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REAL PALEY-WIENER THEOREMS FOR THE INVERSE FOURIER TRANSFORM ON A RIEMANNIAN SYMMETRIC SPACE

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We prove real Paley-Wiener theorems for the inverse Fourier transform on a semisimple Riemannian symmetric space G/K of the noncompact type. The functions on G/K whose Fourier transform has compact support are characterised by a L^2 growth condition. We also obtain real Paley-Wiener theorems for the inverse spherical transform.

1. Introduction.

The classical Fourier transform $\mathcal{F}_{\mathrm{cl}}$ is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^k)$ onto itself. The space $C_c^\infty(\mathbb{R}^k)$ of smooth functions with compact support is dense in $\mathcal{S}(\mathbb{R}^k)$, and the classical Paley-Wiener theorem characterises the image of $C_c^\infty(\mathbb{R}^k)$ under $\mathcal{F}_{\mathrm{cl}}$ as rapidly decreasing functions having an holomorphic extension to $\mathbb{C}^k$ of exponential type. Since $\mathbb{R}^k$ is self-dual, the same theorem also applies to the inverse Fourier transform.

Let G be a noncompact semisimple Lie group and K a maximal compact subgroup of G. The Fourier transform $\mathcal{F}$ on the Riemannian symmetric space $X = G/K$ is an analogue of the classical Fourier transform on $\mathbb{R}^k$. A Paley-Wiener theorem for the Fourier transform $\mathcal{F}$, which characterises the image of $C_c^\infty(X)$ under $\mathcal{F}$ in terms of holomorphic extensions and growth behaviour, as in the classical case, was proved by Helgason, see [**7**]. Furthermore, the L^2-Schwartz space $\mathcal{S}^2(X)$ contains $C_c^\infty(X)$ as a dense subspace and $\mathcal{F}$ is an isomorphism of $\mathcal{S}^2(X)$ onto some generalised Schwartz space, see [**4**].

Unlike the classical case, however, we can not use a duality argument to deduce a Paley-Wiener theorem for the inverse Fourier transform. So how can we characterise the functions whose Fourier transform $\mathcal{F}$ has compact support?

The Fourier transform on X reduces to the spherical transform $\mathcal{H}$ on G when restricted to K-invariant functions. The paper [**8**] provides an answer to the above question for the spherical transform on Schwartz functions in the rank one and complex cases. The characterisation is in analogy with the classical Paley-Wiener theorem given in terms of meromorphic extensions and growth conditions.

In this paper we prove (real) Paley-Wiener theorems for the inverse Fourier transform for general Riemannian symmetric spaces, i.e., we characterise, as a subset of $L^2(X)$, the set of functions f on X whose Fourier transform $\mathcal{F}f$ has compact support. More precisely, $f \in C^\infty(X)$ has to satisfy

$$\lim_{n\to\infty} \|\Delta^n f\|_2^{1/2n} < \infty,$$

where Δ is the Laplace-Beltrami operator (and $(1+|\cdot|)^n f \in L^2(X)$ for all $n \in \mathbb{N}\cup\{0\}$ if we also want the Fourier image to be smooth). Specialising to bi-K-invariant functions yields (real) Paley-Wiener theorems for the inverse spherical transform for general noncompact semisimple Lie groups.

Our approach is based on real analysis techniques developed by H. H. Bang, see [**2**] and [**3**], and V.K. Tuan, see [**10**]. Also see [**11**] for a history and overview of (real) Paley-Wiener theorems for certain transforms (Fourier, Mellin, Hankel...) on $\mathbb{R}$. In particular we use Parseval's formula, intertwining properties of $\mathcal{F}$, and the following characterisation of the radius of the support of a function g on $\mathbb{R}^n$:

$$\sup_{\lambda\in\operatorname{supp} g} \|\lambda\| = \lim_{n\to\infty} \left\{ \int_{\mathbb{R}^n} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n}.$$

For completeness and comparison, we first consider the Fourier transform on $\mathbb{R}^k$. The results here are originally due to H.H. Bang, see [**2**] and [**3**], and V.K. Tuan, see [**10**]. Notice the beautiful symmetry between the (statements of the) results for the various transforms.

2. The Fourier transform on $\mathbb{R}^k$.

For background and details, please see [**9**, Chapter 7]. Let $\mathcal{F}_{\mathrm{cl}}$ denote the classical Fourier transform on $\mathbb{R}^k$:

$$\mathcal{F}_{\mathrm{cl}} f(\lambda) := \int_{\mathbb{R}^k} f(x) e^{-i\lambda\cdot x} dx,$$

defined for nice functions f, for all $\lambda \in \mathbb{C}^k$ for which the above integral makes sense. Let $\Delta = \frac{d^2}{dx_1^2} + \cdots + \frac{d^2}{dx_k^2}$ denote the Laplacian on $\mathbb{R}^k$ and let $\mathcal{S}(\mathbb{R}^k)$ denote the Schwartz space of rapidly decreasing differentiable functions. Then $\mathcal{F}_{\mathrm{cl}}(\Delta f)(\lambda) = -\|\lambda\|^2 \mathcal{F}_{\mathrm{cl}} f(\lambda)$, $(\lambda \in \mathbb{R}^k)$, for all $f \in \mathcal{S}(\mathbb{R}^k)$, and the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^k)$ onto itself, with inverse given by:

$$\mathcal{F}_{\mathrm{cl}}^{-1} g(x) = (2\pi)^{-k} \int_{\mathbb{R}^k} g(\lambda) e^{i\lambda\cdot x} d\lambda, \qquad (x \in \mathbb{R}^k)$$

for $g \in \mathcal{S}(\mathbb{R}^k)$. Parseval's formula states that

$$\begin{aligned}\langle f_1, f_2\rangle &:= \int_{\mathbb{R}^k} f_1(x)\overline{f_2(x)}dx = (2\pi)^{-k}\int_{\mathbb{R}^k} \mathcal{F}_{\rm cl}f_1(\lambda)\overline{\mathcal{F}_{\rm cl}f_2(\lambda)}d\lambda \\ &=: \langle \mathcal{F}_{\rm cl}f_1, \mathcal{F}_{\rm cl}f_2\rangle,\end{aligned}$$

for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^k)$, which implies that $\|f\|_2 = \|\mathcal{F}_{\rm cl}f\|_2$, for all $f \in \mathcal{S}(\mathbb{R}^k)$, and hence that the Fourier transform extends to an isometry from $L^2(\mathbb{R}^k)$ onto itself.

