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# REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

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We study real hypersurfaces $M$ of a complex projective space and show that a condition on the derivative of the Ricci Tensor of $M$ implies $M$ is locally homogeneous with two or three principal curvatures.

## 0. Introduction.

Let $P^n(\mathbb{C})$ be an $n$-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. We consider a real hypersurface $M$ of $P^n(\mathbb{C})$. Let $(\phi,\xi,\eta,g)$ be an almost contact metric structure induced from the complex structure on $P^n(\mathbb{C})$ (§1). If the Ricci transformation of $M$ satisfies

$$SX = aX + b\eta(X)\xi . \tag{0.1}$$

where $a$ and $b$ are constant, we call $M$ a pseudo-Einstein hypersurface [3]. Pseudo-Einstein real hypersurfaces in $P^n(\mathbb{C})$ are completely classified by Kon [3] (see [4]). This result shows that if the Ricci tensor of $M$ has a nice form, then $M$ is determined (see [5]). In this paper, we consider the following problem: If the derivative of the Ricci tensor of $M$ has a nice form, what can we say about $M$?

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We have the following

THEOREM 1. *Let* $M$ *be a real hypersurface of* $P^n(\mathbb{C})$ . *If the Ricci transformation* $S$ *of* $M$ *satisfies*

$$(\nabla_X S)Y = c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} , \tag{0.2}$$

*where* $c$ *is a non-zero constant, and* $A$ *denotes the shape operator* (§1). *Then* $M$ *is locally congruent to a homogeneous hypersurface with two or three distinct principal curvatures.*

We note that pseudo-Einstein hypersurfaces satisfy (0.2). Moreover,

THEOREM 2. *There are no real hypersurfaces with parallel Ricci tensor on which* $\xi$ *is principal.*

## 1. Preliminaries.

Let $M$ be a real hypersurface of $P^n(\mathbb{C})$ . In a neighbourhood of each point, we choose a unit normal vector field $N$ in $P^n(\mathbb{C})$ . The Riemannian connections $\bar{\nabla}$ in $P^n(\mathbb{C})$ and $\nabla$ in $M$ are related by the following formulas for arbitrary vector fields $X$ and $Y$ on $M$ :

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N , \tag{1.1}$$

$$\bar{\nabla}_X N = - AX , \tag{1.2}$$

where $g$ denotes the Riemannian metric on $M$ induced from the Fubini-Study metric $\bar{g}$ on $P^n(\mathbb{C})$ and $A$ is the shape operator of $M$ in $P^n(\mathbb{C})$.

An eigenvector $X$ of the shape operator $A$ is called a principal curvature vector. Also an eigenvalue $\lambda$ of $A$ is called a principal curvature.

It is known that $M$ has an almost contact metric structure induced from the complex structure $J$ on $P^n(\mathbb{C})$, (see [6]), that is, we define a tensor field $\phi$ of type (1.1), a vector field $\xi$ and a 1-form $\eta$ on $M$ by

$$g(\phi X, Y) = \bar{g}(JX, Y) \quad \text{and} \quad g(\xi, X) = \eta(X) = \bar{g}(JX, N).$$

Then we have

$$\phi^2 X = -X + \eta(X)\xi, \; g(\xi, \xi) = 1, \; \phi\xi = 0 . \tag{1.3}$$

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From (1.1), we easily have

$$(1.4) \qquad (\nabla_X \phi) Y = \eta(Y) AX - g(AX, Y)\xi ,$$

$$(1.5) \qquad \nabla_X \xi = \phi AX .$$

Let $\bar{R}$ and $R$ be the curvature tensors of $P^n(\mathbb{C})$ and $M$ respectively. Since the curvature tensor $\bar{R}$ has a nice form, we have the following Gauss and Codazzi equations.

$$(1.6) \qquad \begin{aligned} g(R(X, Y)Z, W) &= g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ &+ g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, Z) \\ &- 2g(\phi X, Y)g(\phi Z, W) + g(AY, Z)g(AX, W) \\ &- g(AX, Z)g(AY, W) \end{aligned}$$

and

$$(1.7) \qquad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi.$$

Using (1.3), (1.5), (1.6) and (1.7), we get

$$(1.8) \qquad SX = (2n+1)X - 3\eta(X)\xi + hAX - A^2X ,$$

$$(1.9) \qquad \begin{aligned} (\nabla_X S)Y = -3\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\} + (Xh)AY + (h - A)(\nabla_X A)Y \\ - (\nabla_X A)AY , \end{aligned}$$

where $h$ = trace $A$ and $S$ denotes the Ricci tensor on $M$ .

