# Real Hypersurfaces of Type $B$ in Complex Two-Plane Grassmannians 

By<br>Young Jin Suh<br>Kyungpook National University, Taegu, Korea<br>Communicated by D. V. Alekseevski

Received September 3, 2004; accepted in final form January 29, 2005
Published online January 29, 2006 © Springer-Verlag 2006


#### Abstract

In this paper we give a characterization of real hypersurfaces of type B, that is, a tube over a totally real totally geodesic $\mathbb{H} P^{n}$ in complex two-plane Grassmannians $G_{2}\left(\mathbb{C}^{m+2}\right), m=2 n$ with the shape operator $A$ satisfying $A \phi+\phi A=k \phi, k$ is non-zero constant, for the structure tensor $\phi$.


2000 Mathematics Subject Classification: 53C40; 53C15
Key words: Complex two-plane Grassmannians, real hypersurfaces of type B, tubes, shape operator, Kähler structure, quaternionic Kähler structure

## 0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_{m}(c)$ or in quaternionic space forms there have been many characterizations of model hypersurfaces of type $A_{1}, A_{2}, B, C, D$ and $E$ in complex projective space $P_{m}(\mathbb{C})$, of type $A_{0}, A_{1}, A_{2}$ and $B$ in complex hyperbolic space $H_{m}(\mathbb{C})$ or $A_{1}, A_{2}, B$ in quaternionic projective space $\mathbb{H} P^{m}$, which are completely classified by Cecil and Ryan [6], Kimura [7], Berndt [2], Martinez and Pérez [8], respectively. Among them there were only a few characterizations of homogeneous real hypersurfaces of type $B$ in complex projective space $P_{m}(\mathbb{C})$. For example, the condition that $A \phi+\phi A=k \phi$, $k$ is non-zero constant, is a model characterization of this kind of type $B$, which is a tube over a real projective space $\mathbb{R} P^{n}$ in $P_{m}(\mathbb{C}), m=2 n$ (see Yano and Kon [13]).

Let $M$ be a $(4 m-1)$-dimensional Riemannian manifold with an almost contact structure $(\phi, \xi, \eta)$ and an associated Riemannian metric $g$. We put

$$
\begin{equation*}
\omega(X, Y)=g(\phi X, Y) \tag{0.1}
\end{equation*}
$$

where $\omega$ defines a 2-form on $M$ and rank $\omega=\operatorname{rank} \phi=4 m-2$.
If there is a non-zero valued function $\rho$ such that

$$
\begin{equation*}
\rho g(\phi X, Y)=\rho \omega(X, Y)=d \eta(X, Y) \tag{0.2}
\end{equation*}
$$

[^0]the rank of the matrix $(\omega)$ being $4 m-2$, we have
$$
\eta \wedge \overbrace{\omega \wedge \cdots \wedge \omega}^{2 m-1 \text { times }}=\eta \wedge \rho^{-(2 m-1)} \overbrace{d \eta \wedge \cdots \wedge d \eta}^{2 m-1 \text { times }} \neq 0 .
$$

Let us denote by $G_{2}\left(\mathbb{C}^{m+2}\right)$ the set of all complex 2-dimensional linear subspaces of $\mathbb{C}^{m+2}$. We call such a set $G_{2}\left(\mathbb{C}^{m+2}\right)$ complex two-plane Grassmannians. This Riemannian symmetric space $G_{2}\left(\mathbb{C}^{m+2}\right)$ has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $\mathfrak{J}=\operatorname{Span}\left\{J_{1}, J_{2}, J_{3}\right\}$ not containing $J$. In other words, $G_{2}\left(\mathbb{C}^{m+2}\right)$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold (see Berndt and Suh [4], [5]).

Now we consider a $(4 m-1)$-dimensional real hypersurface $M$ in complex twoplane Grassmannians $G_{2}\left(\mathbb{C}^{m+2}\right)$. Then from the Kähler structure of $G_{2}\left(\mathbb{C}^{m+2}\right)$ there exists an almost contact structure $\phi$ on $M$. If the non-zero function $\rho$ satisfies (0.2), we call $M$ a contact hypersurface of the Kähler manifold. Moreover, it can be easily verified that a real hypersurface $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$ is contact if and only if there exists a non-zero constant function $\rho$ defined on $M$ such that

$$
\begin{equation*}
\phi A+A \phi=k \phi, \quad k=2 \rho \tag{*}
\end{equation*}
$$

The formula $\left(^{*}\right)$ means that

$$
g((\phi A+A \phi) X, Y)=2 d \eta(X, Y)
$$

where the exterior derivative $d \eta$ of the 1-form $\eta$ is defined by

$$
d \eta(X, Y)=\left(\nabla_{X} \eta\right) Y-\left(\nabla_{Y} \eta\right) X
$$

for any vector fields $X, Y$ on $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$.
On the other hand, in $G_{2}\left(\mathbb{C}^{m+2}\right)$ we are able to consider two kinds of natural geometric conditions for real hypersurfaces $M$ that $[\xi]=\operatorname{Span}\{\xi\}$ or $\mathfrak{D}^{\perp}=$ Span $\left\{\xi_{1}, \xi_{2}, \xi_{3}\right\}, \xi_{i}=-J_{i} N, i=1,2,3$, where $N$ denotes a unit normal to $M$, is invariant under the shape operator $A$ of $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$. The first result in this direction is the classification of real hypersurfaces in $G_{2}\left(\mathbb{C}^{m+2}\right)$ satisfying both two conditions. Namely, Berndt and the present author [4] have proved the following

Theorem A. Let $M$ be a connected real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right), m \geqslant 3$. Then both $[\xi]$ and $\mathfrak{D}^{\perp}$ are invariant under the shape operator of $M$ if and only if
(A) $M$ is an open part of a tube around a totally geodesic $G_{2}\left(\mathbb{C}^{m+1}\right)$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$, or
(B) $m$ is even, say $m=2 n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{H} P^{n}$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$.

In Theorem A the vector $\xi$ contained in the one-dimensional distribution $[\xi]$ is said to be a Hopf vector when it becomes a principal vector for the shape operator $A$ of $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$. Moreover in such a situation $M$ is said to be a Hopf hypersurface. Besides this, a real hypersurface $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$ also admits the 3-dimensional distribution $\mathfrak{D}^{\perp}$, which is spanned by almost contact 3 -structure vector fields $\left\{\xi_{1}, \xi_{2}, \xi_{3}\right\}$, such that $T_{x} M=\mathfrak{D} \oplus \mathfrak{D}^{\perp}$. Also in the paper [5] due to Berndt and
the present author we have given a characterization of real hypersurfaces of type $A$ when the shape operator $A$ of $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$ commutes with the structure tensor $\phi$, which is equivalent to the condition that the Reeb flow on $M$ is isometric. Moreover, in the paper due to the present author [12] we have also given a characterization of type $A$ by vanishing Lie derivative of the shape operator $A$ in the direction of the structure vector field $\xi$.

Real hypersurfaces of type $B$ in Theorem A is just the case that the one dimensional distribution $[\xi]$ is contained in $\mathfrak{D}^{\perp}$. It was shown in the paper [11] that the tube of type $B$ satisfies the following formula on the orthogonal complement of the one-dimensional distribution $[\xi]$

$$
A \phi_{\nu}-\phi_{\nu} A=0, \quad \nu=1,2,3 .
$$

From this view point, the present author [11] has given a characterization that the almost contact 3 -structure tensors $\left\{\phi_{1}, \phi_{2}, \phi_{3}\right\}$ and the shape operator $A$ of a real hypersurface $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$ commute with each other as follows:

Theorem B. Let $M$ be a Hopf real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$ satisfying $\left(^{*}\right)$ on the orthogonal complement of the one-dimensional distribution $[\xi]$. Then $M$ is locally congruent to an open part of a tube around a totally geodesic $₫ \Vdash^{m}$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$, where $m=2 n$.

Now in this paper as another characterization of real hypersurfaces of type $B$ in complex two-plane Grassmannians $G_{2}\left(\mathbb{C}^{m+2}\right)$ in terms of the contact hypersurface we want to assert the following remarkable fact:

Theorem. Let $M$ be a real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$ with constant mean curvature satisfying

$$
A \phi+\phi A=k \phi
$$

where the function $k$ is non-zero and constant. Then $M$ is congruent to an open part of a tube around a totally geodesic $\uplus P^{n}$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$, where $m=2 n$.

## 1. Riemannian Geometry of $G_{2}\left(\mathbb{C}^{m+2}\right)$

In this section we summarize basic material about $G_{2}\left(\mathbb{C}^{m+2}\right)$, for details we refer to [3], [4] and [5]. The special unitary group $G=\mathrm{SU}(m+2)$ acts transitively on $G_{2}\left(\mathbb{C}^{m+2}\right)$ with stabilizer isomorphic to $K=S(U(2) \times U(m)) \subset G$. Then $G_{2}\left(\mathbb{C}^{m+2}\right)$ can be identified with the homogeneous space $G / K$, which we equip with the unique analytic structure for which the natural action of $G$ on $G_{2}\left(\mathbb{C}^{m+2}\right)$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{f}$ the Lie algebra of $G$ and $K$, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{f}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $B$ of $\mathfrak{g}$. Then $\mathfrak{g}=\mathfrak{f} \oplus \mathfrak{m}$ is an $\operatorname{Ad}(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o=e K$ and identify $T_{o} G_{2}\left(\mathbb{C}^{m+2}\right)$ with $\mathfrak{m}$ in the usual manner. Since $B$ is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $\operatorname{Ad}(K)$-invariance of $B$ this inner product can be extended to a $G$-invariant Riemannian metric $g$ on $G_{2}\left(\mathbb{C}^{m+2}\right)$. In this way $G_{2}\left(\mathbb{C}^{m+2}\right)$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space.