Let $f \in C^\infty(\mathbb{R}^k)$ such that $\Delta^n f \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N}\cup\{0\}$ and let $f_2 \in C_c^\infty(\mathbb{R}^k)$. Then:

$$\begin{aligned}\langle \mathcal{F}_{\rm cl}(\Delta f), \mathcal{F}_{\rm cl}f_2\rangle &= \langle \Delta f, f_2\rangle = \langle f, \Delta f_2\rangle = \langle \mathcal{F}_{\rm cl}f, \mathcal{F}_{\rm cl}(\Delta f_2)\rangle \\ &= \langle \mathcal{F}_{\rm cl}f, -\|\lambda\|^2\mathcal{F}_{\rm cl}f_2\rangle = \langle -\|\lambda\|^2\mathcal{F}_{\rm cl}f, \mathcal{F}_{\rm cl}f_2\rangle,\end{aligned}$$

and we conclude that $\mathcal{F}_{\rm cl}(\Delta f)(\lambda) = -\|\lambda\|^2\mathcal{F}_{\rm cl}f(\lambda)$ a.e., by a density argument, whence $\mathcal{F}_{\rm cl}(\Delta^n f)(\lambda) = (-1)^n\|\lambda\|^{2n}\mathcal{F}_{\rm cl}f(\lambda)$ a.e., and

$$\int_{\mathbb{R}^k} |\Delta^n f(x)|^2 dx = (2\pi)^{-k}\int_{\mathbb{R}^k} \|\lambda\|^{4n}|\mathcal{F}_{\rm cl}f(\lambda)|^2 d\lambda, \tag{1}$$

for all $\in \mathbb{N}\cup\{0\}$.

We define the support, $\operatorname{supp} g$, of $g \in L^2(\mathbb{R}^k)$ to be the smallest closed set, outside which the function g vanishes almost everywhere, and $R_g := \sup_{\lambda\in\operatorname{supp} g}\|\lambda\|$ to be the radius of the support of g; R_g is finite if, and only if, g has compact support.

Lemma 2.1. *Let $g \in L^2(\mathbb{R}^k)$ such that $\|\lambda\|^{2n}g(\lambda) \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N}\cup\{0\}$. Then*

$$R_g = \lim_{n\to\infty}\left\{\int_{\mathbb{R}^k}\|\lambda\|^{4n}|g(\lambda)|^2 d\lambda\right\}^{1/4n}.$$

Proof. Assume g has compact support with $R_g > 0$. Then:

$$\begin{aligned}&\limsup_{n\to\infty}\left\{\int_{\mathbb{R}^k}\|\lambda\|^{4n}|g(\lambda)|^2 d\lambda\right\}^{1/4n} \\ &\le R_g \limsup_{n\to\infty}\left\{\int_{\|\lambda\|\le R_g}|g(\lambda)|^2 d\lambda\right\}^{1/4n} = R_g.\end{aligned}$$

On the other hand,

$$\int_{R_g-\varepsilon\le\|\lambda\|\le R_g}|g(\lambda)|^2 d\lambda > 0,$$

for any $\varepsilon > 0$, hence

$$\begin{aligned}
&\liminf_{n\to\infty}\left\{\int_{\mathbb{R}^k}\|\lambda\|^{4n}|g(\lambda)|^2 d\lambda\right\}^{1/4n} \\
&\geq \liminf_{n\to\infty}\left\{\int_{R_g-\varepsilon\leq\|\lambda\|\leq R_g}\|\lambda\|^{4n}|g(\lambda)|^2 d\lambda\right\}^{1/4n} \\
&\geq (R_g-\varepsilon)\liminf_{n\to\infty}\left\{\int_{R_g-\varepsilon\leq\|\lambda\|\leq R_g}|g(\lambda)|^2 d\lambda\right\}^{1/4n} = R_g-\varepsilon,
\end{aligned}$$

and thus

$$\lim_{n\to\infty}\left\{\int_{\mathbb{R}_n}\|\lambda\|^{4n}|g(\lambda)|^2 d\lambda\right\}^{1/4n} = R_g.$$

Now assume that g has unbounded support. Then

$$\int_{\|\lambda\|\geq N}|g(\lambda)|^2 d\lambda > 0,$$

for any $N > 0$, so:

$$\begin{aligned}
&\liminf_{n\to\infty}\left\{\int_{\mathbb{R}^k}\|\lambda\|^{4n}|g(\lambda)|^2 d\lambda\right\}^{1/4n} \\
&\geq \liminf_{n\to\infty}\left\{\int_{\|\lambda\|\geq N}\|\lambda\|^{4n}|g(\lambda)|^2 d\lambda\right\}^{1/4n} \\
&\geq N\liminf_{n\to\infty}\left\{\int_{\|\lambda\|\geq N}|g(\lambda)|^2 d\lambda\right\}^{1/4n} = N,
\end{aligned}$$

for arbitrary $N > 0$, and we conclude that

$$\liminf_{n\to\infty}\left\{\int_{\mathbb{R}^k}\|\lambda\|^{4n}|g(\lambda)|^2 d\lambda\right\}^{1/4n} = \infty.$$

□

Let $L^2_c(\mathbb{R}^k)$ denote the subspace of $L^2(\mathbb{R}^k)$ of functions with compact support and let $L^2_R(\mathbb{R}^k) := \{g \in L^2_c(\mathbb{R}^k) \mid R_g = R\}$. Let also $C^\infty_R(\mathbb{R}^k) := \{g \in C^\infty_c(\mathbb{R}^k) \mid R_g = R\}$.

