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REAL HYPERSURFACES WITH CONSTANT TOTALLY REAL SECTIONAL CURVATURE IN A COMPLEX SPACE FORM

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Abstract. We characterize real hypersurfaces with constant holomorphic sectional curvature of a non flat complex space form as the ones which have constant totally real sectional curvature.

Keywords: real hypersurfaces, holomorphic and totally real sectional curvature

MSC 2000: 53C55, 53C42

1. Introduction

The sectional curvature gives a lot of information of the intrinsic geometry of a Riemannian manifold. For instance, manifolds which have constant sectional curvature have been a great source of study. In complex manifolds, the holomorphic sectional curvature and the totally real sectional curvature arise naturally and it is very well-known that the constancy of holomorphic sectional curvature is equivalent to the constancy of totally real sectional curvature. The main target of this paper is to study whether this fact is inherited to real hypersurfaces of complex space forms $\mathbb{C}M^m(c)$, $c \neq 0$, with constant holomorphic sectional curvature c. Real hypersurfaces with constant holomorphic sectional curvature of $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geqslant 3$, have been classified by Kimura in [2] when $c > 0$, i.e., in the complex projective space $\mathbb{C}P^m(c)$, and by the authors in [6] and [7] when $c < 0$, i.e., in the complex hyperbolic space $\mathbb{C}H^m(c)$.

Let M be a connected real hypersurface of $\mathbb{C}M^m(c)$, $c \neq 0$, N a local unit normal vector field to M, If J is the almost complex structure of $\mathbb{C}M^m(c)$, $c \neq 0$, we

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will denote $\xi = -JN$. Given a vector field X tangent to M, we will write $JX = \varphi X + \eta(X)N$, where φX and $\eta(X)N$ are the tangential and the normal component of JX respectively. We recall that M is ruled if the distribution $\mathbb{D}(p) = \{X \in T_pM \colon X \perp \xi\}$, $p \in M$, is integrable and its leaves are totally geodesic $\mathbb{C}M^{m-1}(c)$.

If π is a 2-plane included in $\mathbb{D}(p)$, where $p \in M$, we will say that π is totally real if $\varphi\pi$ is orthogonal to π. We denote by $T(\pi) = T(X,Y)$ the sectional curvature of any totally real 2-plane $\pi = \mathrm{Span}\{X,Y\}$ included in $\mathbb{D}(p)$, $p \in M$, and we will call it the totally real sectional curvature of M. If $T(\pi)$ is constant for any π included in $\mathbb{D}(p)$ and any $p \in M$, we will say that M has constant totally real sectional curvature. If the complex dimension of the complex space form is $m = 2$, there are no totally real 2-planes tangent to M. Therefore, the totally real sectional curvature is meaningful when $m \geqslant 3$.

We need to write $c = 4\varepsilon/k^2$, where $\varepsilon = \pm 1$ is the sign of c and $k \neq 0$ is a real constant. Our results are:

Theorem. *Let M be a real hypersurface of $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geqslant 3$, on which T is constant. Then M is one of the following:*

a) *ruled, $c = 4T$,*

b) *a real hypersurface which admits a foliation of codimension two such that each leaf is contained in a totally geodesic $\mathbb{C}M^{m-1}(c)$, $c \neq 0$, as a ruled real hypersurface, $c = 4T$,*

c) *if $\varepsilon = 1$, an open subset of a geodesic hypersphere of radius $r > 0$, i.e., an open subset of a tube of radius $r > 0$ over a point, $T = \frac{c}{4} + \frac{\cot^2(r)}{k^2}$,*

d) *if $\varepsilon = -1$ then M is an open subset of either*

   d.1) *a tube of radius $r > 0$ over a totally geodesic $\mathbb{C}M^{m-1}(c)$, $\frac{c}{4} < T = \frac{c}{4} + \frac{\tanh^2(r)}{k^2} < \frac{c}{4} + \frac{1}{k^2}$,*

   d.2) *a Montiel tube, $T = \frac{c}{4} + \frac{1}{k^2}$,*

   d.3) *a geodesic hypersphere of radius $r > 0$, $\frac{c}{4} + \frac{1}{k^2} < \frac{c}{4} + \frac{\coth^2(r)}{k^2}$.*

