**REAL POSITIVITY AND APPROXIMATE IDENTITIES
IN BANACH ALGEBRAS**

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Blecher and Read recently introduced and studied a new notion of positivity in operator algebras, with an eye to extending certain C^*-algebraic results and theories to more general algebras. In the present paper, we generalize some part of this, and some other facts, to larger classes of Banach algebras.

1. Introduction

An *operator algebra* is a closed subalgebra of $B(H)$ for a complex Hilbert space H. Blecher and Read [2011; 2013a; 2014] and Read [2011] recently introduced and studied a new notion of positivity in operator algebras (see also [Blecher and Neal 2012a; 2012b; Bearden et al. 2014; Blecher et al. 2008]), with an eye to extending certain C^*-algebraic results and theories to more general algebras. Over the last several years, we have mentioned in lectures on this work that most of the results of those papers make sense for bigger classes of Banach algebras, and

that many of the tools and techniques exist there. In the present paper we initiate this direction. Thus we generalize a number of the main results from the series of papers mentioned above, and some other facts, to a larger class of Banach algebras. In the process we give simplifications of several facts in these earlier papers. We will also point out some of the main results from the series of papers mentioned above which do not seem to generalize, or are less tidy if they do. (We will not spend much time discussing aspects from that series concerning noncommutative peak interpolation, or generalizations of noncommutative topology such as the noncommutative Urysohn lemma; these seem unlikely to generalize much farther.)

Before we proceed we make an editorial/historical note: the preprint [Blecher and Read 2013b], which contains many of the basic ideas and facts we use here, has been split into several papers, which have each taken on a life of their own (e.g., [Blecher and Read 2014] which focuses on operator algebras, and the present paper in the setting of Banach algebras).

In this paper we are interested in Banach algebras A (over the complex field) with a bounded approximate identity (bai). In fact, often there will be a contractive approximate identity (cai), and, in this case, we call A an *approximately unital* Banach algebra. A Banach algebra with an identity of norm 1 will be called *unital*. Most of our results are stated for approximately unital algebras. Frequently this is simply because algebras in this class have an especially nice "multiplier unitization" A^1, defined below, and a large portion of our constructs are defined in terms of A^1. Also, approximately unital algebras constitute a strong platform for the simultaneous generalization of as much as possible from the series of papers referenced above. However, as one might expect, for algebras without any kind of approximate identity it is easy to derive variants of a large portion of our results (namely, almost all of Sections 3, 4, and 7), by viewing the algebra as a subalgebra of a unital Banach algebra (any unitization, for example). We will discuss this point in more detail in Section 9 and in a forthcoming conference proceedings survey article [Blecher 2015].

Indeed many of our results are stated for special classes of Banach algebras, for example, for Banach algebras with a sequential cai or which are Hahn–Banach smooth in a sense defined later. Several of the results are sharper for *M-approximately unital Banach algebras*, which means that A is an M-ideal in its multiplier unitization A^1 (see Section 2). This is equivalent to saying that A is approximately unital and for all $x \in A^{**}$, we have $\|1 - x\|_{(A^1)^{**}} = \max\{\|e - x\|_{A^{**}}, 1\}$. Here e is the identity for A^{**}, if it has one (otherwise it is a "mixed identity" of norm 1 — see below for the definition of this). However, as will be seen from the proofs, some of the results involving the M-approximately unital hypothesis will work under weaker assumptions, for example, *strong proximinality* of A in A^1 at 1 (that is, given $\epsilon > 0$,

there exists a $\delta > 0$ such that if $y \in A$ with $\|1 - y\| < 1 + \delta$ then there is a $z \in A$ with $\|1 - z\| = 1$ and $\|y - z\| < \epsilon$).

We now outline the structure of this paper, describing each section briefly. Because our paper is rather diverse, to help the readers focus we will also mention at least one highlight from each section. In Section 2 we discuss unitization and states, and also introduce some classes of Banach algebras. A key result in this section ensures the existence of a "real positive" cai in Banach algebras with a countable cai satisfying a reasonable extra condition. We also characterize this extra condition, and the related property that the quasistate space be weak* closed and convex. In the latter setting, by the bipolar theorem, there exists a "Kaplansky density theorem". (Conversely, such a density result often immediately gives a real positive approximate identity by weak* approximating an identity in the bidual by real positive elements in A, and using, e.g., Lemma 2.1 below.) Section 3 starts by generalizing many of the basic ideas from the papers of Blecher and Read cited above involving cais, roots, and positivity. With these in place, we give several applications of the kind found in those papers; for example, we characterize when xA is closed in terms of the "generalized invertibility" of the real positive element x, and show that these are the right ideals qA for a real positive idempotent q in A. We also list several examples illustrating some of the things from the cited series of papers that will break down without further restrictions on the class of Banach algebras considered. The main advance in Section 4 is the introduction of the concept of *hereditary subalgebras* (HSAs), an important tool in C^*-algebra theory, to Banach algebras, and establishing the basics of their theory. In particular, we study the relationship between HSAs and one-sided ideals with one-sided approximate identities. Some aspects of this relationship are problematic for general Banach algebras, but it works much better in separable algebras, as we shall see. We characterize the HSAs, and the associated class of one-sided ideals, as increasing unions of "principal" ones; and indeed in the separable case they are exactly the "principal" ones. Indeed it is obvious that in a Banach algebra A, every closed right ideal with a real positive left bai is of the form $\overline{EA}$ for a set E of real positive elements of A. Section 4 contains an Aarnes–Kadison-type theorem for Banach algebras, and related results that use the Cohen's factorization proof technique. Some similar results and ideas have been found by Sinclair (in [Sinclair 1978], for example), but these are somewhat different, and were not directly connected to "positivity". It is interesting though that Sinclair was inspired by papers of Esterle based on the Cohen's factorization proof technique, and one of these does have some connection to our notion of positivity [Esterle 1978].

In Section 5 we consider the better behaved class of M-approximately unital Banach algebras. The main result here is the generalization of Read's theorem [Read 2011] to this class. That is, such algebras have cais (e_t) satisfying $\|1 - 2e_t\| \leq 1$. This may be the class to which the most results from our previous operator algebra

papers will generalize, as we shall see at points throughout our paper. In Section 6 we show that basic aspects and notions from the classical theory of ordered linear spaces correspond to interesting facts about our positivity for our various classes of approximately unital Banach algebras (for example, for M-approximately unital algebras, or certain algebras with a sequential cai). Indeed the highlight of this section is the revealing of interesting connections between Banach algebras and this classical ordered linear theory (see also [Blecher and Read 2014] for more, and clearer, such connections if the algebras are in addition operator algebras). In the process we generalize several basic facts about C^*-algebras. For example, we give the aforementioned variant of Kaplansky's density theorem, and variants of several well-known order-theoretic properties of the unit ball of a C^*-algebra and its dual.

In Sections 7 and 8 we find variants for approximately unital Banach algebras of several other results about two-sided ideals from [Blecher and Read 2011; 2013a; 2014]. In Section 7 we assume that A is commutative, and in this case we are able to establish the converse of the last result mentioned in our description of Section 4 above. Thus closed ideals having a real positive bai, in a commutative Banach algebra A, are precisely the spaces $\overline{EA}$ for sets E of real positive elements of A. In Section 8 we only consider ideals that are M-ideals in A (this does generalize the operator algebra case at least for two-sided ideals, since the closed two-sided ideals with cais in an operator algebra are exactly the M-ideals [Effros and Ruan 1990]). The lattice theoretic properties of such ideals behave considerably more like the C^*-algebra case and are related to faces in the quasistate space. Section 8 may be considered to be a continuation of the study of M-ideals in Banach algebras initiated in [Smith and Ward 1978; 1979; Smith 1979] and, e.g., [Harmand et al. 1993, Chapter V]. At the end of this section, we give a "noncommutative peak interpolation" result reminiscent of Tietze's extension theorem, which is based on a remarkable result of Chui, Smith, Smith, and Ward [Chui et al. 1977]. This solves an open problem from [Blecher and Read 2013b], or earlier, concerning real positive elements in a quotient. Finally, in Section 9 we discuss which results from earlier sections generalize to algebras without a cai; more details on this are given in [Blecher 2015]. The latter is a survey article which also contains a few additional details on some of the material in the present paper, as well as some small improvements found after this paper was in press.

We now list some of our notation and general facts: We write $\mathrm{Ball}(X)$ for the set $\{x \in X : \|x\| \le 1\}$. If E, F are sets then EF denotes the span of products xy for $x \in E, y \in F$. If $x \in A$ for a Banach algebra A, then $\mathrm{ba}(x)$ denotes the closed subalgebra generated by x. For two spaces X, Y which are in duality, for a subset E of X, we use the polar $E^\circ = \{y \in Y : \langle x, y \rangle \ge -1 \text{ for all } x \in E\}$.

For us, Banach algebras satisfy $\|xy\| \le \|x\|\|y\|$. We recall that a nonunital Banach algebra A is Arens regular if and only if its unitization is Arens regular (any

unitization will do here). In the rest of this paragraph, we consider an Arens regular approximately unital Banach algebra A. For such an algebra, we will always write e for the unique identity of A^{**}. Indeed if A is an Arens regular Banach algebra with cai (e_t), and $e_{t_\mu} \to \eta$ weak* in A^{**}, then $e_{t_\mu} a \to \eta a$ weak* for all $a \in A$. So $\eta a = a$, and similarly $a\eta = a$. Therefore η is the unique identity e of A^{**}, and $e_t \to e$ weak*. We will show at the end of this section that the multiplier unitization A^1 is isometrically isomorphic to the subalgebra $A + \mathbb{C}e$ of A^{**}.

If A is a Banach algebra which is not Arens regular, then the multiplication we usually use on A^{**} is the "second Arens product" ($\diamond$ in the notation of [Dales 2000]). This is weak* continuous in the second variable. If A is a nonunital, not necessarily Arens regular, Banach algebra with a bai, then A^{**} has a so-called "mixed identity" [Dales 2000; Palmer 1994; Doran and Wichmann 1979], which we will again write as e. This is a right identity for the first Arens product, and a left identity for the second Arens product. A mixed identity need not be unique; indeed, mixed identities are just the weak* limit points of bais for A.

We will also use the theory of M-ideals. These were invented by Alfsen and Effros, and [Harmand et al. 1993] is the basic text for their theory. We recall, a subspace E of a Banach space X is an M-ideal in X if $E^{\perp\perp}$ is complemented in X^{**} via a contractive projection P so that $X^{**} = E^{\perp\perp} \oplus^\infty \operatorname{Ker} P$. In this case, there is a unique contractive projection onto $E^{\perp\perp}$. M-ideals have many beautiful properties, some of which will be mentioned below.

We will need the following result several times:

Lemma 1.1. *Let X be a Banach space, and suppose that (x_t) is a bounded net in X with $x_t \to \eta$ weak* in X^{**}. Then*

$$\|\eta\| = \lim_t \inf\{\|y\| : y \in \operatorname{conv}\{x_j : j \geq t\}\}.$$

Proof. It is easy to see that $\|\eta\| \leq \lim_t \inf\{\|y\| : y \in \operatorname{conv}\{x_j : j \geq t\}\}$, for example, by using the weak* semicontinuity of the norm, and noting that for every t and any choice $y_t \in \operatorname{conv}\{x_j : j \geq t\}$, we have $y_t \to \eta$ weak*. By way of contradiction, suppose that

$$\|\eta\| < C < \lim_t \inf\{\|y\| : y \in \operatorname{conv}\{x_j : j \geq t\}\}.$$

Then there exists t_0 such that the norm closure of $\operatorname{conv}\{x_j : j \geq t\}$ is disjoint from $C \operatorname{Ball}(X)$ for all $t \geq t_0$. By the Hahn–Banach theorem, there exists $\varphi \in X^*$ with

$$C \|\varphi\| < K < \operatorname{Re} \varphi(x_j), \quad j \geq t,$$

so that $C \|\varphi\| < K \leq \operatorname{Re} \varphi(\eta)$. This contradicts $\|\eta\| < C$. $\qquad \square$

Any nonunital operator algebra has a unique operator algebra unitization (see [Blecher and Le Merdy 2004, Section 2.1]), but of course this is not true for Banach

algebras. We will choose to use the unitization that typically has the smallest norm among all unitizations, and which we now describe. If A is an approximately unital Banach algebra, then the left regular representation embeds A isometrically in $B(A)$. We will always write A^1 for the *multiplier unitization* of A; that is, we identify A^1 isometrically with $A + \mathbb{C}I$ in $B(A)$. For $a \in A, \lambda \in \mathbb{C}$, we have

$$\|a + \lambda 1\| = \sup\{\|ac + \lambda c\| : c \in \text{Ball}(A)\} = \sup_t \|ae_t + \lambda e_t\| = \lim_t \|ae_t + \lambda e_t\|,$$

(see [loc. cit., A.4.3], for example). If A is actually nonunital then the map $\chi_0(a + \lambda 1) = \lambda$ on A^1 is contractive, as is any character on a Banach algebra. We call this the *trivial character*. Below, 1 will almost always denote the identity of A^1, if A is not already unital. Note that the multiplier unitization also makes sense for the so-called *self-induced* Banach algebras, namely those for which the left regular representation embeds A isometrically in $B(A)$.

If A is a nonunital, approximately unital Banach algebra then the multiplier unitization A^1 may also be identified with a subalgebra of A^{**}. Indeed if e is a mixed identity of norm 1 for A^{**} then $A + \mathbb{C}e$ is then a unitization of A (by basic facts about the Arens product). To see that this is isometric to A^1 above, note that for any $c \in \text{Ball}(A), a \in A, \lambda \in \mathbb{C}$, we have

$$\|ac + \lambda c\| \leq \|a + \lambda e\|_{A^{**}} = \|e(a + \lambda 1)\|_{(A^1)^{**}} \leq \|a + \lambda 1\|_{A^1}.$$

Thus by the displayed equation in the last paragraph, $\|a + \lambda e\|_{A^{**}} = \|a + \lambda 1\|_{A^1}$ as desired.

2. Unitization and states

If A is an approximately unital Banach algebra, then we may view A in its multiplier unitization A^1, and write

$$\mathfrak{F}_A = \{a \in A : \|1 - a\| \leq 1\} = \{a \in A : \|e - a\| \leq 1\},$$

where e is as in the last paragraph (or set $e = 1$ if A is unital). So

$$\tfrac{1}{2}\mathfrak{F}_A = \{a \in A : \|1 - 2a\| \leq 1\}.$$

If $x \in \frac{1}{2}\mathfrak{F}_A$ then $x, 1 - x \in \text{Ball}(A^1)$. Also, $\mathfrak{F}_A = \mathfrak{F}_{A^1} \cap A$, and $\mathfrak{F}_A$ is closed under the quasiproduct $a + b - ab$. (It is interesting that cones containing $\mathfrak{F}_A$ were used to obtain nice results about "order" in unital Banach algebras and their duals in Section 1 of the historically important paper [Kelley and Vaught 1953], based on a 1951 ICM talk. Slightly earlier, $\mathfrak{F}_A$ also appeared in a memoir by Kadison.)

If $\eta \in A^{**}$ then an expression such as $\lambda 1 + \eta$ will usually need to be interpreted as an element of $(A^1)^{**}$, with 1 interpreted as the identity for A^1 and $(A^1)^{**}$. Thus $\|1 - \eta\|$ denotes $\|1 - \eta\|_{(A^1)^{**}}$. We define

$$\mathfrak{F}_{A^{**}} = \{\eta \in A^{**} : \|1 - \eta\| \leq 1\} = A^{**} \cap \mathfrak{F}_{(A^1)^{**}}.$$

We write $\mathfrak{r}_A$ for the set of $a \in A$ whose numerical range in A^1 is contained in the right half-plane. That is,

$$\mathfrak{r}_A = \{a \in A : \operatorname{Re} \varphi(a) \geq 0 \text{ for all } \varphi \in S(A^1)\},$$

where $S(A^1)$ denotes the states on A^1. Note that $\mathfrak{r}_A$ is a closed cone in A, but it is not proper (hence it is what is sometimes called a *wedge*). We write $a \preceq b$ if $b - a \in \mathfrak{r}_A$. It is easy to see that $\mathbb{R}^+ \mathfrak{F}_A \subset \mathfrak{r}_A$. Conversely, if A is a unital Banach algebra and $a \in \mathfrak{r}_A$ then $a + \epsilon 1 \in \mathbb{R}^+ \mathfrak{F}_A$ for every $\epsilon > 0$. Indeed $a + \epsilon 1 \in C \mathfrak{F}_A$, where $C = \|a\|^2/\epsilon + \epsilon$, as can be easily seen from the well-known fact that the numerical range of a is contained in the right half-plane if and only if $\|1 - ta\| \leq 1 + t^2 \|a\|^2$ for all $t > 0$ (see, e.g., [Magajna 2009, Lemma 2.1]).

One main reason why we almost always assume that A is approximately unital in this paper is that $\mathfrak{F}_A$ and $\mathfrak{r}_A$ are well-defined as above. However, as we said in the introduction, if A is not approximately unital, it is easy to see how to proceed in a large number of our results (namely in almost all of Sections 3, 4, and 7), and this is discussed briefly in Section 9.

The following is no doubt in the literature, but we do not know of a reference that proves all that is claimed. It follows from it that mixed identities in A^{**} are just the weak* limits of bais for A, when these limits exist.

Lemma 2.1. *If A is a Banach algebra, and if a bounded net $x_t \in A$ converges weak* to a mixed identity $e \in A^{**}$, then a bai for A can be found with weak* limit e, and formed from convex combinations of the x_t.*

Proof. Given $\epsilon > 0$ and a finite set $F \subset A^*$, there exists $t_{F,\epsilon}$ such that

$$|\varphi(x_t) - e(\varphi)| < \epsilon, \quad t \geq t_{F,\epsilon}, \varphi \in F.$$

Given a finite set $E = \{a_1, \ldots, a_n\} \subset A$, we have that $x_t a_k \to a_k$ and $a_k x_t \to a_k$ weakly. So there is a convex combination y of the x_t for $t \geq t_{F,\epsilon}$ with

$$\|y a_k - a_k\| + \|a_k y - a_k\| \leq \epsilon.$$

We also have $|\varphi(y) - e(\varphi)| \leq \epsilon$ for $\varphi \in F$. Write this y as y_λ, where $\lambda = (E, F, \epsilon)$. Given $\epsilon_0 > 0$ and $a \in A$, if $\epsilon \leq \epsilon_0$ and $\{a\} \subset E$, then $\|y_\lambda a - a\| + \|a y_\lambda - a\| \leq \epsilon \leq \epsilon_0$ for $\lambda = (E, F, \epsilon)$ with any F. So (y_λ) is a bai. Also if $\varphi \in F$ then $|\varphi(y_\lambda) - e(\varphi)| < \epsilon$. So $y_\lambda \to e$ weak*.   □

Remark. The "sequential version" of the last result is false. For example, consider the usual cai $(n\chi_{[-1/(2n),1/(2n)]})$ of $L^1(\mathbb{R})$ with convolution product. A subnet of this converges weak* to a mixed identity $e \in L^1(\mathbb{R})^{**}$. However, there can be no weak* convergent sequential bai for $L^1(\mathbb{R})$, since $L^1(\mathbb{R})$ is weakly sequentially complete.

For a general approximately unital nonunital Banach algebra A with cai (e_t), the definition of "state" is problematic. There are many natural notions, for example: (i) a contractive functional φ on A with $\varphi(e_t) \to 1$ for some fixed cai (e_t) for A, (ii) a contractive functional φ on A with $\varphi(e_t) \to 1$ for all cai (e_t) for A, and (iii) a norm 1 functional on A that extends to a state on A^1, where A^1 is the multiplier unitization above. If A is not Arens regular then (i) and (ii) can differ; that is, whether $\varphi(e_t) \to 1$ depends on which cai for A we use. And if e is a mixed identity then the statement $\varphi(e) = 1$ may depend on which mixed identity one considers. In this paper, for simplicity, and because of its connections with the usual theory of numerical range and accretive operators, we will take (iii) above as the definition of a *state* of A. We shall also often consider states in the sense of (i), and will usually ignore (ii) since in some sense it may be treated as a "special case" of (i) (that is, almost all computations in the paper involving the class (i) are easily tweaked to give the "(ii) version"). We define $S(A)$ to be the set of states in the sense of (iii) above. This is easily seen to be norm closed, but will not be weak* closed if A is nonunital. We define

$$\mathfrak{c}_{A^*} = \{\varphi \in A^* : \operatorname{Re}\varphi(a) \ge 0 \text{ for all } a \in \mathfrak{r}_A\},$$

and note that this is a weak* closed cone containing $S(A)$. These are called the *real positive functionals* on A. If $\mathfrak{e} = (e_t)$ is a fixed cai for A, define

$$S_\mathfrak{e}(A) = \{\varphi \in \operatorname{Ball}(A^*) : \lim_t \varphi(e_t) = 1\}$$

(this corresponds to (i) above). Note that $S_\mathfrak{e}(A)$ is convex but $S(A)$ may not be (as in, e.g., Example 3.16). An argument in the next proof shows that $S_\mathfrak{e}(A) \subset S(A)$. Finally we remark that for any $y \in A$ of norm 1, if $\varphi \in \operatorname{Ball}(A^*)$ satisfies $\varphi(y) = 1$, then $x \mapsto \varphi(yx)$ is in $S_\mathfrak{e}(A)$ for all cais $\mathfrak{e}$ of A.

We recall that a subspace E of a Banach space X is called Hahn–Banach smooth in X if every functional on E has a unique Hahn–Banach extension to X. Any M-ideal in X is Hahn–Banach smooth in X. See [Harmand et al. 1993] and references therein for more on this topic.

Lemma 2.2. *For approximately unital Banach algebras A which are Hahn–Banach smooth in A^1, and therefore for M-approximately unital Banach algebras, and $\varphi \in A^*$ with norm 1, the following are equivalent*:

 (i) *φ is a state on A (that is, extends to a state on A^1).*

 (ii) *$\varphi(e_t) \to 1$ for every cai (e_t) for A.*

 (iii) *$\varphi(e_t) \to 1$ for some cai (e_t) for A.*

 (iv) *$\varphi(e) = 1$ whenever $e \in A^{**}$ is a weak* limit point of a cai for A (that is, whenever e is a mixed identity of norm 1 for A^{**}).*

Proof. Clearly (ii) implies (iii). If $\varphi \in \text{Ball}(A^*)$, write $\tilde{\varphi}$ for its canonical weak* continuous extension to A^{**}. If (e_t) is a cai for A with weak* limit point e and $\varphi(e_t) \to 1$, then $\tilde{\varphi}(e) = 1$. It follows that $\tilde{\varphi}_{|A^1}$ is a state on A^1. So (iii) implies (i). To see that (i) implies (iv), suppose that A is Hahn–Banach smooth in A^1, and that φ is a norm 1 functional on A that extends to a state ψ on A^1. If (e_t) is a cai for A with weak* limit point e, then also $\tilde{\varphi}_{|A+\mathbb{C}e}$ is a norm-1 functional extending φ so that $\tilde{\varphi}_{|A+\mathbb{C}e} = \psi$, and for some subnet,

$$\varphi(e) = \lim_t \varphi(e_{t_\mu}) = \tilde{\varphi}(e) = \psi(1) = 1.$$

We leave the remaining implication as an exercise. □

Under certain conditions on an approximately unital Banach algebra A, we shall see in Corollary 2.8 that $S(A^1)$ is the convex hull of the trivial character χ_0 and the set of states on A^1 extending states of A, and that the weak* closure of $S(A)$ equals $\{\varphi_{|A} : \varphi \in S(A^1)\}$.

The numerical range $W(a)$ (or $W_A(a)$) of $a \in A$, if A is an approximately unital Banach algebra, will be defined to be $\{\varphi(a) : \varphi \in S(A)\}$. If A is Hahn–Banach smooth in A^1 then it follows from Lemma 2.2 that $S(A)$ is convex, and hence so is $W(a)$. We shall see in Corollary 2.8 that under the condition mentioned in the last paragraph, we have $\overline{W_A(a)} = \text{conv}\{0, W_A(a)\} = W_{A^1}(a)$.

The following is related to results from [Smith and Ward 1979] or [Harmand et al. 1993, Section V.3] or [Arias and Rosenthal 2000; Davidson and Power 1986].