For computational reasons we normalize $g$ such that the maximal sectional curvature of $\left(G_{2}\left(\mathbb{C}^{m+2}\right), g\right)$ is eight. Since $G_{2}\left(\mathbb{C}^{3}\right)$ is isometric to the two-dimensional complex projective space $\mathbb{C} P^{2}$ with constant holomorphic sectional curvature eight we will assume $m \geqslant 2$ from now on. Note that the isomorphism $\operatorname{Spin}(6) \simeq \operatorname{SU}(4)$ yields an isometry between $G_{2}\left(\mathbb{C}^{4}\right)$ and the real Grassmann manifold $G_{2}^{+}\left(\mathbb{R}^{6}\right)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^{6}$.

The Lie algebra $\mathfrak{f}$ has the direct sum decomposition $\mathfrak{f}=\mathfrak{s u}(m) \oplus \mathfrak{s u}(2) \oplus \mathfrak{R}$, where $\mathbb{R}$ is the center of $\mathfrak{f}$. Viewing $\mathfrak{f}$ as the holonomy algebra of $G_{2}\left(\mathbb{C}^{m+2}\right)$, the center $\mathfrak{R}$ induces a Kähler structure $J$ and the $\mathfrak{s u}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_{2}\left(C^{m+2}\right)$. If $J_{1}$ is any almost Hermitian structure in $\mathfrak{I}$, then $J J_{1}=J_{1} J$, and $J J_{1}$ is a symmetric endomorphism with $\left(J J_{1}\right)^{2}=I$ and $\operatorname{Tr}\left(J J_{1}\right)=0$. This fact will be used frequently throughout this paper.

A canonical local basis $J_{1}, J_{2}, J_{3}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures $J_{\nu}$ in $\mathfrak{I}$ such that $J_{\nu} J_{\nu+1}=J_{\nu+2}=-J_{\nu+1} J_{\nu}$, where the index is taken modulo 3. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $\left(G_{2}\left(\mathbb{C}^{m+2}\right), g\right)$, there exist for any canonical local basis $J_{1}, J_{2}, J_{3}$ of $\mathfrak{I}$ three local one-forms $q_{1}, q_{2}, q_{3}$ such that

$$
\begin{equation*}
\bar{\nabla}_{X} J_{\nu}=q_{\nu+2}(X) J_{\nu+1}-q_{\nu+1}(X) J_{\nu+2} \tag{1.1}
\end{equation*}
$$

for all vector fields $X$ on $G_{2}\left(\mathbb{C}^{m+2}\right)$.
Let $p \in G_{2}\left(\mathbb{C}^{m+2}\right)$ and $W$ a subspace of $T_{p} G_{2}\left(\mathbb{C}^{m+2}\right)$. We say that $W$ is a quaternionic subspace of $T_{p} G_{2}\left(\mathbb{C}^{m+2}\right)$ if $J W \subset W$ for all $J \in \mathfrak{J}_{p}$. And we say that $W$ is a totally complex subspace of $T_{p} G_{2}\left(\mathbb{C}^{m+2}\right)$ if there exists a one-dimensional subspace $\mathfrak{B}$ of $\mathfrak{J}_{p}$ such that $J W \subset W$ for all $J \in \mathfrak{B}$ and $J W \perp W$ for all $J \in \mathfrak{B}^{\perp} \subset \mathfrak{J}_{p}$. Here, the orthogonal complement of $\mathfrak{B}$ in $\mathfrak{J}_{p}$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_{2}\left(\mathbb{C}^{m+2}\right)$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_{2}\left(\mathbb{C}^{m+2}\right)$.

The Riemannian curvature tensor $\bar{R}$ of $G_{2}\left(\mathbb{C}^{m+2}\right)$ is locally given by

$$
\begin{align*}
\bar{R}(X, Y) Z= & g(Y, Z) X-g(X, Z) Y+g(J Y, Z) J X-g(J X, Z) J Y-2 g(J X, Y) J Z \\
& +\sum_{\nu=1}^{3}\left\{g\left(J_{\nu} Y, Z\right) J_{\nu} X-g\left(J_{\nu} X, Z\right) J_{\nu} Y-2 g\left(J_{\nu} X, Y\right) J_{\nu} Z\right\} \\
& +\sum_{\nu=1}^{3}\left\{g\left(J_{\nu} J Y, Z\right) J_{\nu} J X-g\left(J_{\nu} J X, Z\right) J_{\nu} J Y\right\} \tag{1.2}
\end{align*}
$$

where $J_{1}, J_{2}, J_{3}$ is any canonical local basis of $\mathfrak{I}$.

## 2. Some Fundamental Formulas for Real Hypersurfaces in $G_{2}\left(\mathbb{C}^{m+2}\right)$

In this section we derive some fundamental formulas which will be used in the proof of our main theorem. Let $M$ be a real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$, that is, a hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$ with real codimension one. The induced Riemannian
metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal field of $M$ and $A$ the shape operator of $M$ with respect to $N$.

The Kähler structure $J$ of $G_{2}\left(\mathbb{C}^{m+2}\right)$ induces on $M$ an almost contact metric structure $(\phi, \xi, \eta, g)$. Furthermore, let $J_{1}, J_{2}, J_{3}$ be a canonical local basis of $\mathfrak{I}$. Then each $J_{\nu}$ induces an almost contact metric structure $\left(\phi_{\nu}, \xi_{\nu}, \eta_{\nu}, g\right)$ on $M$. Using the above expression (1.2) for the curvature tensor $\bar{R}$, the Gauss and the Codazzi equations are respectively given by

$$
\begin{aligned}
R(X, Y) Z= & g(Y, Z) X-g(X, Z) Y+g(\phi Y, Z) \phi X-g(\phi X, Z) \phi Y-2 g(\phi X, Y) \phi Z \\
& +\sum_{\nu=1}^{3}\left\{g\left(\phi_{\nu} Y, Z\right) \phi_{\nu} X-g\left(\phi_{\nu} X, Z\right) \phi_{\nu} Y-2 g\left(\phi_{\nu} X, Y\right) \phi_{\nu} Z\right\} \\
& +\sum_{\nu=1}^{3}\left\{g\left(\phi_{\nu} \phi Y, Z\right) \phi_{\nu} \phi X-g\left(\phi_{\nu} \phi X, Z\right) \phi_{\nu} \phi Y\right\} \\
& -\sum_{\nu=1}^{3}\left\{\eta(Y) \eta_{\nu}(Z) \phi_{\nu} \phi X-\eta(X) \eta_{\nu}(Z) \phi_{\nu} \phi Y\right\} \\
& -\sum_{\nu=1}^{3}\left\{\eta(X) g\left(\phi_{\nu} \phi Y, Z\right)-\eta(Y) g\left(\phi_{\nu} \phi X, Z\right)\right\} \xi_{\nu} \\
& +g(A Y, Z) A X-g(A X, Z) A Y
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\nabla_{X} A\right) Y-\left(\nabla_{Y} A\right) X= & \eta(X) \phi Y-\eta(Y) \phi X-2 g(\phi X, Y) \xi \\
& +\sum_{\nu=1}^{3}\left\{\eta_{\nu}(X) \phi_{\nu} Y-\eta_{\nu}(Y) \phi_{\nu} X-2 g\left(\phi_{\nu} X, Y\right) \xi_{\nu}\right\} \\
& +\sum_{\nu=1}^{3}\left\{\eta_{\nu}(\phi X) \phi_{\nu} \phi Y-\eta_{\nu}(\phi Y) \phi_{\nu} \phi X\right\} \\
& +\sum_{\nu=1}^{3}\left\{\eta(X) \eta_{\nu}(\phi Y)-\eta(Y) \eta_{\nu}(\phi X)\right\} \xi_{\nu}
\end{aligned}
$$

where $R$ denotes the curvature tensor of a real hypersurface $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$.
The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$
\begin{align*}
\phi_{\nu+1} \xi_{\nu} & =-\xi_{\nu+2}, \quad \phi_{\nu} \xi_{\nu+1}=\xi_{\nu+2} \\
\phi \xi_{\nu} & =\phi_{\nu} \xi, \quad \eta_{\nu}(\phi X)=\eta\left(\phi_{\nu} X\right)  \tag{2.1}\\
\phi_{\nu} \phi_{\nu+1} X & =\phi_{\nu+2} X+\eta_{\nu+1}(X) \xi_{\nu} \\
\phi_{\nu+1} \phi_{\nu} X & =-\phi_{\nu+2} X+\eta_{\nu}(X) \xi_{\nu+1}
\end{align*}
$$

Then in this section let us give some basic formulas which will be used later.