A 2-plane π tangent to M is called holomorphic if it admits an orthonormal basis of the form $\{X, \varphi X\}$. The holomorphic sectional curvature is the sectional curvature of any holomorphic 2-plane tangent to M. We will denote it by $H(\pi) = H(X)$. The next corollary gives an affirmative answer to the main question of this paper.

Corollary. *Let M be a real hypersurface of $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geqslant 3$. Then M has constant holomorphic sectional curvature if and only if M has constant totally real sectional curvature.*

2. Preliminaries

Let M be a real hypersurface of $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geqslant 3$. Let ∇ be the Levi-Civita connection of M. In the introduction we wrote $JX = \varphi X + \eta(X)N$ for all $X \in TM$. Thus, φ is a skew-symmetric tensor field of type (1,1) of M and η is a 1-form on M. We will denote by g both the metric on $\mathbb{C}M^m(c)$ and the induced metric on M. Now it is easy to see $\eta(X) = g(X, \xi)$. The set (φ, ξ, η, g) is called an almost contact metric structure on M and its elementary properties are

$$\tag{1} \begin{gathered} \varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0 \\ g(\varphi X, Y) + g(X, \varphi Y) = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \end{gathered}$$

for any $X, Y \in TM$, where A is the Weingarten endomorphism associated to N. We will denote by $\mathbf{U}\mathbb{D}(p) = \{X \in T_pM\colon g(X, \xi) = 0, \|X\| = 1\}$, $p \in M$. The Gauss equation allows us to compute the following expressions of the totally real and holomorphic sectional curvature of M

$$\tag{2} T(X, Y) = \frac{c}{4} + g(AX, X)g(AY, Y) - g(AX, Y)^2$$

where $X, Y \in \mathbf{U}\mathbb{D}$ and $g(X, Y) = g(\varphi X, Y) = 0$.

$$\tag{3} H(X) = c + g(AX, X)g(A\varphi X, \varphi X) - g(AX, \varphi X)^2$$

for any $X \in \mathbf{U}\mathbb{D}$. Finally, we need the following results to prove ours.

Theorem 1. ([2]) *Let M be a real hypersurface of $\mathbb{C}P^m(c)$, $c > 0$, $m \geqslant 3$, which has constant holomorphic sectional curvature H. Then M is one of the following cases:*

a) *an open subset of a geodesic hypersphere, $H > c$,*
b) *ruled, $H = c$,*
c) *a real hypersurface which admits a foliation of codimension two such that each leaf is contained in a totally geodesic hyperplane $\mathbb{C}P^{m-1}(c)$ as a ruled real hypersurface, $H = c$.*

Theorem 2. ([6] and [7]) *Let M be a real hypersurface of $\mathbb{C}H^m(c)$, $c < 0$, $m \geqslant 3$, which has constant holomorphic sectional curvature H. Then M is one of the following cases:*

a) *an open subset of a geodesic hypersphere of radius $r > 0$, $c + \frac{1}{k^2} < H = c + \frac{\coth^2(r)}{k^2}$,*
b) *an open subset of a Montiel tube, $H = c + \frac{1}{k^2}$,*

c) *an open subset of a tube of radius* $r > 0$ *over a hyperplane* $\mathbb{C}H^{m-1}(c)$, $c < H = c + \frac{\tanh^2(r)}{k^2} < c + \frac{1}{k^2}$,

d) *ruled,* $H = c$,

e) *a real hypersurface which admits a foliation of codimension two such that each leaf is contained in a totally geodesic hyperplane* $\mathbb{C}H^{m-1}(c)$ *as a ruled real hypersurface,* $H = c$.