Definition 2.2. We define the L^2-Paley-Wiener space $\mathrm{PW}^2(\mathbb{R}^k)$ to be the space of all functions $f \in C^\infty(\mathbb{R}^k)$ satisfying:

(a) $\Delta^n f \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N}\cup\{0\}$.

(b) $R^\Delta_f := \lim_{n\to\infty}\|\Delta^n f\|_2^{1/2n} < \infty$.

Let also $\mathrm{PW}^2_R(\mathbb{R}^k) := \{f \in \mathrm{PW}^2(\mathbb{R}^k) \mid R^\Delta_f = R\}$, for $R \geq 0$.

The proof of Theorem 2.3 below shows that the limit in (b) above is well-defined. The real version of the L^2-Paley-Wiener theorem for the inverse Fourier transform can now be formulated as follows:

Theorem 2.3. *The inverse Fourier transform $\mathcal{F}_{\mathrm{cl}}^{-1}$ is a bijection of $L^2_c(\mathbb{R}^k)$ onto $\mathrm{PW}^2(\mathbb{R}^k)$, mapping $L^2_R(\mathbb{R}^k)$ onto $\mathrm{PW}^2_R(\mathbb{R}^k)$.*

Proof. Let $g \in L^2_R(\mathbb{R}^k)$. Then $\|\lambda\|^n g(\lambda) \in L^1(\mathbb{R}^k)$ for all $n \in \mathbb{N} \cup \{0\}$, and $\mathcal{F}_{\mathrm{cl}}^{-1} g \in C^\infty_o(\mathbb{R}^k)$. We also have $\Delta^n(\mathcal{F}_{\mathrm{cl}}^{-1} g) = \mathcal{F}_{\mathrm{cl}}^{-1}((-1)^n \|\lambda\|^{2n} g) \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N} \cup \{0\}$, by the formula for $\mathcal{F}_{\mathrm{cl}}^{-1}$, and (1) thus yields:

$$\lim_{n\to\infty} \left\{ \int_{\mathbb{R}^k} |\Delta^n(\mathcal{F}_{\mathrm{cl}}^{-1} g)(x)|^2 dx \right\}^{1/4n}$$
$$= \lim_{n\to\infty} \left\{ (2\pi)^{-k} \int_{\mathbb{R}^k} \|\lambda\|^{4n} |g(\lambda)|^2 d\lambda \right\}^{1/4n} = R,$$

whence $\mathcal{F}_{\mathrm{cl}}^{-1} g \in \mathrm{PW}^2_R(\mathbb{R}^k)$.

Let now $f \in \mathrm{PW}^2_R(\mathbb{R}^k)$. Then $\mathcal{F}_{\mathrm{cl}}(\Delta^n f)(\lambda) = (-1)^n \|\lambda\|^{2n} \mathcal{F}_{\mathrm{cl}} f(\lambda) \in L^2(\mathbb{R}^k)$ for all $n \in \mathbb{N}$, and another application of (1) shows that

$$\lim_{n\to\infty} \left\{ (2\pi)^{-k} \int_{\mathbb{R}^k} \|\lambda\|^{4n} |\mathcal{F}_{\mathrm{cl}} f(\lambda)|^2 d\lambda \right\}^{1/4n}$$
$$= \lim_{n\to\infty} \left\{ \int_{\mathbb{R}^k} |\Delta^n f(x)|^2 dx \right\}^{1/4n} = R,$$

and we conclude that $\mathcal{F}_{\mathrm{cl}} f$ has compact support with $R_{\mathcal{F}_{\mathrm{cl}} f} = R$. □

Remark 2.4. The classical (complex) L^2-Paley-Wiener theorem implies that $\mathrm{PW}^2_R(\mathbb{R}^k)$ exactly consists of those $L^2(\mathbb{R}^k)$ functions that can be extended to holomorphic functions of exponential type R on $\mathbb{C}^k$.

Remark 2.5. Let $f \in \mathrm{PW}^2(\mathbb{R})$. Then $\frac{d^n}{dx^n} f \in L^p(\mathbb{R})$ for all $n \in \mathbb{N} \cup \{0\}$, and:

$$\lim_{n\to\infty} \left\| \frac{d^n}{dx^n} f \right\|_p^{1/n} = R_{\mathcal{F}_{\mathrm{cl}} f} = R^\Delta_f,$$

for all $1 \leq p \leq \infty$. This follows from [**2**, Theorem 1]. Similar results hold for $\mathbb{R}^k$, $k > 1$, see [**3**, Theorem 3] and [**10**, Theorem 4].

Definition 2.6. We define the Paley-Wiener space $\mathrm{PW}(\mathbb{R}^k)$ as the space of all functions $f \in C^\infty(\mathbb{R}^k)$ satisfying:

(a) $(1 + |x|)^m \Delta^n f \in L^2(\mathbb{R}^k)$ for all $m, n \in \mathbb{N} \cup \{0\}$.
(b) $R^\Delta_f := \lim_{n\to\infty} \|\Delta^n f\|_2^{1/2n} < \infty$.

Let again $\mathrm{PW}_R(\mathbb{R}^k) := \{f \in \mathrm{PW}(\mathbb{R}^k) \mid R^\Delta_f = R\}$, for $R \geq 0$.

We notice that the only difference between $\mathrm{PW}^2(\mathbb{R}^k)$ and $\mathrm{PW}(\mathbb{R}^k)$ is an extra requirement of polynomial decay, to help ensure that $\mathcal{F}_{\mathrm{cl}}f \in C^\infty(\mathbb{R}^k)$.

The real version of the Paley-Wiener theorem for the inverse Fourier transform is the following:

Theorem 2.7. *The inverse Fourier transform $\mathcal{F}_{\mathrm{cl}}^{-1}$ is a bijection of $C_c^\infty(\mathbb{R}^k)$ onto $\mathrm{PW}(\mathbb{R}^k)$, mapping $C_R^\infty(\mathbb{R}^k)$ onto $\mathrm{PW}_R(\mathbb{R}^k)$.*

Proof. Let $g \in C_R^\infty(\mathbb{R}^k)$, then $\mathcal{F}_{\mathrm{cl}}^{-1}g \in \mathcal{S}(\mathbb{R}^k)$, and $\mathcal{F}_{\mathrm{cl}}^{-1}g \in \mathrm{PW}_R^2(\mathbb{R}^k)$ by Theorem 2.3.

Let $f \in \mathrm{PW}_R(\mathbb{R}^k) \subset \mathrm{PW}_R^2(\mathbb{R}^k)$. Then $\mathcal{F}_{\mathrm{cl}}f \in C^\infty(\mathbb{R}^k)$ since f has polynomial decay, and $\mathcal{F}_{\mathrm{cl}}f$ has compact support with $R_{\mathcal{F}_{\mathrm{cl}}f} = R$ by Theorem 2.3. □

3. Lie group notation.

In this section we introduce the Lie group notation we need in the next sections. We refer to [5], [6] and [7] for further details.

Let G be a real connected noncompact semisimple Lie group with finite center and let θ be a Cartan involution of G. Then the fixed point group $K := G^\theta$ is a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote their Lie algebras, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ into the ± 1 eigenspaces of θ. The Killing form on $\mathfrak{g}$ induces an $\mathrm{Ad}K$-invariant scalar product on $\mathfrak{p}$ and hence a G-invariant Riemannian metric on $X := G/K$. With this structure, X becomes a Riemannian globally symmetric space of the noncompact type.

Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Denote its real dual by $\mathfrak{a}^*$ and its complex dual by $\mathfrak{a}_{\mathbb{C}}^*$. The Killing form of $\mathfrak{g}$ induces a scalar product $\langle \cdot, \cdot \rangle$ and hence a norm $\|\cdot\|$ on $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$. Let $\Sigma \subset \mathfrak{a}^*$ denote the root system of $(\mathfrak{g}, \mathfrak{a})$ and let W be the associated Weyl group. Choose a set $\Sigma_+ \subset \Sigma$ of positive roots, let $\mathfrak{n} := \bigoplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha$ be the corresponding nilpotent subalgebra of $\mathfrak{g}$ and let $\mathfrak{a}_+ := \{H \in \mathfrak{a} \,|\, \alpha(H) > 0 \,\forall\, \alpha \in \Sigma_+\}$ be the positive Weyl chamber with $\overline{\mathfrak{a}_+}$ it's closure. Denote by $\mathfrak{a}_+^*$ and $\overline{\mathfrak{a}_+^*}$ the similar cones in $\mathfrak{a}^*$, and define the element $\rho \in \mathfrak{a}^*$ by: $\rho(H) := \frac{1}{2}\sum_{\alpha \in \Sigma_+} m_\alpha \alpha(H)$, $H \in \mathfrak{a}$, where $m_\alpha = \dim \mathfrak{g}_\alpha$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition of $\mathfrak{g}$ and $G = KAN = NAK$ the corresponding Iwasawa decompositions of G, where A and N are the Lie groups generated by $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Every $g \in G$ can be represented as: $g = K \exp H(g) N = N \exp A(g) K$, where the projections onto the A-parts $A(g) \in \mathfrak{a}$ and $H(g) \in \mathfrak{a}$ are uniquely determined. We note that $A(g) = -H(g^{-1})$. Let $M := Z_K(\mathfrak{a})$, then $B := K/M$ is a compact homogeneous space. We define the vector $A(x, b) \in \mathfrak{a}$ as $A(x, b) := A(k^{-1}g)$, for $x = gK \in X$ and $b = kM \in B$.

Put $A_+ = \exp(\mathfrak{a}_+)$, then $\overline{A_+} = \exp(\overline{\mathfrak{a}_+})$. The Cartan decomposition implies that the natural mapping from $K/M \times A_+ \times K$ into $G = K\overline{A_+}K$ is a diffeomorphism onto its dense open image. We define the norm of an element $g \in G$ as: $|g| = |k_1 \exp(H) k_2| = \|H\|$, with $H \in \overline{\mathfrak{a}_+}$; this is the K-invariant geodesic distance to the origin eK. We denote by $B_R := \{g \in G \mid |g| \leq R\}$ the K-invariant ball of radius R around e.