Lemma 2.3. *If A is an approximately unital Banach algebra, if A^1 is the unitization above, and if e is a weak* limit of a cai (resp. bai in $\mathfrak{F}_A$) for A then $\|1 - 2e\|_{(A^1)^{**}} \leq 1$ if and only if there is a cai (resp. bai in $\mathfrak{F}_A$) (e_i) with weak* limit e and $\limsup_i \|1 - 2e_i\|_{A^1} \leq 1$.*

Proof. One direction follows from Alaoglu's theorem. Suppose $\|1 - 2e\|_{(A^1)^{**}} \leq 1$ and there is a net (x_t) which is a cai (resp. bai in $\mathfrak{F}_A$) for A with $x_t \to e$ weak*. Then $1 - 2x_t \to 1 - 2e$ weak* in $(A^1)^{**}$. By Lemma 1.1, for any $n \in \mathbb{N}$, there exists a t_n such that for every $t \geq t_n$,

$$\inf\{\|1 - 2y\| : y \in \text{conv}\{x_j : j \geq t\}\} < 1 + \frac{1}{2n}.$$

For every $t \geq t_n$, choose such a $y_t^n \in \text{conv}\{x_j : j \geq t\}$ with $\|1 - 2y_t^n\| < 1 + 1/n$. If t does not dominate t_n, define $y_t^n = y_{t_n}^n$. So for all t, we have $\|1 - 2y_t^n\| < 1 + 1/n$. Writing (n, t) as i, we may view (y_t^n) as a net (e_i) indexed by i, with $\|1 - 2y_t^n\| \to 1$. Given $\epsilon > 0$ and $a_1, \ldots, a_m \in A$, there exists a t_1 such that $\|x_t a_k - a_k\| < \epsilon$ and $\|a_k x_t - a_k\| < \epsilon$ for all $t \geq t_1$ and all $k = 1, \ldots, m$. Hence the same assertion is true with x_t replaced by y_t^n. Thus $(y_t^n) = (e_i)$ is a bai for A with the desired property. □

We recall from the introduction that if A is an approximately unital Banach algebra which is an M-ideal in the particular unitization A^1 above, then A is an

M-approximately unital Banach algebra. Any unital Banach algebra is an M-approximately unital Banach algebra (here $A^1 = A$). By [Harmand et al. 1993, Proposition I.1.17(b)], examples of M-approximately unital Banach algebras include any Banach algebra that is an M-ideal in its bidual, and which is approximately unital (or whose bidual has an identity). Several examples of such are given in [loc. cit.], for example, the compact operators on ℓ^p for $1 < p < \infty$. We also recall that the property of being an M-ideal in its bidual is inherited by subspaces, and hence by subalgebras. Not every Banach algebra with a cai is M-approximately unital. By [loc. cit., Proposition II.3.5], $L^1(\mathbb{R})$ with convolution multiplication cannot be an M-ideal in any proper superspace.

We just said that any unital Banach algebra A is M-approximately unital; hence, any finite dimensional unital Banach algebra is Arens regular and M-approximately unital (if one wishes to avoid the redundancy of $A = A^1$ in the discussion below, take the direct sum of A with any Arens regular M-approximately unital Banach algebra, such as c_0). Thus any kind of bad behavior occurring in finite-dimensional unital Banach algebras (resp. unital Banach algebras) will appear in the class of Arens regular M-approximately unital Banach algebras (resp. M-approximately unital Banach algebras). This will have the consequence that several aspects of the Blecher–Read papers will not generalize, for instance, conclusions involving "near positivity". This can also be seen in the examples scattered through our paper, for instance, Examples 3.13–3.16 below.

Suppose that (e_t) is a cai for a Banach algebra A with weak* limit point $e \in A^{**}$. Then left multiplication by e (in the second Arens product) is a contractive projection from $(A^1)^{**}$ onto the ideal $A^{\perp\perp}$ of $(A^1)^{**}$ (note that $(A^1)^{**} = A^{\perp\perp} + \mathbb{C}1 = A^{\perp\perp} + \mathbb{C}(1-e)$). Thus by the theory of M-ideals [loc. cit.], A is an M-ideal in A^1 if and only if left multiplication by e is an M-projection.

Lemma 2.4. *A nonunital approximately unital Banach algebra A is M-approximately unital if and only if for all $x \in A^{**}$, we have*

$$\|1 - x\|_{(A^1)^{**}} = \max\{\|e - x\|_{A^{**}}, 1\}.$$

Here e is a mixed identity for A^{**} of norm 1. If these conditions hold then there is a unique mixed identity for A^{**} of norm 1, it belongs in $\frac{1}{2}\mathfrak{F}_{A^{**}}$, and

$$\|1 - \eta\| = 1 \quad \Longleftrightarrow \quad \|e - \eta\| \le 1, \quad \eta \in A^{**}.$$

Proof. By the statement immediately above the lemma, and by the theory of M-ideals [Harmand et al. 1993], A is an M-ideal in A^1 if and only if left multiplication by e is an M-projection, that is, if and only if

$$\|\eta + \lambda 1\|_{(A^1)^{**}} = \max\{\|\eta + \lambda e\|_{A^{**}}, |\lambda| \|1 - e\|\}, \quad \eta \in A^{**}, \lambda \in \mathbb{C}.$$

If this holds then setting $\lambda = 1$ and $\eta = 0$ shows that $\|1 - e\| \leq 1$. However, by the Neumann lemma we cannot have $\|1 - e\| < 1$. Thus $\|1 - e\| = 1$ if these hold. The statement is tautological if $\lambda = 0$, so we may assume the contrary. Dividing by $|\lambda|$ and setting $x = -\eta/|\lambda|$, one sees that A is M-approximately unital if and only if

$$\|1 - x\|_{(A^1)^{**}} = \max\{\|e - x\|_{A^{**}}, 1\}, \quad x \in A^{**}.$$

In particular, $\|1 - 2e\|_{(A^1)^{**}} = \max\{\|e\|, 1\} = 1$. The final assertion is now clear too. The uniqueness of the mixed identity follows from the next result.   □

Remark. Indeed if B is any unitization of a nonunital approximately unital Banach algebra A, and if A is an M-ideal in B, then the first few lines of the last proof, with A^1 replaced by B, show that $B = A^1$, the multiplier unitization of A.

Thus A is M-approximately unital if and only if $\|1 - x\|_{(A^1)^{**}} = \|e - x\|_{A^{**}}$ for all $x \in A^{**}$, unless the last quantity is less than 1, in which case $\|1 - x\|_{(A^1)^{**}} = 1$.

We will show later that for M-approximately unital Banach algebras, there is a cai (e_t) for A with $\|1 - 2e_t\|_{A^1} \leq 1$ for all t.

Lemma 2.5. *Let A be a closed ideal, and also an M-ideal, in a unital Banach algebra B. If e and f are two weak* limit points in A^{**} of two cai for A, then $e = f$. Thus A^{**} has a unique mixed identity of norm 1. In particular, if A is M-approximately unital then A^{**} has a unique mixed identity of norm 1.*

Proof. As in the discussion above Lemma 2.4, left multiplications by e or f, in the second Arens product, are contractive projections onto the ideal $A^{\perp\perp}$ of $(A^1)^{**}$. So these maps equal the M-projection [Harmand et al. 1993], and hence are equal. So $e = f$. Thus every cai for A converges weak* to e, so that A^{**} has a unique mixed identity.   □

If A is an approximately unital Banach algebra, but A^{**} has no identity then we define $\mathfrak{r}_{A^{**}} = A^{**} \cap \mathfrak{r}_{(A^1)^{**}}$. If A is an approximately unital Banach algebra then $\mathfrak{F}_{A^{**}}$ and $\mathfrak{r}_{A^{**}}$ are weak* closed. Indeed the $\mathfrak{F}_{A^{**}}$ case of this is obvious. By [Magajna 2009], $\mathfrak{r}_{(A^1)^{**}}$ is weak* closed, hence so is $\mathfrak{r}_{A^{**}} = A^{**} \cap \mathfrak{r}_{(A^1)^{**}}$.

Remark. Note that if A^{**} has a mixed identity of norm 1 then we can define states of A^{**} to be norm-1 functionals φ with $\varphi(e) = 1$ for all mixed identities e of A^{**} of norm 1. Then one could define $\mathfrak{r}_{A^{**}}$ to be the elements $x \in A^{**}$ with $\text{Re } \varphi(x) \geq 0$ for all such states of A^{**}. This coincides with the definition of $\mathfrak{r}_{A^{**}}$ above the remark if A is M-approximately unital. Indeed such states φ on A^{**} extend to states $\varphi(e \cdot)$ of $(A^1)^{**}$. Conversely if A is an M-approximately unital Banach algebra, then given a state φ of $(A^1)^{**}$, we have

$$1 = \|\varphi\| = \|\varphi \cdot e\| + \|\varphi \cdot (1 - e)\| \geq |\varphi(e)| + |\varphi(1 - e)| \geq \varphi(1) = 1 = \varphi(e) + \varphi(1 - e).$$

It follows from this that $\|\varphi e\| = |\varphi(e)| = \varphi(e)$. Hence if $\eta \in \mathrm{Ball}(A^{**})$ then

$$|\varphi(\eta)| = |\varphi e(\eta)| \leq \|\varphi e\| = \varphi(e),$$

so that the restriction of φ to A^{**} is either zero or is a positive multiple of a state on A^{**}. Thus for M-approximately unital Banach algebras, the two notions of $\mathfrak{r}_{A^{**}}$ under discussion coincide.

Let $Q(A)$ be the quasistate space of A, namely $Q(A) = \{t\varphi : t \in [0, 1], \varphi \in S(A)\}$. Similarly, $Q_{\mathfrak{e}}(A) = \{t\varphi : t \in [0, 1], \varphi \in S_{\mathfrak{e}}(A)\}$. We set

$$\mathfrak{r}_A^{\mathfrak{e}} = \{x \in A : \mathrm{Re}\,\varphi(x) \geq 0 \text{ for all } \varphi \in S_{\mathfrak{e}}(A)\},$$

$$\mathfrak{c}_{A^*}^{\mathfrak{e}} = \{\varphi \in A^* : \mathrm{Re}\,\varphi(x) \geq 0 \text{ for all } x \in \mathfrak{r}_A^{\mathfrak{e}}\}.$$

Note that $\mathfrak{r}_A \subset \mathfrak{r}_A^{\mathfrak{e}}$ since $S_{\mathfrak{e}}(A) \subset S(A)$.

Lemma 2.6. *Let A be a nonunital Banach algebra with a cai $\mathfrak{e}$.*

(1) *Then 0 is in the weak* closure of $S_{\mathfrak{e}}(A)$. Hence 0 is in the weak* closure of $S(A)$. Thus $Q(A)$ is a subset of the weak* closure of $S(A)$, and similarly $Q_{\mathfrak{e}}(A) \subset \overline{S_{\mathfrak{e}}(A)}^{\mathrm{w}*}$.*

(2) *The weak* closure of $S_{\mathfrak{e}}(A)$ is contained in $\mathfrak{c}_{A^*}^{\mathfrak{e}} \cap \mathrm{Ball}(A^*)$. It is also contained in $S(A^1)_{|A}$, and both of the latter two sets are subsets of $\mathfrak{c}_{A^*} \cap \mathrm{Ball}(A^*)$.*

Proof. (1) For every t, there exists $s(t) \geq t$ such that $\|e_{s(t)} - e_t\| \geq 1/2$ (or else taking the limit over $s > t$, we get the contradiction $\|1 - e_t\| < 1$, which is impossible by the Neumann lemma, or since the trivial character χ_0 is contractive). Take a norm-1 $\psi_t \in A^*$ such that $\psi_t(e_{s(t)} - e_t) = \|e_{s(t)} - e_t\|$. Let $\Phi_t(x) = \psi_t((e_{s(t)} - e_t)x)/\|e_{s(t)} - e_t\|$. Then $\Phi_t \in S_{\mathfrak{e}}(A)$ because it has norm 1 and $\lim_s \Phi_t(e_s) = 1$. One has $\lim_t \Phi_t(x) = 0$ for all $x \in A$. To see this, given $\epsilon > 0$, choose t_0 such that $\|e_t x - x\| < \epsilon$ for all $t \geq t_0$. For such t, we have

$$\frac{|\psi_t((e_{s(t)} - e_t)x)|}{\|e_{s(t)} - e_t\|} \leq 2\|\psi_t\|\|(e_{s(t)} - e_t)x\| < 4\epsilon.$$

Thus $\Phi_t \to 0$ weak*. The rest is obvious.

(2) The first assertion is clear by the definitions and since $\mathfrak{c}_{A^*}^{\mathfrak{e}} \cap \mathrm{Ball}(A^*)$ is weak* closed. Similarly, that the weak* closure is contained in $S(A^1)_{|A}$ follows since $S_{\mathfrak{e}}(A) \subset S(A)$ as we saw above, and because $S(A^1)$ and hence $S(A^1)_{|A}$, are weak* closed. We leave the rest as an exercise using $\mathfrak{r}_A \subset \mathfrak{r}_A^{\mathfrak{e}}$. □

We will say that an approximately unital Banach algebra A is *scaled* (resp. $\mathfrak{e}$-*scaled*) if every f in $\mathfrak{c}_{A^*}$ (resp. in $\mathfrak{c}_{A^*}^{\mathfrak{e}}$) is a nonnegative multiple of a state, that is, if and only if $\mathfrak{c}_{A^*} = \mathbb{R}^+ S(A)$ (resp. $\mathfrak{c}_{A^*}^{\mathfrak{e}} = \mathbb{R}^+ S_{\mathfrak{e}}(A)$), equivalently, if and only if $\mathfrak{c}_{A^*} \cap \mathrm{Ball}(A^*) = Q(A)$ (resp. $\mathfrak{c}_{A^*}^{\mathfrak{e}} \cap \mathrm{Ball}(A^*) = Q_{\mathfrak{e}}(A)$). Examples of scaled Banach algebras include M-approximately unital Banach algebras (see Proposition 6.2)

and $L^1(\mathbb{R})$ with convolution product. One can show that $L^1(\mathbb{R})$ is not $\mathfrak{e}$-scaled if $\mathfrak{e}$ is the usual cai (see the remark after Lemma 2.1 and Example 3.16).

Lemma 2.7. *Let A be an approximately unital Banach algebra.*

(1) *Suppose that $\mathfrak{e} = (e_t)$ is a cai for A. Then $Q_{\mathfrak{e}}(A)$ is weak* closed in A^* if and only if A is $\mathfrak{e}$-scaled. If these hold then $Q_{\mathfrak{e}}(A)$ is a weak* compact convex set in $\mathrm{Ball}(A^*)$, and $S_{\mathfrak{e}}(A)$ is weak* dense in $Q_{\mathfrak{e}}(A)$.*

(2) *If $S(A)$ or $Q(A)$ is convex then $Q(A)$ is weak* closed in A^* if and only if A is scaled.*

Proof. (1) By the bipolar theorem, $\mathfrak{c}^{\mathfrak{e}}_{A^*} = \overline{\mathbb{R}^+ s_{\mathfrak{e}}(a)}^{w^*}$. So $\mathbb{R}^+ S_{\mathfrak{e}}(A)$ is weak* closed if and only if $\mathfrak{c}^{\mathfrak{e}}_{A^*} = \mathbb{R}^+ S_{\mathfrak{e}}(A)$, that is, if and only if A is $\mathfrak{e}$-scaled. By the Krein–Smulian theorem this happens if and only if $\mathrm{Ball}(\mathbb{R}^+ S_{\mathfrak{e}}(A)) = Q_{\mathfrak{e}}(A)$ is weak* closed. The weak* density assertion follows from Lemma 2.6.

(2) This follows by a similar argument to (1) if $Q(A)$ is convex (and this is implied by $S(A)$ being convex). $\qquad\square$

Corollary 2.8. *If A is a nonunital approximately unital Banach algebra, then the following are equivalent*:

(i) *A is scaled.*

(ii) *$S(A^1)$ is the convex hull of the trivial character χ_0 and the set of states on A^1 extending states of A.*

(iii) *$Q(A) = \{\varphi_{|A} : \varphi \in S(A^1)\}$.*

(iv) *$Q(A)$ is convex and weak* compact.*

If these hold then $Q(A) = \overline{S(A)}^{w^*}$, and the numerical range satisfies

$$\overline{W_A(a)} = \mathrm{conv}\{0, W_A(a)\} = W_{A^1}(a), \quad a \in A.$$

Proof. (i) $\Rightarrow$ (ii): Clearly the convex hull in (ii) is a subset of $S(A^1)$. Conversely, if $\varphi \in S(A^1)$ then $\varphi_{|A}$ is real positive, so that by (i) we have $\varphi_{|A} = t\psi$ for $t \in (0, 1]$ and $\psi \in S(A)$. Then $\varphi = t\hat{\psi} + (1-t)\chi_0$, where $\hat{\psi}$ is the state extending ψ.

(ii) $\Rightarrow$ (iii): We leave this as an exercise.

(iii) $\Rightarrow$ (iv): Suppose that (φ_t) is a net in $S(A^1)$ whose restrictions to A converge weak* to $\psi \in A^*$. A subnet (φ_{t_λ}) converges weak* to $\varphi \in S(A^1)$, and $\psi = \varphi_{|A}$, clearly. This gives the weak* compactness in (iv), and the convexity is easier.

(iv) $\Rightarrow$ (i): This follows from (2) of the previous lemma.

Assume that these hold. Since $S(A) \subset Q(A)$, that $Q(A) = \overline{S(A)}^{w^*}$ is now clear from the fact from Lemma 2.6 that $Q(A) \subset \overline{S(A)}^{w^*}$. Since A is nonunital, we have $0 \in W_{A^1}(a)$. Clearly $W_A(a) \subset W_{A^1}(a)$, so that $\mathrm{conv}\{0, W_A(a)\} \subset W_{A^1}(a)$. The converse inclusion follows easily from the above, so $\mathrm{conv}\{0, W_A(a)\} = W_{A^1}(a)$.

Also, clearly $\overline{W_A(a)} \subset W_{A^1}(a)$, and the converse inclusion follows since $S(A^1)_{|A} = Q(A) = \overline{S(A)}^{w*}$. $\hfill\square$

Remark. (1) Thus if $S(A) = S_{\mathfrak{e}}(A)$ for some cai $\mathfrak{e}$ of A, then A is scaled if and only if $Q(A)$ is weak* closed.

(2) In particular, if A is unital then conditions (i) and (iv) in the previous result are automatically true. Indeed $S(A)$ is weak* closed, and hence $Q(A)$ is too, and the rest follows from Lemma 2.7. Item (i) also follows from the proof of [Magajna 2009, Theorem 2.2].

Theorem 2.9. *Let $\mathfrak{e} = (e_n)$ be a sequential cai for a Banach algebra A. If $Q_{\mathfrak{e}}(A)$ is weak* closed, then A possesses a sequential cai in $\mathfrak{r}_A^{\mathfrak{e}}$. Moreover, for every $a \in A$ with $\inf\{\mathrm{Re}\,\varphi(a) : \varphi \in S_{\mathfrak{e}}(A)\} > -1$, there is a sequential cai (f_n) in $\mathfrak{r}_A^{\mathfrak{e}}$ such that $f_n + a \in \mathfrak{r}_A^{\mathfrak{e}}$ for all n.*

Proof. We first state a general fact about compact spaces K. If (f_n) is a bounded sequence in $C(K, \mathbb{R})$ such that $\lim_n f_n(x)$ exists for every $x \in K$ and is nonnegative, then for every $\epsilon > 0$, there is a function $f \in \mathrm{conv}\{f_n\}$ such that $f \geq -\epsilon$ on K. Indeed if this were not true, then $\overline{\mathrm{conv}}\{f_n\}$ and $C(K)_+$ would be disjoint. By a Hahn–Banach separation argument and the Riesz–Markov theorem, there is a probability measure m such that $\sup_n \int_K f_n\, dm < 0$. This is a contradiction since $\lim_n \int_K f_n\, dm \geq 0$ by Lebesgue's dominated convergence theorem.

Set K to be the weak* closure of $S_{\mathfrak{e}}(A)$ in A^* (so that $K = Q_{\mathfrak{e}}(A)$ by Lemma 2.6), and let $f_n(\varphi) = \mathrm{Re}\,\varphi(e_n)$ for $\varphi \in K$. Since $\lim_n \mathrm{Re}\,\varphi(e_n) \geq 0$ for all $\varphi \in Q_{\mathfrak{e}}(A)$, we can apply the previous paragraph to find an $x \in \mathrm{conv}\{e_n\}$ such that $\inf_{\varphi \in K} \varphi(x) > -\epsilon$. Similarly, choose $y_1 \in \mathrm{conv}\{e_n\}$ such that $\inf_{\varphi \in K} \varphi(x + \epsilon y_1) > -\epsilon/2$. Continue in this way, choosing $y_n \in \mathrm{conv}\{e_n\}$ such that

$$\inf_{\varphi \in K} \varphi\left(x + \epsilon \sum_{k=1}^{n} 2^{1-k} y_k\right) > -\epsilon/2^n.$$

Set $u = \sum_{k=1}^{\infty} 2^{-k} y_k \in \overline{\mathrm{conv}}\{e_n\}$, and $z = x + 2\epsilon u$. This is in $\mathfrak{r}_A^{\mathfrak{e}}$, and $\|z - x\| < 2\epsilon$.

Choose a subsequence (e_{k_n}) of (e_n) such that

$$\|e_{k_n} e_n - e_n\| + \|e_n e_{k_n} - e_n\| < 2^{-n}.$$

For each $m \in \mathbb{N}$, apply the last paragraph to $(e_{k_n})_{n \geq m}$, with ϵ replaced by 2^{-m}, to find $x_m, u_m \in \overline{\mathrm{conv}}\{e_{k_n} : n \geq m\}$ with $z_m = x_m + 2^{1-m} u_m \in \mathfrak{r}_A^{\mathfrak{e}}$. Then

$$\|x_m e_m - e_m\| + \|e_m x_m - e_m\| < 2^{-m}.$$

From this it is easy to see that (x_m) is a cai for A. It is also easy to see now that $e'_m = (1/\|z_m\|)z_m$ is a bai (hence also a cai) for A in $\mathfrak{r}_A^{\mathfrak{e}}$.

The case for the "moreover" is similar. Suppose that

$$\inf\{\operatorname{Re} \varphi(a) : \varphi \in S_{\mathfrak{e}}(A)\} > -1.$$

We may assume the infimum is negative, and choose $t > 1$ so that the infimum is still greater than -1 with a replaced by ta. We now begin to follow the argument in previous paragraphs, with the same K, but starting from a cai (e'_n) in $\mathfrak{r}^{\mathfrak{e}}_A$. Since $\lim_n \operatorname{Re} \varphi(ta + e'_n) \geq 0$ for all $\varphi \in Q_{\mathfrak{e}}(A)$, we can apply the above to find an element $x \in \operatorname{conv}\{e'_n\} \subset \mathfrak{r}^{\mathfrak{e}}_A$ such that $\inf_{\varphi \in K} \varphi(ta + x) > -\epsilon$. Continue as above to find $u \in \overline{\operatorname{conv}}\{e'_n\} \subset \mathfrak{r}^{\mathfrak{e}}_A$ so that $z = ta + x + 2\epsilon u$ is in $\mathfrak{r}^{\mathfrak{e}}_A$, with $\|z - x - ta\| < 2\epsilon$. For each $m \in \mathbb{N}$, there exists such $x_m, u_m \in \mathfrak{r}^{\mathfrak{e}}_A$ so that $z_m = ta + x_m + 2^{1-m}u_m$ is in $\mathfrak{r}^{\mathfrak{e}}_A$, with $\|z_m - x_m - ta\| \leq 2^{1-m}$, and such that (x_m) is a cai for A. Note that $z_m - ta \in \mathfrak{r}^{\mathfrak{e}}_A$, and hence $f_m = (1/\|z_m - ta\|)(z_m - ta) \in \mathfrak{r}^{\mathfrak{e}}_A$. Also (f_m) is a bai (hence a cai) for A in $\mathfrak{r}^{\mathfrak{e}}_A$. There exists an N such that $t/\|z_m - ta\| > 1$ for $m \geq N$. Thus $f_m + a \in \mathfrak{r}^{\mathfrak{e}}_A$ for $m \geq N$, since this is a convex combination of f_m and $f_m + ta/\|z_m - ta\| = z_m/\|z_m - ta\|$. $\qquad\square$

Corollary 2.10. *Let $\mathfrak{e} = (e_n)$ be a sequential cai for a Banach algebra A. Assume that $S(A) = S_{\mathfrak{e}}(A)$ (which is the case, for example, if A is Hahn–Banach smooth). If $Q(A)$ is weak* closed, then A possesses a sequential cai in $\mathfrak{r}_A$. Moreover, for every $a \in A$ with $\inf\{\operatorname{Re} \varphi(a) : \varphi \in S(A)\} > -1$, there is a sequential cai (f_n) in $\mathfrak{r}_A$ such that $f_n \succeq -a$ for all n. If, in addition, A has a sequential cai in $\mathfrak{F}_A$ then the sequential cai (f_n) in the last line can also be chosen to be in $\mathfrak{F}_A$.*

Proof. By the last result, A has a sequential cai in $\mathfrak{r}_A$ satisfying the first two assertions. Suppose that A has a sequential cai, (e'_n) say, in $\mathfrak{F}_A$. One then follows the last paragraph of the last proof. Now $x_m, u_m \in \mathfrak{F}_A$. Define f_m as before, but the desired cai is

$$\frac{\|x_m + 2^{1-m}u_m\|}{1 + 2^{1-m}} f_m,$$

which is easy to see is a convex combination of x_m and u_m, and hence is in $\mathfrak{F}_A$. Moreover a tiny modification of the argument above shows that the sum of this cai and a is in $\mathfrak{r}_A$ for m large enough. $\qquad\square$

Remark. Under the conditions of Corollary 2.10, and if A has a sequential approximate identity in $\frac{1}{2}\mathfrak{F}_A$ (resp. $\mathfrak{F}_A$), then a slight variant of the last proof shows that for any $a \in A$ with $\inf\{\operatorname{Re} \varphi(a) : \varphi \in S(A)\} > -1$, there is a sequential bai (f_n) in $\frac{1}{2}\mathfrak{F}_A$ (resp. $\mathfrak{F}_A$) such that $f_n \succeq -a$ for all n. By Corollary 3.9 (and the remark after it) below, if A has a sequential bai in $\mathfrak{r}_A$ then A does have a sequential bai in $\mathfrak{F}_A$.

