

Real hypersurfaces of a complex hyperbolic space

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1. Introduction.

During the last years the study of real hypersurfaces of Kaehlerian manifolds has been an important subject in geometry of submanifolds, especially when the ambient space is a complex space form.

One of the first results in this way (see [12]) was to state that any real hypersurface M of a complex space form $\bar{M}(c)$ with holomorphic sectional curvature $c \neq 0$ is not totally umbilical. This is a direct consequence of classical Codazzi's equation for such a hypersurface. From that equation, also one can deduce that there does not exist real hypersurfaces M of $\bar{M}(c)$, $c \neq 0$, with parallel second fundamental form H . So, it seems interesting to describe and characterize real hypersurfaces of $\bar{M}(c)$, $c \neq 0$, with a few principal curvatures or with derivative ∇H of the second fundamental form of short length. These problems have been solved, in the case $c > 0$, in [2], [6], [10], [11] and other works.

On the other hand, Kon, in [5], stated that there are no Einstein real hypersurfaces in $\bar{M}(c)$, $c > 0$, and he studied a less restrictive condition for the Ricci tensor of these hypersurfaces: the pseudo-Einstein condition (see also [6]). In fact, he classified the pseudo-Einstein real hypersurfaces of the complex projective space using Takagi's works [10] and [11].

Finally, Cecil and Ryan generalized in [2] some results of [10] and [5]. They described in terms of tubes over complex submanifolds the real hypersurfaces of the complex projective space which appear in the literature.

Now we are interested in these problems when $c < 0$, that is, when $\bar{M}(c)$ is the complex hyperbolic space CH^m (for convenience we will assume $c = -4$). So, A. Romero and the author have classified in [7] all complete real hypersurfaces of CH^m which admit a S^1 -principal bundle which is a parallel Lorentzian hypersurface of the anti-De Sitter space H_1^{2m+1} . These real hypersurfaces have the least length for ∇H as we will show in a forthcoming paper.

In this paper we construct some examples of real hypersurfaces of CH^m (Section 6) and we give a complete classification of the real hypersurfaces of CH^m with at most two principal curvatures at each point. In this classification we

obtain tubes over complex and totally real submanifolds of CH^m and a real hypersurface M_m^* of CH^m which has no focal points and which is congruent to all its parallel hypersurfaces. In fact, we will prove

THEOREM 7.4. *If M is a complete real hypersurface of CH^m , $m \geq 3$, with at most two principal curvatures at each point, then M is congruent to one of the following spaces:*

- a) *A geodesic hypersphere.*
- b) *A tube of arbitrary radius over a complex hyperbolic hyperplane.*
- c) *A "self-tube" M_m^* .*
- d) *A tube of radius $\log((1+\sqrt{3})/\sqrt{2})$ over a totally real hyperbolic hyperplane.*

Also, we will obtain the following characterization of the space M_m^* :

COROLLARY 7.5. *The only complete real hypersurface of CH^m , $m \geq 3$, which has no focal points and which is congruent to all its parallel hypersurfaces and such that $J\xi$ is a principal vector is the space M_m^* , where ξ is a unit vector normal to the hypersurface.*

It is necessary to remark that the real hypersurface appearing in Theorem 7.4, d), has exactly two constant principal curvatures at each point and, however, it is not totally η -umbilical (see [5] for definition). This cannot hold when the ambient space is the complex projective space (see [2]).

In Section 8 we will deal with pseudo-Einstein real hypersurfaces of CH^m . We will state:

COROLLARY 8.2. *There are no Einstein real hypersurfaces in CH^m , $m \geq 3$.*

In this way, we will prove the following classification result:

THEOREM 8.1. *The only complete real hypersurfaces of CH^m , $m \geq 3$, which are pseudo-Einstein are (up to congruences) the spaces a), b) or c) in Theorem 7.4.*

This last result asserts that a real hypersurface of CH^m is pseudo-Einstein if and only if it is totally η -umbilical.

Our main tool in this paper is based on [2]. Given a real hypersurface M of CH^m , we will "displace" it parallelly following a normal direction to obtain a submanifold $\phi_r M$ of CH^m which is complex or anti-holomorphic. Then we will relate the respective second fundamental forms of M and $\phi_r M$.

2. Preliminaries.

The Bergmann metric tensor g and the complex structure J of the complex hyperbolic space CH^m can be obtained as follows (see [4]): we consider the Hermitian form $(\ , \)$ on the complex vector space C^{m+1} given by

$$(z, w) = -z_0\bar{w}_0 + \sum_{j=1}^m z_j\bar{w}_j$$

where $z = (z_0, z_1, \dots, z_m)$, $w = (w_0, w_1, \dots, w_m) \in \mathbb{C}^{m+1}$. The inner product

$$\langle z, w \rangle = \text{Re}(z, w)$$

is an indefinite metric of index 2 on \mathbb{C}^{m+1} . Then, the hypersurface H_1^{2m+1} of \mathbb{C}^{m+1} defined by

$$H_1^{2m+1} = \{z \in \mathbb{C}^{m+1} \mid (z, z) = -1\}$$

endowed with the induced metric tensor from $\langle \cdot, \cdot \rangle$ is the well known anti-De Sitter space, which is a Lorentzian manifold of constant sectional curvature -1 . Moreover, if $z \in H_1^{2m+1}$, the tangent space $T_z H_1^{2m+1}$ is identifiable with the subspace of \mathbb{C}^{m+1}

$$\{w \in \mathbb{C}^{m+1} \mid \langle z, w \rangle = 0\}.$$

Now, H_1^{2m+1} is a principal S^1 -bundle over CH^m with projection map $\pi : H_1^{2m+1} \rightarrow CH^m$ such that $\text{Ker}(\pi_*)_z = \text{span}\{V_z\}$ with $V_z = \sqrt{-1}z \in T_z H_1^{2m+1}$. So, the tangent space $T_{\pi(z)} CH^m$ can be identified with the subspace of \mathbb{C}^{m+1}

$$T'_z = \{w \in \mathbb{C}^{m+1} \mid (z, w) = 0\}.$$

Now, the complex structure J of CH^m is induced from the multiplication by the imaginary unity $\sqrt{-1}$, that is,

$$JX = (\pi_*)_z(\sqrt{-1}X')$$

where $X \in T_{\pi(z)} CH^m$ and $X' \in T'_z$, $(\pi_*)_z(X') = X$ (horizontal lift). Also, the Bergmann metric tensor g of constant holomorphic sectional curvature -4 can be obtained from the relation

$$g(X, Y) = \langle X', Y' \rangle$$

where $X, Y \in T_{\pi(z)} CH^m$.

In this way, the projection $\pi : H_1^{2m+1} \rightarrow CH^m$ is a metric submersion in the sense of [8] with fundamental tensor J . So, if ∇' and $\bar{\nabla}$ are the metric connections of H_1^{2m+1} and CH^m respectively, we have

$$(2.1) \quad \nabla'_X Y' = (\bar{\nabla}_X Y)' + g(JX, Y)V_z \quad \nabla'_{V_z} X' = \nabla'_X V_z = \sqrt{-1}X'$$

for all $X, Y \in T_{\pi(z)} CH^m$.

Now, let M be a real hypersurface of CH^m and let ξ be a unit normal field defined near $x = \pi(z) \in M$. Then, if $X \in T_x M$, one has

$$JX = \phi X + f(X)\xi$$

tangent and normal components respectively. So, ϕ is a $(1, 1)$ -tensor and f is

a 1-form. Moreover, $f(X)=g(X, U)$ with $U=-J\xi$ and (ϕ, f) determines a metric almost contact structure on M (see [5] for more details).

We denote by H the Weingarten map on T_xM associated to ξ . Then the Codazzi and Gauss equations for M are (see [5], p. 341)

$$(2.2) \quad (\nabla_X H)Y - (\nabla_Y H)X = -f(X)\phi Y + f(Y)\phi X - 2g(X, \phi Y)U$$

$$(2.3) \quad R(X, Y)Z = -g(Y, Z)X + g(X, Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y \\ + 2g(X, \phi Y)\phi Z + g(HY, Z)HX - g(HX, Z)HY$$

where $X, Y, Z \in T_xM$, ∇ is the metric connection of the induced metric g on M and R is the curvature operator of ∇ .