We identify functions on X with right-K-invariant functions on G. We normalise the invariant measure on X as:

$$\int_X f(x)dx = \int_K \int_{\mathfrak{a}_+} \int_K f(k_1 \exp(H) k_2) J(H) dk_1 dH dk_2,$$

for $f \in C_c^\infty(X)$, where the Jacobian J is given by: $J(H) = \prod_{\alpha \in \Sigma_+} (e^{\alpha(H)} - e^{-\alpha(H)})^{m_\alpha}$, dH is the Lebesgue measure on $\mathfrak{a}$ and dk is the measure on K such that $\int_K dk = 1$. We notice that $0 \leq J(H) \leq Ce^{2\rho(H)}$, for $H \in \overline{\mathfrak{a}_+}$, where C is a positive constant.

The spherical functions φ_λ, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, on G are defined as:

$$\varphi_\lambda(g) := \int_K e^{(i\lambda+\rho)A(k^{-1}g)} dk = \int_K e^{-(i\lambda+\rho)H(g^{-1}k)} dk.$$

We note that φ_λ is Weyl group invariant, $\varphi_{w\lambda} = \varphi_\lambda$, $w \in W$. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. We write $Df(g)$ for the action of $D \in U(\mathfrak{g})$ on $f \in C^\infty(G)$ from the left at $g \in G$. The L^p-Schwartz space $\mathcal{S}^p(X)$, $0 < p \leq 2$, is defined as the space of all functions $f \in C^\infty(X)$ such that:

$$\sup_{g \in G} (1 + |g|)^m \varphi_o(g)^{-\frac{2}{p}} |Df(g)| < \infty,$$

for all $D \in U(\mathfrak{g})$ and $m \in \mathbb{N} \cup \{0\}$. We can also characterise $\mathcal{S}^p(X)$ as the space of all functions $f \in C^\infty(X)$ satisfying:

$$(1 + |g|)^m Df(g) \in L^p(X),$$

for all $D \in U(\mathfrak{g})$ and $m \in \mathbb{N} \cup \{0\}$. We note that $\mathcal{S}^p(X) \not\subset L^q(X)$ for $0 < q < p \leq 2$.

4. The Fourier transform.

In this section, we recall some facts and theorems for the Fourier transform on a noncompact semisimple Riemannian symmetric space, see [7, Chapter 3] for details and references.

The Fourier transform of a function f on X is defined as:

$$\mathcal{F}f(\lambda, b) := \int_X f(x) e^{(-i\lambda+\rho)(A(x,b))} dx,$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $b \in B$ for which the integral exists. In particular, $\mathcal{F}f$ extends to a smooth function on $\mathfrak{a}_{\mathbb{C}}^* \times B$, holomorphic in the first variable, for $f \in C_c^\infty(X)$, see also below.

The plane wave eigenfunction

$$e_{\lambda,b}(x) := e^{(i\lambda+\rho)(A(x,b))}, \tag{2}$$

is a joint eigenfunction of $\mathbb{D}(X)$, the commutative algebra of G-invariant differential operators on X, for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $b \in B$, or, more precisely

$$De_{\lambda,b} = \Gamma(D)(i\lambda)e_{\lambda,b}, \qquad \forall D \in \mathbb{D}(X), \qquad (\lambda \in \mathfrak{a}^*_{\mathbb{C}},\, b \in B)$$

where $\Gamma : \mathbb{D}(X) \to S(\mathfrak{a}^*)^W$ is the Harish-Chandra isomorphism. In particular,

$$\Delta e_{\lambda,b} = -(\langle\lambda, \lambda\rangle + \|\rho\|^2)e_{\lambda,b}, \qquad (\lambda \in \mathfrak{a}^*_{\mathbb{C}},\, b \in B)$$

for the Laplace-Beltrami operator Δ on X, and hence

$$\mathcal{F}(\Delta f)(\lambda, b) = -(\langle\lambda, \lambda\rangle + \|\rho\|^2)\mathcal{F}f(\lambda, b), \qquad (\lambda \in \mathfrak{a}^*_{\mathbb{C}},\, b \in B)$$

for all $f \in C_c^\infty(X)$, by self-adjointness of Δ, see also (6).

A C^∞-function $\psi(\lambda, b)$ on $\mathfrak{a}^*_{\mathbb{C}} \times B$, holomorphic in λ, is called a holomorphic function of uniform exponential type R, if there exists a constant $R \geq 0$, such that, for each $N \in \mathbb{N}$, we have:

$$\sup_{\lambda\in\mathfrak{a}^*_{\mathbb{C}},\, b\in B} e^{-R|\Im\lambda|}(1 + |\lambda|)^N|\psi(\lambda, b)| < \infty.$$

The space of holomorphic functions of uniform exponential type R will be denoted $\mathcal{H}_R(\mathfrak{a}^*_{\mathbb{C}} \times B)$ and we denote by $\mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)$ their union over all $R > 0$. Let furthermore $\mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)^W$ denote the space of all functions $\psi \in \mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)$ satisfying the symmetry condition:

$$\int_B e^{(iw\lambda+\rho)(A(x,b))}\psi(w\lambda, b)db = \int_B e^{(i\lambda+\rho)(A(x,b))}\psi(\lambda, b)db, \tag{3}$$

for $w \in W$ and all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $x \in X$.

The Paley-Wiener theorem states that the Fourier transform is a bijection of the space $C_c^\infty(X)$ onto the space $\mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)^W$, with the following inversion formula:

$$f(x) = \int_{\mathfrak{a}^*_+} \int_B e^{(i\lambda+\rho)(A(x,b))}\mathcal{F}f(\lambda, b)|c(\lambda)|^{-2}d\lambda db, \qquad (x \in X) \tag{4}$$

where $c(\lambda)$ is the Harish-Chandra c-function, for $f \in C_c^\infty(X)$. Moreover, $\mathcal{F}f \in \mathcal{H}_R(\mathfrak{a}^*_{\mathbb{C}} \times B)^W$ if, and only if, $\operatorname{supp} f \subset B_R$. We note that $|c(\lambda)|^{-2}$ is bounded by some polynomial for $\lambda \in \mathfrak{a}^*$.