We also remark that Corollary 3.4 of [Blecher 2015] generalizes the first assertion of Corollary 2.10 above to nonsequential cais.

Proposition 2.11. *If A is a scaled approximately unital Banach algebra then the weak* closure of $\mathfrak{r}_A$ is $\mathfrak{r}_{A^{**}}$.*

Proof. It is easy to see from the definitions that $\mathfrak{r}_A \subset \mathfrak{r}_{A^{**}}$. Clearly $\mathfrak{r}_A^\circ = \mathfrak{c}_{A^*}$, so the result will follow from the bipolar theorem if we can show that

$$(\mathfrak{c}_{A^*})^\circ = \mathfrak{r}_{A^{**}} = \mathfrak{r}_{(A^1)^{**}} \cap A^{**}.$$

Since $\mathfrak{r}_A \subset \mathfrak{r}_{A^{**}}$, it is clear that $(\mathfrak{r}_{A^{**}})_\circ \subset \mathfrak{c}_{A^*}$. If $\varphi \in \mathfrak{c}_{A^*}$ then $\varphi = t\psi$ for $t > 0$, $\psi \in S(A)$. Then ψ extends to a state $\hat{\psi}$ on A^1, and to a weak* continuous state ρ on $(A^1)^{**}$. If $\eta \in \mathfrak{r}_{A^{**}}$, we have

$$\operatorname{Re} \eta(\psi) = \operatorname{Re} \eta(\hat{\psi}) = \operatorname{Re} \rho(\eta) \geq 0.$$

That is, $\varphi \in (\mathfrak{r}_{A^{**}})_\circ$. Then $(\mathfrak{r}_{A^{**}})_\circ = \mathfrak{c}_{A^*}$, and hence by the bipolar theorem, $(\mathfrak{c}_{A^*})^\circ = \mathfrak{r}_{A^{**}}$.   □

We remark that if an approximately unital Banach algebra A is scaled then any mixed identity e for A^{**} of norm 1 is lower semicontinuous on $Q(A)$. For if $\varphi_t \to \varphi$ weak* and $\varphi_t(e) = \|\varphi_t\| \leq r$ for all t, then $\|\varphi\| = \varphi(e) \leq r$. A similar assertion holds in the $\mathfrak{e}$-scaled case.

3. Positivity and roots in Banach algebras

Proposition 3.1. *If B is a closed subalgebra of a nonunital Banach algebra A, and if A and B have a common cai, then $B^1 \subset A^1$ isometrically and unitally, $S(B^1) = \{f_{|B^1} : f \in S(A^1)\}$, and $\mathfrak{F}_B = B \cap \mathfrak{F}_A$ and $\mathfrak{r}_B = B \cap \mathfrak{r}_A$. Moreover, in this case, if A is M-approximately unital then so is B.*

Proof. We leave the first part of this as an exercise. The last assertion follows using [Harmand et al. 1993, Proposition I.1.16], since in this case multiplying by e leaves $(B^1)^\perp$ invariant inside $(A^1)^{**}$.   □

Remark. Similarly, in the situation of Proposition 3.1 we have $\mathfrak{r}_B^\mathfrak{e} = B \cap \mathfrak{r}_A^\mathfrak{e}$ if $\mathfrak{e}$ is the common cai.

Proposition 3.2. *Suppose that J is a closed approximately unital ideal in an approximately unital Banach algebra A, and that J is also an M-ideal in A. Then:*

(1) *$\mathfrak{F}_J = J \cap \mathfrak{F}_A$ and $\mathfrak{r}_J = J \cap \mathfrak{r}_A$, and states on J extend to states on A.*

(2) *If J is nonunital then $J^1 \subset A^1$ isometrically and unitally, and*

$$S(J^1) = \{f_{|J^1} : f \in S(A^1)\}.$$

(3) *If A is M-approximately unital, then so is J.*

(4) *If $\mathfrak{e} = (e_i)$ is a cai of A, then there is a cai $\mathfrak{h} = (h_j)$ of J such that $\varphi_{|J} \in Q_\mathfrak{h}(J)$ whenever $\varphi \in S_\mathfrak{e}(A)$.*

Proof. (2) For $a \in J$ and $\lambda \in \mathbb{C}$, we have

$$\|a + \lambda 1\|_{A^1} = \sup\{\|ax + \lambda x\|_A : x \in \text{Ball}(A)\} \geq \|a + \lambda 1\|_{J^1}.$$

Let f be a mixed identity of J^{**} of norm 1, which is the limit of a cai (f_i). For every $x \in \text{Ball}(A)$, one has

$$\|ax + \lambda x\|_A = \max\{\|fax + \lambda fx\|, \|\lambda(1 - f)x\|\}.$$

Setting $a = 0$ temporarily, we see that $\|\lambda(1 - f)x\| \leq |\lambda| \leq \|a + \lambda 1\|_{J^1}$. For any $a \in J$, we have $fax = ax$ and $ax + \lambda fx = \text{w}^* \lim_i a f_i x + \lambda f_i x$, so that

$$\|fax + \lambda fx\| \leq \liminf_i \|af_i x + \lambda f_i x\| \leq \|a + \lambda 1\|_{J^1}.$$

Thus $\|a + \lambda 1\|_{A^1} = \|a + \lambda 1\|_{J^1}$.

(1) If J is nonunital then by (2) and the Hahn–Banach theorem, we have $S(J^1) = \{f_{|J^1} : f \in S(A^1)\}$, and so states on J extend to states on A. If J is unital, an extension of states is given by $\varphi \mapsto \varphi(1_J \cdot)$. It also is clear from (1) that $\mathfrak{F}_J = J \cap \mathfrak{F}_A$ in the nonunital case, and we leave the unital case as an exercise (using the fact that multiplication by the identity of J is an M-projection). The identity $\mathfrak{r}_J = J \cap \mathfrak{r}_A$ is handled similarly. Indeed, clearly $J \cap \mathfrak{r}_A \subset \mathfrak{r}_J$ since states on J extend to states on A^1. We leave the converse inclusion as an exercise (for example, it follows from $\mathfrak{F}_J = J \cap \mathfrak{F}_A \subset J \cap \mathfrak{r}_A$, and Proposition 3.5 below).

(3) We can assume J is nonunital. It follows from [Harmand et al. 1993, Proposition 1.17(b)] that if J is an M-ideal in A, and A is an M-ideal in A^1, then J is an M-ideal in A^1. By the same result, J is an M-ideal in J^1.

(4) Let e denote a weak* limit point in A^{**} of (e_i). Let (g_k) be any cai for J, with weak* limit point g in $J^{\perp\perp}$. Then $(h_j) = (g_k e_i)$ (indexed first by i and then j) is a cai for J. Then $h = ge$ is a weak* limit point of (h_j). We have $(1 - g)e = e - h$. Since left multiplication by g is the M-projection of A^{**} onto $J^{\perp\perp}$, as we have seen several times above, one has $\|e - h\| \leq 1$. Let $\varphi \in S_e(A)$ be given. We claim that if $\varphi(h) = 0$ then $\varphi_{|J} = 0$; and if $\varphi(h) \neq 0$ then $\varphi(h \cdot)/\varphi(h)$ is a state on J^1. Note that if $\varphi(h) \neq 0$ then

$$1 = \varphi(e) = \varphi(h) + \varphi((1-g)e) \leq |\varphi(h)| + |\varphi((1-g)e)| \leq \|\varphi(g \cdot)\| + \|\varphi((1-g) \cdot)\|,$$

which equals 1 due to the L-decomposition in A^*. Thus we must have $\varphi(h) \geq 0$. Let $a + \lambda 1 \in \text{Ball}(J^1)$ be given. Then for any unimodular scalar γ, one has

$$\|\gamma(ha + \lambda h) + e - h\|_{A^{**}} = \max\{\|ha + \lambda h\|, \|e - h\|\} \leq 1.$$

Therefore,

$$\left|\varphi(\gamma(ha + \lambda h) + e - h)\right| = \left|\gamma\varphi(ha + \lambda h) + 1 - \varphi(h)\right| \leq 1$$

for all such γ. So for some such γ,

$$|\varphi(ha + \lambda h)| + 1 - \varphi(h) = \varphi(\gamma(ha + \lambda h) + e - h) \le 1,$$

so that $|\varphi(ha + \lambda h)| \le \varphi(h)$. $\square$

Proposition 3.3 (Esterle). *If A is a unital Banach algebra then $\mathfrak{F}_A$ is closed under* (*principal*) *t-th powers for any $t \in [0, 1]$. Thus if A is an approximately unital Banach algebra then $\mathfrak{F}_A$ and $\mathbb{R}^+\mathfrak{F}_A$ are closed under t-th powers for any $t \in (0, 1]$.*

Proof. This is in [Esterle 1978, Proposition 2.4] (see also [Blecher and Read 2011, Proposition 2.3]), but for convenience we repeat the construction. If $\|1 - x\| \le 1$, define

$$x^t = \sum_{k=0}^{\infty} \binom{t}{k}(-1)^k(1 - x)^k, \quad t > 0.$$

For $k \ge 1$, the sign of $\binom{t}{k}(-1)^k$ is always negative, and $\sum_{k=1}^{\infty} \binom{t}{k}(-1)^k = -1$. It follows that the series for x^t above is a norm-limit of polynomials in x with no constant term. Also, $1 - x^t = \sum_{k=1}^{\infty} \binom{t}{k}(-1)^k(1 - x)^k$, which is a convex combination in $\mathrm{Ball}(A^1)$. So $x^t \in \mathfrak{F}_A$.

Using the Cauchy product formula in Banach algebras in a standard way, one deduces that $(x^{1/n})^n = x$ for any positive integer n. $\square$

From [Esterle 1978, Proposition 2.4], if $x \in \mathfrak{F}_A$ then we also have $(x^t)^r = x^{tr}$ for $t \in [0, 1]$ and any real r, and that if $ax_n \to a$, where $a \in A$ and (x_n) is a sequence with $\|x_n - 1\| < 1$, then $ax_n^t \to a$ with n for all real t.

If A is a unital Banach algebra then we define the $\mathfrak{F}$-transform to be $\mathfrak{F}(x) = x(1 + x)^{-1} = 1 - (1 + x)^{-1}$ for $x \in \mathfrak{r}_A$. Then $\mathfrak{F}(x) \in \mathrm{ba}(x)$. The inverse transform takes y to $y(1 - y)^{-1}$.

Lemma 3.4. *If A is an approximately unital Banach algebra then $\mathfrak{F}(\mathfrak{r}_A) \subset \mathfrak{F}_A$.*

Proof. This is because by a result of Stampfli and Williams [1968, Lemma 1],

$$\|1 - x(1 + x)^{-1}\| = \|(1 + x)^{-1}\| \le d^{-1} \le 1,$$

where d is the distance from -1 to the numerical range of x. $\square$

If A is also an operator algebra then we have shown elsewhere [Blecher and Read 2014, Lemma 2.5] that the range of the $\mathfrak{F}$-transform is exactly the set of strict contractions in $\frac{1}{2}\mathfrak{F}_A$.

Proposition 3.5. *If A is an approximately unital Banach algebra then $\overline{\mathbb{R}^+\mathfrak{F}_A} = \mathfrak{r}_A$.*

Proof. As in [Blecher and Read 2013a, Theorem 3.3], it follows that if $x \in \mathfrak{r}_A$ then $x = \lim_{t \to 0^+}(1/t)tx(1 + tx)^{-1}$. By Lemma 3.4, $tx(1 + tx)^{-1} \in \mathfrak{F}_A$. So $\mathbb{R}^+\mathfrak{F}_A$ is dense in $\mathfrak{r}_A$. $\square$

In the following results we will use the fact that if A is an approximately unital Banach algebra, then the "regular representation" $A \to B(A)$ is isometric. Thus we can view an accretive $x \in A$ and its principal roots as operators in $B(A)$. These are sectorial of angle $\pi/2$, and so we can use the theory of roots (fractional powers) from, e.g., [Haase 2006, Section 3.1] or [Li et al. 2003; Sz.-Nagy et al. 2010]. Basic properties of such (principal) powers include: $x^s x^t = x^{s+t}$, $(cx)^t = c^t x^t$ for positive scalars c, s, t, and $t \to x^t$ is continuous. See also, for example, [Yosida 1965, Chapter IX, Section 11], [Blecher 2015], [Blecher and Read 2014, Lemma 1.1(1)] or [Esterle 1978, p. 64]. Also $x^t = \lim_{t \to 0+} (x + \epsilon 1)^t$ for $t > 0$, and the latter can be taken to be with respect to the usual Riesz functional calculus (see [Haase 2006, Proposition 3.1.9]). Principal n-th roots of accretive elements are unique for any positive integer n (see [Li et al. 2003]).

Remark. It is easy to see from the last fact that the definitions of x^t given in [Haase 2006] and [Li et al. 2003, Theorem 1.2] coincide. A similar argument shows that if $x \in \mathfrak{F}_A$ then the definitions of x^t given in [Haase 2006] and Proposition 3.3 coincide, if $t > 0$. Indeed for the latter we may assume that $0 < t \leq 1$ and work in $B(A)$ as above (and we may assume A unital). Then the two definitions of y^t coincide if $y = (1/(1 + \epsilon))(x + \epsilon I)$, since both equal the t-th power of y as given by the Riesz functional calculus. However $\sum_{k=0}^{\infty} \binom{t}{k}(-1)^k (1 - y)^k$ converges uniformly to $\sum_{k=0}^{\infty} \binom{t}{k}(-1)^k (1 - x)^k$, as $\epsilon \to 0^+$, since the norm of the difference of these two series is dominated by

$$\sum_{k=1}^{\infty} \binom{t}{k}(-1)^k \left(\frac{1}{1+\epsilon} - 1 \right) \|(1-x)^k\| \leq \frac{\epsilon}{1+\epsilon} \to 0.$$

See [Blecher 2015] for more details concerning the last remark, and also for a better estimate in the next result in the operator algebra case.

Lemma 3.6. *Let A be an approximately unital Banach algebra. If $\|x\| \leq 1$ and $x \in \mathfrak{r}_A$, then*

$$\|x^{1/m}\| \leq \frac{2m^2}{(m-1)\pi} \sin\left(\frac{\pi}{m}\right) \leq \frac{2m}{m-1}$$

for $m \geq 2$. More generally,

$$\|x^\alpha\| \leq \frac{2 \sin(\alpha\pi)}{\pi\alpha(1-\alpha)} \|x\|^\alpha$$

if $0 < \alpha < 1$ and $x \in \mathfrak{r}_A$. If A is also an operator algebra then one may remove the 2s in these estimates.

Proof. This follows from the well-known A. V. Balakrishnan representation of powers,

$$x^\alpha = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty t^{\alpha-1} (t + x)^{-1} x \, dt$$

(see, e.g., [Haase 2006]). We use the simple fact that $\|(t + x)^{-1}\| \le 1/t$ for accretive x and $t > 0$, and so

$$\|(t + x)^{-1}x\| = \left\|\left(1 + \frac{x}{t}\right)^{-1}\frac{x}{t}\right\| = \left\|\mathfrak{F}\left(\frac{x}{t}\right)\right\| \le 2,$$

and is even less than or equal to 1 in the operator algebra case by the observation after Lemma 3.4. Then the norm of x^α is dominated by

$$\frac{2\sin(\alpha\pi)}{\pi}\left(\int_0^1 t^{\alpha-1} \cdot 1 \, dt + \int_1^\infty t^{\alpha-1}\frac{1}{t} \, dt\right) = \frac{2\sin(\alpha\pi)}{\pi\alpha(1-\alpha)}.$$

The rest is clear from this.   □

We will sometimes use the fact from [Li et al. 2003, Corollary 1.3] that the n-th root function is continuous on $\mathfrak{r}_A$.

Lemma 3.7. *There is a nonnegative sequence (c_n) in c_0 such that for any unital Banach algebra A, and $x \in \mathfrak{F}_A$ or $x \in \text{Ball}(A) \cap \mathfrak{r}_A$, we have $\|x^{1/n}x - x\| \le c_n$ for all $n \in \mathbb{N}$.*

Proof. We follow the proof of [Blecher and Read 2013a, Theorem 3.1], taking $R = 3$ there. This is based on the Banach algebra construction from [Li et al. 2003], so it will be valid in the present generality. There an estimate $\|x^{1/n}x - x\| \le Dc_n$ is given, for a nonnegative sequence (c_n) in c_0. We need to know that D does not depend on A or x. This follows if $\|\lambda(\lambda 1 - x)^{-1}\|$ is bounded independently of A or x on the curve Γ there. On the piece of the curve Γ_2, this follows by using [Stampfli and Williams 1968, Lemma 1] that $\|(\lambda 1 - x)^{-1}\| \le d^{-1}$, where d is the distance from λ to $W(x)$. On the other part of Γ, we have $\lambda = te^{i\theta}$ for $0 \le t \le R$, and for a fixed θ with $\pi/2 < |\theta| < \pi$. However, by the same result of Stampfli and Williams, $\|(\lambda 1 - x)^{-1}\| \le d^{-1}$ if $\lambda \ne 0$, where d is the distance from λ to the y-axis. Thus the quantity will be bounded since $|\lambda|/d = \csc(\theta - \pi/2)$.   □

The following (essentially from [Macaev and Palant 1962]) is a related result:

Lemma 3.8. *Let A be a unital Banach algebra. If $\alpha \in (0, 1)$ then there exists a constant K such that if $a, b \in \mathfrak{r}_A$, and $ab = ba$, then $\|(a^\alpha - b^\alpha)c\| \le K\|(a-b)c\|^\alpha$ for any $c \in \text{Ball}(A)$.*

Proof. By the Balakrishnan representation in the proof of Lemma 3.6, if $c \in \text{Ball}(A)$, we have

$$(a^\alpha - b^\alpha)c = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty t^{\alpha-1}\left((t + a)^{-1}a - (t + b)^{-1}b\right)c \, dt.$$

By the inequality $\|(t + x)^{-1}\| \le 1/t$ for accretive x, we have

$$\left\|\left((t + a)^{-1}a - (t + b)^{-1}b\right)c\right\| = \left\|(t + a)^{-1}(t + b)^{-1}(a - b)tc\right\| \le \frac{1}{t}\|(a - b)c\|,$$

and so as in the proof of Lemma 3.6, $\left\| \int_0^\infty t^{\alpha-1} \big((t+a)^{-1}a - (t+b)^{-1}b \big) c \, dt \right\|$ is dominated by

$$4 \int_0^\delta t^{\alpha-1} \, dt + \int_\delta^\infty t^{\alpha-2} \, dt \, \|(a-b)c\| = \frac{4}{\alpha} \delta^\alpha + \frac{\delta^{\alpha-1}}{1-\alpha} \|(a-b)c\|$$

for any $\delta > 0$. We may now set $\delta = \|(a-b)c\|$ to obtain our inequality. $\qquad\square$

Corollary 3.9. *An approximately unital Banach algebra with a left bai (resp. right bai, bai) in $\mathfrak{r}_A$ has a left bai (resp. right bai, bai) in $\mathfrak{F}_A$.*

Proof. If (e_t) is a left bai in $\mathfrak{r}_A$, let $b_t = \mathfrak{F}(e_t) \in \mathfrak{F}_A$. If $a \in A$ then

$$b_t^{1/n} a = b_t^{1/n}(a - e_t a) + (b_t^{1/n} e_t - e_t)a + e_t a.$$

The first term here converges to 0 with t since $(b_t^{1/n})$ is in $\mathfrak{F}_A$, and hence is bounded. Similarly, the middle term can be seen to converge to 0 with n by rewriting it as $(b_t^{1/n} b_t - b_t)(1 + e_t)a$. Working in A^1 and applying Lemma 3.7, we have

$$\|(b_t^{1/n} b_t - b_t)(1 + e_t)a\| \le c_n \|1 + e_t\| \|a\| \le K c_n \to 0$$

for a constant K independent of t. The third term converges to a with t. So $(b_t^{1/n})$ is a left bai. Similarly in the right and two-sided cases. $\qquad\square$

Remark. If the bai in the last result is sequential, then so is the one constructed in $\mathfrak{F}_A$.

Corollary 3.10. *If A is an approximately unital Banach algebra then $\mathfrak{r}_A$ is closed under n-th roots for any positive integer n.*

Proof. From the proof of Proposition 3.5, we know that if $x \in \mathfrak{r}_A$, then $x = \lim_{t \to 0+} (1/t) tx(1 + tx)^{-1}$ and $tx(1 + tx)^{-1} \in \mathfrak{F}_A$. Thus by [Li et al. 2003, Corollary 1.3], we have that $x^r = \lim_{t \to 0+} 1/t^r (tx(1 + tx)^{-1})^r$ for $0 < r < 1$. By Proposition 3.3, the latter powers are in $\mathbb{R}^+ \mathfrak{F}_A$, so that $x^r \in \overline{\mathbb{R}^+ \mathfrak{F}_A} = \mathfrak{r}_A$. $\qquad\square$

Proposition 3.11. *If A is an approximately unital Banach algebra and $x \in \mathfrak{r}_A$ then $\mathrm{ba}(x) = \mathrm{ba}(\mathfrak{F}(x))$, and so $\overline{xA} = \overline{\mathfrak{F}(x)A}$.*

Proof. This follows from the elementary spectral theory of unital Banach algebras, applied in A^1. Below we compute the spectrum in $\mathrm{ba}(x)^1$. Since $0 \notin \mathrm{Sp}(1+x)$, we have $(1+x)^{-1} \in \mathrm{ba}(1, x)$, so that $\mathfrak{F}(x) \in \mathrm{ba}(x)$. Any character of $\mathrm{ba}(x)^1$ applied to $\mathfrak{F}(x)$ gives a number of the form $z = w(1+w)^{-1}$ in the open unit disk, and in fact also inside the circle $|z - \frac{1}{2}| \le \frac{1}{2}$ if $\mathrm{Re}(w) \ge 0$. Since $1 \notin \mathrm{Sp}(\mathfrak{F}(x))$, we have $(1 - \mathfrak{F}(x))^{-1} \in \mathrm{ba}(1, \mathfrak{F}(x))$, so that $x = -\mathfrak{F}(x)(1 - \mathfrak{F}(x))^{-1} \in \mathrm{ba}(\mathfrak{F}(x))$. The rest is clear. $\qquad\square$

Lemma 3.12. *If p is an idempotent in a unital Banach algebra A then $p \in \mathfrak{F}_A$ if and only if $p \in \mathfrak{r}_A$. If p is an idempotent in A^{**} for an approximately unital Banach algebra A then $p \in \mathfrak{F}_{A^{**}}$ if and only if $p \in \mathfrak{r}_{A^{**}}$.*

Proof. The first follows from the well-known Lumer–Phillips characterization of accretiveness in terms of $\|\exp(-tp)\| \leq 1$ for all $t > 0$ (see, e.g., [Bonsall and Duncan 1971, Theorem 6, p. 30]). If p is idempotent then $\exp(-tp) = 1 - (1 - e^{-t})p$, and if this is contractive for all $t > 0$ then $\|1 - p\| \leq 1$. For the second, work in $(A^1)^{**}$ and use facts above.   □

However, one cannot say that the idempotents in the last result are also in $\frac{1}{2}\mathfrak{F}_A$, as is the case for operator algebras. The following examples illustrate this, and other "bad behavior" not seen in the class of operator algebras.

Example 3.13. Let ℓ_4^1 be identified with the l^1-semigroup algebra of the abelian semigroup $\{1, a, b, c\}$ with relations making a, b, c idempotent, and $ab = ac = bc = c$. Then $p = 1 - a, q = 1 - b \in \mathfrak{F}_A \setminus \frac{1}{2}\mathfrak{F}_A \subset \mathfrak{r}_A$. For such p, set $x = \frac{1}{2}p \in \frac{1}{2}\mathfrak{F}_A$, and notice that $x^{1/n} = 2^{-1/n}p$ which is not always in $\frac{1}{2}\mathfrak{F}_A$ (if it were, then we get the contradiction that its limit p is in $\frac{1}{2}\mathfrak{F}_A$). So we see that $\frac{1}{2}\mathfrak{F}_A$ is not closed under n-th roots. We also see that if $x \in \frac{1}{2}\mathfrak{F}_A$ then $\overline{xA}$ need not have a left cai (even if A is commutative). It does have a left bai of norm at most 2, and indeed a left bai in $\mathfrak{F}_A$ by Corollary 3.18.

In this example, $pq = p^{\frac{1}{2}}q^{\frac{1}{2}} = 1 - a - b + c \notin \mathfrak{r}_A$ (as can be seen by considering states $f(\alpha a + \beta b + \gamma c + \lambda 1) = \gamma z + \lambda + \alpha + \beta$ for $|z| \leq 1$). So $x^{1/2}y^{1/2}$ need not be in $\mathfrak{r}_A$ even if $x, y \in \frac{1}{2}\mathfrak{F}_A$. This shows that the main results about roots in [Bearden et al. 2014] fail in more general M-approximately unital Arens regular Banach algebras. Note too that if $J_1 = pA$ and $J_2 = qA$, then $J_1 \cap J_2 = \mathbb{C}d = dA$, where $d = pq$, but dA has no identity or bai in $\mathfrak{r}_A$. This shows that, unlike in the operator algebra case, finite intersections of extremely nice closed ideals need not be "nice" in the sense of the theory developed in this paper. See, however, Section 8 for a context in which finite intersections will behave well.

