on V,

$$g(AX, U_1)U_1 + g(AX, AU_1)AU_1 = g(AX, U_1)A^2U_1. \tag{3.11}$$

Taking the scalar product of (3.11) with U_1, then $g(AX, U_1)(1 - g(AU_1, AU_1)) = 0$ for all $X \in \mathbb{D}$ on V. As M is not curvature-adapted at any point of V, $g(AX, U_1) \neq 0$ for some $X \in \mathbb{D}$ and then $g(AU_1, AU_1) = 1$. This allows us to write $AU_1 = X_1$, where $X_1 \in \mathbb{D}$ is a unit tangent vector to V. Then from (3.11) we obtain $g(AX, X_1) = 0$ for any $X \in \mathbb{D} \cap \mathrm{Span}\{X_1\}^\perp$ and $AX_1 = U_1 + g(AX_1, X_1)X_1$.

Take unit $X_2, X_3 \in \mathbb{D} \cap \mathrm{Span}\{X_1\}^\perp$ on V. We must have $R_{X_2}(R_{X_3}(U_1)) = R_{X_3}(R_{X_2}(U_1))$. By (2.5),

$$g(AX_3, X_3)\sum_{k=1}^{3} g(X_1, \phi_k X_2)\phi_k X_2 = g(AX_2, X_2)\sum_{k=1}^{3} g(X_1, \phi_k X_3)\phi_k X_3. \tag{3.12}$$

Take $X_3 \in Q(X_1)^\perp \cap \mathbb{D}$, and $X_2 = \phi_1 X_1$, and insert them in (3.12). Now we get $g(AX_3, X_3) = 0$. If we take $X_2, X_3 \in Q(X_1) \cap \mathrm{Span}\{X_1\}^\perp$, from (3.12) we obtain $g(AX_3, X_3)X_1 = g(AX_2, X_2)X_1$, that is to say, $g(AX_3, X_3) = g(AX_2, X_2)$. Lemma 3.1 readily gives now a contradiction. Therefore, M is curvature-adapted on G.

Case 4. Suppose $g(AU_i, U_i) \neq 0$, $i = 1, 2, 3$, on G. From (3.3) we know $g(AU_i, AU_j) = 0$ for any $i, j \in \{1, 2, 3\}$, $i \neq j$.

Case 4.1. If $g(AU_i, U_i) \neq g(AU_j, U_j)$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$. From (3.4), all U_i, $i \in \{1, 2, 3\}$, are principal. Then, G is a curvature-adapted real hypersurface in $\mathbb{Q}P^m$ with three distinct principal curvatures on $\mathbb{D}^\perp$. But this is impossible due to Theorem A.

Case 4.2. We suppose $g(AU_1, U_1) = g(AU_2, U_2) \neq g(AU_3, U_3)$ on G.

If M is curvature-adapted on G, we simply resort to Theorem A. Thus, we suppose that there is a point $q \in G$ where M is not curvature-adapted. From (3.4), U_1, U_2 are principal with non-vanishing principal curvature γ, defined on G. Then, U_3 is not principal, and there is an open subset $V \subset G$ where U_3 is not principal. We put $AU_3 = \gamma_3 U_3 + \epsilon X_3$, where $\gamma_3, \epsilon \in \mathbb{C}^\infty(V)$, and $X_3 \in \mathbb{D}$ is a unit tangent vector to V.

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By (3.6), taking a cyclic permutation of (1, 2, 3) we get

$$g(AX, U_3)A^2U_3 = g(AX, U_3)U_3 + g(AX, AU_3)AU_3,$$

for any $X \in \mathbb{D}$ on V. If we choose $X \in \mathbb{D} \cap \mathrm{Span}\{X_3\}^{\perp}$ and insert it in the above equation, then $\epsilon g(AX, X_3)AU_3 = 0$, and as $\epsilon \neq 0$, $X_3 \neq 0$, then $g(AX, X_3) = 0$. This case is finished by a similar reasoning as in the case 3.

Case 4.3. $g(AU_i, U_i) = g(AU_j, U_j) \neq 0$ for any $i, j, \in \{1, 2, 3\}$.

From (3.3), $g(AU_i, AU_j) = 0$ for any $i, j \in \{1, 2, 3\}, i \neq j$. We denote the $\mathbb{D}$-component of a tangent vector X to M by $(X)_*$. If we call $\mathbb{D}_1 = \mathrm{Span}\{(AU_i)_* : i = 1, 2, 3\}$, we will discuss on the dimension of $\mathbb{D}_1$.