If p is a point of M, the rank of A at p is called the type number of M at p, and it will be denoted by $t(p)$.

Theorem 3. ([6] and [8]) *Let* M *be a real hypersurface of* $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geqslant 3$, *which satisfies* $t(p) \leqslant 2$ *for all* $p \in M$. *Then* M *is ruled.*

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3. Proof of the Theorem

Let p be a point of M. Let $X, Y, Z \in \mathbf{U}\mathbb{D}(p)$ such that $\mathrm{Span}\{X, Y\}$ and $\mathrm{Span}\{X, Z\}$ are totally real and $g(X, Z) = 0$. There is a curve $X(t)$, $t \in (-\delta, \delta)$, such that $X(t) \in \mathbb{D}(p)$, $\mathrm{Span}\{Y, X(t)\}$ is totally real, $X(0) = X$ and $X'(0) = Z$. By (2)

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g(AX(t), Y)g(AY, Y) - g(AX(t), Y)^2 = 0.$$

A straightforward computation shows

$$0 = g(AY, Y)g(AX, Z) - g(AX, Y)g(AY, Z) \tag{4}$$

for any $X, Y, Z \in \mathbf{U}\mathbb{D}(p)$ such that $\mathrm{Span}\{X, Y\}$ and $\mathrm{Span}\{Y, Z\}$ are totally real, and $g(X, Z) = 0$. In the sequel, we will denote by $(*)_{\mathbb{D}}$ the component of $(*)$ in $\mathbb{D}$. Take $\{\xi, E_1, \ldots, E_{2m-2}\}$ an orthonormal basis of T_pM such that

$$(AE_i)_{\mathbb{D}} = a_i E_i \quad i = 1, \ldots, 2m-2 \tag{5}$$

where a_i are functions on M. Choose $i \in \{1, \ldots, 2m-2\}$. If we substitute $Y = E_i$ in (4),

$$0 = a_i g(AX, Z) \tag{6}$$

where $X, Z \in \mathbf{U}\mathbb{D}(p)$, $\mathrm{Span}\{X, Z\} \perp \mathrm{Span}\{E_i, \varphi E_i\}$ and $g(X, Z) = 0$. Now we have to discuss the following three cases:

Case 1. At least two of the a_i are not zero. We can suppose without losing any generality that a_1, a_2 are not zero. Let $(\Omega) = \{p \in M\colon a_1(p) \neq 0, a_2(p) \neq 0\}$. During this case, $p \in (\Omega)$ unless otherwise stated. From this and (6), $g(AX, Z) = 0$ for any $X, Z \in \mathbf{U}\mathbb{D}(p)$ such that $g(X, Z) = 0$ and $\mathrm{Span}\{X, Z\}$ is orthogonal to $\mathrm{Span}\{E_1, \varphi E_1\}$. Therefore there exists a local orthonormal basis $\{\xi, E_1, \varphi E_1, F_2, \varphi F_2, \dots, F_{m-1}, \varphi F_{m-1}\}$ of $T(\Omega)$ such that

$$\begin{aligned} (7) \qquad & (AE_1)_{\mathbb{D}} = a_1 E_1, \quad (A\varphi E_1)_{\mathbb{D}} \in \mathrm{Span}\{E_1\}^{\perp} \cap \mathbb{D} \\ & (AF_k)_{\mathbb{D}} \in \mathrm{Span}\{\varphi E_1, F_k\}, \quad (A\varphi F_k)_{\mathbb{D}} \in \mathrm{Span}\{\varphi E_1, \varphi F_k\}, \quad k = 1, \dots, m-1. \end{aligned}$$