Let f_1, $f_2 \in C_c^\infty(X)$, then Parseval's formula for $\mathcal{F}$ is given by:

$$\int_X f_1(x)\overline{f_2(x)}dx = \int_{\mathfrak{a}^*_+} \int_B \mathcal{F}f_1(\lambda, b)\overline{\mathcal{F}f_2(\lambda, b)}|c(\lambda)|^{-2}d\lambda db. \tag{5}$$

We conclude that the Fourier transform extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}^*_+ \times B, |c(\lambda)|^{-2}d\lambda db)$. In the following we adopt the convention $L^2(\mathfrak{a}^*_+ \times B) := L^2(\mathfrak{a}^*_+ \times B, |c(\lambda)|^{-2}d\lambda db)$.

Let $f \in C^\infty(X)$ such that $\Delta^n f \in L^2(X)$ for all $n \in \mathbb{N} \cup \{0\}$ and let $f_2 \in C_c^\infty(X)$. Then self-adjointness of the Laplace-Beltrami operator Δ:

$$\int_X \Delta^n f(x) f_2(x) dx = \int_X f(x) \Delta^n f_2(x) dx, \tag{6}$$

Parseval's formula (5) and density of $C_c^\infty(X)$ imply, as in the classical case, that

$$\mathcal{F}(\Delta^n f)(\lambda, b) = (-1)^n (\|\lambda\|^2 + \|\rho\|^2)^n \mathcal{F} f(\lambda, b), \tag{7}$$

a.e., for all $n \in \mathbb{N} \cup \{0\}$.

5. The inverse Fourier transform.

We define the inverse Fourier transform $\mathcal{F}^{-1} g$ of a function g on $\mathfrak{a}_+^* \times B$ via (4):

$$\mathcal{F}^{-1} g(x) := \int_{\mathfrak{a}_+^*} \int_B e^{(i\lambda + \rho)(A(x,b))} g(\lambda, b) |c(\lambda)|^{-2} d\lambda db,$$

for all $x \in X$ for which the integral exists.

We define the support, $\operatorname{supp} g$, of $g \in L^2(\mathfrak{a}_+^* \times B)$ to be the smallest closed set in $\mathfrak{a}_+^* \times B$, outside which the function g vanishes almost everywhere, and $R_g := \sup_{(\lambda,b) \in \operatorname{supp} g} \|\lambda\|$ to be the 'radius' of the support of g.

Lemma 5.1. *Let $g \in L^2(\mathfrak{a}_+^* \times B)$ such that $\|\lambda\|^{2n} g(\lambda, b) \in L^2(\mathfrak{a}_+^* \times B)$ for all $n \in \mathbb{N} \cup \{0\}$. Then*

$$R_g = \lim_{n \to \infty} \left\{ \int_{\mathfrak{a}_+^*} \int_B \|\lambda\|^{4n} |g(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right\}^{1/4n}.$$

Proof. As for Lemma 2.1. □

Let $L_c^2(\mathfrak{a}_+^* \times B)$ denote the subspace of $L^2(\mathfrak{a}_+^* \times B)$ of functions with bounded support and let $L_R^2(\mathfrak{a}_+^* \times B) := \{g \in L_c^2(\mathfrak{a}_+^* \times B) \mid R_g = R\}$.

Definition 5.2. We define the L^2-Paley-Wiener space $\mathrm{PW}^2(X)$ as the space of all functions $f \in C^\infty(X)$ satisfying:

(a) $\Delta^n f \in L^2(X)$ for all $n \in \mathbb{N} \cup \{0\}$.
(b) $R_f^\Delta := \lim_{n\to\infty} \|(\Delta + \|\rho\|^2)^n f\|_2^{1/2n} < \infty$.

Let also $\mathrm{PW}_R^2(X) := \{f \in \mathrm{PW}^2(X) \mid R_f^\Delta = R\}$, for $R \geq 0$.

The real L^2-Paley-Wiener theorem for the inverse Fourier transform can now be formulated as follows:

Theorem 5.3. *The inverse Fourier transform $\mathcal{F}^{-1}$ is a bijection of $L_c^2(\mathfrak{a}_+^* \times B)$ onto $\mathrm{PW}^2(X)$, mapping $L_R^2(\mathfrak{a}_+^* \times B)$ onto $\mathrm{PW}_R^2(X)$.*

Proof. Let $g \in L^2_R(\mathfrak{a}^*_+ \times B)$. Then $\mathcal{F}^{-1}g \in C^\infty(X)$ by Lebesgue's dominated convergence theorem. Equation (2) gives, for $D \in \mathbb{D}(X)$,

$$D(\mathcal{F}^{-1}g)(x) = \int_{\mathfrak{a}^*_+} \int_B \Gamma(D)(i\lambda) e^{(i\lambda+\rho)(A(x,b))} g(\lambda, b)|c(\lambda)|^{-2} d\lambda db,$$

which in particular shows that $(\Delta + \|\rho\|)^n \mathcal{F}^{-1}g = \mathcal{F}^{-1}((-1)^n \|\lambda\|^{2n} g) \in L^2(X)$ for all $n \in \mathbb{N} \cup \{0\}$. Parseval's formula (5) with

$$f_1 = f_2 = \mathcal{F}^{-1}((-1)^n \|\lambda\|^{2n} g)$$

yields:

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \int_X |(\Delta + \|\rho\|)^n (\mathcal{F}^{-1}g)(x)|^2 dx \right\}^{1/4n} \\ &= \lim_{n\to\infty} \left\{ \int_{\mathfrak{a}^*_+} \int_B \|\lambda\|^{4n} |g(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right\}^{1/4n} = R, \end{aligned}$$

whence $\mathcal{F}^{-1}g \in \mathrm{PW}^2_R(X)$.