Case 4.3.1. Let $q \in G$ be a point where $\dim\mathbb{D}_1 = 0$. M is curvature-adapted at q.

Case 4.3.2. Let $q \in G$ be a point where $\dim\mathbb{D}_1 = 1$. There is an open neighbourhood of q contained in G where we can assume $AU_1 = \gamma U_1 + \delta X_1$, γ, δ being non-vanishing C^{∞} functions defined on V, and $X_1 \in \mathbb{D}$ a unit tangent vector to V. Moreover, $AU_2 = \gamma U_2$ and $AU_3 = \gamma U_3$. Once again, a similar reasoning as above making use of (3.6) makes us get a contradiction.

Case 4.3.3. Let $q \in G$ be a point where $\dim\mathbb{D}_1 = 2$. There is an open neighbourhood V of q contained in G where we can assume $AU_1 = \gamma U_1 + \delta_1 X_1$ and $AU_2 = \gamma U_2 + \delta_2 X_2$, where $\gamma, \delta_1, \delta_2$ are non-vanishing C^{∞} functions defined on V, and $X_1, X_2 \in \mathbb{D}$ are orthonormal tangent vectors to V. As a consequence, $AU_3 = \gamma U_3$ on V. From (3.6) we have $g(AX, U_1)U_1 + g(AX, U_2)A^2U_2 + g(AX, AU_1) AU_1 = g(AX, U_2)U_2 + g(AX, U_1)A^2U_1 + g(AX, AU_2)AU_2$. If we take $X \in \mathbb{D} \cap \mathrm{Span}\{X_1\}^{\perp}$ and insert it in the above expression, and taking the scalar product and U_1, we obtain $\gamma\delta_1 g(AX, X_1) = 0$. This yields $g(AX, X_1) = 0$. Similarly, for any $X \in \mathbb{D} \cap \mathrm{Span}\{X_2\}^{\perp}$ we see

$$g(AX, X_i) = 0, \quad \text{for any } X \in \mathbb{D}, \quad g(X, X_i) = 0, \quad i = 1, 2, \text{ on } V. \tag{3.13}$$

Take a unit $Y \in \mathbb{D} \cap \mathrm{Span}\{X_1, X_2\}^{\perp}$. Developing $R_{X_2}(R_Y(U_1)) = R_Y(R_{X_2}(U_1))$ we get

$$g(AY, Y)\sum_{i=1}^{3} g(X_1, \phi_i X_2)\phi_i X_2 = g(AX_2, X_2)\sum_{i=1}^{3} g(X_1, \phi_i Y)\phi_i Y. \tag{3.14}$$

If we choose $X \in \mathbb{D}$ such that $g(X, X_2) = 0$, and we insert it in (3.7) we obtain

$$0 = \gamma \sum_{i=1}^{3} g(X_2, \phi_i X)\phi_i X + g(AX, X)X_2. \tag{3.15}$$

According to this equation, $g(AX_1, X_1) \neq 0$ if and only if $Q(X_1) = Q(X_2)$ and $g(AX_1, X_1) = 0$ if and only if $Q(X_1) \perp Q(X_2)$. Once again, we need to discuss two subcases:

Case 4.3.3.1. There is a point $x \in V$ where $g(AX_1, X_1) \neq 0$. Then, there is an open subset $W \subset V$ where $g(AX_1, X_1) \neq 0$, so that $g(AX, X) = -\gamma$ for any unit $X \in Q(X_1)$ which is orthogonal to X_2 on W. Similarly, we obtain $g(AX_2, X_2) = -\gamma$. By (3.14), $g(AY, Y) = 0$ for any $Y \in \mathbb{D} \cap Q(X_1)^{\perp}$. This implies $g(AY, Z) = 0$ for any orthogonal $Y, Z \in \mathbb{D} \cap Q(X_1)^{\perp}$. We choose unit $Z \in Q(X_1)$ and unit $Y \in Q(X_1)^{\perp} \cap \mathbb{D}$. From our

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hypothesis, $0 = R_Y(R_Z(Y))$ and by (2.5) we have

$$0 = g(AZ, Z)(1 - \|AY\|^2)AY - g(AZ, Y)\{AZ - g(AZ, Y)Y + 3\sum_{i=1}^{3} g(AZ, \phi_i Y)\phi_i Y - g(AZ, AY)AY\} \tag{3.16}$$

If we insert $Z = X_1$ in (3.16), then $0 = -\gamma(1 - \|AY\|^2)AY$, so that $AY = \|AY\|^2 AY$. As $g(AY, Z) = 0$ for any $Y, Z \in \mathbb{D} \cap Q(X_1)^\perp$, we know $AY \in Q(X_1) \cap \mathrm{Span}\{X_1, X_2\}^\perp$. Thus, at most there are two linearly independent tangent vector fields $Y_1, Y_2 \in \mathbb{D} \cap Q(X_1)^\perp$ such that $\|AY_i\| = 1$, $i = 1, 2$, and if $Z \in \mathbb{D} \cap Q(X_1)^\perp \cap \mathrm{Span}\{Y_1, Y_2\}^\perp$, then $AZ = 0$.