Example 3.14. In the Banach algebra $A = l^1(\mathbb{Z}_2)$ with convolution multiplication, we know that $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ is a contractive idempotent in $\frac{1}{2}\mathfrak{F}_A$ with numerical range $\overline{B}\left(\frac{1}{2}, \frac{1}{2}\right)$. The states in this example are the functionals $(a, b) \mapsto a + bz$ for $|z| \leq 1$. All of the principal n-th roots of p obviously have the same numerical range. So the numerical range of $p^{1/n}$ does not "converge" to the x-axis. Thus we cannot expect statements in the Blecher–Read papers involving "near positivity" to generalize (unless A is a Hermitian Banach *-algebra satisfying the conditions in the latter part of [Li et al. 2003], in which case the numerical ranges of $x^{1/n}$ do "converge" to the x-axis if x is accretive). Note also in this example that p is not an M-projection in A. Thus we cannot expect support projections to be associated with M-projections in general. In this example, it is easy to see that $x = (a, b) \in \mathfrak{r}_A$ if and only if $|b| \leq \mathrm{Re}\, a$, whereas $x \in \frac{1}{2}\mathfrak{F}_A$ if and only if $|b|^2 - |b| \leq \mathrm{Re}\, a - |a|^2$. In this example, the Cayley transform does not take $\mathfrak{r}_A$ into the set of contractions, so that $x(1+x)^{-1}$ need not be in $\frac{1}{2}\mathfrak{F}_A$.

This example also serves to show that if B is an approximately unital closed ideal in a commutative finite-dimensional approximately unital Banach algebra, then $\mathfrak{r}_B$ and $\mathfrak{F}_B$ need not be related to $\mathfrak{r}_A$ and $\mathfrak{F}_A$, unlike the setting of operator algebras (where there is a very strong relationship between these, even in the case that B is a subalgebra). Indeed let $B = \mathbb{C}(1, 1)$ inside the last example. Then we have $1_B = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $\mathfrak{r}_B = \{(a, a) : \operatorname{Re} a \geq 0\}$ and $\mathfrak{F}_B = \left\{(a, a) : a \in \overline{B\left(\frac{1}{2}, \frac{1}{2}\right)}\right\}$.

For a state φ on an operator algebra A and $x \in \mathfrak{F}_A$, it is the case that $\varphi(s(x)) = 0$ if and only if $\varphi(x) = 0$ if and only if $\varphi \in \mathrm{ba}(x)^\perp$. Here $s(x)$ is the support projection of x from [Blecher and Read 2011]. In Example 3.14, if $x = \left(\frac{1}{2}, \frac{i}{2}\right)$ and $\varphi((a, b)) = a + ib$ then $x \in \operatorname{Ker} \varphi$ but x^2 and $s(x) = 1$ are not in $\operatorname{Ker} \varphi$. Thus much of the theory of "strictly real positive" elements from [loc. cit.] and its sequels breaks down.

A slight variant of this example is the same algebra, but with norm $\|\|(a, b)\|\| = |a| + 2|b|$. Here $J = \mathbb{C}\left(\frac{1}{2}, \frac{1}{2}\right)$ is an ideal equal to xA for $x \in \mathfrak{F}_A$, but this ideal has no cai.

Example 3.15. The unital Banach algebra $l^1(\mathbb{N})$, with convolution product, is easily seen to be equal to $\mathrm{ba}(x)$ where $x = 1 + \frac{1}{2}\vec{e}_2 \in \mathfrak{F}_A$. However $l^1(\mathbb{N})$ is not Arens regular; thus its second dual is not commutative in either one of the Arens products [Palmer 1994, §1.4.9]. Thus $\mathrm{ba}(x)^{**}$ need not be commutative if $x \in \mathfrak{F}_A$. In this example, it is easy to compute $\mathfrak{F}_A$ and $\mathfrak{r}_A$. C. A. Bearden has verified that in this example, unlike the operator algebra case [Bearden et al. 2014], $(x^{1/n})$ need not increase in the "real positive ordering" with n for $x \in \frac{1}{2}\mathfrak{F}_A$.

Example 3.16. The approximately unital Banach algebra $A = L^1(\mathbb{R})$ with convolution product has multiplier unitization $A^1 = A \oplus^1 \mathbb{C}$. This can be seen from Wendel's result that the measure algebra $M(\mathbb{R})$ embeds canonically in $B(L^1(\mathbb{R}))$ isometrically [Dales 2000], so that $L^1(\mathbb{R})^1$ can be identified with $L^1(\mathbb{R}) + \mathbb{C}\delta_0$, where δ_0 is the point mass at 0. Thus $S(A)$ corresponds to the set of $f \in L^\infty(\mathbb{R})$ of norm 1. It follows immediately that $\mathfrak{F}_A = \mathfrak{r}_A = (0)$ in this case. This algebra is not Arens regular. Note that any norm-1 functional on $L^1(\mathbb{R})$ extends to a state on $L^1(\mathbb{R})^1$ clearly. However, there are many norm-1 functions $g \in L^\infty(\mathbb{R})$ with $1 \neq \lim_{t \to 0+} \int_{\mathbb{R}} g e_t$ for the usual positive cai $\mathfrak{e} = (e_t)$ of $L^1(\mathbb{R})$ (the one in the remark after Lemma 2.1), for example, if g takes only negative values. This shows that Lemma 2.2 fails for more general Banach algebras. For this same cai $\mathfrak{e}$, we remark that $S_{\mathfrak{e}}(A)$ corresponds to the set of $f \in \mathrm{Ball}(L^\infty(\mathbb{R}))$ for which the mean value of f at 0 (this mean value is the limit with n of the (integral) average of f over the interval of width $1/n$ centered at 0) exists and equals 1. From this it is easy to see that $\mathfrak{r}_A^{\mathfrak{e}} = (0)$ and $\mathfrak{c}_{A*}^{\mathfrak{e}} = A^*$.

Because of the above examples and the considerations mentioned after Lemma 2.3 above, the following result cannot be improved, even for M-approximately unital Arens regular Banach algebras:

Proposition 3.17. *If $x \in \mathfrak{r}_A$ then* $\mathrm{ba}(x)$ *has a bai in* $\mathfrak{F}_A$, *and hence any weak* limit point of this bai is a mixed identity residing in* $\mathfrak{F}_{A^{**}}$. *Indeed* $(x^{1/n})$ *is a bai for* $\mathrm{ba}(x)$ *in* $\mathfrak{r}_A$, *and* $(\mathfrak{F}(x)^{1/n})$ *is a bai for* $\mathrm{ba}(x)$ *in* $\mathfrak{F}_A$.

Proof. Note that $x^{1/n}x \to x$ by Lemma 3.7. That $(x^{1/n})$ is bounded follows from Lemma 3.6. Thus $(x^{1/n})$ is a bai for $\mathrm{ba}(x)$ in $\mathfrak{r}_A$.

In the case that $x \in \mathfrak{F}_A$, we have $(x^{1/n})$ is in $\mathfrak{F}_A$ (using Proposition 3.3). We remark that the proof of [Blecher and Read 2011, Lemma 2.1] (see also [Blecher et al. 2008]) displays a different, and often useful, bai in $\mathfrak{F}_A$. In the general case, note that if $x \in \mathfrak{r}_A$ then $\mathrm{ba}(x) = \mathrm{ba}(\mathfrak{F}(x))$ by Proposition 3.11, and so $(\mathfrak{F}(x)^{1/n})$ is a bai for $\mathrm{ba}(x)$. $\qquad \square$

For an approximately unital Banach algebra A and $x \in \mathfrak{r}_A$, by Proposition 3.11 we have $\mathrm{ba}(x) = \mathrm{ba}(\mathfrak{F}(x))$ and $\overline{xA} = \overline{\mathfrak{F}(x)A}$. If A is not Arens regular then Example 3.15 shows that $\mathrm{ba}(x)$ need not be Arens regular if $x \in \mathfrak{F}_A$. (However, it is Arens semiregular as is any commutative Banach algebra [Palmer 1994].) Thus $\mathrm{ba}(x)^{**}$ need not be commutative. We write $s(x)$ for the weak* Banach limit of $(x^{1/n})$ in A^{**}. That is $s(x)(f) = \mathrm{LIM}_n f(x^{1/n})$ for $f \in A^*$, where LIM is a Banach limit. It is easy to see that $xs(x) = s(x)x = x$, by applying these to $f \in A^*$. Hence $s(x)$ is a mixed identity of $\mathrm{ba}(x)^{**}$ and is idempotent. By the Hahn–Banach theorem, it is easy to see that $s(x) \in \overline{\mathrm{conv}(\{x^{1/n} : n \in \mathbb{N}\})}^{w*}$. By Corollary 3.10 and Lemma 3.12, and the fact below Lemma 2.5 that $\mathfrak{F}_{A^{**}}$ is weak* closed, we see that $s(x)$ resides in $\mathfrak{F}_{A^{**}}$. If $\mathrm{ba}(x)$ is Arens regular then $s(x)$ will be the identity of $\mathrm{ba}(x)^{**}$. Therefore in this case, or more generally if $\mathrm{ba}(x)^{**}$ has a unique left identity in the second Arens product, $s(x)$ is also the weak* limit of $(\mathfrak{F}(x)^{1/n})$. Indeed in this case we can set $s(x)$ to be the weak* limit of any bai for $\mathrm{ba}(x)$. This is the case, for example, if $\mathrm{ba}(x)$ is M-approximately unital (that is, if it is an M-ideal in $\mathrm{ba}(x)^1$), by Lemma 2.5.

Remark. Note that if $x \in \mathfrak{r}_A$ then $\mathrm{ba}(x)$ is M-approximately unital if A is M-approximately unital and $\mathrm{ba}(x)^1 \subset A^1$ isometrically (by the argument in Proposition 3.1). It is claimed in [Smith 1979] that the support projection of an M-ideal in a commutative Banach algebra is central. We did not follow this proof (and its author confirmed that at present there seemed to him to be a gap), but this would imply that if $\mathrm{ba}(x)$ is M-approximately unital then $s(x)$ is central in $\mathrm{ba}(x)^{**}$, and thus is actually a (unique) two-sided identity for $\mathrm{ba}(x)^{**}$.

We call $s(x)$ above a *support* idempotent of x, or a (left) support idempotent of $\overline{xA}$ (or a (right) support idempotent of $\overline{Ax}$). The reason for this name is the following result.

Corollary 3.18. *If A is an approximately unital Banach algebra, and $x \in \mathfrak{r}_A$ then $\overline{xA}$ has a left bai in $\mathfrak{F}_A$ and $x \in \overline{xA} = s(x)A^{**} \cap A$ and $(xA)^{\perp\perp} = s(x)A^{**}$. (These products are with respect to the second Arens product.)*

Proof. Indeed if $J = \overline{xA}$ then $J = \overline{\mathfrak{F}(x)A}$ by Proposition 3.5. So we may assume that $x \in \mathfrak{F}_A$. Since $\overline{xA}$ contains $\overline{x\,\mathrm{ba}(x)}$, which in turn contains (actually, is equal to) $\mathrm{ba}(x)$, it contains x and $x^{1/n}$. So $(x^{1/n})$ is a left bai in $\mathfrak{F}_A$ for $\overline{xA}$. We have $s(x) \in J^{\perp\perp}$, and $J^{\perp\perp} \subset s(x)A^{**} \subset J^{\perp\perp}$, since $J^{\perp\perp}$ is a right ideal in A^{**}. Hence $J^{\perp\perp} = s(x)A^{**}$, so that $J = s(x)A^{**} \cap A$. $\square$

As in [Blecher and Read 2011, Lemma 2.10] we have:

Corollary 3.19. *If A is an approximately unital Banach algebra, and $x, y \in \mathfrak{r}_A$, then $\overline{xA} \subset \overline{yA}$ if and only if $s(y)s(x) = s(x)$. In this case, $\overline{xA} = A$ if and only if $s(x)$ is a left identity for A^{**}. (These products are with respect to the second Arens product.)*

Proof. This is essentially just as in the proof of Lemma 2.10 (and Corollary 2.6) of [loc. cit.]. For example, if $\overline{xA} \subset \overline{yA}$ then, since $x \in \overline{xA}$, we have $s(y)x = x$. Hence $s(y)z = z$ for all $z \in \mathrm{ba}(x)$, and so $s(y)s(x) = s(x)$, since as we said earlier $s(x) \in \overline{\mathrm{ba}(x)}^{w*}$. $\square$

As in [loc. cit., Corollary 2.7] we have:

Corollary 3.20. *Suppose that A is a closed approximately unital subalgebra of an approximately unital Banach algebra B, and that $\mathfrak{r}_A \subset \mathfrak{r}_B$. If $x \in \mathfrak{r}_A$, then the support projection of x computed in A^{**} is the same, via the canonical embedding $A^{**} \cong A^{\perp\perp} \subset B^{**}$, as the support projection of x computed in B^{**}.*

We recall that x is pseudo-invertible in A if there exists $y \in A$ with $xyx = x$. The following result (and several of its corollaries below) should be compared with the C^*-algebraic version of the result due to Harte and Mbekhta [1992; 1993], and to the earlier version of the result in the operator algebra case (see particularly [Blecher and Read 2011, Section 3; 2014, Subsection 2.4]).

Theorem 3.21. *Let A be an approximately unital Banach algebra A, and $x \in \mathfrak{r}_A$. The following are equivalent*:

  (i) $s(x) \in A$.

 (ii) xA *is closed.*

(iii) Ax *is closed.*

(iv) x *is pseudo-invertible in A.*

 (v) x *is invertible in* $\mathrm{ba}(x)$.

Moreover, these conditions imply that

(vi) 0 *is isolated in, or absent from,* $\mathrm{Sp}_A(x)$.

Finally, if $\mathrm{ba}(x)$ *is semisimple then* (i)–(vi) *are equivalent.*

Proof. We recall that $(x^{1/m})_{m \in \mathbb{N}}$ is a bai for $\mathrm{ba}(x)$, by Proposition 3.17, and it has weak* limit point $s(x) \in \mathrm{ba}(x)^{\perp\perp} \subset A^{**}$.

(ii) $\Rightarrow$ (i): Suppose xA is closed. Then

$$x^{1/2} \in \mathrm{ba}(x) \subset \overline{x\,\mathrm{ba}(x)} \subset \overline{xA} = xA,$$

so $x^{1/2} = xy$ for some $y \in A$. Thus if $z = x^{1/2}y \in A$ then $x = x^{1/2}xy = xz$, and so $a = az$ for every $a \in \mathrm{ba}(x)$. Now $s(x)z = z$ since $x^{1/2} \in \mathrm{ba}(x)$, for example. On the other hand, $s(x)z = s(x)$ since $x^{1/n}z = x^{1/n}$ so that

$$(s(x)z)(f) = fs(x)(z) = \mathrm{LIM}_n\, f(x^{1/n}z) = \mathrm{LIM}_n\, f(x^{1/n}) = s(x)(f), \quad f \in A^*.$$

Thus $s(x) = z \in A$. (Of course, in this case $x^{1/n} \to s(x)$ in norm.)

(i) $\Rightarrow$ (iv): Recall $s(x)$ is a left identity of $\mathrm{ba}(x)^{**}$ in the second Arens product, and if (i) holds, it is an identity, and $\mathrm{ba}(x)$ is unital. This implies, by the Neumann lemma, that x is invertible in $\mathrm{ba}(x)$, and hence that x is pseudo-invertible in A.

(iv) $\Rightarrow$ (ii): Item (iv) implies that $xA = xyA$ is closed since xy is idempotent.

That (iii) is equivalent to the others follows from (ii) and the symmetry in (i) or (iv). That (v) is equivalent to (i) is now obvious from the above.

For the equivalences with (vi), by the definition of spectrum, and because of the form of (v), we may assume A is unital. That (iv) implies (vi) may be proved similarly to the analogous argument in [Blecher and Read 2011, Theorem 3.2], but replacing $B(H)$ and $B(K)$ with $B(A)$ and $B(xA)$. We can assume that $0 \in \mathrm{Sp}_A(x)$, so that x is not invertible. Then $xA \neq A$, for if $xA = A$ then $s(x)$ is a left identity for A. It is also a right identity since if (e_t) is a cai for A then $s(x)e_t = e_t \to s(x)$. Then the inverse of x in $\mathrm{ba}(x)$ is an inverse in A, contradicting the fact that x is not invertible in A^1. It may be simpler to prove the equivalent fact that 0 is isolated in the spectrum of $x^{1/2}$. By the argument in [loc. cit., Theorem 3.2] it is enough to prove that 0 is isolated in the spectrum of L in $B(A)$, where L is left multiplication by $x^{1/2}$. We note that

$$x^{1/2}A \subset xA \subset eA \subset x^{1/2}A,$$

where $e = x^{1/2}y = s(x)$ and y is the pseudo-inverse of x. So these subspaces coincide; call this space K. It follows that K is an invariant subspace for L, indeed $R = L_{|K}$ is continuous, surjective and one-to-one (since $x^{1/2}x^{1/2}a = 0$ implies that $x^{1/2}a = 0$, since $x^{1/2}$ is a limit of polynomials in x with no constant term). Thus $0 \notin \mathrm{Sp}_{B(K)}(R)$; hence $R + zI_K$ is invertible for z in a small disk centered at 0. Since $A = eA \oplus (1-e)A$, it is easy to argue that $L + zI_A = (L + zI)e \oplus z(1-e)$ is invertible in $B(A)$ for such z if $z \neq 0$. So 0 is isolated in the spectrum of L in $B(A)$.

The last assertion follows just as in [loc. cit., Theorem 3.2]. $\qquad\square$

Remark. We have been informed by Matthias Neufang that he and M. Mbekhta have also generalized the analogous result from [Blecher and Read 2011; 2013b], or a variant of it, to the class of Banach algebras that are ideals in their bidual.

The next result is an analogue of [Blecher and Read 2011, Theorem 2.12]:

Proposition 3.22. *If A is an approximately unital Banach algebra, a subalgebra of a unital Banach algebra B with $\mathfrak{r}_A \subset \mathfrak{r}_B$, and $x \in \mathfrak{r}_A$, then x is invertible in B if and only if $1_B \in A$ and x is invertible in A, and if and only if $\mathrm{ba}(x)$ contains 1_B; and in this case $s(x) = 1_B$.*

Proof. It is clear by the Neumann lemma that if $\mathrm{ba}(x)$ contains 1_B then x is invertible in $\mathrm{ba}(x)$, and hence in A. Conversely, if x is invertible in B (or in A) then by the equivalences (i)–(iv) proved in the last theorem, we have $s(x) \in B$, and this is the identity of $\mathrm{ba}(x)$. If $xy = 1_B$, then $1_B = xy = s(x)xy = s(x) \in \mathrm{ba}(x) \subset A$. $\square$

Corollary 3.23. *Let A be an approximately unital Banach algebra. A closed right ideal J of A is of the form xA for some $x \in \mathfrak{r}_A$ if and only if $J = qA$ for an idempotent $q \in \mathfrak{F}_A$.*

Proof. If xA is closed for a nonzero $x \in \mathfrak{r}_A$ then by Theorem 3.21, $q = s(x) \in \mathfrak{F}_A$. Hence it is easy to see that $xA = qA$. The other direction is trivial. $\square$

Corollary 3.24. *If a nonunital approximately unital Banach algebra A contains a nonzero $x \in \mathfrak{r}_A$ with xA closed, then A contains a nontrivial idempotent in $\mathfrak{F}_A$.*

Proof. By the above, $xA = qA$ for a nontrivial idempotent q in $\mathfrak{F}_A$. $\square$

Corollary 3.25. *If an approximately unital Banach algebra A has no left identity, then $xA \neq A$ for all $x \in \mathfrak{r}_A$.*

Remark. If A is a Banach algebra such that $\frac{1}{2}\mathfrak{F}_A$ is closed under n-th roots then one may also generalize other parts of the theory in [Blecher and Read 2011]. For example, in this case, if $x \in \mathfrak{F}_A$ then the support projection $s(x)$ is a bicontractive projection, and $\mathrm{ba}(x)$ has a cai in $\frac{1}{2}\mathfrak{F}_A$.

4. One-sided ideals and hereditary subalgebras

At the outset it should be said there seems to be no completely satisfactory theory of hereditary subalgebras. This can already be seen in finite-dimensional unital examples where one may have $pA = qA$ for projections $p, q \in \mathfrak{F}_A$, but no good relation between pAp and qAq. For example, one could take the opposite algebra to the one in Example 4.3. Another example arises when one considers various mixed identities in the second dual A^{**}, with the second Arens product, inside $(A^1)^{**}$. In this section we will investigate what initial parts of the theory do work. We shall see that things work considerably better if A is separable.

We define an *inner ideal* in A to be a closed subalgebra D with $DAD \subset D$. To see what kinds of results one might hope for, note that in the unital example in the last paragraph, given an idempotent $p \in A$, the right ideal $J = pA$ contains a unital inner ideal $D = pAp$ of A. Conversely, if $D = pAp$ then $J = DA = pA$ is a right ideal with a left identity.

In nonunital examples things become more complicated. One may define a hereditary subalgebra to be an inner ideal D of A which has a bai. This then induces a right ideal $J = DA$ with a left bai, and a left ideal $K = AD$ with a right bai. We shall call these the *induced* one-sided ideals. We have $JK = J \cap K = D$ just as in [Blecher et al. 2008, Corollary 2.6]. However, unlike the previous paragraph, without further conditions one cannot in general obtain a hereditary subalgebra from a right ideal with a left bai. The following example illustrates some of what can go wrong.

Example 4.1. One of the main results in [Blecher et al. 2008] is that if J is a closed right ideal with a left cai in an operator algebra A, then there exists an associated hereditary subalgebra D of A, in particular, a closed approximately unital subalgebra $D \subset J$ with $J = DA$. This is false without further conditions in more general Banach algebras. Indeed, suppose that $J = A$ is a separable Banach algebra with a sequential left cai, but no commuting bounded left approximate identity. See [Dixon 1978] for such an example. By way of contradiction, suppose that there is a closed subalgebra $D \subset J$ with a bai, such that $J = DA$. By [Sinclair 1978], D has a commuting bounded approximate identity, and this will be a commuting bounded left approximate identity for J, a contradiction.

This example also shows that if J is a closed right ideal with a left cai, we cannot rechoose another left cai (e_t) with $e_s e_t \to e_s$ with t for all s. This is critical in the operator algebra theory in, e.g., [Blecher et al. 2008, Section 2].

In order to obtain a working theory, we now impose the condition that the bais considered are in $\mathfrak{r}_A$. Thus we define a *right $\mathfrak{F}$-ideal* (resp. left $\mathfrak{F}$-ideal) in an approximately unital Banach algebra A to be a closed right (resp. left) ideal with a left (resp. right) bai in $\mathfrak{F}_A$ (or equivalently, by Corollary 3.9, in $\mathfrak{r}_A$). Henceforth in this section, by a *hereditary subalgebra* (HSA) of A we will mean an inner ideal D with a two-sided bai in $\mathfrak{F}_A$ (or equivalently, by Corollary 3.9, in $\mathfrak{r}_A$). Perhaps these should be called $\mathfrak{F}$-HSAs to avoid confusion with the notation in [Blecher et al. 2008; Blecher and Read 2011] where one uses cais instead of bais, but for brevity we shall use the shorter term. Also it is shown in [Blecher 2015] that in an operator algebra A these two notions coincide, and that right $\mathfrak{F}$-ideals in A are just the r-ideals of [Blecher et al. 2008] (and similarly in the left case).

Note that an HSA D induces a pair of right and left $\mathfrak{F}$-ideals $J = DA$ and $K = AD$. As we pointed out a few paragraphs back, it is not clear that the converse holds, namely that every right $\mathfrak{F}$-ideal comes from an HSA in this way. In fact, the

main results of this section are, firstly, that if A is separable then this is true, and indeed all HSAs and $\mathfrak{F}$-ideals are of the form in the next lemma. Secondly, we shall prove (see Corollaries 4.6 and 4.11) that if A is not necessarily separable then the HSAs and $\mathfrak{F}$-ideals in A are just the closures of increasing unions of ones of the form in this lemma:

Lemma 4.2. *If A is an approximately unital Banach algebra, and $z \in \mathfrak{F}_A$, set $J = \overline{zA}$, $D = \overline{zAz}$, and $K = \overline{Az}$. Then D is an HSA in A and J and K are the induced right and left $\mathfrak{F}$-ideals mentioned above.*

Proof. By Cohen factorization, $D = D^4 \subset JK \subset J \cap K$, and if $x \in J \cap K$ then $x = \lim_n z^{1/n} x z^{1/n} \in D$. So $z \in D = JK = J \cap K$. Also $J = pA^{**} \cap A$ by Corollary 3.18, and $D = pA^{**}p \cap A$ is an HSA in A, and $K = A^{**}p \cap A$, where $p = s(z)$. To see this, note that $pz = z = zp$, so that $K \subset A^{**}p \cap A$. If $a \in A^{**}p \cap A$, then $az^{1/n}$ has weak* limit point $ap = a$. Hence a convex combination converges in norm, so that $a \in K$, and then $K = A^{**}p \cap A$. A similar argument works for D. Finally, $DA = J$, since $zA \subset DA \subset J$, and similarly $AD = K$. $\qquad\square$

Remark. (1) In general D and K are determined by the particular z used above, and not by J alone.