Now let us put

$$
J X=\phi X+\eta(X) N, \quad J_{\nu} X=\phi_{\nu} X+\eta_{\nu}(X) N
$$

for any tangent vector $X$ of a real hypersurface $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$, where $N$ denotes a normal vector of $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$. Then from this and the formulas in Section 1 we have that

$$
\begin{array}{r}
\left(\nabla_{X} \phi\right) Y=\eta(Y) A X-g(A X, Y) \xi, \quad \nabla_{X} \xi=\phi A X \\
\nabla_{X} \xi_{\nu}=q_{\nu+2}(X) \xi_{\nu+1}-q_{\nu+1}(X) \xi_{\nu+2}+\phi_{\nu} A X \\
\left(\nabla_{X} \phi_{\nu}\right) Y=-q_{\nu+1}(X) \phi_{\nu+2} Y+q_{\nu+2}(X) \phi_{\nu+1} Y+\eta_{\nu}(Y) A X-g(A X, Y) \xi_{\nu} \tag{2.4}
\end{array}
$$

Summing up these formulas, we know the following

$$
\begin{align*}
\nabla_{X}\left(\phi_{\nu} \xi\right) & =\left(\nabla_{X} \phi_{\nu}\right) \xi+\phi_{\nu}\left(\nabla_{X} \xi\right) \\
& =-q_{\nu+1}(X) \phi_{\nu+2} \xi+q_{\nu+2}(X) \phi_{\nu+1} \xi+\eta_{\nu}(\xi) A X-g(A X, \xi) \xi_{\nu}+\phi_{\nu} \phi A X \tag{2.5}
\end{align*}
$$

Moreover, from $J J_{\nu}=J_{\nu} J, \nu=1,2,3$, it follows that

$$
\begin{equation*}
\phi \phi_{\nu} X=\phi_{\nu} \phi X+\eta_{\nu}(X) \xi-\eta(X) \xi_{\nu} \tag{2.6}
\end{equation*}
$$

## 3. Some Key Propositions

Before going to give the proof of our main Theorem in the introduction let us check that "What kind of model hypersurfaces given in Theorem A satisfy the formula $\left({ }^{*}\right)$." In other words, it will be an interesting problem to know whether there exist any real hypersurfaces in $G_{2}\left(\mathbb{C}^{m+2}\right)$ satisfying the condition $\left(^{*}\right)$.

In this section we will show that only real hypersurfaces of type $B$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$, that is, a tube over a quaternionic projective space $\mathbb{H}^{n}$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$ satisfies the formula $A \phi+\phi A=k \phi, m=2 n$, where the function $k$ is non-zero and constant.

Now in order to solve such a problem let us recall some Propositions given by Berndt and the present author [4] as follows:

For a tube of type $A$ in Theorem A we have the following
Proposition A. Let $M$ be a connected real hypersurface of $G_{2}\left(\mathbb{C}^{m+2}\right)$. Suppose that $A \mathfrak{D} \subset \mathfrak{D}, A \xi=\alpha \xi$, and $\xi$ is tangent to $\mathfrak{D}^{\perp}$. Let $J_{1} \in \mathfrak{J}$ be the almost Hermitian structure such that $J N=J_{1} N$. Then $M$ has three (if $r=\pi / 2 \sqrt{8}$ ) or four (otherwise) distinct constant principal curvatures

$$
\alpha=\sqrt{8} \cot (\sqrt{8} r), \quad \beta=\sqrt{2} \cot (\sqrt{2} r), \quad \lambda=-\sqrt{2} \tan (\sqrt{2} r), \quad \mu=0
$$

with some $r \in(0, \pi / 4)$. The corresponding multiplicities are

$$
m(\alpha)=1, \quad m(\beta)=2, \quad m(\lambda)=2 m-2=m(\mu)
$$

and the corresponding eigenspaces we have

$$
\begin{aligned}
& T_{\alpha}=\mathbb{R} \xi=\mathbb{R} J N=\mathbb{R} \xi_{1}, \\
& T_{\beta}=\mathbb{C}^{\perp} \xi=\mathbb{C}^{\perp} N=\mathbb{R} \xi_{2} \oplus \mathbb{R} \xi_{3}, \\
& T_{\lambda}=\left\{X \mid X \perp \mathbb{H} \xi, J X=J_{1} X\right\}, \\
& T_{\mu}=\left\{X \mid X \perp \mathbb{H} \xi, J X=-J_{1} X\right\},
\end{aligned}
$$

where $\mathbb{R} \xi, \mathbb{C} \xi$ and $\mathbb{Q} \xi$ repectively denotes real, complex and quaternionic span of the structure vector $\xi$ and $\mathbb{C}^{\perp} \xi$ denotes the orthogonal complement of $\mathbb{C} \xi$ in $\mathbb{H} \xi$.

For such kind of real hypersurfaces of type $A$ mentioned above let us check whether this type satisfies the formula $\left({ }^{*}\right)$ or not.

Now let us assume that real hypersurfaces of type $A$ satisfies the formula $\left(^{*}\right)$. In Proposition A let us put $X=\xi_{2} \in T_{\beta}, \beta=\beta_{2}=\beta_{3}$, and $\xi=\xi_{1}$. Then by the formula (2.1) we have

$$
\begin{aligned}
A \phi \xi_{2}+\phi A \xi_{2} & =A \phi_{2} \xi_{1}+\phi A \xi_{2} \\
& =-A \xi_{3}+\beta_{2} \phi \xi_{2} \\
& =-\beta_{3} \xi_{3}-\beta_{2} \xi_{3} \\
& =-2 \sqrt{2} \cot \sqrt{2} r \xi_{3}
\end{aligned}
$$

From this, together with the formula $\left(^{*}\right.$ ) we know

$$
-k \xi_{3}=k \phi \xi_{2}=-2 \sqrt{2} \cot \sqrt{2} r \xi_{3}
$$

which means $k=2 \sqrt{2} \cot \sqrt{2} r$.
On the other hand, by the paper [5] of Berndt and the present author we know that the distributions $T_{\lambda}$ and $T_{\mu}$ in Proposition A are $\phi$-invariant, that is $\phi T_{\lambda} \subset T_{\lambda}$ and $\phi T_{\mu} \subset T_{\mu}$ respectively. By virtue of this fact we know that for any $X \in T_{\lambda}$, $\lambda=-\sqrt{2} \tan \sqrt{2} r$

$$
A \phi X+\phi A X=-2 \sqrt{2} \tan \sqrt{2} r \phi X
$$

Then in this time $k=-2 \sqrt{2} \tan \sqrt{2} r$. From this, together with the above formula we get $\cot ^{2} \sqrt{2} r=-1$, which makes a contradiction. So real hypersurfaces of type $A$ can not satisfy the formula $\left(^{*}\right)$.

Moreover, for a tube of type $B$ in Theorem A we introduce the following
Proposition B. Let $M$ be a connected real hypersurface of $G_{2}\left(\mathbb{C}^{m+2}\right)$. Suppose that $A \mathfrak{D} \subset \mathfrak{D}, A \xi=\alpha \xi$, and $\xi$ is tangent to $\mathfrak{D}$. Then the quaternionic dimension $m$ of $G_{2}\left(\mathbb{C}^{m+2}\right)$ is even, say $m=2 n$, and $M$ has five distinct constant principal curvatures

$$
\alpha=-2 \tan (2 r), \quad \beta=2 \cot (2 r), \quad \gamma=0, \quad \lambda=\cot (r), \quad \mu=-\tan (r)
$$

with some $r \in(0, \pi / 4)$. The corresponding multiplicities are

$$
m(\alpha)=1, \quad m(\beta)=3=m(\gamma), \quad m(\lambda)=4 n-4=m(\mu)
$$

and the corresponding eigenspaces are

$$
T_{\alpha}=\mathbb{R} \xi, \quad T_{\beta}=\mathfrak{J} J \xi, \quad T_{\gamma}=\mathfrak{J} \xi, T_{\lambda}, T_{\mu}
$$

where

$$
T_{\lambda} \oplus T_{\mu}=(\mathbb{H C} \mathcal{C})^{\perp}, \quad \mathfrak{J} T_{\lambda}=T_{\lambda}, \quad \mathfrak{J} T_{\mu}=T_{\mu}, \quad J T_{\lambda}=T_{\mu}
$$

Of course we have proved that all of the principal curvatures and its eigenspaces of the tube of type $A$ (resp. the tube of type $B$ ) in Theorem A satisfies all of the properties in Proposition A (resp. Proposition B).

Now by using this Proposition B we show that a tube of type $B$ in Theorem A, that is, a tube over a totally geodesic $\Vdash P^{n}$ in $G_{2}\left(\mathbb{C}^{m+2}\right), m=2 n$ satisfies the formula $\left({ }^{*}\right)$ for a constant $k=2 \cot 2 r$ as follows:

For any $\xi \in T_{\alpha}, \alpha=-2 \tan 2 r$, we have

$$
A \phi \xi+\phi A \xi=0=k \phi \xi
$$

For any $\xi_{\nu} \in T_{\beta}, \beta=2 \cot 2 r$, the eigen space $T_{\gamma}=\mathfrak{J} \xi$ gives $\phi \xi_{\nu} \in T_{\gamma}$. This implies $A \phi \xi_{\nu}=0$ for any $\nu=1,2,3$. From this we have the following for $k=2 \cot 2 r$

$$
A \phi \xi_{\nu}+\phi A \xi_{\nu}=2 \cot 2 r \phi \xi_{\nu}
$$

For any $X \in T_{\lambda}, \lambda=$ cot $r$ we know that $J T_{\lambda}=T_{\mu}$ gives

$$
A \phi X+\phi A X=-\tan r \phi X+\cot r \phi X=2 \cot 2 r \phi X
$$

This means that the formula $\left(^{*}\right)$ holds for $k=2 \cot 2 r$.
Finally, for the case $\phi \xi_{\nu} \in T_{\gamma}, \nu=1,2,3$, the formula $\left({ }^{*}\right)$ also holds for $k=2 \cot 2 r$.