We choose X_3, X_4 such that $Q(X_1) = \mathrm{Span}\{X_1, X_2, X_3, X_4\}$ is an orthonormal basis satisfying $g(AX_3, X_4) = 0$. If we insert $Z = X_3$ in (3.16), we get

$$0 = g(AX_3, Y)\{AX_3 - g(AX_3, Y)Y + 3\sum_{i=1}^{3} g(AX_3, \phi_i Y)\phi_i Y - g(AX_3, AY)AY\},$$

for any $Y \in \mathbb{D} \cap Q(X_1)^\perp$. Suppose that there is a unit tangent vector field $Y_1 \in \mathbb{D} \cap Q(X_1)^\perp$ on W such that $g(AX_3, Y_1) \neq 0$. Then

$$0 = AX_3 - g(AX_3, Y_1)Y_1 + 3\sum_{i=1}^{3} g(AX_3, \phi_i Y_1)\phi_i Y_1 - g(AX_3, AY_1)AY_1.$$

If we take the scalar product of the above equation and $\phi_i Y_1$, we get $0 = g(AX_3, \phi_i Y_1)$. This yields

$$0 = AX_3 - g(AX_3, Y_1)Y_1 - g(AX_3, AY_1)AY_1. \tag{3.17}$$

Taking the scalar product of (3.17) and X_3 (respectively, X_4), we obtain $0 = -\gamma - g(AX_3, AY_1)g(AY_1, X_3)$, $0 = g(AX_3, AY_1)g(AY_1, X_4)$. From here, $g(AY_1, X_4) = 0$, that is to say, $AY_1 \in Q(X_1) \cap \mathrm{Span}\{X_1, X_2, X_4\}^\perp$, and therefore $AY_1 = g(AY_1, X_3)X_3$, but since $AY_1 \neq 0$, we know $\|AY_1\| = 1$ and up to a change of sign, $g(AX_3, Y_1) = 1$. Thus, $AX_3 = -\gamma X_3 + Y_1$, $AY_1 = X_3$. We obtain a similar result if we exchange X_3 and X_4.

Firstly, if X_3 and X_4 are not principal on a certain open subset contained on W, there are orthonormal $Y_1, Y_2 \in \mathbb{D} \cap Q(X_1)^\perp$ such that

$$\begin{gathered} AX_3 = -\gamma X_3 + Y_1, \quad AY_1 = X_3, \\ AX_4 = -\gamma X_4 + Y_2, \quad AY_2 = X_4, \\ AZ = 0 \text{ for all } Z \in \mathbb{D} \cap Q(X_1)^\perp \cap \mathrm{Span}\{Y_1, Y_2\}^\perp. \end{gathered} \tag{3.18}$$

The dimension of ker A is $\dim \ker A = \dim \mathbb{D} \cap Q(X_1)^\perp \cap \mathrm{Span}\{Y_1, Y_2\}^\perp = 4m - 10 \geq 2$. Given $Z_1, Z_2 \in \ker A$, by (2.6), $2g(Z_1, \phi_3 Z_2) = g((\nabla_{Z_1}A)Z_2 - (\nabla_{Z_2}A)Z_1, U_3) = \gamma g([Z_2, Z_1], U_3) = \gamma\{g(Z_2, \phi_3 AZ_1) + g(Z_1, \phi_3 AZ_2)\} = 0$, that is to say, $\phi_3 \ker A \subset (\ker A)^\perp$. But by (2.3), $\phi_3 \ker A \subset \mathrm{Span}\{Y_1, Y_2\}$, and then $4m - 10 = \dim \ker A \leq 2$, which implies $m \leq 3$. Moreover, $\phi_3 \ker A = \mathrm{Span}\{Y_1, Y_2\}$, yielding $\ker A = \mathrm{Span}\{\phi_3 Y_1, \phi_3 Y_2\}$. Now, we insert $\phi_3 Y_1$ and X_3 in (2.6), bearing in mind (3.18),