(2) We note that if $z \in \mathfrak{F}_A$ then with the notation in the last proof, $K^{\perp\perp} = \overline{A^{**}p}^{w*}$ and $D^{\perp\perp} = \overline{pA^{**}p}^{w*}$. (The weak* closure here is not necessary if A is Arens regular.) Indeed $K^{\perp\perp} \subset \overline{A^{**}p}^{w*}$. Also $p \in \text{ba}(z)^{\perp\perp} \subset D^{\perp\perp} \subset K^{\perp\perp}$, so that $A^{**}p \subset K^{\perp\perp}$. Thus $K^{\perp\perp} = \overline{A^{**}p}^{w*}$. It is well known that $J + K$ is closed, which implies, as in the proof of [Blecher and Zarikian 2006, Lemma 5.29], that $(J \cap K)^{\perp} = \overline{J^{\perp} + K^{\perp}}$, so that $D^{\perp\perp} = J^{\perp\perp} \cap K^{\perp\perp} = \overline{pA^{**}p}^{w*}$.

Example 4.3. The following example illustrates some other issues that arise for left ideals in general Banach algebras, which obstruct following the r-ideal and hereditary subalgebra theory of operator algebras [Blecher et al. 2008; Blecher and Read 2011]. First, for $E \subset \mathfrak{F}_A$, it may be that $\overline{EA}$ has no left cai. Even if E has two elements this may fail, and, in this case, $\overline{EA}$ may not even equal $\overline{aA}$ for any $a \in A$. Thus, in general, the class of right $\mathfrak{F}$-ideals in noncommutative algebras is not closed under either finite sums or finite intersections (see Example 3.13). Also, it need not be the case that EAE has a bai if $E \subset \mathfrak{F}_A$. A simple three-dimensional example illustrating all of these points is the set of lower triangular 2×2 matrices with its norm as an operator on ℓ_2^1 (see [Smith and Ward 1978, Example 4.1]), and $E = \{E_{11} \pm E_{21}\}$.

Theorem 4.4. *Suppose that J is a right $\mathfrak{F}$-ideal in an approximately unital Banach algebra A. For every compact subset $K \subset J$, there exists $z \in J \cap \mathfrak{F}_A$ with $K \subset zJ \subset zA$.*

Proof. We may assume that A is unital, and follow the idea in the proof of Cohen's factorization theorem (see, e.g., [Pedersen 1998, Theorem 4.1] or [Dales 2000]).

For any $f_1, f_2, \ldots \in J \cap \mathfrak{F}_A$, define $z_n = \sum_{k=1}^n 2^{-k} f_k + 2^{-n} \in J + \mathbb{C}1$. We have

$$\|1 - z_n\| = \left\| \sum_{k=1}^n 2^{-k}(1 - f_k) \right\| \leq \sum_{k=1}^n 2^{-k} = 1 - 2^{-n},$$

and so by the Neumann lemma, $z_n^{-1} \in J + \mathbb{C}1$ and $\|z_n^{-1}\| \leq 2^n$.

Let (e_t) be a left cai for J in $\mathfrak{F}_A$, set $z_0 = 1$, and choose $\epsilon > 0$. For each $x \in K$, we have $\lim_t \|(1 - e_t)z_n^{-1}x\| = 0$. Thus by the Arzelà–Ascoli theorem, and passing repeatedly to subnets, we can inductively choose a subsequence (f_n) of (e_t), and use these to inductively define z_n by the formula above, so that

$$\max_{x \in K} \left\| (1 - f_{n+1})z_n^{-1}x \right\| \leq 2^{-n}\epsilon, \quad n \geq 0.$$

Set $z = \sum_{k=1}^\infty 2^{-k} f_k \in \overline{\mathrm{conv}}(e_n) \subset J \cap \mathfrak{F}_A$. If $x \in K$, set $x_n = z_n^{-1}x$. Then

$$\|x_{n+1} - x_n\| = \|z_{n+1}^{-1}(z_n - z_{n+1})z_n^{-1}x\| = \|2^{-n-1}z_{n+1}^{-1}(1 - f_{n+1})z_n^{-1}x\| \leq 2^{-n}\epsilon.$$

Hence $w = \lim_n x_n$ exists and $zw = x$. Note also that

$$\|x_n - x\| \leq \sum_{k=1}^n \|x_k - x_{k-1}\| \leq 2\epsilon,$$

so that $\|w - x\| \leq 2\epsilon$ if one wishes for that (so that $\|w\| \leq \|x\| + \epsilon$). $\square$

Remark. In the case of operator algebras, or in the commutative case considered in Section 7, one can choose the z in the last result in $\mathrm{conv}(K)$, if K is, for example, a finite set in $J \cap \mathfrak{F}_A$. If A is noncommutative, this fails as we saw in Example 4.3.

Corollary 4.5. *Let A be an approximately unital Banach algebra. The closed right ideals with a countable left bai in $\mathfrak{r}_A$ are precisely the "principal right ideals" $\overline{zA}$ for some $z \in \mathfrak{F}_A$. Every separable right $\mathfrak{F}$-ideal is of this form.*

Proof. The one direction is easy since $(z^{1/n})$ is a left bai for $\overline{zA}$ (see the proof of Corollary 3.18). Conversely, if (e_n) is a countable left bai in $\mathfrak{r}_A$ for right ideal J, set $K = \{1/ne_n\}$ and apply Theorem 4.4.

For the last assertion, if $\{d_n\}$ is a countable dense set in a right $\mathfrak{F}$-ideal J, apply Theorem 4.4, with $K = \{d_n/(n\|d_n\|)\}$. There exists $z \in J \cap \mathfrak{F}_A$ with $K \subset \overline{zA}$. Hence $J \subset \overline{zA} \subset J$. $\square$

Corollary 4.6. *The right $\mathfrak{F}$-ideals in an approximately unital Banach algebra A are precisely the closures of increasing unions of closed right $\mathfrak{F}$-ideals of the form $\overline{zA}$ for some $z \in \mathfrak{F}_A$.*

Proof. Suppose that J is an arbitrary right $\mathfrak{F}$-ideal in A. Let $\epsilon > 0$ be given (this is not needed for the proof but will be useful elsewhere). Let E be the left bai in $\mathfrak{F}_A$ considered as a set, and let Λ be the set of finite subsets of E ordered by inclusion. Define $z_G = x$ if $G = \{x\}$ for $x \in E$. For any two element set $G = \{x_1, x_2\}$ in Λ,

one can apply Theorem 4.4 to obtain an element $z_G \in \mathfrak{F}_A$ with $GA \subset z_G A$, and, moreover, such that $x_k = z_G w_k$ with $\|w_k - x_k\| < \epsilon$ for each k, if one wishes for that. For any three element set $G = \{x_1, x_2, x_3\}$ in Λ, we can similarly choose $z_G \in \mathfrak{F}_A$ with $z_H A \subset z_G A$ for all proper subsets H of G (and with the "moreover" above too). Proceeding in this way, we can inductively choose for any n element set G in Λ an element $z_G \in \mathfrak{F}_A$ with $z_H A \subset z_G A$ for all proper subsets H of G (and, moreover, such that each such z_H can be written as $z_G w$ for some w with $\|w - z_H\| < \epsilon$, if one wishes for that). Thus $(\overline{z_G A})$ is increasing (as sets) with $G \in \Lambda$, and $\overline{\bigcup_{G \in \Lambda} z_G A} = J$.

Conversely, suppose that Λ is a directed set and that $J = \overline{\bigcup_t J_t}$, where $(J_t)_{t \in \Lambda}$ is an increasing net of subspaces of A, and $J_t = \overline{z_t A}$ for $z_t \in \mathfrak{F}_A$. Thus if $t_1 \leq t_2$ then $J_{t_1} \subset J_{t_2}$, so that $s(z_{t_2}) z_{t_1} = z_{t_1}$. Hence $s(z_t) x \to x$ with t for all $x \in J$. Thus a weak* limit point p of $(s(z_t))_{t \in \Lambda}$ acts as a left identity for J, and hence is a left identity for $J^{\perp\perp}$. Thus $J^{\perp\perp} = pA^{**}$. Since this left identity p is in the weak* closure of the convex set $\mathfrak{F}_A \cap J$, the usual argument (see, e.g., p. 81 of [Blecher and Le Merdy 2004]) shows that J has a left bai in $\mathfrak{F}_A \cap J$. So J is a right $\mathfrak{F}$-ideal in A. $\qquad\square$

Remark. (1) Note that $(z_G^{1/n})$ in the last proof is a left bai for the right ideal J there. This net is indexed by $n \in \mathbb{N}$ and $G \in \Lambda$. To see this, suppose $x \in J$ is given, and that $\|z_{G_1} a - x\| < \epsilon$, where $a \in A$. If $G_1 \subset G$ then $z_{G_1} \in z_G A$. By the proof of Corollary 4.6, we can choose w with $z_{G_1} = z_G w$ and $\|w\| \leq 3$. Choose N such that $c_n < \epsilon/3$ for $n \geq N$, where c_n is as in Lemma 3.7. Then by that result, $\|z_G^{1/n} z_{G_1} - z_{G_1}\| = \|z_G^{1/n} z_G w - z_G w\| \leq 3 c_n < \epsilon$. Thus

$$\|z_G^{1/n} x - x\| \leq \|z_G^{1/n} x - z_G^{1/n} z_{G_1} a\| + \|z_G^{1/n} z_{G_1} a - z_{G_1} a\| + \|z_{G_1} a - x\| < (3 + \|a\|)\epsilon$$

for all G containing G_1, and $n \geq N$. So $(z_G^{1/n})$ is a left bai for J.

(2) If $(z_G)_{G \in \Lambda}$ is as above, it is tempting to define $D = \overline{\bigcup_{G \in \Lambda} z_G A z_G}$. However, we do not see that this can be adjusted to make it an HSA.

In the operator algebra case, most of the following result and its proof were first in the preprint [Blecher and Read 2013b] (which, as we said on the first page, has now morphed into several papers). We thank Charles Read for discussions on that result in May 2013, and thank Garth Dales and Tomek Kania for conversations in the same period on algebraically finitely generated ideals in Banach algebras, and in particular, for drawing our attention to the results in [Sinclair and Tullo 1974] (these will not be used in the present proof below, but were used in an earlier version). We say that a right module Z over A is *algebraically countably generated* (resp. *algebraically finitely generated*) over A if there exists a countable (resp. finite) set $\{x_k\}$ in Z such that every $z \in Z$ may be written as a finite sum $\sum_{k=1}^n x_k a_k$ for some $a_k \in A$.

Corollary 4.7. *Let A be an approximately unital Banach algebra. A right $\mathfrak{F}$-ideal J in A is algebraically countably generated as a right module over A if and only if $J = qA$ for an idempotent $q \in \mathfrak{F}_A$. This is also equivalent to J being algebraically countably generated as a right module over A^1.*

Proof. Let J be a right $\mathfrak{F}$-ideal which is algebraically countably generated over A by elements $x_1, x_2, \ldots$ in A. We can assume that $\|x_k\| \to 0$, and so $\{x_k : k \in \mathbb{N}\}$ is compact. By Theorem 4.4, there exists $z \in J$ such that $\{x_k\} \subset zA$. Thus $x_k A \subset zA^2 = zA$ for all k, and so $J \subset zA \subset J$, and $J = zA$. By Corollary 3.23, $J = qA$ for an idempotent $q \in \mathfrak{F}_A$.

If J is algebraically countably generated over A^1 then by the above $J = qA^1$. Clearly $q \in A$, and so $J = \{x \in A : qx = x\} = qA$. $\qquad\square$

Lemma 4.8. *Let A be an approximately unital Banach algebra, with a closed subalgebra D. If D has a bai from $\mathfrak{F}_A$, then for every compact subset $K \subset D$, there is $x \in D \cap \mathfrak{F}_A$ such that $K \subset xDx \subset xAx$.*

Proof. This can be done by adapting the proof of Theorem 4.4 as follows. We can inductively choose a subsequence (f_n) of the bai (e_n) with

$$\max_{x \in K}\big(\|(1 - f_{n+1})z_n^{-1}x\| + \|xz_n^{-1}(1 - f_{n+1})\|\big) \leq 2^{-2n}\epsilon$$

for each n. Choose z as before. If $x \in K$, set $x_n = z_n^{-1}xz_n^{-1} \in D$. Then

$$\|x_{n+1} - x_n\| \leq \|(z_{n+1}^{-1}x - z_n^{-1}x)z_{n+1}^{-1}\| + \|z_n^{-1}(xz_{n+1}^{-1} - xz_n^{-1})\|,$$

which is dominated by $2^{n+1}\|z_{n+1}^{-1}x - z_n^{-1}x\| + 2^n\|xz_{n+1}^{-1} - xz_n^{-1}\|$. Again we have $\|z_{n+1}^{-1}x - z_n^{-1}x\| \leq 2^{-2n}\epsilon$, and similarly $\|xz_{n+1}^{-1} - xz_n^{-1}\| \leq 2^{-2n}\epsilon$. So $\|x_{n+1} - x_n\| \leq (2^{1-n} + 2^{-n})\epsilon < \epsilon/2^{n-2}$. Thus $w = \lim_n x_n$ exists in D, and $zwz = \lim_n z_n x_n z_n = x$ as desired. We also have $\|w - x\| \leq 2\epsilon$ as before, if we wish for this. $\qquad\square$

Remark. The above, and the next couple of results, are closely related to the results of Sinclair [1978], Esterle, and others on the Cohen factorization method, which also shows there is a commuting cai or bai under certain hypotheses. However the result above does not follow from Sinclair's results, and the latter do not directly connect to "positivity" in our sense.

Applying Lemma 4.8 to a suitable scaling of a countable bai in $\mathfrak{F}_A$, as in the proof of Corollary 4.5, we obtain:

Theorem 4.9. *Let A be an approximately unital Banach algebra, and let D be an inner ideal in A. Then D has a countable bai from $\mathfrak{F}_A$ (or equivalently, from $\mathfrak{r}_A$) if and only if there exists an element $z \in D \cap \mathfrak{F}_A$ with $D = \overline{zAz}$. Thus such D has a countable commuting bai from $\mathfrak{F}_A$. Any separable inner ideal in A with a bai from $\mathfrak{r}_A$ is of this form.*

The following is an Aarnes–Kadison-type theorem for Banach algebras. For another result of this type, see [Sinclair 1978].

Corollary 4.10. *If A is a subalgebra of a unital Banach algebra B, and we set $\mathfrak{r}_A = A \cap \mathfrak{r}_B$, then the following are equivalent*:

(i) *A has a sequential (commuting) bai from $\mathfrak{r}_A$.*

(ii) *There exists an $x \in \mathfrak{r}_A$ with $A = \overline{xAx}$.*

(iii) *There exists an $x \in \mathfrak{r}_A$ with $A = \overline{xA} = \overline{Ax}$.*

(iv) *There exists an $x \in \mathfrak{r}_A$ with $s(x)$, a mixed identity for A^{**}.*

Any separable Banach algebra with a bai from $\mathfrak{r}_A$ satisfies all of the above, as does any M-approximately unital Banach algebra which is separable or has a countable bai.

This is clear from earlier results. Indeed the last theorem gives the equivalence of (i) and (ii) above and the separability assertion, and that (ii) implies (iii) follows from Lemma 4.2, for example. Also (iii) implies (i) by considering $(x^{1/n})$, and (iii) is equivalent to (iv) by Corollary 3.19. Again, $\mathfrak{r}_A$ can be replaced by $\mathfrak{F}_A = A \cap \mathfrak{F}_B$ throughout this result, or in any of the items (i) to (iv).

As a consequence of the last results, if D is an HSA in an approximately unital Banach algebra A, and if D has a countable bai from $\mathfrak{F}_A$, then D is of the form in Lemma 4.2. We leave it to the reader to check that doing an "HSA variant" of the proof of Corollary 4.6, using Lemma 4.8 and mixed identities rather than left identities, yields:

Corollary 4.11. *The HSAs in an approximately unital Banach algebra A are exactly the closures of increasing unions of HSAs of the form $\overline{zAz}$ for $z \in \mathfrak{F}_A$.*

Proof. We just sketch the more difficult direction of this since this is so close to the proof of Corollary 4.6. Indeed we proceed as in the proof of Corollary 4.6, taking E to be the bai (e_t). Define Λ and $z_G \in D \cap \mathfrak{F}_A$ for $G \in \Lambda$ as before, but using Lemma 4.8. Note that each e_t is in some $\overline{z_G A z_G}$, which in turn is contained in the closed inner ideal $D' = \overline{\bigcup_{G \in \Lambda} z_G A z_G}$. Since for $x \in D$, we have $x = \lim_t e_t x e_t \in D' \subset D$, the result is now clear. $\square$

Remark. As in the remark after Corollary 4.6, if one takes care with the choice of the z in the last corollary, the n-th roots of these z can be a bai for the HSA.

5. Better cai for M-approximately unital algebras

In this section we consider the better behaved class of M-approximately unital Banach algebras. We will use the fact that M-ideals in Banach spaces are *strongly proximinal*. (Actually the only "proximinality-type" condition we use here is "the strongly proximinal at 1 property" mentioned in the introduction.)

Lemma 5.1. *Let X be a Banach space, and suppose that J is an M-ideal in X, and $x \in X$, $y \in J$, and $\epsilon > 0$, with $\|x - y\| < d(x, J) + \epsilon$. Then there exists a $z \in J$ with $\|y - z\| < 3\epsilon$ and $\|x - z\| = d(x, J)$.*

Proof. This follows from the proof of [Harmand et al. 1993, Proposition II.1.1]. $\square$

Theorem 5.2. *Let A be an M-approximately unital Banach algebra. Then $\mathfrak{F}_A$ is weak* dense in $\mathfrak{F}_{A^{**}}$, and $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$. Thus A has a cai in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. This is easy if A is unital, so we will focus on the nonunital case. Suppose that $\eta \in A^{**}$ with $\|1 - \eta\| \le 1$. Suppose that (x_t) is a bounded net in A with weak* limit η in A^{**}, so that $1 - x_t \to 1 - \eta$ weak* in $(A^1)^{**}$. By Lemma 1.1, for any $n \in \mathbb{N}$, there exists a t_n such that for every $t \ge t_n$,

$$\inf\{\|1 - y\| : y \in \operatorname{conv}\{x_j : j \ge t\}\} < 1 + \frac{1}{2n}.$$

For every $t \ge t_n$, choose such a $y_t^n \in \operatorname{conv}\{x_j : j \ge t\}$ with $\|1 - y_t^n\| < 1 + 1/n$. If t does not dominate t_n, define $y_t^n = y_{t_n}^n$. So for all t, we have $\|1 - y_t^n\| < 1 + 1/n$. Writing (n, t) as i, we may view (y_t^n) as a net indexed by i, with $\|1 - y_t^n\| \to 1$. Given $\epsilon > 0$ and $\varphi \in A^*$, there exists a t_1 such that $|\varphi(x_t) - \eta(\varphi)| < \epsilon$ for all $t \ge t_1$. Hence $|\varphi(y_t^n) - \eta(\varphi)| \le \epsilon$ for all $t \ge t_1$ and all n. Thus $y_t^n \to \eta$ weak* with t. By Lemma 5.1, since $d(1, A) = 1$, we can choose $w_t^n \in A$ with $\|w_t^n - y_t^n\| < 3/n$ and $\|1 - w_t^n\| = 1$. Clearly $w_t^n \to \eta$ weak*.

That $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$ follows from this, and the idea in Proposition 3.5. We omit the details, since this also follows from Propositions 2.11 and 6.2.

Next, let e be the identity of A^{**}. By Lemma 2.4, we have that $e \in \frac{1}{2}\mathfrak{F}_{A^{**}}$. Suppose that (z_t) is a net in $\frac{1}{2}\mathfrak{F}_A$ with weak* limit e in A^{**}. Standard arguments (see, e.g., [Dales 2000, Proposition 2.9.16]) show that convex combinations w_t of the z_t have the property that $a w_t$ and $w_t a$ converge weakly to a for all $a \in A$. The usual argument (see, e.g., the proof of [Blecher et al. 2008, Theorem 6.1]) shows that further convex combinations are a cai in $\frac{1}{2}\mathfrak{F}_A$. $\square$

Remark. For the first statements of Theorem 5.2, we do not need the full strength of the "M-approximately unital" condition, just strong proximinality at 1. For the existence of a cai in $\frac{1}{2}\mathfrak{F}_A$, the argument only uses strong proximinality at 1 and $\|1 - 2e\| \le 1$. Similarly, the existence of a bai in $\mathfrak{F}_A$ will follow from strong proximinality at 1 and $\|1 - e\| \le 1$.

Applied to operator algebras, the latter gives short proofs of a recent theorem of Read [2011] (see also [Blecher 2013]), as well as [Blecher and Read 2011, Lemma 8.1; 2013a, Theorem 3.3]. (We remark though that the proof of Read's theorem in [Blecher 2013] does contain useful extra information that does not seem to follow from the methods of the present paper, as is pointed out, for example, in Remark 2 after Theorem 2.1 in [Blecher and Read 2014].) Several other results

from [Blecher and Read 2011] now follow from the last result, and with otherwise unchanged proofs, for M-approximately unital Banach algebras. For example:

Corollary 5.3 (cf. [Blecher and Read 2011, Corollary 1.5; Smith and Ward 1979, Theorem 2.8]). *If J is a closed two-sided ideal in a unital Arens regular Banach algebra A, and if J is M-approximately unital, and if the support projection of J in A^{**} is central there, then J has a cai (e_t) with $\|1 - 2e_t\| \leq 1$ for all t, which is also quasicentral (that is, $e_t a - a e_t \to 0$ for all $a \in A$).*

Corollary 5.4 (cf. [Blecher and Read 2011, Corollary 1.6]). *Let A be an M-approximately unital Banach algebra. Then A has a countable bai (f_n) if and only if A has a countable cai in $\frac{1}{2}\mathfrak{F}_A$. This is also equivalent (by Theorem 4.9) to $A = \overline{xAx}$ for some $x \in \mathfrak{F}_A$.*

Remark. We can also use the results in this section to develop a slightly different approach to hereditary subalgebras than the one taken in Section 4. For example, the following is a generalization of the phenomenon in the first example in [Blecher et al. 2008, Section 2], which can be interpreted as saying that for any contractive projection p in the multiplier algebra $M(A)$, pAp is an HSA in the sense of that paper. Suppose that A is an M-approximately unital Banach algebra, and that p is an idempotent in $M(A)$ with $\|1 - 2p\| \leq 1$. For simplicity, suppose that A is Arens regular. Define $D = pAp$. Note that D is an inner ideal in A. We claim that D has a bai in $\frac{1}{2}\mathfrak{F}_D$. To see this, note that by the usual arguments, $D^{\perp\perp} = pA^{**}p$. By Theorem 5.2, there is a net w_λ in $\frac{1}{2}\mathfrak{F}_A$ with $w_\lambda \to p$ weak*. Set $d_\lambda = pw_\lambda p$; then $d_\lambda \in \frac{1}{2}\mathfrak{F}_D$, and $d_\lambda \to p$ weak*. By the usual arguments, convex combinations of the d_λ give a cai for D in $\frac{1}{2}\mathfrak{F}_D$. It is easy to see that $\overline{DA} = pA$ and $\overline{AD} = Ap$ are the induced one-sided ideals, and (d_λ) is a one-sided cai for these.

6. Banach algebras and order theory

As we said earlier, $\mathfrak{r}_A$ and $\mathfrak{r}_A^{\mathfrak{c}}$ are closed cones in A, but are not proper in general (and hence are what are sometimes called *wedges*). By the argument at the start of Section 2 in [Blecher and Read 2014], $\mathfrak{c}_A = \mathbb{R}^+\mathfrak{F}_A$ is a proper cone. These cones naturally induce orderings: we write $a \preceq b$ (resp. $a \preceq_{\mathfrak{c}} b$) if $b - a \in \mathfrak{r}_A$ (resp. $b - a \in \mathfrak{r}_A^{\mathfrak{c}}$). These are preorderings, but are not in general antisymmetric. Because of this, some aspects of the classical theory of ordered linear spaces will not generalize. Certainly many books on ordered linear spaces assume that their cones are proper. However, other books (such as [Asimow and Ellis 1980] or [Jameson 1970]) do not make this assumption in large segments of the text, and it turns out that the ensuing theory interacts in a remarkable way with our recent notion of positivity, as we point out in this section and in [Blecher and Read 2014; 2013a]. For example, in the ordered space theory, the cone $\mathfrak{d} = \{x \in X : x \geq 0\}$ in an ordered space X is said to be *generating* if $X = \mathfrak{d} - \mathfrak{d}$. This is sometimes called *positively generating*

or *directed* or *conormal*. If it is not generating, one often looks at the subspace
$\mathfrak{d} - \mathfrak{d}$. In this language, we shall see next that $\mathfrak{r}_A$ and $\mathfrak{c}_A = \mathbb{R}^+ \mathfrak{F}_A$ are generating
cones if A is M-approximately unital, or has a sequential cai and satisfies some
further conditions of the type met in Section 2. We first discuss the order theory of
M-approximately unital algebras.