## 4. Some Key Lemmas and Theorems

In order to give a characterization of type $B$ among the classes of real hypersurfaces $M$ in complex two-plane Grassmannians $G_{2}\left(\mathbb{C}^{m+2}\right)$ we will prepare some lemmas and a proposition as follows:

Lemma 1. Let $M$ be a real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$ satisfying

$$
A \phi+\phi A=k \phi
$$

where the function $k$ is non-zero and constant. Then $\operatorname{Tr} A=\alpha+n k$, where $n=2 m-1$

Proof. Now suppose that $\phi A+A \phi=k \phi$. By applying $\phi$ to the left, we have

$$
\phi^{2} A+\phi A \phi=k \phi^{2}
$$

Then it follows that

$$
A X-\phi A \phi X-k X+(k-\alpha) \eta(X) \xi=0
$$

Now let us take an orthonormal basis $\left\{e_{i} \mid i=1, \ldots, 4 m-1\right\}$ for $M$ in above formula. Then we have

$$
\begin{equation*}
\operatorname{Tr} A-\operatorname{Tr} \phi A \phi-(4 m-1) k+(k-\alpha)=0 \tag{4.1}
\end{equation*}
$$

On the other hand, we know

$$
\operatorname{Tr} \phi A \phi=\operatorname{Tr} A \phi^{2}=-\operatorname{Tr} A+\alpha
$$

Because we have

$$
A \phi^{2} X=-A X+\eta(X) A \xi=-A X+\alpha \eta(X) \xi
$$

From this, together with (4.1), we have

$$
\operatorname{Tr} A=(2 m-1) k+\alpha
$$

which completes the proof of Lemma 1.
Now let us assume that the structure vector $\xi$ is principal and denote by $\mathfrak{H}$ the orthogonal complement of the real span $[\xi]$ of the structure vector $\xi$ in $T M$. Then taking an inner product of the Codazzi equation in section 2 with $\xi$ and using $A \xi=\alpha \xi$ imply

$$
\begin{align*}
-2 g & (\phi X, Y)+2 \sum_{\nu=1}^{3}\left\{\eta_{\nu}(X) \eta_{\nu}(\phi Y)-\eta_{\nu}(Y) \eta_{\nu}(\phi X)-g\left(\phi_{\nu} X, Y\right) \eta_{\nu}(\xi)\right\} \\
& =g\left(\left(\nabla_{X} A\right) Y-\left(\nabla_{Y} A\right) X, \xi\right) \\
& =g\left(\left(\nabla_{X} A\right) \xi, Y\right)-g\left(\left(\nabla_{Y} A\right) \xi, X\right) \\
& =(X \alpha) \eta(Y)-(Y \alpha) \eta(X)+\alpha g((A \phi+\phi A) X, Y)-2 g(A \phi A X, Y) \tag{4.2}
\end{align*}
$$

Putting $X=\xi$, we have

$$
\begin{equation*}
Y \alpha=(\xi \alpha) \eta(Y)-4 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \eta_{\nu}(\phi Y) \tag{4.3}
\end{equation*}
$$

for any tangent vector field $Y$ on $M$. Substituting this formula into (4.2), then we have

$$
\begin{aligned}
& -2 g(\phi X, Y)+2 \sum_{\nu=1}^{3}\left\{\eta_{\nu}(X) \eta_{\nu}(\phi Y)-\eta_{\nu}(Y) \eta_{\nu}(\phi X)-g\left(\phi_{\nu} X, Y\right) \eta_{\nu}(\xi)\right\} \\
& =4 \sum_{\nu=1}^{3}\left\{\eta(X) \eta_{\nu}(\phi Y)-\eta(Y) \eta_{\nu}(\phi X)\right\} \eta_{\nu}(\xi)+\alpha g((A \phi+\phi A) X, Y) \\
& \quad-2 g(A \phi A X, Y)
\end{aligned}
$$

From this formula we are able to assert
Lemma 2. If $A \xi=\alpha \xi$ and $X \in \mathfrak{H}$ with $A X=\lambda X$, then

$$
\begin{aligned}
0= & (2 \lambda-\alpha) A \phi X-(2+\lambda \alpha) \phi X+2 \sum_{\nu=1}^{3}\left\{2 \eta_{\nu}(\xi) \eta_{\nu}(\phi X) \xi-\eta_{\nu}(X) \phi_{\nu} \xi\right. \\
& \left.-\eta_{\nu}(\phi X) \xi_{\nu}-\eta_{\nu}(\xi) \phi_{\nu} X\right\}
\end{aligned}
$$

Lemma 3. Let $M$ be a real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$ satisfying $A \phi+\phi A=k \phi, k$ is non-zero and constant. Then $\xi$ is principal. Moreover, the principal curvature function $\alpha$ is constant provided that $\xi \in \mathfrak{D}^{\perp}$ or $\xi \in \mathfrak{D}$.

Proof. Then by applying the structure vector $\xi$ to the above assumption in the right side, we know $\phi A \xi=0$. This means $A \xi=\alpha \xi$, that is, the structure vector
$\xi$ is principal. Then we are able to use (4.2) and (4.3). The formula (4.3) means that

$$
\begin{equation*}
\operatorname{grad} \alpha=(\xi \alpha) \xi+4 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \phi \xi_{\nu} \tag{4.4}
\end{equation*}
$$

For the case where $\xi \in \mathfrak{D}^{\perp}$. We may put $\xi=\xi_{1}$. Then (4.4) implies

$$
\begin{equation*}
\operatorname{grad} \alpha=(\xi \alpha) \xi \tag{4.5}
\end{equation*}
$$

For the case where $\xi \in \mathfrak{D}$. Then naturally the formula (4.4) gives (4.5). Now differentiating (4.5), we have

$$
\nabla_{X}(\operatorname{grad} \alpha)=X(\xi \alpha) \xi+(\xi \alpha) \phi A X
$$

Then this implies

$$
\begin{aligned}
0 & =g\left(\nabla_{X}(\operatorname{grad} \alpha), Y\right)-g\left(\nabla_{Y}(\operatorname{grad} \alpha), X\right) \\
& =X(\xi \alpha) \eta(Y)-Y(\xi \alpha) \eta(X)+(\xi \alpha) g((\phi A+A \phi) X, Y)
\end{aligned}
$$

This gives

$$
k(\xi \alpha) g(\phi X, Y)=Y(\xi \alpha) \eta(X)-X(\xi \alpha) \eta(Y)
$$

From this, putting $X=\xi$, we have $Y(\xi \alpha)=\xi(\xi \alpha) \eta(Y)$. Then it follows that

$$
k(\xi \alpha) g(\phi X, Y)=0
$$

By virtue of $k \neq 0$, we have $\xi \alpha=0$. From this, together with (4.5) we have $\operatorname{grad} \alpha=0$,
which means that the principal curvature $\alpha$ is constant.
Then by using Lemmas 1 and 3 we have the following Proposition.
Proposition 4. Let $M$ be a real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$ satisfying $A \phi+\phi A=k \phi, k$ is non-zero and constant. Then we have

$$
\begin{aligned}
2 A^{2} X & -2 k A X+(\alpha k+2) X \\
& -\left[\eta(X)\left(2 \alpha^{2}-\alpha k+2\right)+4 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(X)-4 \sum_{\nu} \eta_{\nu}^{2}(\xi) \eta(X)\right] \xi \\
& -2 \sum_{\nu}\left\{\eta_{\nu}(\phi X) \phi \xi_{\nu}-\eta_{\nu}(X) \xi_{\nu}+\eta(X) \eta_{\nu}(\xi) \xi_{\nu}+\eta_{\nu}(\xi) \phi_{\nu} \phi X\right\}=0
\end{aligned}
$$

where $\sum_{\nu}$ denotes the sum from $\nu=1$ to $\nu=3$.
Proof. Now substituting (4.3) into (4.2), we have

$$
\begin{align*}
&-2 g(\phi X, Y)+2 \sum_{\nu}\left\{\eta_{\nu}(X) \eta_{\nu}(\phi Y)-\eta_{\nu}(Y) \eta_{\nu}(\phi X)-g\left(\phi_{\nu} X, Y\right) \eta_{\nu}(\xi)\right\} \\
&=-4 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(\phi X) \eta(Y)+4 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(\phi Y) \eta(X) \\
&+\alpha g((A \phi+\phi A) X, Y)-2 g(A \phi A X, Y) \tag{4.6}
\end{align*}
$$

On the other hand, from the assumption we have

$$
g(A \phi A X, Y)=k g(A \phi X, Y)-g\left(A^{2} \phi X, Y\right)
$$

Then from this together with formula (4.6) we have

$$
\begin{aligned}
& 2 A^{2} \phi X-2 k A \phi X+(\alpha k+2) \phi X-4 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(\phi X) \xi-4 \sum_{\nu} \eta_{\nu}(\xi) \eta(X) \phi \xi_{\nu} \\
& \quad=2 \sum_{\nu}\left\{-\eta_{\nu}(X) \phi \xi_{\nu}-\eta_{\nu}(\phi X) \xi_{\nu}-\eta_{\nu}(\xi) \phi_{\nu} X\right\}
\end{aligned}
$$