This shows $\mathrm{rank}(A_{|\mathbb{D}}) \geqslant 2$ on (Ω). By (2) and (7),

$$(8) \qquad 0 = g(AX, \varphi E_1) g(AZ, \varphi E_1)$$

for any $X, Z \in \mathbf{U}\mathbb{D}(p)$ such that $g(X, Z) = 0$, $\mathrm{Span}\{X, \varphi E_1\}$ and $\mathrm{Span}\{Z, \varphi E_1\}$ are totally real. Given $X, Z \in \mathbf{U}\mathbb{D}(p)$ in these latter conditions, the vectors $X' = \frac{1}{\sqrt{2}}(X + Z)$, $Z' = \frac{1}{\sqrt{2}}(X - Z)$ satisfy the conditions of (8) and introducing them in that formula, we obtain $g(AX, \varphi E_1)^2 = g(AZ, \varphi E_1)^2$. From this and (8) we see that $A\varphi E_1 \in \mathrm{Span}\{\xi, \varphi E_1\}$. Now, bearing in mind (7), there exists a local orthonormal basis $\{\xi, E_1, \varphi E_1, \dots, E_{m-1}, \varphi E_{m-1}\}$ of $T(\Omega)$ such that

$$(9) \qquad (AE_k)_{\mathbb{D}} = d_k E_k, \quad (A\varphi E_k)_{\mathbb{D}} = e_k \varphi E_k, \quad k = 1, \dots, m-1$$

where d_k, e_k are functions on (Ω). By (9), if we take $X = E_j$, $Y = E_i$, $i \neq j$ in (2)

$$(10) \qquad T - \frac{c}{4} = d_j d_i \quad i \neq j.$$

If we put $X = \varphi E_j$, $Y = E_i$, $i \neq j$ in (2), by (9)

$$(11) \qquad T - \frac{c}{4} = e_j d_i \quad i \neq j.$$

And similarly, if $X = \varphi E_i$, $Y = \varphi E_j$

$$(12) \qquad T - \frac{c}{4} = e_j e_i \quad i \neq j.$$

Now we show $4T \neq c$. Indeed, if $4T = c$, then from (10) and (12) we deduce that there is a point $q \in (\Omega)$ such that at most one of the d_i and at most one of the e_j are not zero at q. Equations (11) yield $i = j$. We can suppose $i = j = 1$ without losing any generality. Now we put $X = \frac{1}{\sqrt{2}}(E_1 + E_2)$ and $Y = \frac{1}{\sqrt{10}}(2E_1 + \varphi E_1 - 2E_2 - \varphi E_2)$. It is easy to check that $\mathrm{Span}\{X, Y\}$ is totally real. If we introduce X, Y in (2), by bearing (9) in mind, we obtain $0 = T(X,Y) - \frac{c}{4} = \frac{d_1(q)e_1(q)}{20}$, which yields $d_1(q) = 0$ or $e_1(q) = 0$. This means $\operatorname{rank}(A_{|\mathbb{D}}) \leqslant 1$, which is a contradiction.

Therefore, $4T - c \neq 0$, and introducing this in (10) and (11),

$$0 \neq d_j = e_j \quad j = 1, \ldots, m-1. \tag{13}$$

Choose $i \neq j$. From (9) and (13), the orthonormal system $\{E_i, \varphi E_i, E_j, \varphi E_j\}$ in T_pM, $p \in (\Omega)$, satisfies

$$\begin{aligned} (AE_i)_{\mathbb{D}} &= d_i E_i, \quad (A\varphi E_i)_{\mathbb{D}} = d_i \varphi E_i \\ (AE_j)_{\mathbb{D}} &= d_j E_j, \quad (A\varphi E_j)_{\mathbb{D}} = d_j \varphi E_j. \end{aligned} \tag{14}$$

If $\alpha = 1/\sqrt{3}$, $\beta = \sqrt{2/3}$, we consider the vectors

$$X = \alpha E_i + \beta E_j, \quad Y = \beta E_i - \alpha E_j, \quad Z = \beta\varphi E_i - \alpha\varphi E_j. \tag{15}$$