Theorem 6.1. *Let A be an M-approximately unital Banach algebra. Any $x \in A$
with $\|x\| < 1$ may be written as $x = a - b$ with $a, b \in \mathfrak{r}_A$ and $\|a\| < 1$ and $\|b\| < 1$.
In fact, one may choose such a, b to also be in $\frac{1}{2}\mathfrak{F}_A$.*

Proof. Assume that $\|x\| = 1$. Since $\mathfrak{F}_{A^{**}} = e + \text{Ball}(A^{**})$ by Lemma 2.4, $x = \eta - \xi$
for $\eta, \xi \in \frac{1}{2}\mathfrak{F}_{A^{**}}$. We may assume that A is nonunital (the unital case follows from
the last line with A^{**} replaced by A). By [Blecher and Read 2011, Lemma 8.1], we
deduce that x is in the weak closure of the convex set $\frac{1}{2}\mathfrak{F}_A - \frac{1}{2}\mathfrak{F}_A$. Therefore it is in
the norm closure, so given $\epsilon > 0$, there exists $a_0, b_0 \in \frac{1}{2}\mathfrak{F}_A$ with $\|x - (a_0 - b_0)\| < \epsilon/2$.
Similarly, there exists $a_1, b_1 \in \frac{1}{2}\mathfrak{F}_A$ with $\|x - (a_0 - b_0) - \epsilon/2(a_1 - b_1)\| < \epsilon/2^2$.
Continuing in this manner, one produces sequences $(a_k), (b_k)$ in $\frac{1}{2}\mathfrak{F}_A$. Setting
$a' = \sum_{k=1}^{\infty}(1/2^k)a_k$ and $b' = \sum_{k=1}^{\infty}(1/2^k)b_k$, which are in $\frac{1}{2}\mathfrak{F}_A$ since the latter
is a closed convex set, we have $x = (a_0 - b_0) + \epsilon(a' - b')$. Let $a = a_0 + \epsilon a'$ and
$b = b_0 + \epsilon b'$. By convexity, $(1/(1 + \epsilon))a \in \frac{1}{2}\mathfrak{F}_A$ and $(1/(1 + \epsilon))b \in \frac{1}{2}\mathfrak{F}_A$.

If $\|x\| < 1$, choose $\epsilon > 0$ with $\|x\|(1 + \epsilon) < 1$. Then $x/\|x\| = a - b$ as above,
so that $x = \|x\|a - \|x\|b$. We have

$$\|x\|a = \big(\|x\|(1 + \epsilon)\big) \cdot \Big(\frac{1}{1+\epsilon}a\Big) \in [0, 1) \cdot \tfrac{1}{2}\mathfrak{F}_A \subset \tfrac{1}{2}\mathfrak{F}_A,$$

and similarly $\|x\|b \in \frac{1}{2}\mathfrak{F}_A$.   □

Remark. (1) If A is M-approximately unital then can every $x \in \text{Ball}(A)$ be written
as $x = a - b$ with $a, b \in \mathfrak{r}_A \cap \text{Ball}(A)$? As we said above, this is true if A is unital.
We are particularly interested in this question when A is an operator algebra (or
uniform algebra). We can show that in general $x \in \text{Ball}(A)$ cannot be written as
$x = a - b$ with $a, b \in \frac{1}{2}\mathfrak{F}_A$. To see this let A be the set of functions in the disk
algebra vanishing at -1, an approximately unital function algebra. Let W be the
closed connected set obtained from the unit disk by removing the "slice" consisting
of all complex numbers with negative real part and argument in a small open interval
containing π. By the Riemann mapping theorem, it is easy to see that there is a
conformal map h of the disk onto W taking -1 to 0, so that $h \in \text{Ball}(A)$. By way
of contradiction, suppose that $h = a - b$ with $a, b \in \frac{1}{2}\mathfrak{F}_A$. We use the geometry of
circles in the plane: if $z, w \in \overline{B\big(\frac{1}{2}, \frac{1}{2}\big)}$ with $|z - w| = 1$ then $z + w = 1$. It follows that
$a + b = 1$ on a nontrivial arc of the unit circle, and hence everywhere (by [Hoffman
1962, p. 52]). However, $a(-1) + b(-1) = 0$, which is the desired contradiction.

(2) Applying Theorem 6.1 to ix for $x \in A$, one gets a similar decomposition $x = a - b$ with the "imaginary parts" of a and b positive. One might ask if, as is suggested by the C^*-algebra case, one may write for each ϵ, any $x \in A$ with $\|x\| < 1$ as $a_1 - a_2 + i(a_3 - a_4)$ for a_k with numerical range in a thin horizontal "cigar" of height less than ϵ centered on the line segment $[0, 1]$ in the x-axis. In fact this is false, as one can see in the case that A is the set of upper triangular 2×2 matrices with constant diagonal entries.

A bounded $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$ (resp. $\mathbb{C}$-linear $\varphi : A \to \mathbb{C}$) is called real positive if $\varphi(\mathfrak{r}_A) \subset [0, \infty)$ (resp. $\operatorname{Re} \varphi(\mathfrak{r}_A) \geq 0$). The set of real positive functionals on A is the *real dual cone*, and we write it as $\mathfrak{c}_{A*}^{\mathbb{R}}$. Similarly, the "real version" of $\mathfrak{c}_{A*}^{\mathfrak{e}}$ will be written as $\mathfrak{c}_{A*}^{\mathfrak{e}, \mathbb{R}}$. By the usual trick, for any $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$, there is a unique $\mathbb{C}$-linear $\tilde{\varphi} : A \to \mathbb{C}$ with $\operatorname{Re} \tilde{\varphi} = \varphi$, and clearly φ is real positive if and only if $\tilde{\varphi}$ is real positive.

Proposition 6.2. *Let A be an M-approximately unital Banach algebra. An $\mathbb{R}$-linear $f : A \to \mathbb{R}$ (resp. $\mathbb{C}$-linear $f : A \to \mathbb{C}$) is real positive if and only if f is a nonnegative multiple of the real part of a state (resp. nonnegative multiple of a state). Thus M-approximately unital algebras are scaled Banach algebras.*

Proof. The one direction is obvious. For the other, by the observation above the proposition, we can assume that $f : A \to \mathbb{C}$ is $\mathbb{C}$-linear and real positive. If A is unital then the result follows from the proof of [Magajna 2009, Theorem 2.2]. Otherwise by Proposition 3.2(4) applied to the inclusion $A \subset A^1$, we see that the condition in Corollary 2.8(iii) holds. So A is scaled by Corollary 2.8. (We remark that we had a different proof in an earlier draft.) $\square$

We now turn to other classes of algebras (although we will obtain another couple of results for M-approximately unital algebras later in this section in parts (2) of Corollaries 6.7 and 6.8).

The following is a variant and simplification of [Blecher and Read 2013b, Lemma 2.7 and Corollary 2.9] and [Blecher and Read 2013a, Corollary 3.6].

Proposition 6.3. *Let A be an scaled approximately unital Banach algebra. Then the real dual cone $\mathfrak{c}_{A*}^{\mathbb{R}}$ equals $\{t \operatorname{Re}(\psi) : \psi \in S(A), t \in [0, \infty)\}$. The prepolar of $\mathfrak{c}_{A*}^{\mathbb{R}}$, which equals its real predual cone, is $\mathfrak{r}_A$, and the polar of $\mathfrak{c}_{A*}^{\mathbb{R}}$, which equals its real dual cone, is $\mathfrak{r}_{A**}$.*

Proof. It follows as in Proposition 6.2 that

$$\mathfrak{c}_{A*}^{\mathbb{R}} = \{t \operatorname{Re}(\psi) : \psi \in S(A), t \in [0, \infty)\}.$$

The prepolar of $\mathfrak{c}_{A*}^{\mathbb{R}}$, which equals its real predual cone, is $\mathfrak{r}_A$ by the bipolar theorem. We proved in Proposition 2.11 that $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A**}$. This together with the bipolar theorem gives the last assertion. $\square$

The following is a "Kaplansky density" result for $\mathfrak{r}_{A**}$:

Proposition 6.4. *Let A be an approximately unital Banach algebra such that $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$ (as we saw in Proposition 2.11 was the case for scaled approximately unital algebras). Then the set of contractions in $\mathfrak{r}_A$ is weak* dense in the set of contractions in $\mathfrak{r}_{A^{**}}$. If, in addition, there exists a mixed identity of norm 1 in $\mathfrak{r}_{A^{**}}$, then A has a cai in $\mathfrak{r}_A$.*

Proof. We use a standard kind of bipolar argument from the theory of ordered spaces. If E and F are closed sets in a TVS with E compact, then $E + F$ is closed. By this principle, and by Alaoglu's theorem, $\mathrm{Ball}(A^*) + \mathfrak{c}_{A^*}$ is weak* closed. Its prepolar (resp. polar) certainly is contained in $\mathrm{Ball}(A) \cap \mathfrak{r}_A$ (resp. $\mathrm{Ball}(A^{**}) \cap \mathfrak{r}_{A^{**}}$). This uses the fact that

$$(\mathfrak{c}_{A^*})^\circ = \mathfrak{r}_A^{\circ\circ} = \overline{\mathfrak{r}_A}^{\,\mathrm{w}*} = \mathfrak{r}_{A^{**}}$$

by the bipolar theorem. However, if $a \in \mathrm{Ball}(A) \cap \mathfrak{r}_A$ and $f \in \mathrm{Ball}(A^*)$ and $g \in \mathfrak{c}_{A^*}$, then $\mathrm{Re}(f(a) + g(a)) \geq -1 + 0 = -1$. So the prepolar of $\mathrm{Ball}(A^*) + \mathfrak{c}_{A^*}$ is $\mathrm{Ball}(A) \cap \mathfrak{r}_A$, and similarly its polar is $\mathrm{Ball}(A^{**}) \cap \mathfrak{r}_{A^{**}}$. Thus $\mathrm{Ball}(A) \cap \mathfrak{r}_A$ is weak* dense in $\mathrm{Ball}(A^{**}) \cap \mathfrak{r}_{A^{**}}$ by the bipolar theorem. The last assertion clearly follows from this and Lemma 2.1. $\qquad\square$

The condition in the next result that A^{**} is unital is a bit restrictive (it holds, for example, if A is Arens regular and approximately unital), but the result illustrates some of what one might like to be true in more general situations:

Theorem 6.5. *Let A be a Banach algebra such that A^{**} is unital, and suppose that $\mathfrak{e}$ is a cai for A. Then $\mathfrak{r}_A^{\mathfrak{e}} \subset \mathfrak{r}_{A^{**}}$ if and only if $\mathfrak{r}_A^{\mathfrak{e}} = \mathfrak{r}_A$. Suppose that the latter is true, and that $Q_{\mathfrak{e}}(A)$ is weak* closed. Then A is scaled, $S(A) = S_{\mathfrak{e}}(A)$, and A has a cai in $\mathfrak{r}_A$. Also in this case, $A = \mathfrak{r}_A - \mathfrak{r}_A$. Indeed, any $x \in A$ with $\|x\| < 1$ may be written as $x = a - b$ for $a, b \in \mathfrak{r}_A \cap \mathrm{Ball}(A)$.*

Proof. If $f \in S(A)$ then by viewing $A^1 = A + \mathbb{C}e$, we may extend f to a state $\hat{f}$ of A^{**}. If $x \in \mathfrak{r}_A^{\mathfrak{e}} \subset \mathfrak{r}_{A^{**}}$ then $\mathrm{Re}\, f(x) = \mathrm{Re}\, \hat{f}(x) \geq 0$. Thus $\mathfrak{r}_A^{\mathfrak{e}} \subset \mathfrak{r}_A$, and so these sets are equal. We also see that $\mathfrak{c}_{A^*} = \mathfrak{c}_{A^*}^{\mathfrak{e}}$. If $Q_{\mathfrak{e}}(A)$ is weak* closed then A is $\mathfrak{e}$-scaled by Lemma 2.7, so that $f = tg$ for some $g \in S_{\mathfrak{e}}(A)$ and for some t which must equal 1. It follows that $S(A) = S_{\mathfrak{e}}(A)$. Hence A is scaled, so that the weak* closure of $\mathfrak{r}_A \cap \mathrm{Ball}(A)$ is $\mathfrak{r}_{A^{**}} \cap \mathrm{Ball}(A^{**})$ by Proposition 6.4. Since the latter contains an identity, A has a cai in $\mathfrak{r}_A$ by the observation after that result. The assertion concerning $\|x\| < 1$ follows by a slight variant of the proof of Theorem 6.1. $\qquad\square$

In fact it is not too hard to see, as we shall show in another paper, that if A^{**} is unital (or if it has a unique mixed identity), and A has a cai in $\mathfrak{r}_A$ then A has a cai in $\mathfrak{F}_A$ (and the latter cai can be chosen to be sequential if the first cai is sequential).

We now attempt to prove parts of the last theorem, and some other order theoretic results, in the case that A^{**} is not unital. We will mostly be using the class of states $S_\mathfrak{e}(A)$ with respect to a fixed cai $\mathfrak{e}$, and the matching cones $\mathfrak{r}_A^\mathfrak{e}$ and $\mathfrak{c}_{A*}^\mathfrak{e}$, as opposed to $S(A)$ and its matching cones. The reason for this is that we will want norm additivity

$$\|c_1\varphi_1 + \cdots + c_n\varphi_n\| = c_1 + \cdots + c_n, \quad \varphi_k \in S(A), c_k \geq 0.$$

In many interesting examples, $S(A)$ satisfies this additivity property (for example, if A is Hahn–Banach smooth, by Lemma 2.2), and in this case almost all the rest of the results in this section will be true for the $S(A)$ variants, and with all the subscripts and superscripts and every hyphenated $\mathfrak{e}$ dropped.

Lemma 6.6. *Suppose that* $\mathfrak{e} = (e_t)$ *is a fixed cai for a Banach algebra* A, *and suppose that* $Q_\mathfrak{e}(A)$ *is weak* closed in* A^*.

(1) *The cones* $\mathfrak{c}_{A*}^\mathfrak{e}$ *and* $\mathfrak{c}_{A*}^{\mathfrak{e},\mathbb{R}}$ *are additive* (*that is, the norm on the dual space of* A *is additive on these cones*).

(2) *If* (φ_t) *is an increasing net in* $\mathfrak{c}_{A*}^{\mathfrak{e},\mathbb{R}}$ *which is bounded in norm, then the net converges in norm, and its limit is the least upper bound of the net.*

Proof. (1) If $\psi = c\varphi$ for $\varphi \in S_\mathfrak{e}(A)$ and $c \geq 0$, then

$$\|\psi\| = c\|\varphi\| = \lim_t \psi(e_t).$$

Indeed, for an appropriate mixed identity e of A^{**} of norm 1, we have $\|\varphi\| = \langle e, \varphi \rangle$ for all $\varphi \in \mathfrak{c}_{A*}^{\mathfrak{e},\mathbb{R}}$. It follows that the norm on $B(A, \mathbb{R})$ is additive on $\mathfrak{c}_{A*}^{\mathfrak{e},\mathbb{R}}$. The complex scalar case is similar.

(2) It follows from (1) and [Asimow and Ellis 1980, Proposition 3.2, Chapter 2]. □

We recall that the positive part of the open unit ball of a C^*-algebra is a directed set. The following is a Banach algebra version of this:

Corollary 6.7. (1) *Let* $\mathfrak{e}$ *be a cai for a Banach algebra* A, *and suppose that* $Q_\mathfrak{e}(A)$ *is weak* closed in* A^*. *Then the open unit ball of* A *is a directed set with respect to the* $\preceq_\mathfrak{e}$ *ordering. That is, if* $x, y \in A$ *with* $\|x\|, \|y\| < 1$, *then there exists* $z \in A$ *with* $\|z\| < 1$ *and* $z \in \mathfrak{r}_A^\mathfrak{e}$, *and also* $x \preceq_\mathfrak{e} z$ *and* $y \preceq_\mathfrak{e} z$.

(2) *If* A *is an* M-*approximately unital Banach algebra, then given* $x, y \in A$ *with* $\|x\|, \|y\| < 1$, *a majorant* z *can be chosen as in* (1), *but also with* $z \in \frac{1}{2}\mathfrak{F}_A$.

Proof. (1) By Lemma 6.6(1) together with [Asimow and Ellis 1980, Corollary 3.6, Chapter 2], for any $x, y \in A$ with $\|x\| < 1$ and $\|y\| < 1$, there exists a $w \in A$ with $\|w\| < 1$ and $w - x, w - y \in \mathfrak{r}_A^\mathfrak{e}$. By the last assertion of Theorem 2.9 (setting the

a there to be $-tw$ for some appropriate $t > 1$), we have $w \preceq_{\mathfrak{e}} z$ for some $z \in \mathfrak{r}_A^{\mathfrak{e}}$ with $\|z\| < 1$. So

$$-z \preceq_{\mathfrak{e}} -w \preceq_{\mathfrak{e}} x \preceq_{\mathfrak{e}} w \preceq_{\mathfrak{e}} z.$$

Similarly, y "lies between" z and $-z$, from which it is easy to see that z is in $\mathfrak{r}_A^{\mathfrak{e}}$.

(2) This is similar to (1), but uses the fact that $S(A) = S_{\mathfrak{e}}(A)$ by Lemma 2.2, so every $\mathfrak{e}$ can be dropped. We also use the following principle twice in place of the cited results in the proof above: if $\|z\| < 1$ then by Theorem 6.1 we may write $z = a - b$ for $a, b \in \frac{1}{2}\mathfrak{F}_A$, and then $-b \preceq z \preceq a$. $\qquad\square$

For a C^*-algebra B, a natural ordering on the positive part of the open unit ball of B turns the latter into a net which is a positive cai for B (see [Pedersen 1979]). A similar result holds for operator algebras [Blecher and Read 2014, Proposition 2.6]. We are not sure if there is an analogue of this for the classes of algebras in the last result.

Corollary 6.8. (1) *Let $\mathfrak{e}$ be a cai for a Banach algebra A, and suppose that $Q_{\mathfrak{e}}(A)$ is weak* closed in A^*. For all $x \in A$, there exists an element $z \in A$ with z in $\mathfrak{r}_A^{\mathfrak{e}}$ and $-z \preceq_{\mathfrak{e}} x \preceq_{\mathfrak{e}} z$. Thus $x = a - b$, where $a, b \in \mathfrak{r}_A^{\mathfrak{e}}$. Moreover, if $\|x\| < 1$ then z, a, b can all be chosen in $\mathrm{Ball}(A)$.*

(2) *If A is an M-approximately unital Banach algebra, then given $x \in A$ with $\|x\| < 1$, an element z can be chosen satisfying the inequalities in (1), but also with $z \in \frac{1}{2}\mathfrak{F}_A$.*

Proof. Apply Corollary 6.7 to x and $-x$. Clearly, $a = (z+x)/2$ and $b = (z-x)/2$. $\square$

In the language of [Messerschmidt 2015], item (1) implies that the associated preorder on A is *approximately* 1-*absolutely conormal*, and from the theory of ordered Banach spaces in that reference, this is equivalent to $B(A, \mathbb{R})$ being "absolutely monotone". That is, with respect to the natural induced ordering on $B(A, \mathbb{R})$, if $-\psi \le \varphi \le \psi$ then $\|\varphi\| \le \|\psi\|$.

Corollary 6.9. *Let $\mathfrak{e}$ be a cai for a Banach algebra A, and suppose that $Q_{\mathfrak{e}}(A)$ is weak* closed in A^*. If $f \le g \le h$ in $B(A, \mathbb{R})$ in the natural $\mathfrak{e}_{A^*}^{\mathfrak{e}}$-ordering, then $\|g\| \le \|f\| + \|h\|$.*

Proof. This follows from Corollary 6.8 by [Batty and Robinson 1984, Theorem 1.1.4]. $\qquad\square$

Corollary 6.10. *If A is an approximately unital Banach algebra then the last four results are true with all the subscripts and superscripts and every hyphenated $\mathfrak{e}$ dropped if also $S(A) = S_{\mathfrak{e}}(A)$ for the cai $\mathfrak{e}$ appearing in those results (which holds, for example, if A is Hahn–Banach smooth in A^1).*

Proof. Indeed, in the Hahn–Banach smooth case, $S(A) = S_{\mathfrak{e}}(A)$ by Lemma 2.2, and if the latter holds then all $\mathfrak{e}$ may be dropped. $\qquad\square$

In the part of Corollary 6.10 dealing with Corollary 6.7(1), and with Corollary 6.8 in the $\|x\| < 1$ case, one may often get the majorants z appearing in those corollaries to also be in $\mathfrak{F}_A$ (and even get a sequential cai for A in $\mathfrak{F}_A$ consisting of such majorants z). We will discuss this in another paper, but briefly this follows from the ideas in Corollary 2.10 and the paragraphs after that, and the idea in the paragraph after Theorem 6.5.

Remark. (1) Above we saw that under various hypotheses, a Banach algebra A had a cai in $\mathfrak{r}_A$, and the latter was a generating cone, that is $A = \mathfrak{r}_A - \mathfrak{r}_A$. Conversely, we shall see in Corollary 7.6 that if A is commutative, approximately unital, and $A = \mathfrak{r}_A - \mathfrak{r}_A$, then A has a bai in $\mathfrak{F}_A$.

(2) It is probably never true for an approximately unital operator algebra A that $B(A, \mathbb{R}) = \mathfrak{c}_{A^*}^{\mathbb{R}} - \mathfrak{c}_{A^*}^{\mathbb{R}}$. Indeed, in the case $A = \mathbb{C}$, the latter space has real dimension 1. However, the complex span of the (usual) states of an approximately unital operator algebra A is A^* (the complex dual space). Indeed, by a result of Moore [1971] (see also [Asimow and Ellis 1972]), the complex span of the states of any unital Banach algebra A is A^*. In the approximately unital Banach algebra case, at least if A is scaled, the same fact follows by using a Hahn–Banach extension and Corollary 2.8(iii).

(3) Every element $x \in \frac{1}{2}\mathfrak{F}_A$ need not achieve its norm at a state, even in M_2 (consider $x = (I + E_{12})/2$, for example).

(4) We thank Miek Messerschmidt for calling our attention to the result in [Batty and Robinson 1984] used in Corollary 6.9. Previously we had a cruder inequality in that result.

(5) Note that A is not usually "order-cofinal" in A^1, in the sense of the ordered space literature, even for A any C^*-algebra with no countable cai (and hence no strictly real positive element).

7. Ideals in commutative Banach algebras

Throughout this section, A will be a commutative approximately unital Banach algebra. We will use ideas from [Blecher et al. 2008; Blecher and Read 2011; 2013a] (see [Esterle 1978; Kaniuth et al. 2010] for some other Banach algebra variants of some of these ideas). In the following statement, the "respectively"s are placed correctly, despite first impressions.

Theorem 7.1. *Let A be a commutative approximately unital Banach algebra. The closed ideals in A with a bai in $\mathfrak{r}_A$ (resp. $\mathfrak{F}_A$) are precisely the ideals of the form $\overline{EA}$ for some subset $E \subset \mathfrak{F}_A$ (resp. $E \subset \mathfrak{r}_A$). They are also the closures of increasing unions of ideals of the form $\overline{xA}$ for $x \in \mathfrak{F}_A$ (resp. $x \in \mathfrak{r}_A$).*

Proof. Suppose that $E \subset \mathfrak{r}_A$, and we will prove that $\overline{EA}$ has a bai in $\mathfrak{F}_A$. We may assume that $E \subset \mathfrak{F}_A$ since $\overline{EA} = \overline{\mathfrak{F}(E)A}$, as may be seen using Proposition 3.11. We will first suppose that E has two elements, and here we will include a separate argument if A is Arens regular since the computations are interesting. Then we will discuss the case where E has n elements, and then the general case.

If $x, y \in \mathfrak{r}_A$ then $\overline{xA}$ and $\overline{yA}$ are ideals with bais in $\mathfrak{F}_A$ by Corollary 3.18. Their support idempotents $s(x)$ and $s(y)$ are in $\mathfrak{F}_{A^{**}}$. Indeed if $J = \overline{xA}$ then by Corollary 3.18, we have $J^{\perp\perp} = s(x)A^{**}$, and $J = s(x)A^{**} \cap A$. (In the non-Arens regular case we are using the second Arens product here.) In the rest of this paragraph, we assume that A is Arens regular. Set

$$s(x, y) = s(x) + s(y) - s(x)s(y) = 1 - (1 - s(x))(1 - s(y)),$$

where $s(x, y)$ is an idempotent dominating both $s(x)$ and $s(y)$ in the sense that $s(x, y)s(x) = s(x)$ and $s(x, y)s(y) = s(y)$. If f is another idempotent dominating both $s(x)$ and $s(y)$ then $fs(x, y) = s(x, y)$, so that $s(x, y)$ is the "supremum" of $s(x)$ and $s(y)$ in this ordering. Then notice that $\|(1-x^{1/n})(1-y^{1/m})\| \leq 1$, and also

$$\left\|(1 - s(x))(1 - s(y))\right\| = \|1 - s(x, y)\| \leq 1.$$

Notice too that $\overline{xA + yA}$ has a bai in $\mathfrak{F}_A$ with terms of form

$$x^{1/n} + y^{1/m} - x^{1/n}y^{1/m} = 1 - (1 - x^{1/n})(1 - y^{1/m}),$$

which has bound 2. A double weak* limit point of this bai from $\mathfrak{F}_A \cap \overline{EA}$ is $s(x, y)$. So as usual $\overline{xA + yA} = \{a \in A : s(x, y)a = a\}$.