Replacing $X$ by $\phi X$, we have

$$
\begin{aligned}
2 A^{2} X= & 2 \eta(X) A^{2} \xi+2 k A X-2 k \eta(X) A \xi-(\alpha k+2) X \\
& +\eta(X)(\alpha k+2) \xi+4 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(X) \xi-4 \sum_{\nu} \eta_{\nu}^{2}(\xi) \eta(X) \xi \\
& +2 \sum_{\nu}\left\{\eta_{\nu}(\phi X) \phi \xi_{\nu}-\eta_{\nu}(X) \xi_{\nu}+\eta(X) \eta_{\nu}(\xi) \xi_{\nu}+\eta_{\nu}(\xi) \phi_{\nu} \phi X\right\} \\
= & {\left[\eta(X)\left(2 \alpha^{2}-\alpha k+2\right)+4 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(X)-4 \sum_{\nu} \eta_{\nu}^{2}(\xi) \eta(X)\right] \xi } \\
& -(\alpha k+2) X+2 k A X+2 \sum_{\nu}\left\{\eta_{\nu}(\phi X) \phi \xi_{\nu}\right. \\
& \left.-\eta_{\nu}(X) \xi_{\nu}+\eta(X) \eta_{\nu}(\xi) \xi_{\nu}+\eta_{\nu}(\xi) \phi_{\nu} \phi X\right\}
\end{aligned}
$$

Now we are going to prove a key Lemma which will be useful in the proof of our Main Theorem.

Lemma 5. Under the same assumption as in Proposition 4 we have

$$
X(k-\operatorname{Tr} A)=\eta(X) \xi(k-\operatorname{Tr} A)+4 \sum_{\nu} \eta_{\nu}(\phi X) \eta_{\nu}(\xi) .
$$

Proof. Differentiating $(\phi A+A \phi) X=k \phi X$ covariantly, we have

$$
\left(\nabla_{Y} \phi\right) A X+\phi\left(\nabla_{Y} A\right) X+\left(\nabla_{Y} A\right) \phi X+A\left(\nabla_{Y} \phi\right) X=(Y k) \phi X+k\left(\nabla_{Y} \phi\right) X .
$$

Then substituting the formula (2.2) into the above equation, we have

$$
\begin{aligned}
\eta(X)\{ & \left.A^{2} Y+\alpha A Y-k A Y\right\}-g\left(A^{2} X+\alpha A X-k A X, Y\right) \xi+\phi\left(\nabla_{Y} A\right) X \\
& +\left(\nabla_{Y} A\right) \phi X=(Y k) \phi X
\end{aligned}
$$

From this, using Proposition 4, we have

$$
\begin{array}{r}
\eta(X)\left[\alpha A Y-\frac{\alpha k+2}{2} Y+\left\{\eta(Y)\left(\alpha^{2}-\frac{\alpha}{2} k+1\right)\right.\right. \\
\left.+2 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(Y)-2 \sum_{\nu} \eta_{\nu}^{2}(\xi) \eta(Y)\right\} \xi
\end{array}
$$

$$
\begin{aligned}
& \left.+\sum_{\nu}\left\{\eta_{\nu}(\phi Y) \phi \xi_{\nu}-\eta_{\nu}(Y) \xi_{\nu}+\eta(Y) \eta_{\nu}(\xi) \xi_{\nu}+\eta_{\nu}(Y) \phi_{\nu} \phi Y\right\}\right] \\
& -g\left(\alpha A X-\frac{\alpha k+2}{2} X, Y\right) \xi \\
& +g\left(\left\{\eta(X)\left(\alpha^{2}-\frac{\alpha}{2} k+1\right)+2 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(X)-2 \sum_{\nu} \eta_{\nu}^{2}(\xi) \eta(X)\right\} \xi, Y\right) \xi \\
& +\sum_{\nu} g\left(\left\{\eta_{\nu}(\phi X) \phi \xi_{\nu}-\eta_{\nu}(X) \xi_{\nu}+\eta(X) \eta_{\nu}(\xi) \xi_{\nu}+\eta_{\nu}(\xi) \phi_{\nu} \phi X\right\}, Y\right) \xi \\
& +\phi\left(\nabla_{Y} A\right) X+\left(\nabla_{Y} A\right) \phi X \\
& =(Y k) \phi X
\end{aligned}
$$

From this, contracting, we have

$$
\begin{aligned}
\sum_{i}\left(E_{i} k\right) \phi E_{i}= & \alpha A \xi-\frac{\alpha k+2}{2} \xi-\alpha \sum_{i} g\left(A E_{i}, E_{i}\right) \xi \\
& +\frac{\alpha k+2}{2} \sum_{i} g\left(E_{i}, E_{i}\right) \xi-\sum_{i, \nu} \eta_{\nu}\left(\phi E_{i}\right) g\left(\phi \xi_{\nu}, E_{i}\right) \xi \\
& +\sum_{i, \nu} \eta_{\nu}\left(E_{i}\right) \eta_{\nu}\left(E_{i}\right) \xi-\sum_{i, \nu} \eta\left(E_{i}\right) \eta_{\nu}(\xi) \eta_{\nu}\left(E_{i}\right) \xi \\
& -\sum_{i, \nu} \eta_{\nu}(\xi) g\left(\phi_{\nu} \phi E_{i}, E_{i}\right) \xi+\phi\left(\nabla_{E_{i}} A\right) E_{i}+\left(\nabla_{E_{i}} A\right) \phi E_{i}
\end{aligned}
$$

where $\sum_{i}\left(\right.$ resp. $\left.\sum_{\nu}\right)$ denotes the sum from $i=1$ to $i=4 m-1$ (resp. from $\nu=1$ to $\nu=3$ ). Then by virtue of formulas defined in (2.1) we have

$$
\begin{align*}
\sum_{i}\left(E_{i} k\right) \phi E_{i}= & \alpha A \xi-\frac{\alpha k+2}{2} \xi-\alpha(\operatorname{Tr} A) \xi+\frac{\alpha k+2}{2}(4 m-1) \xi+6 \xi-2 \sum_{\nu} \eta_{\nu}^{2}(\xi) \xi \\
& -\sum_{\nu} \eta_{\nu}(\xi)\left(\operatorname{Tr} \phi_{\nu} \phi\right) \xi+\sum_{i} \phi\left(\nabla_{E_{i}} A\right) E_{i}+\sum_{i}\left(\nabla_{E_{i}} A\right) \phi E_{i} \tag{4.7}
\end{align*}
$$

On the other hand, the first term in the fourth line of (4.7) becomes

$$
\sum_{i} g\left(\phi\left(\nabla_{E_{i}} A\right) E_{i}, X\right)=-\sum_{i} g\left(\left(\nabla_{E_{i}} A\right) E_{i}, \phi X\right)=-\sum_{i} g\left(E_{i},\left(\nabla_{E_{i}} A\right) \phi X\right)
$$

Also by virtue of the Codazzi equation in section 2 the last term of the above equation can be changed into

$$
\begin{align*}
\sum_{i} g & \left(\left(\nabla_{E_{i}} A\right) \phi X-\left(\nabla_{\phi X} A\right) E_{i}, E_{i}\right) \\
= & \sum_{\nu} \eta(X) \eta_{\nu}^{2}(\xi)-\sum_{\nu} \eta_{\nu}(X) \eta_{\nu}(\xi)+\sum_{\nu} \eta_{\nu}(X) \operatorname{Tr} \phi_{\nu} \phi \\
& \quad-\eta(X) \sum_{\nu} \eta_{\nu}(\xi) \operatorname{Tr} \phi_{\nu} \phi-\sum_{\nu} \eta_{\nu}(X) \eta_{\nu}(\xi)+\eta(X) \sum_{\nu} \eta_{\nu}^{2}(\xi) . \tag{4.8}
\end{align*}
$$

Also let us use the Codazzi equation in the final term of the fourth line of (4.7). Then it follows that

$$
\begin{align*}
\sum_{i} g\left(\phi E_{i},\left(\nabla_{E_{i}} A\right) X\right)= & \sum_{i} g\left(\phi E_{i},\left(\nabla_{X} A\right) E_{i}\right)-\eta(X) \sum_{i} g\left(\phi E_{i}, \phi E_{i}\right) \\
& +\sum_{\nu, i} \eta_{\nu}\left(\phi E_{i}\right) g\left(\phi_{\nu} \phi X, \phi E_{i}\right)+\sum_{\nu, i}\left\{\eta_{\nu}\left(E_{i}\right) g\left(\phi_{\nu} X, \phi E_{i}\right)\right. \\
& \left.-\eta_{\nu}(X) g\left(\phi_{\nu} E_{i}, \phi E_{i}\right)-2 g\left(\phi_{\nu} E_{i}, X\right) \eta_{\nu}\left(\phi E_{i}\right)\right\} \\
& +\sum_{\nu, i}\left\{\eta\left(E_{i}\right) \eta_{\nu}(\phi X)-\eta(X) \eta_{\nu}\left(\phi E_{i}\right)\right\} \eta_{\nu}\left(\phi E_{i}\right) \tag{4.9}
\end{align*}
$$