Using (2.2) and the fact that CH^m is Kaehlerian, Kon has stated in [5], p. 342:

LEMMA 2.1. *Let M be a real hypersurface of CH^m and we suppose that $J\xi$ is a principal vector on M , that is, $HJ\xi = \mu J\xi$. Then, we have*

- a) $2\phi = \mu(\phi H + H\phi) - 2H\phi H$
 - b) $X \cdot \mu = (U \cdot \mu)f(X)$ for all X tangent to M , and $(U \cdot \mu)(\phi H + H\phi) = 0$ on M .
- Also, from (2.2) we have immediately (see [12]):

LEMMA 2.2. *Let M be a real hypersurface of CH^m , $m > 1$. Then M is not totally umbilical.*

Now, if S is the Ricci tensor of M we have from (2.3) (see [5], p. 341):

$$(2.4) \quad S(X, Y) = -(2m+1)g(X, Y) + 3f(X)f(Y) + (\text{tr}H)g(HX, Y) - g(H^2X, Y)$$

for all X, Y tangent to M .

Finally, given the real hypersurface M of CH^m , one can construct (see [7]) a Lorentzian hypersurface M' of H_1^{2m+1} which is a principal S^1 -bundle over M with time-like totally geodesic fibers and projection $\pi': M' \rightarrow M$ in such a way that the square

$$\begin{array}{ccc} M' & \xrightarrow{i'} & H_1^{2m+1} \\ \pi' \downarrow & & \downarrow \pi \\ M & \xrightarrow{i} & CH^m \end{array}$$

is commutative (i, i' being the respective immersions), and, thus, if $z \in M'$, then $V_z \in T_zM'$ and $\text{Ker}(\pi'_*)_z = \text{span}\{V_z\}$. Moreover, if ξ is a unit field normal to M defined near $x = \pi'(z)$, the horizontal lift ξ' of ξ by π' provides a unit field normal to M' defined near z . If H' denotes the Weingarten map on T_zM' as-

For studying the kernel of $(F_*)_{r\xi} : T_{r\xi}NM \rightarrow T_{F(r\xi)}CH^m$, we can suppose from (3.1) $r \geq 0$, taking $-\xi$ instead of ξ if necessary. Moreover, as one has that $(F_*)_0$ is an isomorphism, we put $r > 0$.

On the other hand, if ξ is a unit normal field defined on a neighbourhood W of $x \in M$, we have the following local trivialization of NM , taking into account that $\eta = \lambda\xi$ with $\lambda \in \mathbf{R}$ for all $\eta \in NM$

$$(3.2) \quad T_{NM}|_W = TW \times \text{span} \{ \partial / \partial \lambda \}.$$

Thus, similar computations as in [2] provide us

$$(3.3) \quad (F_*)_{r\xi}(\partial / \partial \lambda) = (\pi_*)_z(\sinh r w + \cosh r \xi')$$

$$(3.4) \quad (F_*)_{r\xi}(X) = (\pi_*)_z \{ \cosh r X' - \sinh r [(HX)' + \langle X', \sqrt{-1} \xi' \rangle \sqrt{-1} w] \}$$

for all $X \in T_x M$ and where $z = \cosh r w + \sinh r \xi' \in H_1^{2m+1}$. It is easy to see from (3.3) and (3.4) :

PROPOSITION 3.1. *If M is a real hypersurface of CH^m with $HJ\xi = \mu J\xi$ where ξ is a unit normal field defined near $x \in M$, then, with the local trivialization (3.2), we have*

a) $(F_*)_{r\xi}(\partial / \partial \lambda) = (\pi_*)_z(\sinh r w + \cosh r \xi')$

b) $(F_*)_{r\xi}(J\xi) = (\pi_*)_z(\cosh 2r - (1/2)\mu \sinh 2r)(\sinh r \sqrt{-1} w + \cosh r \sqrt{-1} \xi')$

c) $(F_*)_{r\xi}(X) = (\pi_*)_z(\cosh r X' - \sinh r (HX)')$

where $X \in T_x M$ with $g(X, J\xi) = 0$ and $z = \cosh r w + \sinh r \xi'$, $\pi(w) = x$.

REMARK. Computations for getting b) have been made in such a way that the vector on the right side lies in T'_z .

As an immediate consequence we find the focal points of M when $HJ\xi = \mu J\xi$.

PROPOSITION 3.2. *Let M be a real hypersurface of CH^m with $HJ\xi = \mu J\xi$ where the unit normal field ξ is defined near $x \in M$. Then*

a) $\text{Ker}(F_*)_{r\xi} = V_{\coth r}$ if $\mu \neq 2\coth 2r$ at x , where $V_{\coth r}$ is the subspace of $T_x M$ consisting of the orthogonal to $J\xi$ principal vectors corresponding to the principal curvature $\coth r$.

b) $\text{Ker}(F_*)_{r\xi} = V_{\coth r} \oplus \text{span} \{ J\xi \}$ if $\mu = 2\coth 2r$.

4. Parallel hypersurfaces and focal sets of a real hypersurface of CH^m .

Let M be an orientable real hypersurface of CH^m and let ξ be a unit normal field on M . We suppose that $J\xi$ is a principal vector at each point of M , that is, $HJ\xi = \mu J\xi$. For $r > 0$, we define a map $\phi_r : M \rightarrow CH^m$ by $\phi_r(x) = F(r\xi(x))$ where F was defined in (3.1).

When there are no focal points of M in $\phi_r M$, one has, from Propositions 3.1

and 3.2, that ϕ_r has rank $2m-1$ at each point of M . So, $\phi_r M$ is a real hypersurface of CH^m called parallel hypersurface at oriented distance r from M . If $\phi_r M$ contains some focal point of M , then we need some additional assumptions to guarantee that $\phi_r M$ is a submanifold of CH^m . In fact, we have an analogue to Theorem 1 in [2]:

THEOREM 4.1. *Let M be an orientable real hypersurface of CH^m such that $J\xi$ is a principal vector at each point, corresponding to a constant principal curvature μ . Let $r > 0$ and we assume that ϕ_r has constant rank q on M . Then, if $\mu = 2\coth 2r$ (resp. if $\mu \neq 2\coth 2r$), for every $x_0 \in M$ there exists an open neighbourhood W of x_0 such that $\phi_r W$ is a $q/2$ -dimensional complex (resp. q -dimensional anti-holomorphic) submanifold embedded in CH^m . Moreover W lies in a tube of radius r over $\phi_r W$.*

PROOF. Given $x_0 \in M$, let W be an open neighbourhood of x_0 such that $\phi_r W = V$ is a q -dimensional real submanifold embedded in CH^m (utilize the inverse function theorem).

If $p \in V$, one has $p = \phi_r(x) = \pi(z)$ with $z = \cosh r w + \sinh r \xi'$, $x \in W$ and $w \in H_1^{2m+1}$, $\pi(w) = x$. Then $T_p V = \text{span}\{(\phi_r)_*(X) \mid X \in T_x M\}$. Since $HJ\xi = \mu J\xi$, $\text{Ker } f_x$ is an H -invariant subspace of $T_x M$ and, so, we can take an orthonormal basis $X_1, \dots, X_{2m-2}, J\xi$ of $T_x M$ which satisfy $HX_i = \lambda_i X_i$, $X_i \in \text{Ker } f_x$, $i = 1, \dots, 2m-2$. So, we have

$$T_p V = \text{span}\{(\phi_r)_*(J\xi), (\phi_r)_*(X_i), i = 1, \dots, 2m-2\}.$$

From Proposition 3.1, we get

$$(4.1) \quad T_p V = \text{span}\{(\pi_*)_z(\cosh 2r - (1/2)\mu \sinh 2r)(\sinh r \sqrt{-1}w + \cosh r \sqrt{-1}\xi'), \\ (\pi_*)_z(\cosh r - \lambda_i \sinh r)X'_i, i = 1, \dots, 2m-2\}$$

where $z = \cosh r w + \sinh r \xi' \in H_1^{2m+1}$ with $\pi(w) = x$ for each $x \in \phi_r^{-1}(p)$.