$$\begin{aligned} g\big((\nabla_{\phi_3 Y_1}A)X_3 - (\nabla_{X_3}A)\phi_3 Y_1, U_3\big) &= 2g(\phi_3 Y_1, \phi_3 X_3) = 0 \\ &= g\big(\nabla_{\phi_3 Y_1} AX_3, U_3\big) + g\big(A\nabla_{X_3}\phi_3 Y_1, U_3\big) \\ &= -g(AX_3, \phi_3 A\phi_3 Y_1) - \gamma g(\phi_3 Y_1, \phi_3 AX_3) = -\gamma g(Y_1, AX_3) = -\gamma, \end{aligned}$$

which is a contradiction.

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Secondly, we assume that X_3 is not principal and X_4 is principal on a certain open subset included in W. We have now $AZ = 0$ for all $Z \in \mathbb{D} \cap Q(X_1)^{\perp} \cap \mathrm{Span}\{Y_1\}^{\perp}$. In particular, $A\phi_1 Y_1 = A\phi_2 Y_2 = 0$. By (2.6), a similar computation as above considering $g((\nabla_{\phi_1 Y_1})A)\phi_2 Y_1 - (\nabla_{\phi_2 Y_1} A)\phi_1 Y_1, U_3)$ gives a contradiction.

Thirdly, if X_3 and X_4 are principal, we know $g(AX_3, Y) = g(AX_4, Y) = 0$ for all $Y \in \mathbb{D} \cap Q(X_1)^{\perp}$ on the whole W. This together with the fact that $g(AY, Y) = 0$ for all $Y \in \mathbb{D} \cap Q(X_1)^{\perp}$ implies $AY = 0$ for all $Y \in \mathbb{D} \cap Q(X_1)^{\perp}$. Lemma 3.1 gives now a contradiction.

Case 4.3.3.2. $g(AX_1, X_1) = 0$ on the whole V. We already have pointed out that in this case, $Q(X_1) \perp Q(X_2)$. We insert $X \in \mathbb{D} \cap \mathrm{Span}\{X_1, X_2\}^{\perp}$ in (3.7), bearing in mind (3.12), and we obtain $g(AX, X)\gamma U_2 = g(AX, X)AU_2 = g(AX, X)\{\gamma U_2 + \delta_2 X_2\}$, which implies $g(AX, X) = 0$. This together with (3.12) yields $AX = 0$ for all $X \in \mathbb{D} \cap \mathrm{Span}\{X_1, X_2\}^{\perp}$. In particular, $A\phi_1 X_1 = A\phi_2 X_2 = 0$. Developing

$$g\big((\nabla_{\phi_1 X_1})A)\phi_2 X_1 - \big(\nabla_{\phi_2 X_1} A)\phi_1 X_1, U_3\big),$$

we get a contradiction as above

Case 4.3.4. Finally, we study the case in which there is a point $q \in G$ where $\dim \mathbb{D}_1 = 3$. There exist an open neighbourhood V of p contained in G, three orthonormal tangent vectors $X_1, X_2, X_3 \in \mathbb{D}$, and $\delta_1, \delta_2, \delta_3 \in C^{\infty}(V)$ such that $AU_i = \gamma U_i + \delta_i X_i, i = 1, 2, 3$. As above, from (3.6), we get $g(AX, X_i) = 0$ for any $X \in \mathbb{D} \cap \mathrm{Span}\{X_i\}^{\perp}$ and $i \in \{1, 2, 3\}$. This means

$$AU_i = \gamma U_i + \delta_i X_i, \quad AX_i = \delta_i U_i + g(AX_i, X_i)X_i, \quad i = 1, 2, 3 \text{ on V}. \tag{3.19}$$

Now we take a unit vector $Y \in \mathrm{Span}\{X_1, X_2, X_3\}^{\perp} \cap \mathbb{D}$. Developing $R_{X_2}(R_Y(U_1)) = R_Y(R_{X_2}(U_1))$ we obtain a similar formula to (3.14). Given $X \in \mathbb{D} \cap \mathrm{Span}\{X_2\}^{\perp}$, by (3.7), we obtain a similar formula to (3.15). We have to make a similar discussion to the above case.