Now from the third term in the right side of (4.9) let us calculate term by term as follows:

$$
\begin{gathered}
\sum_{\nu} \eta_{\nu}\left(\phi E_{i}\right) g\left(\phi_{\nu} \phi X, \phi E_{i}\right)=-\sum_{\nu} \eta_{\nu}\left(\phi^{2} \phi_{\nu} \phi X\right)=\sum_{\nu} \eta_{\nu}(X) \eta_{\nu}(\xi)-\sum_{\nu} \eta(X) \eta_{\nu}^{2}(\xi) \\
\sum_{\nu} \eta_{\nu}\left(E_{i}\right) g\left(\phi_{\nu} X, \phi E_{i}\right)=-\sum_{\nu} \eta_{\nu}\left(\phi \phi_{\nu} X\right)=3 \eta(X)-\sum_{\nu} \eta_{\nu}(X) \eta\left(\xi_{\nu}\right) \\
-\sum_{i, \nu} \eta_{\nu}(X) g\left(\phi_{\nu} E_{i}, \phi E_{i}\right)=\sum_{\nu} \eta_{\nu}(X) \operatorname{Tr} \phi \phi_{\nu} \\
-2 \sum_{i, \nu} g\left(\phi_{\nu} E_{i}, X\right) \eta_{\nu}\left(\phi E_{i}\right)=-6 \eta(X)+2 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(X)
\end{gathered}
$$

and

$$
-\sum_{\nu} \eta(X) \eta_{\nu}\left(\phi E_{i}\right) \eta_{\nu}\left(\phi E_{i}\right)=-3 \eta(X)+\sum_{\nu} \eta(X) \eta\left(\xi_{\nu}\right)^{2}
$$

Substituting all of these formulas into (4.9), we have the following

$$
\begin{align*}
\sum_{i} g\left(\left(\nabla_{E_{i}} A\right) \phi E_{i}, X\right)= & \sum_{i} g\left(\phi E_{i},\left(\nabla_{X} A\right) E_{i}\right)-(4 m-1) \eta(X) \\
& +3 \eta(X)-\sum_{\nu} \eta_{\nu}(X) \eta\left(\xi_{\nu}\right)+\sum_{\nu} \eta_{\nu}(X) \operatorname{Tr} \phi \phi_{\nu}-6 \eta(X) \\
& +2 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(X)-3 \eta(X)+\sum_{\nu} \eta(X) \eta\left(\xi_{\nu}\right)^{2} \\
& +\sum_{\nu} \eta_{\nu}(X) \eta_{\nu}(\xi)-\sum_{\nu} \eta(X) \eta_{\nu}(\xi)^{2} \tag{4.10}
\end{align*}
$$

Now substituting (4.8) and (4.10) into (4.7), then by Lemma 1 and Lemma 3 we have

$$
\begin{aligned}
\sum_{i}\left(E_{i} k\right) g\left(\phi E_{i}, X\right)= & (4 m+4) \eta(X)-2 \sum_{\nu} \eta_{\nu}^{2}(\xi) \eta(X)-\sum_{\nu} \eta_{\nu}(\xi) \operatorname{Tr}\left(\phi_{\nu} \phi\right) \eta(X) \\
& -\sum_{i} g\left(E_{i},\left(\nabla_{\phi X} A\right) E_{i}\right)-\sum_{\nu} \eta(X) \eta_{\nu}^{2}(\xi)+\sum_{\nu} \eta_{\nu}(X) \eta_{\nu}(\xi)
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{\nu} \eta_{\nu}(X) \operatorname{Tr} \phi_{\nu} \phi+\eta(X) \sum_{\nu} \eta_{\nu}(\xi) \operatorname{Tr} \phi_{\nu} \phi+\sum_{\nu} \eta_{\nu}(X) \eta_{\nu}(\xi) \\
& -\eta(X) \sum_{\nu} \eta_{\nu}(\xi)^{2}+\sum_{i} g\left(\phi E_{i},\left(\nabla_{X} A\right) E_{i}\right) \\
& -(4 m-1) \eta(X)+3 \eta(X)-\sum_{\nu} \eta_{\nu}(X) \eta\left(\xi_{\nu}\right) \\
& +\sum_{\nu} \eta_{\nu}(X) \operatorname{Tr} \phi \phi_{\nu}-6 \eta(X)+2 \sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(X) \\
& -3 \eta(X)+\sum_{\nu} \eta(X) \eta\left(\xi_{\nu}\right)^{2}+\sum_{\nu} \eta_{\nu}(X) \eta_{\nu}(\xi)-\sum_{\nu} \eta(X) \eta_{\nu}(\xi)^{2} \\
= & -\eta(X)-4 \sum_{\nu} \eta_{\nu}^{2}(\xi) \eta(X)+4 \sum_{\nu} \eta_{\nu}(X) \eta_{\nu}(\xi)-\operatorname{Tr}\left(\nabla_{\phi X} A\right),
\end{aligned}
$$

where we have used that the structure vector $\xi$ is principal and $\operatorname{Tr}\left(\nabla_{X} A\right) \phi=0$. Then it can be written as follows:

$$
\phi X(k)=\phi X(\operatorname{Tr} A)+\eta(X)+4 \sum_{\nu} \eta_{\nu}^{2}(\xi) \eta(X)-4 \sum_{\nu} \eta_{\nu}(X) \eta_{\nu}(\xi)
$$

From this, replacing $X$ by $\phi X$, we have

$$
\phi^{2} X(k-\operatorname{Tr} A)=-4 \sum_{\nu} \eta_{\nu}(\phi X) \eta_{\nu}(\xi)
$$

Finally, we have arrived at the following formula

$$
X(k-\operatorname{Tr} A)=\eta(X) \xi(k-\operatorname{Tr} A)+4 \sum_{\nu} \eta_{\nu}(\phi X) \eta_{\nu}(\xi)
$$

From this we complete the proof of Lemma 5.
By Lemma 1 we know that the mean curvature is constant if and only if the function $\alpha$ is constant. By the result in Lemma 5 we know that if the function $k-\operatorname{Tr} A$ is constant, then

$$
\begin{equation*}
\sum_{\nu} \eta_{\nu}(\phi X) \eta_{\nu}(\xi)=0 \tag{4.11}
\end{equation*}
$$

for any $X \in T_{x} M$. Then the formula (4.11) is equal to

$$
\begin{equation*}
\sum_{\nu} \eta_{\nu}(\xi) \phi \xi_{\nu}=0 \tag{4.12}
\end{equation*}
$$

On the other hand, the formula $\sum_{\nu} \eta_{\nu}(\xi) \phi^{2} \xi_{\nu}=0$ is equivalent to

$$
\sum_{\nu} \eta_{\nu}(\xi) \phi \xi_{\nu}=0
$$

because $\sum_{\nu} \eta_{\nu}(\xi) \phi \xi_{\nu}$ is orthogonal to the structure vector field $\xi$. From this, (4.12) is equivalent to

$$
\eta(Y) \sum_{\nu} \eta_{\nu}^{2}(\xi)=\sum_{\nu} \eta_{\nu}(\xi) \eta_{\nu}(Y)=0
$$

for any $Y \in \mathfrak{D}$. By virtue of this formula (4.11) is also equivalent to

$$
\begin{equation*}
\xi \in \mathfrak{D} \quad \text { or } \quad \xi \in \mathfrak{D}^{\perp} . \tag{4.13}
\end{equation*}
$$

Accordingly, by Lemma 5 we know that the constancy of the function $\alpha$ implies the formula (4.13). Moreover, conversely, by Lemma 3 we are able to see that (4.13) implies that the function $\alpha$ is constant. Now we summarize this content as follows:

Theorem 4.1. Let $M$ be a real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$ satisfying the formula

$$
A \phi+\phi A=k \phi
$$

where the function $k$ is non-zero and constant. Then the following are equivalent to each other
(1) the mean curvature is constant,
(2) the function $\alpha$ is constant,
(3) $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^{\perp}$.

By virtue of this theorem we also assert the following
Theorem 4.2. Let $M$ be a real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$ with constant mean curvature satisfying the formula

$$
A \phi+\phi A=k \phi
$$

where the function $k$ is non-zero and constant. Then we have the following
(1) The structure vector field $\xi$ is principal,
(2) The function $\alpha$ is constant,
(3) $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^{\perp}$.

## 5. Proof of the Main Theorem

Let $M$ be a real hypersurface in a two-plane complex Grassmannians $G_{2}\left(\mathbb{C}^{m+2}\right)$ with constant mean curvature. Now let us denote by $\mathfrak{H}$ the orthogonal component of the structure vector $\xi$ in the tangent space of $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$. Then by Theorem 4.2 let us consider the following two cases:

Now we consider the first case $\xi \in \mathfrak{D}^{\perp}$. In this case we may put $\xi=\xi_{1}$. Then by Proposition 4 we have for any $X \in \mathfrak{H}=[\xi]^{\perp}$

$$
\begin{equation*}
2 A^{2} X-2 k A X+(\alpha k+2) X-2 \sum_{\nu}\left\{\eta_{\nu}(\phi X) \phi \xi_{\nu}-\eta_{\nu}(X) \xi_{\nu}+\eta_{\nu}(\xi) \phi_{\nu} \phi X\right\}=0 \tag{5.1}
\end{equation*}
$$

From this formula we are able to assert the following
Proposition 5.1. Let $M$ be a real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$ satisfying the formula $\left({ }^{*}\right)$ with constant mean curvature. Then the principal curvature $\alpha$ is constant and for all $X \in \mathfrak{H}$ with $A X=\lambda X$ one of the following two statements holds:
(1) $2 \lambda^{2}-2 k \lambda+\alpha k=0$ and $\phi \mathfrak{D} X=-\phi_{1} \mathfrak{D} X$,
(2) $2 \lambda^{2}-2 k \lambda+(\alpha k+4)=0$ and $\phi_{1} \mathfrak{D} X=\phi \mathfrak{D} X$.