Now, we define $\eta : \phi_r^{-1}(p) \rightarrow T_p CH^m$ by

$$(4.2) \quad \eta(x) = (\pi_*)_z(\sinh r w + \cosh r \xi'_w)$$

where $w \in H_1^{2m+1}$ and $\pi(w) = x$. Then, $\eta(x)$ is a unit vector which is orthogonal to $T_p V$ from (4.1). This map η can be defined for every $p \in V$ and, hence, we have a map $\eta : W \rightarrow BV$, where BV is the unit normal bundle over V . On the other hand, if $\phi_r : BV \rightarrow CH^m$ is the tube of radius r over V , we have

$$\phi_r(-\eta(x)) = (\cosh r(\cosh r w + \sinh r \xi'_w) - \sinh r \eta'(x)) = x.$$

So, $\phi_r(BV) \subset W$ and $\phi_r \circ (-\eta) = I_W$. Thus, η is a diffeomorphism from W onto an open set $\eta(W) \subset BV$ and W lies in a tube of radius r over V . Moreover, for $p \in V$, $\eta(W) \cap T_p^{-1}V$ is open in $T_p^{-1}V$ and so

$$(4.3) \quad T_p^+V = \text{span} \{(\pi_*)_z(\sinh r w + \cosh r \xi'_w) \mid w \in H_1^{2m+1}, \pi(w) = x \in \phi_r^{-1}(p)\}.$$

Now, if $\mu = 2\coth 2r$, from (4.3) and (4.1) we get $JT_p^+V \subset T_p^+V$ and V is complex. Finally, if $\mu \neq 2\coth 2r$, then from (4.1), the vectors

$$(\pi_*)_z(\sinh r \sqrt{-1} w + \cosh r \sqrt{-1} \xi'_w) = J(\pi_*)_z(\sinh r w + \cosh r \xi'_w)$$

with $z = \cosh r w + \sinh r \xi'$, $\pi(w) = x$ lies in T_pV for every $x \in \phi_r^{-1}(p)$. But these vectors span JT_p^+V from (4.3). So, $JT_p^+V \subset T_pV$ and V is anti-holomorphic (generic in the sense of [13]).

REMARK. It is important to see that a real hypersurface M of CH^m with $HJ\xi = \mu J\xi$, $\mu \in \mathbf{R}$ and $|\mu| \leq 2$ cannot be a tube over a complex submanifold of CH^m (cf. [2]).

A global version of Theorem 4.1 can be obtained by supposing that M is complete from the Palais results in [9] exactly as in [2]:

THEOREM 4.2. *Let M be a complete real hypersurface of CH^m with $HJ\xi = \mu J\xi$, $\mu \in \mathbf{R}$. Let $r > 0$ and we assume that ϕ_r has constant rank q on M . Then, if $\mu = 2\coth 2r$ (resp. $\mu \neq 2\coth 2r$) M is a tube of radius r over the complex (resp. anti-holomorphic) submanifold $\phi_r M$ of CH^m .*

5. Principal curvatures of $\phi_r M$.

As in Theorem 4.1, let M be an orientable real hypersurface of CH^m such that $J\xi$ is a principal field corresponding to a constant principal curvature μ . Moreover, we will suppose that ϕ_r has constant rank q on M , $r > 0$.

We take $x \in M$ and we have W and $\phi_r W = V$ associated to x as in Theorem 4.1. If $p \in V$, from (4.3) we can choose x_1, \dots, x_{2m-q} points of W with $\phi_r(x_i) = p$ and such that $N_i = (\pi_*)_{z_i}(N'_i)$, $N'_i = \sinh r w_i + \cosh r \xi'_{w_i}$, constitute a basis of unit vectors for T_p^+V , where $w_i \in H_1^{2m+1}$, $\pi(w_i) = x_i$ and $z_i = \cosh r w_i + \sinh r \xi'_{w_i}$, $i = 1, \dots, 2m - q$. Now, we distinguish two cases:

A) If $\mu = 2\coth 2r$, for fixed $i \in \{1, \dots, 2m - q\}$ and using (4.1), we can take q vectors

$$T_j^i = (\pi_*)_{z_i}(\cosh r - \lambda_j \sinh r) X'_j$$

where $X_j \in T_{x_i}M$, $g(X_j, J\xi) = 0$ and $HX_j = \lambda_j X_j$, $\lambda_j \neq \coth r$, $j = 1, \dots, q$, which form a basis of T_pV .

If we denote by $H_{r,i}$ the Weingarten map on T_pV associated to N_i , we have

$$(5.1) \quad H_{r,i} T_j^i = \text{tangent component of } -\bar{\nabla}_{T_j^i} N_i.$$

Now, by using the O'Neill formulae (2.1) we have

$$(5.2) \quad \bar{\nabla}_{T_j^i} N_i = (\pi_*)_{z_i} \nabla'_{(\cosh r - \lambda_j \sinh r) X'_j} N'_i.$$

It is easy to see that, if $\alpha(t)$ is a curve on H_1^{2m+1} with $\alpha(0)=w_i$ and $\dot{\alpha}(0)=X'_j$, then the curve $\gamma(t)=\cosh r \alpha(t)+\sinh r \xi'(\alpha(t))$ on H_1^{2m+1} satisfies $\gamma(0)=z_i$ and $\dot{\gamma}(0)=(\cosh r-\lambda_j \sinh r)X'_j$. Moreover, $N'_i(t)=\sinh r \alpha(t)+\cosh r \xi'(\alpha(t))$ is a tangent field to H_1^{2m+1} along $\gamma(t)$ with $N'_i(0)=N'_i$. Hence

$$\begin{aligned} \nabla'_{(\cosh r-\lambda_j \sinh r)X'_j} N'_i &= \text{tangent to } H_1^{2m+1} \text{ component of } \dot{N}'_i(0) \\ &= (\lambda_j \cosh r - \sinh r)X'_j \end{aligned}$$

as follows from a direct calculation. This, jointly with (5.1) and (5.2), gives us

$$(5.3) \quad H_{r,i} T_j^i = \frac{\lambda_j \cosh r - \sinh r}{\cosh r - \lambda_j \sinh r} T_j^i = \frac{\lambda_j \coth r - 1}{\coth r - \lambda_j} T_j^i.$$

So, $T_j^i, j=1, \dots, q$, is a diagonalization basis for $H_{r,i}$.

B) If $\mu \neq 2 \coth 2r$, for fixed $i \in \{1, \dots, 2m-q\}$, we have that $JN_i = (\pi_*)_{z_i}(\sqrt{-1}N'_i)$ lies in $T_p V$ as we have seen in Theorem 4.1 from (4.1). Moreover, there exists $q-1$ vectors

$$T_j^i = (\pi_*)_{z_i}(\cosh r - \lambda_j \sinh r)X'_j$$

of $T_p V$ with $X_j \in T_{x_i} M, g(X_j, J\xi) = 0, HX_j = \lambda_j X_j$ and $\lambda_j \neq \coth r, j=1, \dots, q-1$, which form an orthogonal basis of the orthogonal complement of the line span $\{JN_i\}$ in $T_p V$.