Case 4.3.4.1. There is a point $x \in V$ where $g(AX_1, X_1)g(AX_2, X_2)g(AX_3, X_3) \neq 0$. As in the case 4.3.3.1, $Q(X_1) = Q(X_2) = Q(X_3)$. By repeating the computations, we obtain $g(AX, X) = -\gamma$ for all unit $X \in Q(X_1)$ that is orthogonal to X_2 and $g(AX_2, X_2) = -\gamma$. By (3.14), $g(AY, Y) = 0$ for all $Y \in Q(X_1)^{\perp}$. We also obtain a formula like (3.16). Repeating the computations we have $0 = (1 - \|AY\|^2)AY$ for all $Y \in Q(X_1)^{\perp}$. We extend $\{X_1, X_2, X_3\}$ to an orthonormal basis $\{X_1, X_2, X_3, X_4\}$ of $Q(X_1)$.

Suppose that there is a unit $Y \in Q(X_1)^{\perp}$ such that $\|AY\| = 1$ on V (or in a smaller open subset). The same reasoning as in case 4.3.3.1 shows $AZ = 0$ for all $Z \in Q(X_1)^{\perp} \cap \mathrm{Span}\{Y\}^{\perp}$, $AY = X_4$, $AX_4 = -\gamma X_4 + Y$. If we develop $R_{\phi_1 Y}(R_{U_1}(X_4)) = R_{U_1}(R_{\phi_1 Y}(X_4))$, we get $0 = 3g(AX_4, Y)Y$, that is to say, $Y = 0$, which is a contradiction.

Therefore, given $Y \in Q(X_1)^{\perp}$, $AY = 0$. Take a unit $Y \in Q(X_1)^{\perp}$. By (2.6), developing $(\nabla_Y A)\phi_1 Y - (\nabla_{\phi_1 Y} A)Y$ we obtain

$$A[\phi_1 Y, Y] = -2U_1.$$

If we take scalar product of this equation and U_1 (respectively, X_1), we see $-2 = \delta_1 g([\phi_1 Y, Y], X_1)$, respectively $0 = -\gamma g([\phi_1 Y, Y], X_1)$. These two equations clearly contradict each other.

Case 4.3.4.2. We can assume that there is an open subset W contained in V where $Q(X_1) = Q(X_2) \perp Q(X_3)$. By the same computations as before, we obtain

$g(AX, X) = -\gamma$ for all $X \in Q(X_1)$, but now $AX_3 = \delta_3 U_3$. Again, $g(AY, Y) = 0$ for all $Y \in \mathbb{D} \cap Q(X_1)^{\perp}$. As in the case 4.3.4.1, either $\|AU_3\| = 0$ or 1. As $\delta_3 \neq 0$, $AX_3 = U_3$. By developing $R_{\phi_3 X_3}(R_{U_1}(X_3)) = R_{U_1}(R_{\phi_3 X_3}(X_3))$, we get $\gamma U_3 = 4\gamma U_3$, which contradicts $\gamma \neq 0$.

Case 4.3.4.3. We assume $Q(X_i) \perp Q(X_j)$ for all $i, j \in \{1, 2, 3\}$, $i \neq j$. This implies $m \geq 4$. Taking $X = X_1$ in (3.6) we see

$$-\delta_1 U_1 = -\delta_1 A^2 U_1 + \gamma \delta_1 A U_1.$$

As $\delta_1 \neq 0$ we have $A^2 U_1 = U_1 + \gamma A U_1$. This yields $\delta_1 A X_1 = U_1$, and therefore $AX_1 = (1/\delta_1)U_1$. But since $1/\delta_1 = g(AX, U_1) = g(X_1, AU_1) = \delta_1$, we see $\delta_1^2 = 1$. The same reasoning implies $\delta_2^2 = \delta_3^2 = 1$. Up to a change of sign, we assume

$$AU_i = \gamma U_i + X_i, \quad AX_i = U_i, \quad i = 1, 2, 3. \tag{3.20}$$

Similarly as above cases, from (3.7) we obtain on one hand $g(AZ, Z) = 0$ for all $Z \in Q(X_1)^{\perp}$ and $g(AZ, Z) = -\gamma$ for all $Z \in Q(X_1) \cap \mathrm{Span}\{X_1\}^{\perp}$. But this contradicts (3.20), since $Q(X_1) \perp Q(X_3)$.

All these computations show that there is a dense open subset of M where it is curvature-adapted. In such case, that open subset is locally congruent to one of the real hypersurfaces of Theorem A. A connectedness reasoning bearing in mind the constancy of their principal curvatures show that M is an open subset of one of them.

Finally, let M be one of the real hypersurfaces in Theorem A. By considering a locally defined orthonormal frame of principal vectors, a long but straightforward computation shows that all of them satisfy that any two tangent Jacobi operators commute. This concludes the proof.

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