Proof. In order to prove this Proposition we use the formulas in (2.1) to the formula (5.1). Then for any principal vector $X \in \mathfrak{H}$ such that $A X=\lambda X$ the equation (5.1) can be given by

$$
\begin{equation*}
\left\{2 \lambda^{2}-2 k \lambda+(\alpha k+2)\right\} X+4\left\{\eta_{2}(X) \xi_{2}+\eta_{3}(X) \xi_{3}\right\}-2 \phi_{1} \phi X=0 \tag{5.2}
\end{equation*}
$$

Now we decompose the vector $X \in \mathfrak{Y}$ as follows:

$$
X=\mathfrak{D} X+\eta_{2}(X) \xi_{2}+\eta_{3}(X) \xi_{3}
$$

where $\mathfrak{D} X$ denotes the $\mathfrak{D}$ component of the vector $X \in \mathfrak{H}$. Then by the formula (2.1) again we have

$$
\phi_{1} \phi X=\phi_{1} \phi \mathfrak{D} X+\eta_{2}(X) \xi_{2}+\eta_{3}(X) \xi_{3}
$$

From this, together with (5.2) it follows that

$$
\begin{aligned}
\left\{2 \lambda^{2}\right. & -2 k \lambda+\alpha k+2\} \mathfrak{D} X+\left\{2 \lambda^{2}-2 k \lambda+\alpha k+4\right\} \eta_{2}(X) \xi_{2} \\
& +\left\{2 \lambda^{2}-2 k \lambda+\alpha k+4\right\} \eta_{3}(X) \xi_{2}-2 \phi_{1} \phi \mathfrak{D} X=0
\end{aligned}
$$

From this, together with the fact that $\phi_{1} \phi \mathfrak{D} \subset \mathfrak{D}$ we have the following

$$
\begin{aligned}
\left\{2 \lambda^{2}-2 k \lambda+(\alpha k+2)\right\} \mathfrak{D} X-2 \phi_{1} \phi \mathfrak{D} X & =0, \\
\left\{2 \lambda^{2}-2 k \lambda+(\alpha k+4)\right\} \eta_{2}(X) \xi_{2} & =0, \\
\left\{2 \lambda^{2}-2 k \lambda+(\alpha k+4)\right\} \eta_{3}(X) \xi_{3} & =0 .
\end{aligned}
$$

If $2 \lambda^{2}-2 k \lambda+(\alpha k+4)=0$, from the first equation we know $\mathfrak{D} X=-\phi_{1} \phi \mathfrak{D} X$, that is, $\phi_{1} \mathfrak{D} X=\phi \mathfrak{D} X$. Thus we assert the formula (2) in our Proposition.

Now when we consider $2 \lambda^{2}-2 k \lambda+(\alpha k+4) \neq 0$, then $\eta_{2}(X)=\eta_{3}(X)=0$. This means $X \in \mathfrak{D}$. Then by the first equation we know that $\phi_{1} \phi \mathfrak{D} X$ and $\mathfrak{D} X$ are proportional. From this we have

$$
\phi_{1} \phi \mathfrak{D} X= \pm \mathfrak{D} X .
$$

If $\phi_{1} \phi \mathfrak{D} X=-\mathfrak{D} X$, then $\left(2 \lambda^{2}-2 k \lambda+\alpha k+4\right) \mathfrak{D} X=0$, which makes a contradiction. So $\phi_{1} \phi \mathfrak{D} X=\mathfrak{D} X$, that is $\phi \mathfrak{D} X=-\phi_{1} \mathfrak{D} X$. Then we have our assertion (1). From this we complete the proof of our Proposition.

In this section we have assumed that the mean curvature of $M$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$ is constant. Then by Theorem 4.2 we know that the function $\alpha$ is constant. Accordingly, all principal curvatures satisfying the formulas (1) and (2) in Proposition 5.1 are constant. Also by virtue of these two formulas the number of principal curvatures in the subspace $\mathfrak{G}$ is at most four. Since the function $k$ is given by $2 \rho$ as in the introduction, the formulas in Proposition 5.1 can be written by

$$
\begin{equation*}
\lambda^{2}-k \lambda+\rho \alpha=0 \tag{5.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda^{2}-k \lambda+\rho \alpha+2=0 \tag{5.4}
\end{equation*}
$$

In (5.3) the function $k=2 \rho$ is given by the sum of two roots of the quadratic equation. Then it follows that two roots are equal to each other, that is $\rho=\alpha$. By
virtue of this fact we also know that there cannot exist any roots satisfying the formula (5.4). So we are able to assert that $\mathfrak{H}=T_{\alpha}$, where $T_{\alpha}=$ $\{X \in \mathfrak{G} \mid A X=\alpha X\}$.

Since we know that the structure vector $\xi$ is principal with principal curvature $\alpha$, we assert that $M$ is locally congruent to a totally umbilic hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$. But in a paper [10] due to the present author it is proved that there does not exist such a real hypersurface in $G_{2}\left(\mathbb{C}^{m+2}\right)$. So we conclude here that the first case $\xi \in \mathfrak{D}^{\perp}$ cannot appear.

Now let us consider the second case $\xi \in \mathfrak{D}$. Then by Lemma 2 for any $X \in \mathfrak{H}$ and $A \xi=\alpha \xi$, we have

$$
\begin{equation*}
0=(2 \lambda-\alpha) A \phi X-(2+\lambda \alpha) \phi X-2 \sum_{\nu}\left\{\eta_{\nu}(X) \phi_{\nu} \xi+\eta_{\nu}(\phi X) \xi_{\nu}\right\} \tag{5.5}
\end{equation*}
$$

where $\mathfrak{G}=\left[\xi_{1}, \xi_{2}, \xi_{3}, \phi \xi_{1}, \phi \xi_{2}, \phi \xi_{3}\right] \oplus \mathscr{G}$ and $\mathscr{G}$ is the orthogonal complement of the subspace $\left[\xi_{1}, \ldots, \ldots, \phi \xi_{3}\right]$ in $\mathfrak{H}$. Then any vector $X \in \mathfrak{G}$ can be expressed by

$$
X=\mathscr{G} X+\sum_{\nu} \eta_{\nu}(X) \xi_{\nu}-\sum_{\nu} \eta_{\nu}(\phi X) \phi \xi_{\nu}
$$

where $\mathscr{G} X$ denotes the $\mathscr{G}$-component of the vector $X \in \mathfrak{H}$. If $A X=\lambda X$, then by the assumption $(A \phi+\phi A) X=k \phi X$ we know that $A \phi X=(k-\lambda) \phi X$. From this, together with (5.5) we have

$$
\begin{equation*}
0=\{(2 \lambda-\alpha)(k-\lambda)-(2+\lambda \alpha)\} \phi X-2 \sum_{\nu}\left\{\eta_{\nu}(X) \phi_{\nu} \xi+\eta_{\nu}(\phi X) \xi_{\nu}\right\} \tag{5.6}
\end{equation*}
$$

From this, multiplying $\phi$ and using $\phi_{\nu} \xi=\phi \xi_{\nu}, \nu=1,2,3$, we have

$$
\begin{aligned}
\left\{2 \lambda^{2}\right. & -2 k \lambda+(\alpha k+2)\} \mathscr{G} X+\left\{2 \lambda^{2}-2 k \lambda+(\alpha k+2)+2\right\} \sum_{\nu} \eta_{\nu}(X) \xi_{\nu} \\
& -\left\{2 \lambda^{2}-2 k \lambda+(\alpha k+2)+2\right\} \sum_{\nu} \eta_{\nu}(\phi X) \phi \xi_{\nu}=0
\end{aligned}
$$

where we have used the above decomposition for the expression of $X \in \mathfrak{H}$. Accordingly, we are able to assert the following:

$$
\begin{align*}
&\left\{2 \lambda^{2}-2 k \lambda+(\alpha k+2)\right\} \mathscr{G} X=0, \\
&\left\{2 \lambda^{2}-2 k \lambda+(\alpha k+4)\right\} \eta_{\nu}(X) \xi_{\nu}=0, \quad \nu=1,2,3  \tag{5.7}\\
&\left\{2 \lambda^{2}-2 k \lambda+(\alpha k+4)\right\} \eta_{\nu}(\phi X) \phi \xi_{\nu}=0, \quad \nu=1,2,3 .
\end{align*}
$$

From these equations we know that if $2 \lambda^{2}-2 k \lambda+(\alpha k+4) \neq 0$, then the vector $X$ is orthogonal to $\xi_{\nu}$ and $\phi \xi_{\nu}$ for any $\nu=1,2,3$. Then naturally $X=\mathscr{G} X$. From this fact we know that all of principal curvatures corresponding to eigenspaces in the space $\mathfrak{H}$ satisfy one of the following equations:

$$
2 \lambda^{2}-2 k \lambda+(\alpha k+2)=0 \quad \text { or } \quad 2 \lambda^{2}-2 k \lambda+(\alpha k+4)=0
$$

On the other hand, by Theorem 4.2 the functions $\alpha$ and $k$ are known to be constant. From this together with the above equation all of the principal curvatures are constant.