Now, we will evaluate

$$(5.4) \quad H_{r,i} JN_i = \text{tangent component of } -\nabla'_{JN_i} N_i.$$

By utilizing the O'Neill equalities (2.1), one has

$$\begin{aligned} (5.5) \quad \nabla'_{(\cosh 2r - (1/2)\mu \sinh 2r)JN_i} N_i &= (\pi_*)_{z_i} \nabla'_{(\cosh 2r - (1/2)\mu \sinh 2r)\sqrt{-1}N'_i} N'_i \\ &+ (\pi_*)_{z_i}(\cosh 2r - (1/2)\mu \sinh 2r)(\cosh r \sqrt{-1}w_i + \sinh r \sqrt{-1}\xi'_{w_i}). \end{aligned}$$

But $(\cosh 2r - (1/2)\mu \sinh 2r)(\sqrt{-1}N'_i) = L_i + \langle L_i, V_{z_i} \rangle V_{z_i}$ with $L_i = (\cosh r - \mu \sinh r)\sqrt{-1}\xi'_{w_i} - \sinh r \sqrt{-1}w_i \in T_{z_i} H_1^{2m+1}$. Hence, the first addend on the right side of (5.5) is

$$(5.6) \quad (\pi_*)_{z_i} \{ \nabla'_{L_i} N'_i + \langle L_i, V_{z_i} \rangle \nabla'_{V_{z_i}} N'_i \}.$$

Now, if $\alpha(t)$ is a curve on H_1^{2m+1} with $\alpha(0)=w_i$ and $\dot{\alpha}(0)=\sqrt{-1}\xi'_{w_i}$, then the curve $\gamma(t)=\cosh r \alpha(t)+\sinh r \xi'(\alpha(t))$ on H_1^{2m+1} satisfies $\gamma(0)=z_i$ and $\dot{\gamma}(0)=L_i$. So, $N'_i(t)=\sinh r \alpha(t)+\cosh r \xi'(\alpha(t))$ is a tangent to H_1^{2m+1} field which extends N'_i along $\gamma(t)$. Evaluating $\dot{N}'_i(0)$, taking its tangent to H_1^{2m+1} component and using (2.1), one has that (5.6) becomes

$$(5.7) \quad (\pi_*)_{z_i} \{(\sinh r - \mu \cosh r) \sqrt{-1} \xi'_{w_i} - \cosh r \sqrt{-1} w_i\} \\ + (\pi_*)_{z_i} (\mu \cosh 2r - 2 \sinh 2r) \sqrt{-1} N'_i.$$

Finally, (5.4), (5.5) and (5.7) give us

$$(5.8) \quad H_{r,i} J N_i = 2 \frac{\mu \coth 2r - 2}{2 \coth 2r - \mu} J N_i.$$

Moreover, in the same way as in the case A), we have

$$(5.9) \quad H_{r,i} T_j^i = \frac{\lambda_j \coth r - 1}{\coth r - \lambda_j} T_j^i.$$

The equations (5.3), (5.8) and (5.9) relate the principal curvatures $\lambda_j \neq \coth r$, μ of the real hypersurface M and their corresponding of the focal submanifold $\phi_r M$.

6. Examples.

EXAMPLE 6.1 (see [7]). If p, q are integers with $p + q = m - 1$ and $t \in \mathbf{R}$ with $0 < t < 1$, we consider the Lorentz hypersurface $M_{p,q}(t)$ of H_1^{2m+1} defined by the equations

$$-|z_0|^2 + \sum_{j=1}^m |z_j|^2 = -1 \quad t \left(-|z_0|^2 + \sum_{j=1}^p |z_j|^2 \right) = - \sum_{k=p+1}^m |z_k|^2.$$

It is easy to see that $M_{p,q}(t)$ is isometric to the product

$$H_1^{2p+1}(1/(t-1)) \times S^{2q+1}(t/(1-t))$$

where $1/(t-1)$ and $t/(1-t)$ are the respective squares of the radii.

If $z \in M_{p,q}(t)$, one can see that

$$\xi'(z) = - \frac{1}{\sqrt{t}} (tz_0, \dots, tz_p, z_{p+1}, \dots, z_m)$$

is a unit vector normal to $T_z M_{p,q}(t)$. So, if $a = (a_0, \dots, a_m)$ lies in $T_z M_{p,q}(t) = \{a \in \mathbf{C}^{m+1} | \langle z, a \rangle = 0, \langle \xi'(z), a \rangle = 0\}$ and H' denotes the Weingarten map associated to $\xi'(z)$, we have

$$H' a = -\nabla'_a \xi'(z) = -D_a \xi'(z)$$

where D is the usual connection of \mathbf{C}^{m+1} . Hence

$$(6.1) \quad H' a = \frac{1}{\sqrt{t}} (ta_0, \dots, ta_p, a_{p+1}, \dots, a_m).$$

Now, if we put $M_{p,q}^h(t) = \pi(M_{p,q}(t))$, we have a real hypersurface of $\mathbf{C}H^m$. Since $M_{p,q}(t)$ is S^1 -invariant, $\xi_{\pi(z)} = (\pi_*)_z \xi'(z)$ provides a unit field normal to $M_{p,q}^h(t)$. If we denote by H its associated Weingarten map, we have by using

(6.1) and (2.5)

$$(HU_{\pi(z)})' = H'(-\sqrt{-1}\xi'(z)) + V_z = -(\sqrt{t} + (1/\sqrt{t}))\sqrt{-1}\xi'(z)$$

and so $U = -J\xi$ is a principal field corresponding to the principal curvature $\sqrt{t} + (1/\sqrt{t})$. Then, from (2.5) and (2.6), we know that the principal curvatures of H on the orthogonal complement of the line $\text{span}\{U_{\pi(z)}\}$ agree with those of H' in the orthogonal complement of the Lorentz plane $\text{span}\{V_z, \sqrt{-1}\xi'(z)\}$. Now, from (6.1) one can see that these principal curvatures are \sqrt{t} and $1/\sqrt{t}$ with respective multiplicities $2p$ and $2q$.

So, $M_{p,q}^h(t)$ has constant principal curvatures $\tanh r$, $\coth r$ and $2\coth 2r$ with multiplicities $2p$, $2q$ and 1 respectively and where we have put $\tanh r = \sqrt{t}$. It is necessary to remark that, from (2.6), the Weingarten map H' of $M_{p,q}(t)$ is diagonalizable because $2\coth 2r > 2$.

Now, the map $\phi_r : M_{p,q}^h(t) \rightarrow CH^m$, $r = \text{argtanh}\sqrt{t}$, defined in Section 4, has constant rank $2(m-q-1)$ from Proposition 3.2. So, Theorem 4.2 asserts that $M_{p,q}^h(t)$ is a tube of radius r over a $(m-q-1)$ -dimensional complex submanifold of CH^m . Moreover, from (5.3), this submanifold is totally geodesic. In fact, $M_{p,q}^h(t)$ is a tube of radius r over a space CH^{m-q-1} embedded in CH^m in a totally geodesic way.

Only in the cases $p=0$ or $q=0$ (geodesic hypersphere or tube over a complex hyperbolic hyperplane) $M_{p,q}^h(t)$ is totally η -umbilical (see [3], p. 341 for definition) and only in these cases the Ricci tensor S of $M_{p,q}^h(t)$ is of the form

$$S(X, Y) = ag(X, Y) + bf(X)f(Y)$$

(pseudo-Einstein condition) for some functions a, b . In fact, from (2.4), one can see that $a = -2m + (2m-2)\coth^2 r$, $b = 2m$ if $p=0$ and $a = -2m + (2m-2)\tanh^2 r$, $b = 2m$ if $q=0$.

EXAMPLE 6.2 (see [7]). For fixed $t \in \mathbf{R}$, with $t > 0$, let $N(t)$ be the Lorentz hypersurface of H_1^{2m+1} given by

$$-|z_0|^2 + \sum_{j=1}^m |z_j|^2 = -1 \quad |z_0 - z_1|^2 = t.$$

Then $N(t)$ is clearly S^1 -invariant. Moreover, if $\alpha(s) = (\alpha_0(s), \dots, \alpha_m(s))$ is a curve on $N(t)$ with $\alpha(0) = z \in N(t)$ and $\dot{\alpha}(0) = a = (a_0, \dots, a_m)$, we have

$$\langle z, a \rangle = 0 \quad \text{Re}(\bar{z}_0 - \bar{z}_1)(a_0 - a_1) = 0$$

where \langle , \rangle is the indefinite inner product on \mathbf{C}^{m+1} defined in Section 2. Hence, the tangent space $T_z N(t) \subset T_z H_1^{2m+1}$ is identifiable with

$$\{a \in \mathbf{C}^{m+1} \mid \langle a, z \rangle = 0, \langle a, \eta(z) \rangle = 0\}$$

where we have put $\eta(z) = (z_0 - z_1, z_0 - z_1, 0, \dots, 0)$. So, the vector