It is clear that $X, Y, Z \in \mathbf{U}\mathbb{D}(p)$ and $\operatorname{Span}\{X,Y\}$, $\operatorname{Span}\{X,Z\}$ are totally real. From (2), (14) and (15) it is easy to compute $T(X,Y) - \frac{c}{4} = \frac{2d_i^2 + 5d_id_j + 2d_j^2}{9} - \frac{2(d_i - d_j)^2}{9}$ and $T(X,Y) - \frac{c}{4} = \frac{2d_i^2 + 5d_id_j + 2d_j^2}{9}$. Now it is evident $d_i = d_j$ if $i \neq j$. From this and (13),

$$d_1 = \ldots = d_{m-1} = e_1 = \ldots = e_{m-1} = d. \tag{16}$$

From (10), we see that d must be constant. Besides, d cannot be zero as $\operatorname{rank}(A_{|\mathbb{D}}) \geqslant 2$ on (Ω). By (9) and (16) we have

$$(AX)_{\mathbb{D}} = dX, \qquad \text{for any } X \in \mathbb{D}(p), \text{ any } p \in (\Omega)\ (d \in \mathbb{R} - \{0\}). \tag{17}$$

If we choose $X \in \mathbf{U}\mathbb{D}(p)$, $p \in (\Omega)$, by (3) and (17), $H(X) = c + d^2 > c$, which implies that (Ω) has constant holomorphic sectional curvature. Therefore (Ω) is case a) of Theorem 1 or case a), b) or c) of Theorem 2.

In the sequel, we can suppose that the interior of the set $(\Gamma) = \{p \in M \colon a_1(p) = 0 \text{ or } a_2(p) = 0\}$ is non-empty.

Case 2. Let us suppose $a_1 = \ldots = a_{2m-1} = 0$ on an open subset (Ω) of M, which can be supposed to be included in (Γ). By (5), $t(p) \leqslant 2$ on (Ω). By Theorem 3, (Ω) is a ruled real hypersurface. Bearing in mind that M is connected and by a similar reasoning as in Theorem 1, case (A) in [2], we see $M = (\Omega)$, and therefore M is ruled. By definition, $\mathbb{D}$ is integrable and totally geodesic in M, so that if $X, Y \in \mathbb{D}$, then $\nabla_X \varphi Y \in \mathbb{D}$, that is to say, $0 = g(\xi, \nabla_X \varphi Y)$. By (2), $0 = g(\nabla_X \xi, \varphi Y) = g(\varphi AX, \varphi Y) = g(AX, Y)$ for any $X, Y \in \mathbb{D}$. From this equation and (2), it is clear $T(X,Y) = c/4$, which shows that M has constant totally real sectional curvature.

Case 3. Let us suppose that exactly one of the $a_i \neq 0$ and all the other a_k are zero. We can suppose without losing any generality $i = 1$. Let (Ω) be the open subset of M where $a_1 \neq 0$, which can be supposed to be included in (Γ). During this case, $p \in (\Omega)$ unless otherwise stated. From (5), we obtain

$$(AX)_{\mathbb{D}} = a_1 g(X, E_1)E_1 \tag{18}$$

for any $X \in \mathbb{D}$ in a neigbourhood of each point $p \in (\Omega)$. By (2) and (18), $H(X) = c$ for any $X \in \mathbf{U}\mathbb{D}(p)$ and any $p \in (\Omega)$. By Theorem 1 and Theorem 2, either (Ω) is ruled or (Ω) admits a foliation of codimension two such that each leaf is contained in a totally geodesic hyperplane $\mathbb{C}M^{m-1}(c)$ as a ruled hypersurface. As $a_1 \neq 0$, $t(p) \geqslant 3$ on (Ω), and by Theorem 3, (Ω) cannot be ruled. Bearing in mind that M is connected and by a similar reasoning as in Theorem 1, case (B) in [2], $(\Omega) = M$ and therefore M is either case c) of Theorem 1 or case e) of Theorem 2.

Finally, all model spaces of Theorem 1 and Theorem 2 have constant totally real sectional curvature. This finishes the proof. □

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