Let now $f \in \mathrm{PW}^2_R(X)$. Then $\mathcal{F}((\Delta+\|\rho\|)^n f)(\lambda, b) = (-1)^n \|\lambda\|^{2n} \mathcal{F}f(\lambda, b) \in L^2(\mathfrak{a}^*_+ \times B)$ for all $n \in \mathbb{N}$ by (7). Another application of Parseval's formula as above with $f_1 = f_2 = (\Delta + \|\rho\|)^n f$ shows that $R_{\mathcal{F}f} = R^\Delta_f = R$, and we conclude that $\mathcal{F}f$ has bounded support. □

Corollary 5.4. *Let $f \in C^\infty(X)$ be such that $\Delta^n f \in L^2(X)$ for all $n \in \mathbb{N} \cup \{0\}$. It then follows that $\lim_{n\to\infty} \|\Delta^n f\|_2^{1/2n} < \infty$ if, and only if, $\lim_{n\to\infty} \|(\Delta+\|\rho\|^2)^n f\|_2^{1/2n} < \infty$. Furthermore, $\lim_{n\to\infty} \|\Delta^n f\|_2^{1/2n} = (R^2 + \|\rho\|^2)^{1/2}$, for $f \in \mathrm{PW}^2_R(X)$ with $R > 0$.*

Proof. Let $f \in \mathrm{PW}^2_R(X)$, with $R > 0$, then $\mathcal{F}f \in L^2_R(\mathfrak{a}^*_+ \times B)$. Parseval's formula and an easy adaption of the proof of Lemma 2.1 shows that

$$\begin{aligned} &\lim_{n\to\infty} \|\Delta^n f\|_2^{1/2n} \\ &= \lim_{n\to\infty} \left\{ \int_{\mathfrak{a}^*_+} \int_B (\|\lambda\|^2 + \|\rho\|^2)^{2n} |\mathcal{F}f(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right\}^{1/4n} \\ &= (R^2 + \|\rho\|^2)^{1/2}. \end{aligned}$$

Assume that $\lim_{n\to\infty} \|\Delta^n f\|_2^{1/2n} < \infty$. Then $\mathcal{F}(\Delta^n f)(\lambda, b) = (-1)^n(\|\lambda\|^2 + \|\rho\|^2)^n \mathcal{F}f(\lambda, b) \in L^2(\mathfrak{a}^*_+ \times B)$, for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \left\{ \int_{\mathfrak{a}^*_+} \int_B \|\lambda\|^{4n} |\mathcal{F}f(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right\}^{1/4n}$$

$$\leq \lim_{n\to\infty} \left\{ \int_{\mathfrak{a}^*_+} \int_B (\|\lambda\|^2 + \|\rho\|^2)^{2n} |\mathcal{F}f(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \right\}^{1/4n}$$

$$= \lim_{n\to\infty} \|\Delta^n f\|_2^{1/2n} < \infty,$$

that is, $\mathcal{F}f$ has bounded support. □

Remark 5.5. Assume that $f \in \mathcal{S}^p(X)$, with $0 < p < 2$, then $\mathcal{F}f$ extends to an analytic function on a small tube domain around $\mathfrak{a}^* \times B$ in $\mathfrak{a}^*_{\mathbb{C}} \times B$. Hence $\mathcal{F}f$ cannot have compact support on $\mathfrak{a}^* \times B$ and we conclude that $\mathcal{S}^p(X) \cap \mathrm{PW}^2(X) = \{0\}$ for any $0 < p < 2$.

Definition 5.6. We define the Paley-Wiener space $\mathrm{PW}(X)$ as the space of all functions $f \in C^\infty(X)$ satisfying:

(a) $(1 + |x|)^m \Delta^n f \in L^2(X)$ for all m, $n \in \mathbb{N} \cup \{0\}$.

(b) $R_f^\Delta = \lim_{n\to\infty} \|(\Delta + \|\rho\|^2)^n f\|_2^{1/2n} < \infty$.

Let also $\mathrm{PW}_R(X) := \{f \in \mathrm{PW}(X) \,|\, R_f^\Delta = R\}$, for $R \geq 0$.

Here $|x| := |g|$, for $x = gK \in X$. Again, the only difference between the Paley-Wiener spaces $\mathrm{PW}(X)$ and $\mathrm{PW}^2(X)$ is the polynomial decay condition (a), ensuring that $\mathcal{F}f \in C^\infty(\mathfrak{a}^* \times B)^W$ (see below).

The space $C_c^\infty(\mathfrak{a}^* \times B)^W$ is defined as the subspace of functions $\psi \in C_c^\infty(\mathfrak{a}^* \times B)$ satisfying the symmetry condition (3) for all $w \in W$ and all $\lambda \in \mathfrak{a}^*$, $x \in X$. Let finally $C_R^\infty(\mathfrak{a}^* \times B) := \{F \in C_c^\infty(\mathfrak{a}^* \times B) \,|\, R_g = R\}$.

The real Paley-Wiener theorem for the inverse Fourier transform then is:

Theorem 5.7. *The inverse Fourier transform $\mathcal{F}^{-1}$ is a bijection of $C_c^\infty(\mathfrak{a}^* \times B)^W$ onto $\mathrm{PW}(X)$, mapping $C_R^\infty(\mathfrak{a}^* \times B)^W$ onto $\mathrm{PW}_R(X)$.*

Proof. Let $g \in C_R^\infty(\mathfrak{a}^* \times B)^W$, then $g \in L_R^2(\mathfrak{a}^*_+ \times B)$ and thus $\mathcal{F}^{-1}g \in \mathrm{PW}_R^2(X)$ by Theorem 5.3. We furthermore see that $\mathcal{F}^{-1}g \in \mathcal{S}^2(X)$ by [4, Theorem 4.1.1], whence $\mathcal{F}^{-1}g$ satisfies the polynomial decay condition (a).