$$\xi'(z) = \frac{1}{t} \eta(z) - z$$

satisfies the equalities $\langle \xi'(z), z \rangle = 0$ and $\langle \xi'(z), \xi'(z) \rangle = 1$. Hence, we have

$$T_z N(t) = \{a \in T_z H_1^{2m+1} \mid \langle a, \xi'(z) \rangle = 0\}.$$

Then, $\xi'(z)$ is a unit vector normal to $N(t)$ at z . Now, if H' denotes the associated Weingarten map, we have for each $a \in T_z N(t)$

$$H'a = -\nabla'_a \xi'(z) = -D_a \xi'(z)$$

where D is the usual connection on C^{m+1} . So

$$(6.2) \quad H'a = -\frac{1}{t} (a_0 - a_1, a_0 - a_1, 0, \dots, 0) + a.$$

Now, if $M_m^*(t) = \pi(N(t))$ is the corresponding real hypersurface of CH^m , $\xi_{\pi(z)}$ $= (\pi_*)_z \xi'(z)$ provides a unit field normal to $M_m^*(t)$. If H is the associated Weingarten map, we get by using (6.2) and (2.5)

$$(HU_{\pi(z)})' = H'(-\sqrt{-1}\xi'(z)) + V_z = -2\sqrt{-1}\xi'(z).$$

Hence, $U = -J\xi$ is a principal field corresponding to the principal curvature 2. Moreover, since from (6.2) H' is the identity map on the orthogonal complement of the Lorentz plane $\text{span}\{V_z, \sqrt{-1}\xi'(z)\} \subset T_z N(t)$ (note that this orthogonal complement is given by $\{a \in T_z N(t) \mid a_0 = a_1\}$), then, using the relations (2.5), we have that H is also the identity map on the orthogonal complement of the line $\text{span}\{U_{\pi(z)}\} \subset T_{\pi(z)} M_m^*(t)$.

So, we have seen that $HX = X + f(X)U$ for all X tangent to $M_m^*(t)$, and so $M_m^*(t)$ is totally η -umbilical. By using Corollary 5.3 of [7], we have that $M_m^*(t)$ is congruent to $M_m^*(1) = M_m^*$ for each $t > 0$. In [7], we have shown that $M_m^*(t)$ is a homogeneous space with isometry group neither semisimple nor soluble and that it is diffeomorphic to \mathbf{R}^{2m-1} .

From Proposition 3.2, the map $\phi_r : M_m^*(t) \rightarrow CH^m$ has constant rank $2m-1$ for all $r > 0$. So, every ϕ_r is an immersion. Moreover, the real hypersurface $\phi_r M_m^*(t)$ is also totally η -umbilical with principal curvatures 1 and 2 as follows from (5.8) and (5.9). Again by using Corollary 5.3 of [7], we have that $\phi_r M_m^*(t)$ is congruent to M_m^* (in fact, one can easily prove that $\phi_r M_m^*(t) = M_m^*(te^{2r})$) and, so, it is also congruent to $M_m^*(t)$. For these arguments we will say that M_m^* is a "self-tube".

The equation (2.4) and the previous considerations show that M_m^* is pseudo-Einstein with $a = -2$ and $b = 2m$.

REMARK 1. It is easy to note that $M_m^*(t)$, $t > 0$, provides an isoparametric family of hypersurfaces of CH^m , in such a way that all hypersurfaces of this

family have constant mean curvature $2m$. Moreover, from (2.6), this same occurs for the family $N(t)$ of Lorentz hypersurfaces of H_1^{2m+1} . Again from (2.6) the Weingarten map of each $N(t)$ is not diagonalizable because the Lorentz plane $\text{span}\{V_z, \sqrt{-1}\xi'(z)\} \subset T_zN(t)$ is irreducible for each $z \in N(t)$. So, one cannot use for $N(t)$ the Cartan results in [1]. That is, $N(t)$ is an isoparametric family of hypersurfaces of H_1^{2m+1} which have not an analogue in a Riemannian space form of negative curvature.

REMARK 2. We will characterize M_m^* in Corollary 7.5 as the only self-tube among all complete real hypersurfaces of CH^m such that $J\xi$ is principal.

EXAMPLE 6.3. We take $t \in \mathbf{R}$ with $t > 1$ and let $M(t)$ be the Lorentz hypersurface of H_1^{2m+1} defined by

$$-|z_0|^2 + \sum_{j=1}^m |z_j|^2 = -1 \quad \left| -z_0^2 + \sum_{j=1}^m z_j^2 \right|^2 = t$$

which is clearly S^1 -invariant. Taking curves on $M(t)$ one can easily see in an analogous way to the Example 6.2 that

$$\xi'(z) = \frac{1}{\sqrt{t(t-1)}} [Q(z)\bar{z} + tz]$$

where $Q(z) = -z_0^2 + \sum_{j=1}^m z_j^2$, is a unit vector normal to $M(t)$ at z and we can identify

$$T_zM(t) = \{a \in \mathbf{C}^{m+1} \mid \langle a, z \rangle = \langle a, Q(z)\bar{z} \rangle = 0\}.$$

Now, if H' is the Weingarten map associated to $\xi'(z)$, we have for all $a \in T_zM(t)$

$$H'a = -\nabla'_a \xi'(z) = -D_a \xi'(z)$$

where D is the usual connection of \mathbf{C}^{m+1} . So

$$(6.3) \quad H'a = -\frac{1}{\sqrt{t(t-1)}} [2Q(z, a)\bar{z} + Q(z)\bar{a} + ta]$$

with $Q(z, a) = -a_0z_0 + \sum_{j=1}^m a_jz_j$.

Let $M^h(t) = \pi(M(t))$ the corresponding real hypersurface of CH^m . Then, since $M(t)$ is S^1 -invariant, $\xi_{\pi(z)} = (\pi_*)_z \xi'(z)$ is a unit vector normal to $M^h(t)$ at $\pi(z)$. If H denotes its associated Weingarten map, we have, by using (6.3) and (2.5)

$$(HU_{\pi(z)})' = H'(-\sqrt{-1}\xi'(z)) + V_z = -2\frac{\sqrt{t-1}}{\sqrt{t}}\sqrt{-1}\xi'(z).$$

So, we have that $U = -J\xi$ is a principal field corresponding to the principal curvature $2\sqrt{(t-1)/t}$. Moreover, from (6.3), one can see that $M(t)$ has two principal curvatures $(\sqrt{t-1})/\sqrt{t-1}$ and $(\sqrt{t+1})/\sqrt{t-1}$ both with multiplicities

$m-1$ on the orthogonal complement of the Lorentz plane $\text{span}\{V_z, \sqrt{-1}\xi'(z)\} \subset T_z M(t)$ for all $z \in M(t)$. Hence, from (2.6) and the previous considerations, we have that $M^h(t)$ has constant principal curvatures $\tanh r$, $\coth r$, $2\tanh 2r$ with multiplicities $m-1$, $m-1$ and 1 respectively, where we have put $\cosh^2 2r = t$.

It is necessary to remark that $2\tanh 2r \neq \tanh r$ for all $r > 0$, but that $2\tanh 2r = \coth r$ if and only if $r = \log((1 + \sqrt{3})/\sqrt{2})$, that is, $t = 4$. Hence, $M^h(4)$ has two constant principal curvatures $\sqrt{3}$ and $1/\sqrt{3}$ with multiplicities m and $m-1$ respectively. If $t \neq 4$, $M^h(t)$ has three constant principal curvatures with multiplicities $m-1$, $m-1$ and 1 . So, $M^h(t)$ is not totally η -umbilical for each $t > 1$. Moreover, from (2.4) and the comments above, $M^h(t)$ is not pseudo-Einstein.

Now, if $r = (1/2)\text{argcosh}\sqrt{t}$, Proposition 3.2 assures us that $\phi_r: M^h(t) \rightarrow CH^m$ has constant rank m . By using Theorem 4.2, $M^h(t)$ is a tube of radius r over a totally real m -dimensional submanifold of CH^m . A convenient use of (5.8) and (5.9) shows that this submanifold is totally geodesic and, so, it is a real hyperbolic space RH^m embedded in CH^m .

On the other hand, let $SO^1(m+1)$ be the identity component of the subgroup of $GL(m+1, \mathbf{R})$ which preserves the Lorentzian form $-x_0^2 + x_1^2 + \dots + x_m^2$ on \mathbf{R}^{m+1} . One can see that $SO^1(m+1)$ acts transitively on $M^h(t)$ and that $M^h(t)$ is diffeomorphic to the homogeneous space $SO^1(m+1)/SO(m-1)$. Since $SO^1(m+1)$ has maximal compact subgroups isomorphic to $SO(m)$, $M^h(t)$ has the same homotopy type as a totally geodesic submanifold which is isometric to the symmetric space $SO(m)/SO(m-1)$, that is, to a unit sphere S^{m-1} .

REMARK 3. The relation (2.6) between the Weingarten maps H and H' of $M^h(t)$ and $M(t)$ respectively, asserts that H' is not diagonalizable. So, $M(t)$ is an isoparametric family of hypersurfaces of H_1^{2m+1} which has not an analogue in a Riemannian space form of negative curvature (see [1]).

REMARK 4. The tube $M^h(4)$ of radius $\log((1 + \sqrt{3})/\sqrt{2})$ over $RH^m \subset CH^m$ provides an example of real hypersurface of CH^m with two constant principal curvatures which is not totally η -umbilical. This fact is impossible when the ambient space is the complex projective space (see [10] and [2]).

7. Real hypersurfaces of CH^m with at most two principal curvatures at each point.

Let M be a real hypersurface of CH^m with at most two principal curvatures at each point. The remark after Proposition 5.2 of [2] states

LEMMA 7.1. *Let M be a real hypersurface of CH^m , $m \geq 3$, with exactly two principal curvatures at each point. Then $J\xi$ is a principal vector.*

For real hypersurfaces of the complex projective space a well-known result of Maeda, [6], assures that, if $J\xi$ is principal, then the corresponding principal curvature is locally constant. In this way, we have

LEMMA 7.2. *Let M be a real hypersurface of CH^m , $m \geq 3$, with at most two principal curvatures at each point. Then the principal curvature corresponding to $J\xi$ is locally constant.*

PROOF. From Lemma 7.1, we have $HJ\xi = \mu J\xi$. Let $x \in M$ such that $\phi H + H\phi = 0$ at x . Then, from Lemma 2.1, a), one has $\phi = -H\phi H$ at x . Hence, x is not umbilical and, so, there exists $X \in T_x M$ with $g(X, J\xi) = 0$ and $HX = \lambda X$. Then, $H\phi X = -\lambda\phi X$ and $H\phi X = -(1/\lambda)\phi X$. Thus we get $\lambda^2 = 1$ and $\mu = -\lambda$. On the other hand, in the open set consisting of the points of M where $\phi H + H\phi \neq 0$ we have, from Lemma 2.1, b), that μ is locally constant. So, μ is locally constant on M .

Now, we need a result that Takagi has shown when the ambient space is the complex projective space. Slight modifications in the computations of Lemma 3.4 in [11] provide us

LEMMA 7.3. *If M is a real hypersurface of CH^m , $m \geq 3$, with exactly three constant principal curvatures x, y, z at each point where the line $\text{span}\{J\xi\}$ is the eigenspace associated to z , then, we have one of the following possibilities:*

- a) $\phi V_x \subset V_x, \phi V_y \subset V_y, x + y = z$ and $xy = 1$.
- b) $\phi V_x \subset V_y, \phi V_y \subset V_x, x + y = 4/z$ and $xy = 1$.

Now, we can state

THEOREM 7.4. *Let M be a complete and connected real hypersurface of CH^m , $m \geq 3$, with at most two principal curvatures at each point. Then, M is congruent to one of the following spaces:*

- a) A geodesic hypersphere $M_{0, m-1}^h(\tanh^2 r)$ of radius $r > 0$.
- b) A tube $M_{m-1, 0}^h(\tanh^2 r)$ of radius $r > 0$ over a complex hyperbolic hyperplane.
- c) A self-tube M_m^* .
- d) A tube $M^h(4)$ of radius $\log((1 + \sqrt{3})/\sqrt{2})$ over a totally real hyperbolic hyperplane.

PROOF. From Lemmas 7.1 and 7.2, we know that $HJ\xi = \mu J\xi$ with $\mu \in \mathbf{R}_+$ for each unit local field ξ normal to M . We will distinguish three cases:

A) We suppose $\mu^2 > 4$. In this case M is orientable and we choose an orientation for M such that the associated principal curvature μ is greater than 2. Then we can put $\mu = 2 \coth 2r$ for some $r > 0$.

Let $\phi_r : M \rightarrow CH^m$ be as in Section 4. We denote by ν the least multiplicity on M of the principal curvature $\coth r$. Then, from Proposition 3.2, the set $\Omega = \{x \in M \mid \coth r \text{ has multiplicity } \nu \text{ at } x\}$ consists of the points of M where ϕ_r has maximum rank $2m - 2 - \nu$ and, so, Ω is a non-empty open set of M .

If $x \in \Omega$, from Theorem 4.1, there exists an open neighbourhood W of x such that $\phi_r W = V$ is a complex submanifold embedded in CH^m . If we have $0 < \nu < 2m - 2$, then, since $\mu = 2\coth 2r = \tanh r + \coth r \neq \coth r$, we have two principal curvatures at x , namely, $\mu = 2\coth 2r$ and $\lambda = \coth r$. Moreover $\dim V > 0$ and the discussions in Section 5 say that there exists a basis of unit vectors $N_1, \dots, N_{\nu+2}$ of T_p^+V , $p = \phi_r(x)$, in such a way that their associated Weingarten maps $H_{r,i}$, $i = 1, \dots, \nu+2$, are related with H as in (5.3). From this and since μ is the only principal curvature of M at x with associated eigenspace orthogonal to $J\xi$, as it follows from the assumption $\nu < 2m - 2$, we have that $H_{r,i} = (\mu \coth r - 1)/(\coth r - \mu)I$ where $i = 1, \dots, \nu+2$ and I is the identity map. But V is complex and, so, $H_{r,i} = 0$. Hence, $\mu = \tanh r$ which is impossible.

So, if $\nu > 0$, then $\nu = 2m - 2$. In this case, as $\mu \neq \coth r$ and ν is the least multiplicity of $\coth r$, we have that M has two constant principal curvatures $2\coth 2r$ and $\coth r$ with multiplicities 1 and $2m - 2$ at each point. Now, from Proposition 3.2, $\phi_r: M \rightarrow CH^m$ has constant rank zero and M is a geodesic hypersphere $M_{0,m-1}^h(\tanh^2 r)$ from Theorem 4.2.

On the other hand, if $\nu = 0$, then V is a complex hypersurface of CH^m . If the multiplicity of μ is greater than 1 at x , we could have chosen W with the same property at each point. Hence, using (5.3), V would have exactly two principal curvatures at each point. Since V is complex, V would be a complex Einstein hypersurface of CH^m with a principal curvature $(\mu \coth r - 1)/(\coth r - \mu)$ as follows from (5.3). The Chern result in [3] asserts that V is totally geodesic and, so, this principal curvature is zero, that is, $\mu = \tanh r$, which is impossible.

As conclusion, if $\nu = 0$, then the multiplicity of μ is 1 on Ω . If we denote by λ the other principal curvature on W , we have from (5.3) that the complex hypersurface V of CH^m has one principal curvature $(\lambda \coth r - 1)/(\coth r - \lambda)$ at each point. Since V is complex, then $\lambda = \tanh r$. Hence, Ω has two constant principal curvatures $2\coth 2r$ and $\tanh r$ with multiplicities 1 and $2m - 2$ respectively. So, Ω is closed in M and $\Omega = M$. Now, from Proposition 3.2, Theorem 4.2 and (5.3) we have that M is a tube $M_{m-1,0}^h(\tanh^2 r)$ over a space CH^{m-1} embedded in CH^m as a complex totally geodesic hypersurface.