## 2. Proof of Theorems.

First, we determine the hypersurface $M$ satisfying (0.2). Using (1.9), we see that (0.2) is equivalent to

$$(2.1) \qquad \begin{aligned} (c+3)\{\eta(W)g(\phi AX, Y) + \eta(Y)g(\phi AX, W)\} - (Xh)g(AY, W) \\ + g((A - h)(\nabla_X A)Y + (\nabla_X A)AY, W) = 0 . \end{aligned}$$

Contraction with respect to $Y$ and $W$ , together with (1.3), yields

$$(2.2) \qquad -(Xh)h + \mathrm{trace}(\nabla_X A)(2A - h) = 0 .$$

It follows that $h^2$ - trace $A^2$ is constant. Next, using (1.7), we see that (2.1) becomes

(2.3) $$(c+3)\{\eta(W)g(\phi AX,\ Y) + \eta(Y)g(\phi AX,\ W)\} - (Xh)g(AY,\ W)$$
$$+ g((\nabla_Y A)X + \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,\ Y)\xi,\ (A - h)W)$$
$$+ g((\nabla_{AY} A)X + \eta(X)\phi AY - \eta(AY)\phi X - 2g(\phi X,\ AY)\xi,\ W) = 0.$$

Contraction with respect to $X$ and $W$ yields

(2.4) $$(c+3)g(\phi A\xi,\ Y) - (AY)h + \mathrm{trace}(A - h)(\nabla_Y A) + \eta((A - h)\phi Y)$$
$$+ 2g(\phi Y,\ (A - h)\xi) + \mathrm{trace}(\nabla_{AY} A) + 3\eta(\phi AY) = 0\ .$$

since $\phi$ and $\phi A$ are skew-symmetric. (1.3) and commutativity of contraction and covariant differentiation imply

(2.5) $$-cg(A\xi,\ \phi Y) + \mathrm{trace}(\nabla_Y A)A - h\ \mathrm{trace}(\nabla_Y A) = 0\ .$$

and

(2.6) $$-cg(A\xi,\ \phi Y) + \frac{1}{2}Y(\mathrm{trace}\ A^2 - h^2) = 0\ .$$

so that $g(A\xi,\ \phi Y) = 0$ . Consequently, $\xi$ is principal. Let $A\xi = \mu\xi$. Then Lemma 2.4 of [5] implies $\mu$ is locally constant. If we replace $Y$ by $\xi$ , (2.1) becomes

(2.7) $$(c+3)\phi AX - (Xh)A\xi + (A - h + \mu)(\nabla_X A)\xi = 0\ .$$

by (1.3). From (1.5), we have $(\nabla_X A)\xi = \nabla_X(A\xi) - A\nabla_X\xi = (\mu - A)\phi AX$ . Then (2.7) gives

(2.8) $$\{(A - h + \mu)(\mu - A) + (c + 3)\}\phi AX - \mu(Xh)\xi = 0\ .$$

Since $A\phi(T_x M)$ is orthogonal to $\xi$ , both first term and second term are zero, so that $\mu(Xh) = 0$ .

Let $X$ be a principal vector with principal curvature $\lambda$ , which is orthogonal to $\xi$ . Then Lemma 2.2 of [5] implies that $\phi X$ is a principal vector with principal curvature $(\lambda\mu + 2)/(2\lambda - \mu)$ , and $2\lambda - \mu \neq 0$ . Hence (2.8) gives

(2.9) $$\lambda\{ (\frac{\lambda\mu + 2}{2\lambda - \mu} - h + \mu)(\mu - \frac{\lambda\mu + 2}{2\lambda - \mu}) + c + 3\} = 0\ .$$

If $\mu = 0$ , then $\lambda \neq 0$ , $A\phi X = \lambda^{-1}\phi X$ and $(\lambda h - 1) + \lambda^2(c + 3) = 0$ . Let $A_0$ be the restriction of $A$ to the orthogonal complement

$\xi^{\perp}$ $(= \phi T_x M)$ of $\xi$. Then $A_0$ has at most two distinct eigenvalues, so that $M$ has at most three distinct principal curvalures. From the proof of Theorem 4 of [1], $M$ is a homogeneous hypersurface with 2 or 3 distinct principal curvatures.

If $\mu \neq 0$, then $h$ is constant. From (2.9), $A_0$ has at most three distinct constant eigenvalues so that $M$ has at most four distinct constant principal curvatures. Since $\xi$ is principal, Theorem 1 and Theorem 4 in [2] implies that $M$ is a homogeneous hypersurface with 2 or 3 distinct principal curvatures. Thus Theorem 1 is proved.

The same argument implies that if $M$ has parallel Ricci tensor and $\xi$ is principal, then $M$ is a homogeneous hypersurface with 2 or 3 distinct principal survatures. But there is no homogeneous hypersurface with parallel Ricci tensor. Hence Theorem 2 is proved.

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