Let now $f \in \mathrm{PW}_R(X)$. The basic estimate $\|A(g)\| \leq C|g|$, for all $g \in G$, gives us a polynomial estimate (in x) of the derivatives (with respect to λ) of the plane wave eigenfunctions $e_{\lambda,b}(x)$. It is also well-known that $(1 + |x|)^{-r}\varphi_0 \in L^2(X)$ for some large $r \in \mathbb{N}$. All this, the polynomial decay condition (a), the Cauchy-Schwartz theorem and Lebesgue's dominated convergence theorem imply that $\mathcal{F}f \in C^\infty(\mathfrak{a}^* \times B)^W$. Furthermore $\mathcal{F}f$ has the desired compact support by Theorem 5.3. □

6. The inverse spherical transform.

In this section, we specialise our results to bi-K-invariant functions, that is, we consider the (inverse) spherical transform. We refer to [1], [5] and [6] for background concerning Paley-Wiener theorems for the spherical transform. Let $C^\infty(K\backslash G/K) \subset C^\infty(G)$ denote the subspace of bi-K-invariant differentiable functions on G. We will use similar notation for the L^2, Paley-Wiener and Schwartz spaces of K-invariant differentiable functions.

Let $f \in C_c^\infty(K\backslash G/K)$. The spherical transform $\mathcal{H}f$ of f is defined as:

$$\mathcal{H}f(\lambda) := \int_G f(x)\varphi_{-\lambda}(x)dx,$$

for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We note that $\mathcal{F}f(\lambda, b) = \mathcal{H}f(\lambda)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and all $b \in B$. This follows from left-K-invariance of f, the identity $A(k\cdot x, b) = A(x, k^{-1}\cdot b)$ and integrating over K.

The spherical transform is an isomorphism of $\mathcal{S}^2(K\backslash G/K)$ onto $\mathcal{S}(\mathfrak{a}^*)^W$, the Weyl group invariant Schwartz functions on $\mathfrak{a}^*$. The inversion formula is given by:

$$f(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*} \mathcal{H}f(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda, \qquad (x \in G) \tag{8}$$

for $f \in \mathcal{S}^2(K\backslash G/K)$. We use (8) to define the inverse spherical transform $\mathcal{H}^{-1}g$ for a general function g on $\mathfrak{a}^*$:

$$\mathcal{H}^{-1}g(x) := \frac{1}{|W|}\int_{\mathfrak{a}^*} g(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda.$$

Let $f \in C^\infty(K\backslash G/K)$ be such that $\Delta^n f \in L^2(K\backslash G/K)$ for all $n \in \mathbb{N}\cup\{0\}$. Then $\mathcal{H}((\Delta+\|\rho\|^2)^n f)(\lambda) = (-1)^n\|\lambda\|^{2n}\mathcal{H}f(\lambda)$ a.e., and Parseval's formula for $\mathcal{H}$ gives:

$$\int_G |(\Delta + \|\rho\|^2)^n f(x)|^2 dx = \frac{1}{|W|}\int_{\mathfrak{a}^*} \|\lambda\|^{4n}|\mathcal{H}f(\lambda)|^2|c(\lambda)|^{-2}d\lambda,$$

for all $n \in \mathbb{N}\cup\{0\}$. It also follows that the spherical transform extends to an isometry from $L^2(K\backslash G/K)$ onto $L^2(\mathfrak{a}^*, \frac{1}{|W|}|c(\lambda)|^{-2}d\lambda)^W$, where superscript W denotes Weyl group invariance.

Let $L_c^2(\mathfrak{a}^*)^W$ denote the Weyl group invariant L^2-functions on $\mathfrak{a}^*$ with compact support and let subscript R denote the radius of the support. The real versions of the Paley-Wiener theorems for the inverse spherical transform then becomes:

Theorem 6.1. *The inverse spherical transform $\mathcal{H}^{-1}$ is a bijection of $L_c^2(\mathfrak{a}^*)^W$ onto $\mathrm{PW}^2(K\backslash G/K)$, mapping $L_R^2(\mathfrak{a}^*)^W$ onto $\mathrm{PW}_R^2(K\backslash G/K)$.*

Theorem 6.2. *The inverse spherical transform $\mathcal{H}^{-1}$ is a bijection of $C_c^\infty(\mathfrak{a}^*)^W$ onto $\mathrm{PW}(K\backslash G/K)$, mapping $C_R^\infty(\mathfrak{a}^*)^W$ onto $\mathrm{PW}_R(K\backslash G/K)$.*

Proof. The above theorems are special cases of Theorem 5.3 and Theorem 5.7. We note, however, that we can prove them independently using Parseval's formula and intertwining properties of $\mathcal{H}$. □

Remark 6.3. Let $f \in \mathrm{PW}(K\backslash G/K)$ and consider f as a function on $\mathfrak{a}$ by the application $H \mapsto f(\exp(H))$. Then f does not extend to an entire function on $\mathfrak{a}_{\mathbb{C}}$, due to the poles of the spherical function $\varphi_\lambda(\exp(H))$. There is, however, a description of the Paley-Wiener space $\mathrm{PW}(K\backslash G/K)$ as functions having an explicit meromorphic extension and satisfying some exponential growth conditions for the rank 1 and the complex cases, see [**8**] for details.

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