B) We suppose $\mu^2 = 4$. As in the above case, M is orientable and we choose an orientation for M such that the associated principal curvature μ is 2. We denote by ν the least multiplicity on M of the principal curvature 2. We know that $\nu \geq 1$ and $\nu \leq 2m - 2$ from Lemma 2.2.

If $\nu > 1$, we take r such that $\coth r = 2$, that is, $r = \log \sqrt{3}$. Then, from Proposition 3.2, the set $\Omega = \{x \in M \mid 2 \text{ has multiplicity } \nu \text{ at } x\}$ consists of the points of M where the map $\phi_r: M \rightarrow CH^m$ has maximum rank $2m - \nu$ (note that $\mu = 2 \neq 2\coth 2r = 5/2$). So, Ω is open in M .

By using Theorem 4.1, if $x \in \Omega$, then there exists an open neighbourhood W

of x such that $\phi_r W = V$ is an anti-holomorphic submanifold embedded in CH^m . Since $\nu \leq 2m - 2$, let λ be another principal curvature of M on W . From Lemma 2.1, a), it is easy to see that $\lambda = 1$. Now, from discussions in Section 5, (5.8) and (5.9), there exists a basis of unit vectors N_1, \dots, N_ν of $T_p^\perp V$, $p = \phi_r(x)$, such that their corresponding Weingarten maps satisfy $H_{r,i} JN_i = 2JN_i$ and $H_{r,i}$ is the identity map on the orthogonal complement of JN_i in $T_p V$, $i = 1, \dots, \nu$. So, since we suppose $\nu > 1$, we take $i \neq j$ and we have

$$H_{r,j} JN_i = JN_i + g(N_i, N_j) JN_j.$$

Now, from Lemma 2.1 of [13], one has $H_{r,j} JN_i = H_{r,i} JN_j$. Hence, $g(N_i, N_j) = 1$ which is not possible because N_i, N_j are linearly independent.

As conclusion, we get $\nu = 1$ and, so, $\Omega = \{x \in M \mid HX = X + f(X)U \text{ at } x\}$ is closed in M . Then $\Omega = M$ and M has two constant principal curvatures 2 and 1 with respective multiplicities 1 and $2m - 2$. From Corollary 5.3 of [7], we conclude that M is congruent to a self-tube M_m^* . It is convenient to remark that, from Proposition 3.2, (5.8) and (5.9), the map $\phi_r : M \rightarrow CH^m$, $r > 0$, is always an immersion and that $\phi_r M$ has the same principal curvatures as M .

C) Finally, we suppose $\mu^2 < 4$. If $\mu = 0$ at some $x \in M$, then, from Lemma 2.1, a), we have $\phi = -H\phi H$. So, we would have three principal curvatures 0, λ and $-(1/\lambda)$ at x , which is impossible from our hypothesis. Hence, we can take a unit normal field ξ such that its corresponding principal curvature μ is $2 \tanh 2r$ for some $r > 0$ and M is orientable.

Now, Lemma 2.2, a) asserts that, if α is a principal curvature on M corresponding to a principal vector X with $g(X, J\xi) = 0$, then $\alpha' = (1 - \alpha \tanh 2r) / (\tanh 2r - \alpha)$ is another principal curvature corresponding to ϕX . But $\alpha' = \alpha$ implies the inequality $\tanh^2 2r \geq 1$ which is absurd. So, since M has at most two principal curvatures at each point, there are two principal curvatures α, β with $\alpha' = \beta$, $\beta' = \alpha$ and $\phi V_\alpha = V_\beta$ on the orthogonal complement of the line $\text{span}\{J\xi\}$ at each point of M . Moreover, from our hypothesis, we have $\alpha = \mu$ or $\beta = \mu$. So, we can put $\alpha = \mu$ and, hence

$$(7.1) \quad \beta = 2 \tanh 2r - \coth 2r \quad \text{mult}(\beta) = m - 1 \quad \text{mult}(\mu) = m.$$

Now, if $\beta = \coth r$, then one has $\coth^2 r = 1$ or $\coth^2 r = 1/3$, which is impossible. Hence, from Proposition 3.2, the map $\phi_r : M \rightarrow CH^m$ has constant rank either m (if $2 \tanh 2r = \coth r$, that is, $r = \log((1 + \sqrt{3})/\sqrt{2})$) or $2m - 1$ (if $2 \tanh 2r \neq \coth r$). But, if ϕ_r has constant rank $2m - 1$, then, from (5.8), (5.9) and (7.1), $\phi_r M$ would be a real hypersurface of CH^m with three constant principal curvatures 0, $(\beta \coth r - 1) / (\coth r - \beta)$, $(\mu \coth r - 1) / (\coth r - \mu)$ and this is impossible from Lemma 7.3.

Hence, if $\mu^2 < 4$, then we have $\mu = 2 \tanh 2r$ with $r = \log((1 + \sqrt{3})/\sqrt{2})$. So,

$\mu = \coth r = \sqrt{3}$, $\beta = \tanh r = 1/\sqrt{3}$ and ϕ_r has constant rank m . Using Theorem 4.2, (5.8) and (5.9), we have that M is a tube of radius r over a space \mathbf{RH}^m embedded in \mathbf{CH}^m as a totally real and totally geodesic submanifold.

This last theorem and discussions in Section 5 enable us to prove the announced characterization for the space M_m^* defined in Example 6.2.

COROLLARY 7.5. *The only complete and connected self-tube of \mathbf{CH}^m , $m \geq 3$, such that $J\xi$ is principal is the space M_m^* .*

REMARK. We will call "self-tube" a real hypersurface of \mathbf{CH}^m without focal points and such that all its parallel hypersurfaces are congruent to itself.

PROOF OF COROLLARY. We have $HJ\xi = \mu J\xi$ for some function μ where ξ is a local unit field normal to M . Since M has no focal points, from Proposition 3.2, we have $|\mu| \leq 2$ and all the remaining principal curvatures λ of H satisfy $|\lambda| \leq 1$. Moreover, $\phi_r M$ is always a real hypersurface of \mathbf{CH}^m for each $r > 0$, which has the same principal curvatures at a point $\phi_r(x)$ as M at the point x . By using the relations (5.8) and (5.9) between the principal curvatures of M and $\phi_r M$, it is easily seen that $\mu^2 = 4$ and $\lambda^2 = 1$ for another principal curvature of M different from μ .

We choose on M a unit normal field ξ such that $HJ\xi = 2J\xi$ at each point. So, M is orientable. Moreover, from Lemma 2.1, a), we have that $\lambda = 1$ is a principal curvature at each point of M . Also, taking into account Lemma 7.3, there are no points of M where -1 is a principal curvature. Then one concludes that M has two principal curvatures 2 and 1 with respective multiplicities 1 and $2m - 2$ at each point. From the last theorem, one has that $M = M_m^*$.

8. Pseudo-Einstein real hypersurfaces of \mathbf{CH}^m .

A real hypersurface M of \mathbf{CH}^m is called pseudo-Einstein when its Ricci tensor S satisfies the equation

$$(8.1) \quad S(X, Y) = ag(X, Y) + bf(X)f(Y)$$

for all X, Y tangent to M and some functions a, b (see [6]). From (2.3), if M is pseudo-Einstein, then we have

$$(8.2) \quad H^2X - \alpha HX + (a + 2m + 1)X + (b - 3)f(X)U = 0$$

for all X tangent to M , where H is the Weingarten map associated to a unit normal vector $\xi = JU$ and $\alpha = \text{tr}H$.