In the non-Arens regular case we use the second Arens product below. We show that $\overline{xA + yA} = \overline{((x + y)/2)A} = \overline{aA}$, where $a = (x + y)/2 \in \mathfrak{F}_A$. By the proof of [Blecher and Read 2011, Lemma 2.1], we know that $(1 - 1/n \sum_{k=1}^n (1 - a)^k) \in \mathfrak{F}_A$ is a bai for $ba(a)$, and for $\overline{aA}$. Write $x = 1 - z$, $y = 1 - w$ for contractions $z, w \in A^1$, and let $b = (z + w)/2$. Then $a = 1 - b$. Let r be a weak* limit point of the bai above, which is a mixed identity for $ba(a)^{**}$. Then $ra = a$, so that $(1-r)b = (1-r)$. Note that $s = 1 - r$ is a contractive idempotent, and is an identity for $s(A^1)^{**}s$. Since the identity in a Banach algebra is an extreme point, and since $(sz + sw)/2 = s$, we deduce that $sz = zs = s$. Similarly $sw = ws = s$. Thus $rx = x$, so that $x \in rA^{**} \cap A = \overline{aA}$ (as in Corollary 3.18). This works similarly for y, and thus $\overline{xA + yA} = \overline{((x + y)/2)A}$. Thus if $x, y \in \mathfrak{F}_A$ then the support idempotent $s((x + y)/2)$ for a can be taken to be a "support idempotent" for $\overline{xA + yA}$.

A very similar argument works for three elements $x, y, z \in \mathfrak{F}_A$, using, for example, the fact that $\|(1 - x^{1/n})(1 - y^{1/n})(1 - z^{1/n})\| \leq 1$. Indeed, a similar argument works for any finite collection $G = \{x_1, \ldots, x_m\} \in \mathfrak{F}_A$. We have $\overline{GA} = \overline{x_G A}$, where

$$x_G = \tfrac{1}{m}(x_1 + \cdots + x_m) \in \mathfrak{F}_A \cap \overline{EA}.$$

Let us write $s(G)$ for $s((1/m)(x_1 + \cdots + x_m))$. Then $s(G)$ is the support idempotent of $\overline{GA}$, and $s(G)A^{**} = (GA)^{\perp\perp}$, and thus $\overline{GA} = s(G)A^{**} \cap A$. This has a bai in $\mathfrak{F}_A \cap \overline{EA}$, namely $(1-[(1-x_1^{1/n})\cdots(1-x_m^{1/n})])$, or $(1-[(1-x_1^{1/n_1})\cdots(1-x_m^{1/n_m})])$.

If E is a subset of $\mathfrak{F}_A$, let $J = \overline{EA}$, and let Λ be the collection of finite subsets G of E ordered by inclusion. Writing Λ as a net $(G_i)_{i\in\Lambda}$, we have

$$J = \overline{EA} = \overline{\bigcup_{i\in\Lambda} G_i A} = \overline{\bigcup_{i\in\Lambda} x_{G_i} A},$$

where $x_{G_i} \in \mathfrak{F}_A \cap \overline{EA}$. To see that J has a bai in $\mathfrak{F}_A$, as in [Palmer 1994, Theorem 5.1.2(a)], it is enough to show that given $G \in \Lambda$ and $\epsilon > 0$, there exists $a \in \mathfrak{F}_A \cap J$ with $\|ax - x\| < \epsilon$ for all $x \in G$. However, this is clear since, as we saw above, $\overline{GA}$ has a bai in $\mathfrak{F}_A$.

Conversely, suppose that J is an ideal in A with a bai (x_t) in $\mathfrak{r}_A$. Then $J = \overline{\sum_t x_t A} = \overline{EA}$, where $E = \{\mathfrak{F}(x_t)\} \subset \mathfrak{F}_A$ by Proposition 3.11. The remaining results are clear from what we have proved. $\qquad\square$

Remark. (1) See [Lau and Ülger 2014] for a recent characterization of ideals with bais.

(2) We saw in Example 4.3 that several of the methods used in the last proof fail for noncommutative algebras. First, it is not true there that if $x, y \in \mathfrak{F}_A$ then $\overline{xA + yA} = \overline{((x + y)/2)A}$. Also $\overline{xA + yA}$ may have no left cai. Also, it need not be the case that EAE has a bai if $E \subset \mathfrak{F}_A$.

If E is any subset of $\mathfrak{F}_A$ and $J = \overline{EA}$, and if $s = s_E$ is a weak* limit point of any bai in $\mathfrak{F}_A$ for J, then we call s a *support idempotent* for J. Note that $sA^{**} = J^{\perp\perp}$ as usual, and so $J = sA^{**} \cap A$.

Remark. Suppose that I is a directed set, and that $\{E_i : i \in I\}$ is a family of subsets of $\mathfrak{F}_A$ with $E_i \subset E_j$ if $i \leq j$. Then $\overline{\sum_i E_i A} = \overline{EA}$, where $E = \bigcup_i E_i$. Moreover, if s_i is a support idempotent for $\overline{E_i A}$, and if s_i has weak* limit point s' in A^{**} then we claim that s' is a support idempotent for $J = \overline{EA}$. Indeed, clearly $s' \in (J \cap \mathfrak{F}_A)^{\perp\perp}$, since each s_i resides here. Conversely, if $x \in E_i$ then $s_j x = x$ if $j \geq i$, so that $s'x = x$. Thus $s_i x \to x$ in norm for all $x \in J$, so that $s'x = x$ for all $x \in J$. Hence $s'x = x$ for all $x \in J^{\perp\perp}$. Therefore s' is idempotent, and $J^{\perp\perp} \subset s'A^{**}$, and so $J^{\perp\perp} = s'A^{**}$. As usual, $J = s'A^{**} \cap A$. This concludes the proof of the claim. If (x_t) is a net in $J \cap \mathfrak{F}_A$ with weak* limit s' then we leave it as an exercise that one can choose a net of convex combinations of the x_t, which is a bai for J in $\mathfrak{F}_A$ with weak* limit s'. In particular, if $(G_i)_{i\in\Lambda}$ is as in the proof of Theorem 7.1, then the net $s_i = s(G_i)$ has a weak* limit point which is a support projection for $J = \overline{EA}$.

Let us define an $\mathfrak{F}$-*ideal* to be an ideal of the kind characterized in Theorem 7.1, namely a closed ideal in A with a bai in $\mathfrak{r}_A$.

Theorem 7.2. *Let A be a commutative approximately unital Banach algebra. Any separable $\mathfrak{F}$-ideal in A is of the form $\overline{xA}$ for $x \in \mathfrak{F}_A$. Also, the closure of the sum of a countable set of ideals $\overline{x_k A}$ for $x_k \in \mathfrak{F}_A$, equals $\overline{zA}$, where $z = \sum_{k=1}^{\infty} (1/2^k) x_k$.*

Proof. The first assertion follows from the matching result Corollary 4.5, or from the second assertion as in [Blecher and Read 2011, Theorem 2.16]. For the second assertion, let x_k, z be as in the statement. Inductively one can prove that $x_k \in \overline{zA}$, which is what is needed. One begins by setting $x = x_1$ and $y = \sum_{k=2}^{\infty} (1/2^{k-1}) x_k \in \mathfrak{F}_A$. Then $z = (x + y)/2$, and the third paragraph of the proof of Theorem 7.1 shows that $x = x_1 \in \overline{zA}$, and $y \in \overline{zA}$. One then repeats the argument to show all $x_k \in \overline{zA}$. $\square$

As in Section 4, we obtain again that, for example:

Corollary 7.3. *Let A be a commutative M-approximately unital Banach algebra. Then A has a countable cai if and only if there exists $x \in \mathfrak{F}_A$ with $A = \overline{xA}$ (or equivalently, if and only if $s(x)$ is the unique mixed identity of A^{**} of norm 1).*

With this in hand, one can generalize some part of the theory of left ideals and cais in [Blecher et al. 2008; Blecher and Read 2011; 2013a] to the class of ideals in the last theorem, in the commutative case. This class is not closed under finite intersections. In fact, this fails rather badly (see Example 3.13). One may define an $\mathfrak{F}$-*open idempotent* in A^{**} to be an idempotent $p \in A^{**}$ for which there exists a net (x_t) in $\mathfrak{F}_A$ (or equivalently, as we shall see, in $\mathfrak{r}_A$) with $x_t = px_t \to p$ weak*. Thus a left identity for the second Arens product in A^{**} is $\mathfrak{F}$-open if and only if it is in the weak* closure of $\mathfrak{F}_A$. See [Akemann 1970; Pedersen 1979] for the notion of open projection in a C^*-algebra.

Lemma 7.4. *If A is a commutative approximately unital Banach algebra then the $\mathfrak{F}$-open idempotents in A^{**} are precisely the support idempotents for $\mathfrak{F}$-ideals.*

Proof. If p is an $\mathfrak{F}$-open idempotent then it follows that $p \in \mathfrak{F}_{A^{**}}$, and that $J = \overline{EA}$ is an $\mathfrak{F}$-ideal, where $E = \{x_t\}$ (using Theorem 7.1). Also $px = x$ if $x \in J$, and $p \in J^{\perp\perp}$. So $pA^{**} = J^{\perp\perp}$, from which it is easy to see that p is a support idempotent of J.

The converse is obvious by the definition of support idempotent above, and the fact that $\overline{EA} = s_E A^{**} \cap A$. $\square$

Corollary 7.5. *If A is a commutative approximately unital Banach algebra, and $E \subset \mathfrak{r}_A$, then the closed subalgebra generated by E has a bai in $\mathfrak{F}_A$.*

Proof. In Theorem 7.1 we constructed a bai in $\mathfrak{F}_A$ for $\overline{EA}$, and this bai is clearly in the closed subalgebra generated by E, and is a bai for that subalgebra. $\square$

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If A is any approximately unital commutative Banach algebra, define $A_H = \overline{\mathfrak{F}_A A}$. This is an ideal of the type in Theorem 7.1, and is the largest such (by that result).

If A is an operator algebra, it is proved in [Blecher and Read 2013a] that $A = \mathfrak{r}_A - \mathfrak{r}_A$ if and only if A has a cai. In our setting we at least have:

Corollary 7.6. *If A is a commutative approximately unital Banach algebra which is generated by $\mathfrak{r}_A$ as a Banach algebra (and certainly if $A = \mathfrak{r}_A - \mathfrak{r}_A$), then A has a bai in $\mathfrak{F}_A$.*

Proof. This follows from Corollary 7.5 because A is generated by $\mathfrak{r}_A$ in this case, and hence is generated by $\mathfrak{F}_A$ since $\mathfrak{r}_A = \overline{\mathbb{R}^+ \mathfrak{F}_A}$. □

Conversely, if A is M-approximately unital or has a sequential cai satisfying certain conditions discussed in Section 6, then we saw in Section 6 that $A = \mathfrak{r}_A - \mathfrak{r}_A$. Indeed, we saw in the M-approximately unital case in Theorem 6.1 that

$$A = \mathbb{R}^+ (\mathfrak{F}_A - \mathfrak{F}_A) \subset \mathfrak{r}_A - \mathfrak{r}_A \subset A.$$

We do not know if it is always true if, as in the operator algebra case, for any approximately unital commutative Banach algebra we have $A_H = \mathfrak{r}_A - \mathfrak{r}_A = \mathbb{R}^+ (\mathfrak{F}_A - \mathfrak{F}_A)$.

8. M-ideals which are ideals

We now turn to an interesting class of closed approximately unital ideals in a general approximately unital Banach algebra that generalizes the class of approximately unital closed two-sided ideals in operator algebras. (Unfortunately, we see no way yet to apply the theory in [Blecher and Zarikian 2006] to generalize the results in this section to one-sided ideals.) The study of this class was initiated in [Smith and Ward 1978; 1979; Smith 1979]. We will use basic ideas from these papers (see also Werner's theory of inner ideals in the sense of [Harmand et al. 1993, Section V.3]).

First, let A be a unital Banach algebra. We define an M-*ideal ideal* in A to be a subspace J of A which is an M-ideal in A, such that if P is the M-projection then $z = P1$ is central in A^{**} (the latter is automatic, for example, if A is commutative and Arens regular). Actually it suffices in all the arguments below that simply $za = az$ for $a \in A$, but for convenience we will stick to the "central" hypothesis. By [Smith and Ward 1978, Proposition 3.1], z is a hermitian projection of norm 1 (or 0). It is then a consequence of Sinclair's theorem on hermitians [Sinclair 1971] that z is accretive, indeed $W(z) \subset [0, 1]$. The proof of [Smith and Ward 1978, Proposition 3.4] shows that $(1 - z)J^{\perp\perp} = (0)$ (it is shown there that $z J^{\perp\perp} z \subset J^{\perp\perp} = J_1$ in the notation there, and that $(1-z)J \subset J_2$, but clearly $zJ \subset J_1$ so that $(1 - z)J \subset (J - J_1) \cap J_2 \subset J_1 \cap J_2 = (0)$). It also shows that $z(I - P)A^{**} = 0$, so that P is simply left multiplication by z, and

$J^{\perp\perp} = zA^{**}$. Since the latter is an ideal, $J = J^{\perp\perp} \cap A$ is an ideal in A. Moreover, J is approximately unital since z is a mixed identity for $J^{\perp\perp}$ of norm 1. We call z the support projection of J, and write it as s_J. The correspondence $J \mapsto s_J$ is bijective on the class of M-ideal ideals.

Proposition 8.1. *An M-ideal ideal J in a unital Banach algebra A is M-approximately unital, indeed J has a cai in $\frac{1}{2}\mathfrak{F}_A$. Also J is a two-sided $\mathfrak{F}$-ideal in A, and $J = \overline{EA} = \overline{AE}$ for some subset $E \in J \cap \mathfrak{F}_A$.*

Proof. By Proposition 3.2, J is M-approximately unital, so by Theorem 5.2 it has a cai in $\frac{1}{2}\mathfrak{F}_J = J \cap \frac{1}{2}\mathfrak{F}_A$. (The latter equality follows from Proposition 3.2 applied in A^1.) Thus J is a two-sided $\mathfrak{F}$-ideal. We also deduce from Proposition 3.2 that $J^1 \cong J + \mathbb{C}1_A$. Hence $J = \overline{EA} = \overline{AE}$ for some $E \subset J \cap \mathfrak{F}_A$; for example, take E to be the cai above. $\square$

The converse of the last result fails. Indeed even in a commutative algebra, not every ideal $\overline{EA}$ for a subset $E \in \mathfrak{F}_A$, is an M-ideal ideal, nor need have a cai in $\frac{1}{2}\mathfrak{F}_A$ (see Example 3.14).

Suppose that J_1 and J_2 are M-ideal ideals in A, and that P_1, P_2 are the corresponding M-projections on A^{**} with $z_k = P_k 1$ central in A^{**}. As in Corollary 3.19, $J_1 \subset J_2$ if and only if $z_2 z_1 = z_1$, and the latter equals $z_1 z_2$. So the correspondence $J \mapsto s_J$ is an order embedding with respect to the usual ordering of projections in A^{**}. Then by facts above, $P_1 P_2(1) = P_1(z_2) = z_1 z_2$, and this is central in A^{**}. Similarly, $(P_1 + P_2 - P_1 P_2)1 = z_1 + z_2 - z_1 z_2$, and this is central in A^{**}. Hence $J_1 \cap J_2$ and $J_1 + J_2$ are M-ideal ideals in A.

To describe the matching fact about "joins" of an infinite family of ideals, we introduce some notation. Set N to be A^{**}. We will use the fact that N contains a commutative von Neumann algebra. We recall that the *centralizer $Z(X)$* of a dual Banach space X is a weak* closed subalgebra of $B(X)$, and it is densely spanned in the norm topology by its contractive projections, which are the M-projections (see, e.g., [Harmand et al. 1993] and [Blecher and Zarikian 2006, Section 7.1]). It is also a commutative W^*-algebra in the weak* topology from $B(X)$. By [Harmand et al. 1993, Theorem V.2.1]), the map $\theta : Z(N) \to N$ taking $T \in Z(N)$ to $T(1)$ is an isometric homomorphism, and it is weak* continuous by the definition of the weak* topology on $B(N)$ and hence on $Z(N)$. Therefore by the Krein–Smulian theorem, the range of θ is weak* closed, and θ is a weak* homeomorphism onto its range. Thus $Z(N)$ is identifiable with a weak* closed subalgebra Δ of N, which is a commutative W^*-algebra, via the map $T \mapsto T(1)$. All computations can be done inside this commutative von Neumann algebra. Indeed the ordering of support projections z_1, z_2, and their "meet" and "join", which we met a couple of paragraphs above, are simply the standard operations $z_1 \leq z_2, z_1 \vee z_2, z_1 \wedge z_2$ with projections, computed in the W^*-algebra Δ. Of course, we are specifically interested in the weak* closed

subalgebra consisting of elements in Δ that commute with A. The projections in this subalgebra densely span a commutative von Neumann algebra inside Δ.

Lemma 8.2. *The closure of the span of a family $\{J_i : i \in I\}$ of M-ideal ideals in a unital Banach algebra A is an M-ideal ideal in A.*

Proof. Let $\{P_i : i \in I\}$ be the corresponding family of M-projections on A^{**} with $z_i = P_i 1$ central in A^{**}. Let Λ be the collection of finite subsets of I ordered by inclusion. For $F \in \Lambda$, let $J_F = \sum_{i \in F} J_i$; by the above, this will be an M-ideal ideal in A whose support projection s_{J_F} corresponds to $P_F(1)$, where P_F is the M-projection for J_F. Next suppose that (P_F) has weak* limit P in $Z(N)$; by the theory of M-projections, P is the M-projection corresponding to the M-ideal $J = \sum_i J_i = \overline{\sum_{F \in \Lambda} J_F}$. We have $P(1) = z$ is the weak* limit of the (z_i); this is a contractive hermitian projection in the ideal $J^{\perp\perp}$. For $\eta \in N$, we have $z\eta \in J^{\perp\perp}$ so that

$$z\eta = P(z\eta) = \lim_i P_i(z\eta) = \lim_i z_i z\eta = \lim_i z_i \eta = \lim_i \eta z_i = \eta z.$$

Thus z is central in N, and so J is an M-ideal ideal with support projection z, and z is the supremum $\vee_i z_i$ in Δ. $\square$

Next assume that A is an approximately unital Banach algebra. We define an *M-ideal ideal* in A to be a subspace J of A which is an M-ideal in A^1 such that $z = P1$ is central in A^{**} (or, as we said above, simply that $za = az$ for $a \in A$, which will then allow an M-approximately unital A to always be an M-ideal ideal in itself). We may then apply the theory in the last several paragraphs to A^1; thus $N = (A^1)^{**}$ there. Set Δ' to be the weak* closure in Δ of the span of those projections that happen to be in A^{**}. This is also a commutative W^*-algebra.

Theorem 8.3. *If A is an approximately unital Banach algebra then the class of M-ideal ideals in A forms a lattice; indeed, the intersection of a finite number, or the closure of the sum of any collection, of M-ideal ideals is again an M-ideal ideal. The correspondence between M-ideal ideals J in A and their support projections s_J in $\Delta' \subset A^{**}$ is bijective and preserves order, and preserves finite meets and arbitrary joins . That is, $s_{J_1 \cap J_2} = s_{J_1} s_{J_2}$ for M-ideal ideals J_1, J_2 in A; and if $\{J_i : i \in I\}$ is any collection of M-ideal ideals in A and J is the closure of their span, then s_J is the supremum in $\Delta' \subset A^{**}$ of $\{s_{J_i} : i \in I\}$.*

Proof. This result is essentially a summary of some facts above with these facts applied to A^1 instead of A, and with $N = (A^1)^{**}$. $\square$

Clearly any M-ideal ideal in A is Hahn–Banach smooth in A^1 [Harmand et al. 1993], and hence in A.

If J is an M-ideal ideal then we call s_J above a *central open projection* in A^{**}. Clearly such open projections p are weak* limits of nets $x_t \in \frac{1}{2}\mathfrak{F}_A$ with $px_t =$

$x_t p = x_t$. However, not every projection in A^{**} which is such a weak* limit is the support idempotent of an M-ideal ideal (again, see Example 3.14). Nonetheless we expect to generalize more of the theory in [Blecher et al. 2008; Blecher and Read 2011; 2013a] of open projections and r-ideals to this setting. For a start, it is now clear that suprema of any collection, and infima of finite collections, of central open projections are central open projections. If A is an M-approximately unital Banach algebra then the mixed identity e for A^{**} of norm 1 is a central open projection.

Proposition 8.4. *If A is an approximately unital Banach algebra then any central open projection is lower semicontinuous on $Q(A)$.*

Proof. If A is unital then this result is in [Smith and Ward 1979], and we use this below. Let $\varphi_t \to \varphi$ weak* in $Q(A)$, and suppose that $\varphi_t(p) \le r$ for all t. Write $\varphi_t = c_t \psi_t$ for $\psi_t \in S(A)$, and let $\hat{\psi}_t \in S(A^1)$ be a state extending ψ_t. By replacing by a subnet, we can assume that $c_t \to s \in [0,1]$. A further subnet $\hat{\psi}_{t_\nu}$ converges to $\rho \in S(A^1)$ weak*. Thus $\varphi = s\rho_{|A}$, since

$$\varphi_{t_\nu}(a) = c_{t_\nu} \psi_{t_\nu}(a) = c_{t_\nu} \hat{\psi}_{t_\nu}(a) \to s\rho(a), \quad a \in A.$$

By the result from [loc. cit.] mentioned above,

$$\rho(p) \le \liminf_\nu \hat{\psi}_{t_\nu}(p) = \liminf_\nu \psi_{t_\nu}(p).$$

Hence

$$\varphi(p) = s\rho(p) \le \liminf_\nu s\psi_{t_\nu}(p) = \liminf_\nu c_{t_\nu}\psi_{t_\nu}(p) \le r,$$

as desired. $\square$

Given a central open projection $p \in A^{**}$, we set $F_p = \{\varphi \in Q(A) : \varphi(p) = 0\}$.

Theorem 8.5. *Suppose that A is a scaled approximately unital Banach algebra, and p is a central open projection in A^{**}, and $J = pA^{**} \cap A$ is the corresponding ideal. Then $F_p = Q(A) \cap J^\perp$, and this is a weak* closed face of $Q(A)$. Moreover, the assignment Θ taking $p \mapsto F_p$ (resp. $J \mapsto F_p$) from the set of central open projections (resp. M-ideal ideals of A) into the set of weak* closed faces of $Q(A)$, is one-to-one and is a (reverse) order embedding. Moreover, "suprema" (that is, joins of arbitrary families) are taken by Θ to intersections of the corresponding faces.*

Proof. If $J = pA^{**} \cap A$ and $\varphi \in Q(A) \cap J^\perp$ then $\varphi \in F_p$ since $p \in J^{\perp\perp}$. Conversely, if $\varphi \in F_p$ has norm 1 then we have

$$1 = \|\varphi\| = \|\varphi \cdot p\| + \|\varphi \cdot (1-p)\| \ge |\varphi(1-p)| = 1.$$

Thus $\varphi \cdot p = 0$, and so $\varphi \in Q(A) \cap J^\perp$.

If $\varphi \in F_p$ and $\varphi = t\psi_1 + (1-t)\psi_2$ for $\psi_1, \psi_2 \in Q(A)$ and $t \in [0,1]$, then it is clear that $\psi_1, \psi_2 \in F_p$. So F_p is a face of $Q(A)$. Since $F_p = Q(A) \cap J^\perp$, it is weak* closed.

Write $F_p^1 = \{\varphi \in S(A^1) : \varphi(p) = 0\}$. Suppose that $\varphi_t \to \varphi \in Q(A)$ weak*, with $\varphi_t \in F_p$ and $\varphi \neq 0$. Suppose that $\varphi_t = c_t \psi_t$ with $\psi_t \in S(A)$. We may assume that $\psi_t \in S(A^1)$, and then $\psi_t \in F_p^1$. By [Smith and Ward 1978; 1979], F_p^1 is weak* closed, so we have a weak* convergent subnet $\varphi_{t_\mu} \to \psi \in F_p^1$. A further subnet of the c_{t_μ} converges to $c \in [0, 1]$ say. In fact, $c \neq 0$ or else φ_{t_μ} has a norm null subnet, so that $\varphi = 0$. Now it is clear that $c\psi_{|A} = \varphi \in F_p$. So F_p is weak* closed.

If we have two central open projections $p_1 \leq p_2$ then $w = p_2 - p_1$ is a hermitian projection in $(A^1)^{**}$, so that as we said above $W(z) \subset [0, 1]$. Thus it is clear that $\varphi(p_1) \leq \varphi(p_2)$ for states $\varphi \in S(A)$. Hence $F_{p_2} \subset F_{p_1}$.