Now without loss of generality we may put $\alpha=-2 \tan 2 r$ and $\lambda=\cot r$ for a real number $r$ with $0<r<\frac{\pi}{4}$. Then by (5.6) (also by Lemma 2), we know for any $X \in \mathscr{G}$ with $A X=\lambda X$ that

$$
A \phi X=\mu \phi X, \quad \mu=\frac{\alpha \lambda+2}{2 \lambda-\alpha}
$$

Then the function $\mu=-\tan r$. So it follows that

$$
k=\lambda+\mu=\cot r-\tan r=2 \cot 2 r
$$

which implies $\alpha k=-4$. Then its principal curvatures in $\mathfrak{H}$ satisfy

$$
\lambda^{2}-k \lambda-1=0 \quad \text { or } \quad \lambda^{2}-k \lambda=0
$$

Then, including principal curvature $\alpha$ the real hypersurface has at most five distinct constant principal curvatures. Then by the above formulas and the quadratic equations the other possible principal curvatures are

$$
\beta=2 \cot 2 r, \quad \gamma=0, \quad \lambda=\cot r, \quad \mu=-\tan r .
$$

Note that the principal curvature $\lambda$ and $\mu$ are two different roots of the equation

$$
2 x^{2}-2 k x+(\alpha k+2)=0
$$

where $k=2 \cot 2 r$.
A basic role in the geometry of Riemannian symmetric space is played by the so-called maximal flats. In the case of $G_{2}\left(\mathbb{C}^{m+2}\right)$, a maximal flat is a twodimensional totally geodesic submanifold isometric to some flat two-dimensional torus. A non-zero tangent vector $X$ of $G_{2}\left(\mathbb{C}^{m+2}\right)$ is said to be singular if $X$ is tangent to more than one maximal flat of $G_{2}\left(\mathbb{C}^{m+2}\right)$. In $G_{2}\left(\mathbb{C}^{m+2}\right)$ there are two types of singular tangent vectors $X$ which are characterized by the properties $J X \perp \mathfrak{I} X$ and $J X \in \mathfrak{J} X$. We will have to compute explicitly Jacobi vector fields along geodesics whose tangent vectors are all singular. For this we need the eigenvalues and eigenspaces of the Jacobi operator $\bar{R}_{X}:=\bar{R}(., X) X$, where $\bar{R}$ denotes the curvature tensor of $G_{2}\left(\mathbb{C}^{m+2}\right)$ mentioned in Section 1. If $J N \perp \mathfrak{J} N$ then the eigenvalues and eigenspaces of $\bar{R}_{N}$ are given by (see Berndt and Suh [4])

$$
\begin{array}{rlrl}
0 & \mathbb{R} N \oplus \mathfrak{J} J N & =N \oplus\left[\phi_{1} \xi, \phi_{2} \xi, \phi_{3} \xi\right] \\
1 & (\mathbb{H C} N)^{\perp} & =\left[N, \xi, \xi_{1}, \xi_{2}, \xi_{3}, \phi_{1} \xi, \phi_{2} \xi, \phi_{3} \xi\right]^{\perp}  \tag{5.8}\\
4 & \mathbb{R} J N \oplus \mathfrak{I} N & =\mathbb{R} \xi \oplus\left[\xi_{1}, \xi_{2}, \xi_{3}\right]
\end{array}
$$

where $\mathbb{H C} N=\mathbb{R} N \oplus \mathbb{R} J N \oplus \mathfrak{I} N \oplus \mathfrak{J} J N$ and $[\ldots, \ldots, \ldots]$ denotes the linear real span of the given vectors.

For $p \in M$ denote by $c_{p}$ the geodesic in $G_{2}\left(\mathbb{C}^{m+2}\right)$ with $c_{p}(0)=p$ and $\dot{c}_{p}(0)=N_{p}$, and by $F$ the smooth map

$$
F: M \rightarrow G_{2}\left(\mathbb{C}^{m+2}\right), \quad p \mapsto c_{p}(r)
$$

Geometrically, $F$ is the displacement of $M$ at distance $r$ in direction of the normal field $N$. For each $p \in M$ the differential $d_{p} F$ of $F$ at $p$ can be computed by means of Jacobi vector fields by

$$
d_{p} F(X)=Z_{X}(r)
$$

Here, $Z$ is the Jacobi vector field along $c_{p}$ with initial values $Z_{X}(0)=X$ and $Z_{X}^{\prime}(0)=-A X$. Using the explicit descriptions (5.8) of the Jacobi operator $\bar{R}_{N}$ for the case $J N \perp \mathfrak{J} N$ mentioned above and of the shape operator $A$ of $M$ we get

$$
Z_{X}(r)= \begin{cases}\left\{\cos (2 r)-\frac{\rho}{2} \sin (2 r)\right\} E_{X}(r) & \text { if } \rho \in\{\alpha, \beta\} \\ \{\cos (r)-\rho \sin (r)\} E_{X}(r) & \text { if } \rho \in\{\lambda, \mu\} \\ \{1-\rho\} E_{X}(r) & \text { if } \rho \in\{\gamma\}\end{cases}
$$

where $E_{X}$ denotes the parallel vector field along $c_{p}$ with $E_{X}(0)=X$. This shows that the kernel of $d F$ is $T_{\beta} \oplus T_{\lambda}=\Im N \oplus T_{\lambda}$ and that $F$ is of constant rank $\operatorname{dim}\left(T_{\alpha} \oplus T_{\gamma} \oplus T_{\mu}\right)=4 n$. So, locally, $F$ is a submersion onto a $4 n$-dimensional submanifold $B$ of $G_{2}\left(\mathbb{C}^{m+2}\right)$. Moreover, the tangent space of $B$ at $F(p)$ is obtained by parallel translation of $\left(T_{\alpha} \oplus T_{\gamma} \oplus T_{\mu}\right)(p)=\left(\mathbb{H} \xi \oplus T_{\mu}\right)(p)$, which is a quaternionic and real subspace of $T_{p} G_{2}\left(\mathbb{C}^{m+2}\right)$.

Since both $J$ and $\mathfrak{J}$ are parallel along $c_{p}$, also $T_{F(p)} B$ is a quaternionic and real subspace of $T_{F(p)} G_{2}\left(\mathbb{C}^{m+2}\right)$. Thus $B$ is a quaternionic and real submanifold of $G_{2}\left(\mathbb{C}^{m+2}\right)$. Since $B$ is quaternionic, it is totally geodesic in $G_{2}\left(\mathbb{C}^{m+2}\right)$ (see Alekseevski [1]). The only quaternionic totally geodesic submanifolds of $G_{2}\left(\mathbb{C}^{m+2}\right), m=2 n \geqslant 4$, of half dimension are $G_{2}\left(\mathbb{C}^{n+2}\right)$ and $\mathbb{H} P^{n}$ (see Berndt [3]). But only $\mathbb{H} P^{n}$ is embedded in $G_{2}\left(\mathbb{C}^{m+2}\right)$ as a real submanifold. So we conclude that $B$ is an open part of a totally geodesic $\mathbb{H} P^{n}$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$. Rigidity of totally geodesic submanifolds finally implies that $M$ is an open part of the tube with radius $r$ around a totally geodesic $\mathbb{H} P^{n}$ in $G_{2}\left(\mathbb{C}^{m+2}\right)$. Thus we have proved our main Theorem.

## References

[1] Alekseevski DV (1968) Compact quaternion spaces. Funct Anal Appl 2: 106-114
[2] Berndt J (1991) Real hypersurfaces in quaternionic space forms. J Reine Angew Math 419: 9-26
[3] Berndt J (1997) Riemannian geometry of complex two-plane Grassmannians. Rend Sem Mat Univ Politec Torino 55: 19-83
[4] Berndt J, Suh YJ (1999) Real hypersurfaces in complex two-plane Grassmannians. Monatsh Math 127: 1-14
[5] Berndt J, Suh YJ (2002) Isometric flows on real hypersurfaces in complex two-plane Grassmannians. Monatsh Math 137: 87-98
[6] Cecil TE, Ryan PJ (1982) Focal sets and real hypersurfaces in complex projective space. Trans Amer Math Soc 269: 481-499
[7] Kimura M (1986) Real hypersurfaces and complex submanifolds in complex projective space. Trans Amer Math Soc 296: 137-149
[8] Martinez A, Pérez JD (1986) Real hypersurfaces in quaternionic projective space. Ann Math Pura Appl 145: 355-384
[9] Pérez JD, Suh YJ (1997) Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_{i}} R=0$. Diff Geom Appl 7: 211-217
[10] Suh YJ (2003) Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator. Bull Austral Math Soc 67: 493-502
[11] Suh YJ (2003) Real hypersurfaces in complex two-plane Grassmannians with commuting shape operator. Bull Austral Math Soc 68: 379-393
[12] Suh YJ (2006) Real hypersurfaces in complex two-plane Grassmannians with vanishing Lie derivatives. Canadian Math Bull, to appear
[13] Yano K, Kon M (1983) CR-Submanifolds of Kaehlerian and Sasakian Manifolds. Basel: Birkhäuser

Author's address: Y. J. Suh, Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea, e-mail: yjsuh@mail.knu.ac.kr


[^0]:    This work was supported by grant Proj. No. R14-2002-003-01001-0 from the Korea Research Foundation.