Now, from (8.2), it is easily seen that, at those points of M where $b \neq 3$, the operator $K = H^2 - \alpha H$ has two eigenvalues $-(a + 2m + 1)$ and $-(a + b + 2m - 2)$ and

the line $\text{span}\{J\xi\}$ is the eigenspace corresponding to the last one. Where $b=3$, K has an only eigenvalue $-(a+2m+1)$. Hence, if $x \in M$ and $\lambda_1, \lambda_2, \dots, \lambda_{2m-1}$ are the principal curvatures of M at x , we have that $\lambda_i^2 - \alpha\lambda_i$ is an eigenvalue of K at x . So

$$(8.3) \quad \lambda_i^2 - \alpha\lambda_i + (a+2m+1) = 0 \quad i=1, \dots, 2m-1 \quad \text{if } b(x)=3,$$

$$(8.4) \quad \begin{aligned} &\lambda_i^2 - \alpha\lambda_i + (a+2m+1) = 0 \quad i=2, \dots, 2m-1 \quad \text{and} \\ &\lambda_1^2 - \alpha\lambda_1 + (a+b+2m-2) = 0 \quad V_{\lambda_1} = \text{span}\{J\xi\} \quad \text{if } b(x) \neq 3. \end{aligned}$$

After these observations we can state :

THEOREM 8.1. *Let M be a complete and connected real hypersurface of CH^m , $m \geq 3$, which is pseudo-Einstein. Then M is congruent to one of the following spaces :*

- a) *A geodesic hypersphere $M_{0,m-1}^h(\tanh^2 r)$ of radius $r > 0$.*
- b) *A tube $M_{m-1,0}^h(\tanh^2 r)$ of radius $r > 0$ over a complex hyperbolic hyperplane.*
- c) *A self-tube M_m^* .*

PROOF. From (8.3) and (8.4), we know that M has at most three principal curvatures at each point. If M has at most two principal curvatures at each point, we conclude from Theorem 7.4 and the fact that the tube $M^h(4)$ defined in Example 6.3 is not pseudo-Einstein. Thus we will suppose that the set Σ consisting of the points of M where there are exactly three principal curvatures $\lambda_1, \lambda_2, \lambda_3$ is open and non-empty. From (8.3) and (8.4), we have $b \neq 3$ on Σ and, so, $V_{\lambda_1} = \text{span}\{J\xi\}$.

Now, if $x \in \Sigma$ and $\phi H + H\phi = 0$ at x , then, from Lemma 2.1, a), one has $\phi = -H\phi H$ and, hence, $\lambda_i^2 = 1, i=2, 3, \lambda_2 = -\lambda_3$ and $\phi V_{\lambda_2} = V_{\lambda_3}$ at x . So, $\alpha(x) = \text{tr} H_x = (m-1)(\lambda_2 + \lambda_3) + \lambda_1 = \lambda_1$. Moreover, from (8.4), we have $\alpha(x) = \lambda_2 + \lambda_3 = 0$ and, so, we have $\lambda_1 = 0$ at those points of M where $\phi H + H\phi = 0$. Since, from Lemma 2.1, b), λ_1 is locally constant on the open set of M where $\phi H + H\phi \neq 0$, we conclude that λ_1 is locally constant on Σ .

Let $y \in \Sigma$ and Σ_0 the component of Σ with $y \in \Sigma_0$. We know that λ_1 is constant on Σ_0 . We will suppose $\lambda_1 \geq 0$ by reversing the orientation if necessary. Let Ω denote the subset of Σ_0 consisting of the points where the principal curvature $\coth r$ appears with its least multiplicity ν , for some $r > 0$. Since λ_1 is constant on Σ_0 , from Proposition 3.2, we have $\Omega = \{x \in \Sigma_0 \mid \phi_r \text{ has maximum rank}\}$ and, so, Ω is open and non-empty. Now, we will distinguish three cases:

A) If $\lambda_1 > 2$, we take $r > 0$ with $\lambda_1 = 2\coth 2r$. Let $x \in \Omega$ and let W be as in Theorem 4.1. If $\nu = 0$, from Proposition 3.2, we have that $\phi_r W = V$ is a complex hypersurface of CH^m . Moreover, using (5.3), we have that V has at each point two principal curvatures $(\lambda_i \coth r - 1) / (\coth r - \lambda_i), i=2, 3$. Hence, V is

Einstein and the Chern result in [3] asserts that V is totally geodesic, that is, $\lambda_2 = \lambda_3 = \tanh r$. But it is impossible.

Hence, we have $0 < \nu < 2m - 2$ and we can put $\lambda_2 = \coth r$. Again, from (5.3), the $(m - 1 - \nu/2)$ -dimensional complex submanifold V has at each point one principal curvature $(\lambda_3 \coth r - 1)/(\coth r - \lambda_3)$. Since V is complex, this principal curvature vanishes and, so, $\lambda_3 = \tanh r$. Thus, there are on Ω three principal curvatures $2\coth 2r$, $\coth r$ and $\tanh r$ with multiplicities 1, ν and $2m - 2 - \nu$ respectively. As in Theorem 7.4, Ω is closed in M and, so, $\Omega = M$. By using Theorem 4.2 and (5.3), we conclude that M is the tube $M_{m-1-\nu/2, \nu/2}^h(\tanh^2 r)$, $0 < \nu/2 < m - 1$. But any tube of this tube is not pseudo-Einstein.

B) If $\lambda_1 = 2$ on Σ_0 , from Lemma 2.1, a), we have $\phi = H\phi + \phi H - H\phi H$. Now, if $x \in \Sigma_0$ and $X \in T_x M$ with $HX = \lambda_i X$, $i = 2, 3$, we get $(1 - \lambda_i)H\phi X = (1 - \lambda_i)\phi X$. So, either $\lambda_i = 1$ or ϕX is a principal vector corresponding to the principal curvature 1. In any case, we can put $\lambda_2 = 1$ and $\phi V_{\lambda_3} \subset V_{\lambda_2}$ at x .

Let p denote the multiplicity of λ_3 at x . We have $0 < p \leq m - 1$. So, $\alpha(x) = \text{tr} H_x = p\lambda_3 + 2m - p$. On the other hand, from (8.4), one has $\alpha(x) = 1 + \lambda_3$. Hence, $(p - 1)\lambda_3 = 1 + p - 2m$. Since $m \geq 3$, then $p \neq 1$ and, finally, one has $\lambda_3 = (1 + p - 2m)/(p - 1)$. So, λ_3 is locally constant and we can apply Lemma 7.3. In this way, one sees that $\lambda_1 = 2$ is also impossible.

C) Finally, if $\lambda_1 < 2$, then, by using Lemma 2.1, a), it can be easily seen that, for $x \in \Sigma_0$, $X \in T_x M$ with $HX = \lambda_i X$, $i = 2, 3$, ϕX is a principal vector corresponding to the principal curvature $\lambda'_i = (2 - \lambda_1 \lambda_i)/(\lambda_1 - 2\lambda_i)$. But $\lambda'_i = \lambda_i$ implies $\lambda_i^2 \geq 4$ which is not possible. Hence, we have

$$(8.5) \quad \lambda_3 = (2 - \lambda_1 \lambda_i)/(\lambda_1 - 2\lambda_i) \quad \text{and} \quad \phi V_{\lambda_2} = V_{\lambda_3}.$$

Now, from (8.4) and (8.5), we have $(m - 2)(\lambda_2 + \lambda_3) = -\lambda_1$. So, $\lambda_2 + \lambda_3$ is constant on Σ_0 . Again from (8.5), we get $2\lambda_2 \lambda_3 = \lambda_1(\lambda_2 + \lambda_3) - 2$ and $\lambda_2 \lambda_3$ is also constant on Σ_0 . Then, we can use Lemma 7.3 and we can put $\lambda_1 = 2 \tanh 2r$, $\lambda_2 = \coth r$ and $\lambda_3 = \tanh r$ for some $r > 0$. Moreover, $\nu = m - 1$ and $\Omega = \Sigma_0$ is closed in M . Thus, $\Omega = M$. Now, we apply Theorem 4.2, (5.8) and (5.9) and we have that M is the tube $M^h(\cosh^2 2r)$ defined in Example 6.3. But any tube of this type is pseudo-Einstein. So, the proof is concluded.

We found in Section 6 that the function b of (8.1) is the constant $2m$ for all spaces appearing in Theorem 8.1. So, it is immediate

COROLLARY 8.2. *There are no Einstein real hypersurfaces in CH^m , $m \geq 3$.*

Moreover, taking into account Corollary 5.3 of [7], we can state

COROLLARY 8.3. *A complete and connected real hypersurface of CH^m , $m \geq 3$, is pseudo-Einstein if and only if it is totally η -umbilical.*

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