Conversely, suppose that $F_{p_2} \subset F_{p_1}$. If $\varphi \in F_{p_2}^1$ and φ is nonzero on A then, since it is real positive on A, it will be a positive multiple of a state ψ on A. We have $\psi \in F_{p_2} \subset F_{p_1}$, so that $\varphi \in F_{p_1}^1$. That is, $F_{p_2}^1 \subset F_{p_1}^1$. We are now in the setting of [Smith and Ward 1978; 1979], from where we see that these are split faces of $S(A^1)$ and are weak* closed. Let $N_1 \subset N_2$ be the complementary split faces. We may view p_1, p_2 as affine lower semicontinuous functions f_1, f_2 on $S(A^1)$. As in those references, we have $f_k = 0$ on $F_{p_k}^1$, and $f_k = 1$ on N_k. From this and the theory of split faces [Alfsen 1971, Section II.6], it is easy to see that $f_1 \leq f_2$. That is, $\varphi(p_2 - p_1) \geq 0$ for all $\varphi \in S(A^1)$. By [Magajna 2009], this is also true if $\varphi \in S((A^1)^{**})$, and hence if $\varphi \in S(\Delta)$. Therefore $p_1 \leq p_2$ in Δ, so that indeed $p_1 \leq p_2$ in the usual ordering of projections in A^{**}.

The last assertion follows from the identity

$$Q(A) \cap \left(\sum_i J_i \right)^{\perp} = \bigcap_i (Q(A) \cap J_i^{\perp}). \qquad \square$$

Note that the support projection $s(x) \notin \Delta$ in general if $x \in \mathfrak{F}_A$. This can be overcome by restricting to the class where this is true — but unfortunately this class seems often only to be interesting if A is commutative. Thus if A is an approximately unital Banach algebra, write $\mathfrak{F}'_A$ for the set of $x \in \mathfrak{F}_A$ such that multiplying on the left by $s(x)$ in the second Arens product is an M-projection on $N = (A^1)^{**}$, and $s(x)$ is commutes with A^1 (again the latter is automatic if A is commutative and Arens regular). (Note that if A is M-approximately unital then multiplying on the left by $s(x)$ is an M-projection on A^{**} if and only if it is an M-projection on $(A^1)^{**}$.) Define an m-ideal in A to be an ideal of form $\overline{EA}$ for a subset $E \subset \mathfrak{F}'_A$. If A is also a commutative operator algebra then the m-ideals in A are exactly the closed ideals with a cai, by the characterization of r-ideals in [Blecher and Read 2011] (see also [Effros and Ruan 1990]), since in this case $\mathfrak{F}'_A = \mathfrak{F}_A$.

Proposition 8.6. *If A is an approximately unital Banach algebra then any m-ideal in A is an M-ideal ideal in A.*

Proof. Suppose that $x \in \mathfrak{F}'_A$. Setting $J_x = \overline{xA} \subset s(x)A^{**} \cap A$, we have $J_x^{\perp\perp} = s(x)A^{**} = s(x)N$, as in the proof of Corollary 3.18. So $J_x = s(x)A^{**} \cap A$ is an M-ideal ideal. Then $\overline{EA} = \overline{\sum_{x \in E} xA}$ is also an M-ideal ideal by Theorem 8.3. $\square$

The above class is perhaps also a context to which there is a natural generalization of some of the results in [Blecher et al. 2008; Blecher and Read 2011; 2013a; Hay 2007] related to noncommutative peak interpolation, and noncommutative peak and p-sets (see [Blecher 2013] for a short survey of this topic). However, one should not expect the ensuing theory to be particularly useful for noncommutative algebras since the projections in this section are all "central".

Indeed it is unlikely that one could generalize to general Banach algebras the main noncommutative peak interpolation results surveyed in [Blecher 2013], or see [Hay 2007; Blecher et al. 2008; Blecher and Read 2013a; 2014]. However, we end with one nice noncommutative peak interpolation result concerning M-ideal ideals in general Banach algebras, which can also be viewed as a "noncommutative Tietze theorem". In particular, it also solves a problem that arose at the time of [Blecher and Read 2013a], and was mentioned in [Blecher and Read 2013b], namely whether $\mathfrak{r}_{A/J} = q_J(\mathfrak{r}_A)$ when J is an approximately unital ideal in an operator algebra A, and $q_J : A \to A/J$ is the quotient map. In [Blecher and Read 2011], it was shown that $\mathfrak{F}_{A/J} = q_J(\mathfrak{F}_A)$, and it is easy to see that $q_J(\mathfrak{r}_A) \subset \mathfrak{r}_{A/J}$. In fact a much more general fact is true. The main new ingredient needed is [Chui et al. 1977, Theorem 3.1]. Their proof of this result, while remarkable and deep, clearly contains misstatements. However, we were able to confirm that (a small modification of) their proof works at least in the case of unital Banach algebras. For the reader's interest, we will give a rather different, and more direct, proof of their full result.

Let (X, e) be a pair consisting of a Banach space X and an element $e \in X$ such that $\|e\| \leq 1$. Let

$$S_e(X) = \left\{ \varphi \in X^* : \|\varphi\| = 1 = \varphi(e) \right\} \quad \text{and} \quad W(x) = W_X^e(x) = \{ \varphi(x) : \varphi \in S_e(X) \}$$

denote respectively the state space and the numerical range of $x \in X$, relative to e. Of course, these are empty if $\|e\| < 1$. Below we write $B(\lambda, r)$ for the closed disk centered at λ of radius r. The following formula in the Banach algebra case is attributed to Williams in [Bonsall and Duncan 1973], and it may be proved by a tiny modification of the proof at the end of page 1 there.

Lemma 8.7 (Williams formula). *For every $x \in X$, one has*

$$W(x) = \bigcap_{\lambda \in \mathbb{C}} B(\lambda, \|x - \lambda e\|).$$

In particular, $W_X^e(x) = W_{X^{**}}^e(x)$ for every $x \in X$.

Theorem 8.8 (Chui, Smith, Smith, and Ward). *Let (X, e) be as above. Suppose that J is an M-ideal in X and $x \in X$ is such that $W_{X/J}^{Q(e)}(Q(x)))$ has nonempty interior, where $Q \colon X \to X/J$ is the quotient map. Then there exists $y \in J$ such that*

$$\|x - y\|_X = \|Q(x)\|_{X/J} \quad and \quad W_X^e(x - y) = W_{X/J}^{Q(e)}(Q(x)).$$

Proof. For a bounded convex subset $C \subset \mathbb{C}$, $\alpha \in C$, and $\epsilon > 0$, we define

$$N(C, \alpha, \epsilon) = \{\alpha + (1 + \epsilon)(\gamma - \alpha) : \gamma \in C\}.$$

It is an exercise to show that the $N(C, \alpha, \epsilon)$ are open convex neighborhoods of C if $\alpha \in \text{int}(C)$, and they shrink as ϵ decreases.

Let $x \in X$ be given, and fix $\alpha \in \text{int}(W_{X/J}^{Q(e)}(Q(x)))$. Then $|\alpha| < \|Q(x)\|$. Now

$$N(W_{X/J}^{Q(e)}(Q(x)), \alpha, 1)$$

is an open neighborhood of the compact subset $W_{X/J}^{Q(e)}(Q(x))$. By Lemma 8.7, the latter equals $\bigcap_{\lambda \in \mathbb{C}} B(\lambda, \|Q(x - \lambda e)\|_{X/J})$, and so we can find $0 = \lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$, and $\delta > 0$, such that

$$\bigcap_i B(\lambda_i, \|Q(x - \lambda_i e)\|_{X/J} + \delta) \subset N(W_{X/J}^{Q(e)}(Q(x)), \alpha, 1).$$

Let $z_0 = P(x - \alpha e) \in J^{\perp\perp}$ and $\lambda \in \mathbb{C}$. Since P is an M-projection,

$$\|x - z_0 - \lambda e\| = \max\{\|P((\alpha - \lambda)e)\|, \|(I - P)(x - \lambda e + y)\|\}, \quad y \in J,$$

which is dominated by

$$\max\{|\lambda - \alpha|, \|Q(x - \lambda e)\|_{X/J}\} = \|Q(x - \lambda e)\|_{X/J}$$

since $\alpha \in \bigcap_{\lambda \in \mathbb{C}} B(\lambda, \|Q(x - \lambda e)\|_{X/J})$. Thus $\|x - z_0 - \lambda_i e\| < r_i$ for each i, where $r_i = \|Q(x - \lambda_i e)\|_{X/J} + \delta$. Hence by Lemma 1.1, there exists $y_0 \in J$ such that $\|x - y_0 - \lambda_i e\| < r_i$ for all i. Indeed using that lemma similarly to some other proofs in our paper, if $x' \in X$ and $z \in J^{\perp\perp}$ are such that $\|z + x'\|_{X^{**}} < r$, and if $\{y_i\}$ is a net in J which converges to z weak*, one can find a net $\{y_j'\}$ of convex combinations of the y_j such that $y_j' \to z$ and $\|y_j' + x'\|_X < r$. One can iterate this procedure and obtain the same conclusion for any finite sequence $x_1', \ldots, x_m' \in X$ such that $\|z + x_i'\|_{X^{**}} < r_i$ for all $i = 1, \ldots, m$.

It follows that $x_0 = x - y_0$ satisfies $\|x_0\| < \|Q(x)\|_{X/J} + \delta$, and

$$|\varphi(x_0) - \lambda_i| = |\varphi(x - y_0 - \lambda_i e)| \le \|Q(x - \lambda_i e)\|_{X/J} + \delta, \quad \varphi \in S_e(X).$$

This implies

$$W_X(x_0) \subset \bigcap_i B(\lambda_i, \|Q(x - \lambda_i e)\|_{X/J} + \delta) \subset N(W_{X/J}^{Q(e)}(Q(x)), \alpha, 1).$$

Now we iterate the above process, controlling the increments. If $\epsilon > 0$, let $N(\epsilon)$ denote the set of those $x' \in x + J \subset X$ such that

$$\|x'\|_X \le \|Q(x)\|_{X/J} + \frac{\epsilon}{1-\epsilon}(\|Q(x)\|_{X/J} - |\alpha|),$$

and such that $W_X(x') \subset N(W_{X/J}^{Q(e)}(Q(x)), \alpha, \epsilon)$. Note that $x_0 \in N(1)$ (the first condition in the definition of $N(1)$ we treat as being vacuous).

Claim. For any $n = 0, 1, 2, \dots$ and $x_n \in N(2^{-n})$, there is $x_{n+1} \in N(2^{-(n+1)})$ such that $\|x_{n+1} - x_n\| \le 3 \cdot 2^{-n}\|Q(x)\|$ when $n \ge 1$.

Before we prove the claim, we finish the proof of the theorem. Note that if $n \ge 1$ then $\|x_n\| \le 2\|Q(x)\|_{X/J}$ by the first clause in the definition of $N(\epsilon)$. It follows from this and the inequality in the claim that the norm-limit $v = \lim x_n$ exists in $x + J$. It satisfies $\|v\| \le \|Q(x)\|_{X/J}$ by the first clause in the definition of $N(2^{-n})$, and $W_X(v) \subset W_{X/J}(Q(x))$ since by the second clause in that definition,

$$\varphi(v) = \lim \varphi(x_n) \in \bigcap_n N\big(W_{X/J}^{Q(e)}(Q(x)), \alpha, 2^{-n}\big) = W_{X/J}(Q(x)), \quad \varphi \in S_e(X).$$

That $W_{X/J}(Q(x)) \subset W_X(v)$ is an easy exercise. This completes the proof of the theorem.

To prove the claim, let $z = 2^{-n} P(x_n - \alpha e) \in J^{\perp\perp}$. Using the first clause in the definition of $x_n \in N(2^{-n})$, we have

$$\|z\| \le 2^{-n}(\|x_n\| + |\alpha|) < 3 \cdot 2^{-n}\|Q(x)\|.$$

Also, $P(x_n - z) = (1 - 2^{-n})x_n + 2^{-n}\alpha$, so by an argument similar to the M-projection argument in the second paragraph of the proof, we have

$$\|x_n - z\| \le \max\big\{(1 - 2^{-n})\|x_n\| + 2^{-n}|\alpha|, \|Q(x)\|_{X/J}\big\}.$$

The latter equals $\|Q(x)\|_{X/J}$, using the first clause in the definition of $x_n \in N(2^{-n})$.

Suppose that $\varphi_1 \in S_e(X^{**})$ with $\varphi_1 \circ P = \varphi_1$. There exists $\gamma \in W_{X/J}^{Q(e)}(Q(x))$ such that $\varphi_1(x_n) = \alpha + (1 + 2^{-n})(\gamma - \alpha)$, by the second clause in the definition of $x_n \in N(2^{-n})$. Hence, one has

$$\varphi_1(x_n - z) = \alpha + (1 - 2^{-n})(\varphi_1(x_n) - \alpha) = \alpha + (1 - 2^{-2n})(\gamma - \alpha),$$

and the latter is in $W_{X/J}^{Q(e)}(Q(x))$ since it is a convex combination of α and γ. Next, suppose that $\varphi_2 \in S_e(X^{**})$ with $\varphi_2 \circ P = 0$. Then φ_2 induces a "state" on $(X/J)^{**} \cong X^{**}/J^{\perp\perp}$, so that

$$\varphi_2(x_n - z) = \varphi_2(x_n) \in W_{(X/J)^{**}}^{Q(e)}(Q(x)) = W_{X/J}^{Q(e)}(Q(x)).$$

Thus $W_{X^{**}}^e(x_n - z) \subset W_{X/J}^{Q(e)}(Q(x))$, since any $\varphi \in S_e(X^{**})$ is a convex combination of $\varphi_1 = \varphi \circ P$ and $\varphi_2 = \varphi \circ (I - P)$ as above. Here we are using the

L-projection argument we have seen several times, relying on

$$1 = \varphi(e) = \varphi_1(e) + \varphi_2(e) \leq \|\varphi_1\| + \|\varphi_2\| = 1.$$

By the Williams formula (Lemma 8.7),

$$\bigcap_{\lambda \in \mathbb{C}} B(\lambda, \|x_n - z - \lambda e\|_{X^{**}}) = W^e_{X^{**}}(x_n - z) \subset W^{Q(e)}_{X/J}(Q(x)).$$

Let $\delta = 2^{-(n+1)}$. By the argument at the start of the proof, one can choose a finite sequence $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ such that

$$\bigcap_i B(\lambda_i, \|x_n - z - \lambda_i e\|) \subset N\big(W^{Q(e)}_{X/J}(Q(x)), \alpha, \delta\big).$$

Choose $r_i > \|x_n - z - \lambda_i e\|$ with $\bigcap_i B(\lambda_i, r_i) \subset N(W^{Q(e)}_{X/J}(Q(x)), \alpha, \delta)$. By the argument using Lemma 1.1 in the second paragraph of the proof, we can replace z in these inequalities by an element in J. Thus there exists $y \in J$ such that $\|y\| < 3 \cdot 2^{-n} \|Q(x)\|$,

$$\|x_n - y\| \leq \|Q(x)\|_{X/J} + \frac{\delta}{1-\delta}(\|Q(x)\|_{X/J} - |\alpha|),$$

and

$$W(x_n - y) \subset \bigcap_i B(\lambda_i, \|x_n - y - \lambda_i e\|) \subset \bigcap_i B(\lambda_i, r_i) \subset N\big(W^{Q(e)}_{X/J}(Q(x)), \alpha, \delta\big).$$

Hence $x_{n+1} = x_n - y \in N(\delta)$, which completes the proof of the claim. $\square$

We next deal with the exceptional case when $W^{Q(e)}_{X/J}(Q(x)))$ has empty interior, which by convexity happens exactly when it is a line segment or point.

Corollary 8.9. *Suppose that J is an M-ideal ideal (or simply an ideal which is an M-ideal) in a unital Banach algebra A. Let $x \in A/J$ with $K = W_{A/J}(x)$. Then*

(1) *If K is a point, then there exists $a \in A$ with $\|a\| = \|x\|$ and with $W_A(a) = W_{A/J}(x)$.*

(2) *If $K = W_{A/J}(x)$ is a nontrivial line segment then (1) is true "within epsilon". More precisely, in this case, let $\hat{K}$ be any thin triangle with K as one of the sides (so contained in a thin rectangle with side K). Then there exists $a \in A$ with $\|a\| = \|x\|$ and with $K \subset W_A(a) \subset \hat{K}$.*

Proof. If K is a point, then x is a scalar multiple of 1, so this case is obvious. For (2), if K is a nontrivial line segment, choose λ within a small distance ϵ of the midpoint of the line. Then replace A by $B = A \oplus^\infty \mathbb{C}$, replace J by $I = J \oplus (0)$, and consider $(x, \lambda) \in B/I$. It is easy to see that $W_{B/I}((x, \lambda))$ is the convex hull $\hat{K}$ of K and λ. By Theorem 8.8 there exists $(a, \lambda) \in B$ with $W_B((a, \lambda)) = \hat{K}$. If ϵ is small enough, we also have $\|a\| = \|x\|$ (since then $|\lambda|$ is dominated by the maximum of the moduli of two numbers in the numerical range, which is dominated by $\|x\| \leq \|a\|$).

However, similarly $W_B((a, \lambda))$ is the convex hull of $W_A(a)$ and λ, which makes the rest of the proof of (2) an easy exercise in the geometry of triangles. $\square$

We remark that in a previous version of our paper, the last result (and Theorem 8.8 in the unital Banach algebra case) was stated as a claim , not as a theorem. Thus it is referred to in [Blecher and Read 2014] as "the Claim at the end of" the present paper.

We can now answer the open question referred to above Theorem 8.8.

Corollary 8.10. *If A is an approximately unital Banach algebra, and if J is an M-ideal ideal in A, then $\mathfrak{r}_{A/J} = q_J(\mathfrak{r}_A)$. In particular, $\mathfrak{r}_{A/J} = q_J(\mathfrak{r}_A)$ for approximately unital closed two-sided ideals J in any (not necessarily approximately unital) operator algebra A.*

Proof. First suppose that A is unital. We leave it as an exercise that $q_J(\mathfrak{r}_A) \subset \mathfrak{r}_{A/J}$. The converse inclusion follows from Theorem 8.8 and Corollary 8.9 (in the line situation take the triangle above and/or to the right of K). Next suppose that A is a nonunital approximately unital Banach algebra, and that A/J is also nonunital. Then by the last paragraph of A.4.3 in [Blecher and Le Merdy 2004], the inclusion $A/J \subset A^1/J$ induces an isometric isomorphism $A^1/J \cong (A/J)^1$. The result then follows by applying the unital case to the canonical map from A^1 onto $(A/J)^1$. If A/J is unital then one can reduce to the previous case where it is not, by considering the ideal $J \oplus^\infty K$ in $A \oplus^\infty B$, where K is an approximately unital ideal in (e.g., a commutative C^*-algebra) B such that B/J is not unital. For this latter trick, one needs to know that $\mathfrak{r}_{A \oplus^\infty B} = \{(x, y) \in A \oplus^\infty B : x \in \mathfrak{r}_A, y \in \mathfrak{r}_B\}$ for approximately unital Banach algebras, but this is an easy exercise (and a similar relation holds for $\mathfrak{F}_{A \oplus^\infty B}$).

Finally, suppose that A is any nonunital operator algebra and J is an approximately unital closed ideal in A. Then J is an M-ideal in A^1 by [Effros and Ruan 1990]. Also, by the uniqueness of the unitization of an operator algebra mentioned in the introduction, we have $A^1/J \cong (A/J)^1$ completely isometrically if A/J is nonunital (see also [Blecher and Read 2014, Lemma 4.11]). Then the result follows again by applying the unital case to the canonical map from A^1 onto $(A/J)^1$. If A/J is unital, we can reduce to the case where it is not by the trick in the last paragraph. $\square$

By the assertion about the norms in Theorem 8.8 and Corollary 8.9, we can lift elements in $\mathfrak{r}_{A/J}$ to elements in $\mathfrak{r}_A$, keeping the same norm, in the situations considered in the corollary.

As we said, these results may be viewed as noncommutative peak interpolation or noncommutative Tietze theorems. For in the case that A is a uniform algebra on a compact Hausdorff set Ω, the M-ideals J are well known to be the closed ideals with a cai, and are exactly the functions in A vanishing on some p-set $E \subset \Omega$ (see [Smith 1979] and [Harmand et al. 1993, Theorem V.4.2]). Then q_J is identifiable with the restriction map $f \mapsto f_{|E}$, and $A/J \cong \{f_{|E} : f \in A\} \subset C(E)$. The lifting result in Theorems 8.8 and 8.9 in this case say that if $f \in A$ with $f(E) \subset C$ for a

compact convex set C in the plane, then there exists a function $g \in A$ which agrees with f on E, which has norm $\|g\|_{\Omega} = \|f_{|E}\|_E$, and which has range $g(\Omega) \subset C$ (or $g(\Omega) \subset \hat{K}$ if $\operatorname{conv}(f(E))$ is a line segment K, where $\hat{K}$ is a thin triangle given in advance, one of whose sides is K).

9. Banach algebras without cai

If A is a Banach algebra without a cai, or without any kind of bai, we briefly indicate here how to obtain nearly all the results from Sections 3, 4, and 7. We give more details in a forthcoming conference proceedings survey article [Blecher 2015]; however, the interested reader will have no trouble reconstructing this independently from the discussion below. Namely, if B is any unital Banach algebra containing A, for example, any unitization of A, one can define $\mathfrak{F}_A^B = \{a \in A : \|1_B - a\| \leq 1\}$ and $\mathfrak{r}_A^B$ to be the set of $a \in A$ whose numerical range in B is contained in the right half-plane. Also one can define $\mathfrak{F}_A$ (resp. $\mathfrak{r}_A$) to be the union of the $\mathfrak{F}_A^B$ (resp. $\mathfrak{r}_A^B$) over all B as above. Unfortunately it is not clear to us that $\mathfrak{F}_A$ and $\mathfrak{r}_A$ are always convex, which is needed in Sections 4 and 7 (indeed we often need them closed too there). Of course, $\mathfrak{F}_A$ and $\mathfrak{r}_A$ are convex and closed if there is an "extremal" unitization B of A such that $\mathfrak{F}_A^B = \mathfrak{F}_A$ (resp. $\mathfrak{r}_A^B = \mathfrak{r}_A$). This is the case with B equal to the multiplier unitization if A is approximately unital, or more generally if the left regular representation embeds A isometrically in $B(A)$.

Most of the results in Sections 3, 4, and 7 of our paper then work without the approximately unital hypothesis if $\mathfrak{F}_A^B$ and $\mathfrak{r}_A^B$ are used. In particular, we mention the results 3.3–3.6, 3.9–3.11, 3.17–3.19, 3.21, 3.23–3.25, and all lemmas, theorems, and corollaries in Sections 4 and 7 not concerning M-approximately unital algebras. Every one of the statements of these results is still correct if one drops the approximately unital hypothesis, but uses $\mathfrak{F}_A^B$ and $\mathfrak{r}_A^B$ in place of $\mathfrak{F}_A$ and $\mathfrak{r}_A$. Indeed the results just mentioned in Section 3 (and also the first lemma in Section 4) are also correct for general Banach algebras if one uses $\mathfrak{F}_A$ or $\mathfrak{r}_A$ as defined in the last paragraph (the other results in Sections 4 and 7 would seem to need $\mathfrak{F}_A$ and $\mathfrak{r}_A$ (as defined in the last paragraph) being closed and convex).

Some of the results asserted in the last paragraph are obvious from the unital case of the result, and some follow by the obvious modification of the given proof of the result. However, in some of these results, one also needs to know that $\overline{EA} = \overline{EB}$, where B is a unitization of A and E is a subset of $\mathfrak{F}_A^B$ or $\mathfrak{r}_A^B$. This follows from the following fact: if $x \in \mathfrak{r}_A$ as defined in the last paragraph then

$$x \in \overline{xA} = \overline{\operatorname{ba}(x)A} = \overline{xB}$$

for any unitization B of A. Indeed this is clear since by Cohen factorization, $x \in \operatorname{ba}(x) = \operatorname{ba}(x)^2 \subset \overline{xA}$. We also need to know that the $\mathfrak{F}$-transform, and n-th

roots, are independent of the particular unitization used, but this is easy to see using the fact that all unitization norms are equivalent.

Acknowledgments

We thank Charles Read for useful discussions, and for allowing us to take out some of the material in [Blecher and Read 2013b] for inclusion here. We thank the referee for his careful reading of the manuscript, which was plagued by innumerable typos, and for his suggestions. We were also supported as participants in the Thematic Program on Abstract Harmonic Analysis, Banach and Operator Algebras 2014 at the Fields Institute, for which we thank the Institute and the organizers of that program. As we said earlier, the survey article [Blecher 2015] contains a few additional details on some of the material in the present paper, as well as some small improvements found while this paper was in press.

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DAVID P. BLECHER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HOUSTON
HOUSTON, TX 77204-3008
UNITED STATES

dblecher@math.uh.edu

NARUTAKA OZAWA
RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY
KYOTO 606-8502
JAPAN

narutaka@kurims.kyoto-u.ac.jp

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Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu