Proceedings of the 1997 IEEE
International Conference on Robotics and Automation
Albuquerque, New Mexico - April 1997

Real-Time Path Planning Using Harmonic Potentials In Dynamic Environments

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Abstract

Motivated by fluid analogies, artificial harmonic potentials can eliminate local minima problems in robot path planning. In this paper, simple analytical solutions to planar harmonic potentials are derived using tools from fluid mechanics, and are applied to two-dimensional planning among multiple moving obstacles. These closed-form solutions enable real-time computation to be readily achieved.

1 Introduction

This paper develops fast algorithms for two-dimensional path planning, where the task is to find a trajectory which will bring a mobile robot from an initial position $\mathbf{x}_i$, to a final position $\mathbf{x}_f$ while avoiding moving obstacles.

Robot path planning has been studied extensively, and can use local or global computations. Combination of local and global algorithms have also been suggested, where a local method is assisted by a global planner when needed (that is, when the robot gets stuck) as in [1], or paths from all points in space to the goal are defined by some potential field method [2, 3]. In the potential field methods, the position and shape of all obstacles in a given region is assumed to be known and the potential function is constructed using this information. Thus, only local calculations at each point in space are required for the robot to find the direction it should move in. However, potential field methods often have the drawback that a robot could get stuck in local minima [2, 4, 5]. This problem may be solved by forcing local potential extrema to lie on the boundaries of obstacles through the use of *harmonic* potentials, that is, of functions ϕ satisfying Laplace's equation, $\nabla^2\phi = 0$, often motivated by a fluid or electrostatic analogy [6, 7, 8, 9]. The disadvantage of these methods is that Laplace's equation has to be solved numerically over the whole state space, a slow process making it extremely hard to find solutions in real-time for dynamic environments – on an $L \times L$ grid the computation time scales as L^4.

Two notable exceptions to the use of numerical solutions of harmonic potentials are described in [10, 11]. In [10], analytical solutions are developed for simple shaped objects. However, they only consider static environments and the object closest to the robot, with the drawback that the robot is not influenced by any other objects in space. Similarly, [11] use the panel method from computational hydrodynamics to obtain approximate closed-form solutions over the entire space given arbitrarily shaped polygonal obstacles. It is also restricted to static environments.

In this paper, we extend the harmonic potential field method to *dynamic* environments for real-time path planning in two dimensions. We also introduce analytical solutions for *multiple* moving obstacles. In section 2 we detail the analogy between fluid flow and path planning in two dimensions. We also introduce several methods for defining objects in an analytical form, and a new method for defining the goal position of the robot. In section 3, the approach is extended to dynamic environments. Section 4 offers concluding remarks.

2 The Fluid Analogy

Harmonic potentials have the great advantage that they achieve their extremum only on the boundary of objects. Thus no local minima will occur in the admissible configuration space, and a path generated by following a steepest gradient descent is guaranteed to reach the goal without hitting any objects in the domain, if the goal is reachable. This property of harmonic potentials is the extremum property [12]. Such a path is in fluid mechanics termed a streamline, and defines the path of a fluid particle.[1] In using harmonic potentials in two-dimensional path planning it is useful to draw the analog to invicid incompressible irrotational fluid flow, termed *potential flow*, and thereby use different powerful tools and properties from hydrodynamics [6, 11, 12, 13] and intuition. Let the fluid velocity at $\mathbf{x}$ be $\mathbf{U}(\mathbf{x})$. The assumption that the flow is irrotational states that the vorticity vanishes, that is:

$$\nabla \times \mathbf{U} = 0 \;\Leftrightarrow\; \mathbf{U} = \nabla\phi(\mathbf{x}) \,, \tag{1}$$

where $\phi(\mathbf{x})$ is a velocity potential. Since the fluid is assumed to be incompressible, the continuity equation can be written:

$$\nabla \cdot \mathbf{U} = 0 \,. \tag{2}$$

Combining equation (1) and (2) we arrive at the Laplace equation for the velocity potential:

$$\nabla^2\phi = 0 \,. \tag{3}$$

On obstacle boundaries, the boundary condition of Laplace's equation is given by the impenetrability of obstacle boundaries, called von Neumann boundary conditions, which can be expressed as

$$\mathbf{n} \cdot \mathbf{U} = \mathbf{n} \cdot \nabla\phi = 0 \,, \tag{4}$$

where $\mathbf{n}$ is the unit normal vector on the boundary. In addition, there are different flows describing potential flows, such as, sources, sinks, and uniform flows. Potential flow represents an *ideal* fluid flow, where viscosity is ignored.

In section 2.1 we describe several basic methods for defining objects in an analytical form. In section 2.2 we describe our methods and approximations made for using closed form solutions of Laplace's equation to handle multiple objects. Section 2.3 introduces a new method for defining the initial and goal position of of the robot.

2.1 Modeling of Objects

In this section we outline the different methods used for modeling of objects in potential flows.

From standard fluid texts on potential flows, such as the book by Milne-Thomson [12], solutions for various shaped objects can be found in closed form. In fluid dynamics, it is convenient to work with *complex* potentials, rather than just the velocity potential in order to utilize the properties of conformal mapping of complex variables for two-dimensional problems. The complex potential consist of the velocity potential $\phi(\mathbf{x})$, as defined above, and a function called the stream function $\psi(\mathbf{x})$, which is constant on a path of a fluid particle, that is, on a streamline.[2] The complex potential is defined as

$$w = \phi + i\psi = f(z) \,, \quad z = x + iy \,; \tag{5}$$

where $i^2 = -1$.

The velocity field, $\mathbf{U} = [u \;\; v]$ can be found from either ϕ, ψ or w by

$$\begin{aligned} u &= \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = -\mathrm{Re}\left(\frac{dw}{dz}\right) \,, \\ v &= \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} = \mathrm{Im}\left(\frac{dw}{dz}\right) \,. \end{aligned} \tag{6}$$

We here outline the four major methods to define various potential flows: simple flows, use of specific theorems, conformal mapping and a panel method. We here use U to represent a velocity and r to represent a length.

1. SIMPLE FLOWS: The potential for some simple flows can be found by trial and error, such as that of a uniform flow

   $$w = -Uze^{-i\alpha} \,. \tag{7}$$

   Using equation (6) we find $u = U\cos\alpha$, $v = U\sin\alpha$, which we see is the uniform flow of magnitude U making an angle α with the positive x-axis.

2. SPECIFIC THEOREMS: Laplace's equation has been studied extensively, as it appears in many physical problems. Thus, many special theorems for modeling of objects apply. For example, the complex potential

   $$w = -U\left(ze^{-i\alpha} + \frac{r^2e^{i\alpha}}{z - z_0}\right) \,. \tag{8}$$

   represent the flow past a circular cylinder centered at $z_0 = x_0 + iy_0$, with radius r in a uniform stream with velocity U inclined at an angle α with the positive x-direction. This result follows directly from the circle theorem described in [12].

[1] There is one streamline for every object in a flow where the velocity vanishes, termed the stagnation line and the velocity vanishes at the stagnation point on the boundary of the object. However, this line has zero measure, and the stagnation point is a saddle point, so these points, in the absence of large friction, do not cause a problem when using streamlines as robot paths.

[2] Streamlines (ψ =constant), are orthogonal to the equivpotential lines of ϕ (ϕ =constant).

3. CONFORMAL MAPPING: A powerful technique for obtaining flows around objects of more complex shape in two dimension, is by the use of conformal mappings [12, 14, 15]. One of the most useful mappings is the Joukowski transformation:

$$z = \xi + \frac{r^2}{4\xi}, \tag{9}$$

by which we can map the ξ-plane to the z-plane and vice versa. If we take a circle of radius $r = \frac{1}{2}(a+b)$ in the ξ-plane, this transformation will map the circle into an ellipse with major axis a and minor axis b in the z-plane. That is, the circle in the ξ-plane is given by equation (8) as:

$$w = -U\left(\xi e^{-i\alpha} + \frac{(a+b)^2 e^{i\alpha}}{4\xi}\right), \tag{10}$$

and solving equation (9) for ξ we get

$$\xi = \frac{1}{2}(z \pm \sqrt{z^2 - r^2}), \quad r^2 = a^2 - b^2. \tag{11}$$

From these two equations we can compute dw/dz and thus the velocity field around an ellipse.

4. PANEL METHOD: In panel methods [11, 16, 17] a body of unspecified shape may be generated by adding to a uniform flow a linear combination of singularities including sources, sinks, doublets, and vortices. This is done by approximating the shape of the object by a finite number of line segments (in two-dimensions)[3] called *panels*, each of which consists of a uniform distribution of singularities of a certain kind. The distribution of singularities and their magnitude can be determined through a set of linear equation, so that an oncoming uniform flow is deflected around the object. In this paper we have used a panel method based on a uniform distribution of sources, first introduced by Hess and Smith [18], in some of the simulations shown.[4] However, due to limited space we will not derive the theory, but refer the reader to the cited books and articles. This method requires an inversion of an $N \times N$ matrix where N is the number of panels. For not very complex shapes, less than $N = 20$ panels gives a very good approximation and keeps the computational cost low. Unlike the previous methods described, this is an *approximate* method.

[3]Panel methods also exist for three dimensions. Here the surface of the body is covered by a set of small areas.

[4]We also found the method introduced by [11] of having $\mathbf{n} \cdot \mathbf{U} \geq 0$ on the boundaries to be appropriate.

2.2 Multiple Objects

All the methods for modeling of objects outlined in section 2.1, with some exceptions, only gives the analytical solution for *one* object in a uniform flow. Due to the linearity of Laplace's equation, superposition of solutions will also be a solution. However, the superposition of solutions to Laplace's equation for objects in a flow will deform the contour where equation (4) is satisfied from the actual boundaries of the objects, which again may cause intersection of robot path and the objects. Thus, modifications have to be performed in order to use the methods of section 2.1 for robot path planning among multiple objects . We here present a new method for path planning in an environment with several objects where the path is influenced by all known objects in the environment, and allows for moving obstacles and obstacles that change size.

This problem can be solved as follows: If the robot is very close to an object the robot must first of all avoid that object. This can be achieved by considering the flow field near an object as the resulting flow around the object in a fictitious uniform flow, for which the solution is known exactly. When the robot is far away from any objects, the superposition of solutions is close to the exact solution, while superposition and the extremum property ensures that Laplace's equation is satisfied and that the robot will not get stuck in a local minimum. In between these two regions, a transition from the field close to the object and far from the object is used. Thus, by this method, a harmonic potential is obtained that satisfies the boundary conditions. However, this potential is an approximate solution to the potential flow. This approximation does not cause a problem as the analog to potential flow is merely a tool for intuition, and gives us tools from fluid mechanics for solving Laplace's equation.

More explicitly, let $\phi_{i-1} = \phi_e + \sum_{k=1}^{i-1} \phi_{o_k}$ define the velocity potential before the introduction of object i. This potential is composed of ϕ_e which is the "external" field which will define the goal position, and $\sum_{k=1}^{i-1} \phi_{o_k}$ which defines the potential for all the $i-1$ objects that already exist. Let ϕ_{u_i} be the potential of a uniform flow, and let ϕ_{ou_i} be the potential which defines the flow exactly around the object in the uniform flow ϕ_{u_i}. Then $\phi_{o_i} \equiv \phi_{ou_i} - \phi_{u_i}$ is the potential associated with the object in the absence of the uniform flow. We first place a region around the object where we impose that the potential is ϕ_{ou_i} (region A).[5] Thus, once inside this region, the solution is exact for the given object geometry, that is, equation (4) is satisfied on

[5]In implementation, this region is typically chosen to be the region defined by the distance the robot can move in one time step around the objects boundary.

the object boundary, and there are no local extrema by the extremum property. Outside this region, we design an outer region[6] where we make a continuous linear transition from the exact potential of the object in the uniform flow, ϕ_{ou_i}, to the sum of the potential prior to the introduction of object i and the potential of the object without the uniform flow, $\phi_{i-1} + \phi_{o_i}$ (region B).[7] Outside this transition region, the potential is then $\phi_i = \phi_{i-1} + \phi_{o_i}$ (region C). By superposition, since ϕ_{i-1}, ϕ_{o_i} and ϕ_{u_i} are all solutions to Laplace's equation, any linear combination of them is also solution. Thus, the potential is a harmonic potential in all three regions, and by the extremum property there are no local extrema other than on the boundary of the object.

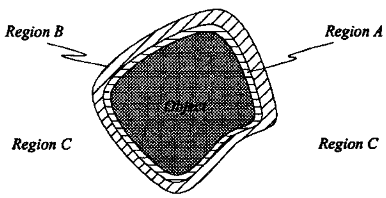

Figure 1: An object and the different zones used for enabling the use of several objects in a field.

The potential in region C is an approximation of the potential of a fluid. Since, by definition, a robot in region C is at a safe distance from any object, the approximated solution for the fluid flow is appropriate for a robot's path. The sectioning of the regions around an object is depicted in figure 1.

If two objects are so close that their region B overlap, one can let the path be determined by the mean of the field from both objects. This will work since the objects are far enough away so that the robot can navigate robustly in the area, by definition of region B. If two objects are so close together that their regions A overlap, it is not safe for a robot to operate in this area, since it is less than the distance the robot can move in one time step from the objects, thus making it liable to collide with one of the objects. In such cases, to make the planner more robust, we view all objects that are so close that their A regions overlap as a single object.

[6]In implementation, this region is typically chosen to be of width of the length a robot can move in a couple of time steps.

[7]We have chosen to make the transition as a sinusoid on the velocities in order to have continuous first order derivatives of the velocities. That is, in this region, $u = \frac{1}{2}(1 + \sin \frac{\pi(r(x,y) - R - \Delta R/2)}{\Delta R})\frac{\partial \phi}{\partial x}$. Here $r(x,y)$ is the distance from the surface of the object, R is the distance from where to start the transition to the external field, and ΔR is the distance over which the transition takes place. The transition is similarly defined for v.

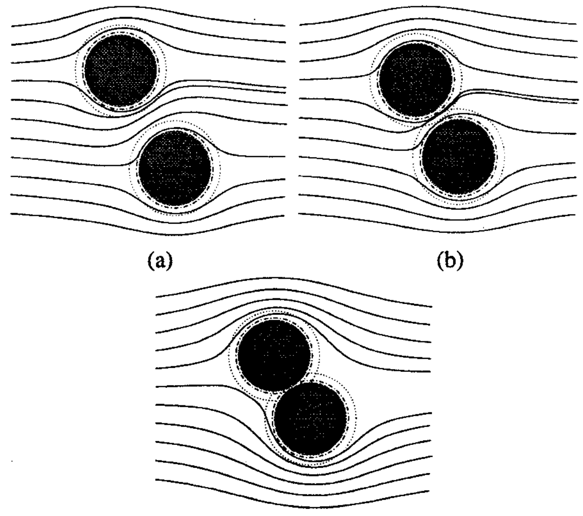

Figure 2: Robot paths, from left to right, past objects for different initial positions.

In figure 2 we show several paths for a robot whose goal is just to move from the left to the right given different initial vertical position. These paths are analog to the stream lines for an ideal fluid. The dotted lines shows the outer border of region B, while the dashed-dotted region shows the outer border of region A.[8] Figure 2(a) shows the paths when the two objects are so far away from each other that they do not have overlapping B regions. (b) shows the robot paths if the B regions (but not A regions) overlap. (c) shows the paths when the objects are so close that they can be considered as a single object (that is, their A regions overlap or the objects touch).

2.3 Defining The External Field

Traditionally, a point source has been placed at the robot's initial position while a point sink has been place at the robot's goal position in order to make the goal the global minimum and the initial position the global maximum of the operating space [6, 7, 8]. However, this method has the disadvantage that the field can become arbitrary small when the robot is far away from its goal and initial position, while the field tends to infinity at the goal and initial position, thus, making the method vulnerable to numerical noise. Kim

[8]Note, that in figure 2 (a) and (b) are *not* identical to the streamlines for a fluid, as we are solving only for one cylinder in a uniform field and superimposing solutions. Part (c) is, in this special case, the same as the streamlines in a fluid, since it is viewed as just one object (obtained by a conformal mapping [12]).

and Khosla [11] used a uniform field directed from the initial point to the goal sink, avoiding a point source at the initial position. This eliminates the near zero field in between goal and robot, but not the singularity of the goal position.

We approach this problem by the use of a uniform field always directed from the robot's current position to the goal position with magnitude U. More precisely, we use the complex potential

$$w = -Uze^{-i\alpha}\ , \ \alpha = \tan^{-1}\left(\frac{y_f - y}{x_f - x}\right)\ , \qquad (12)$$

as the underlying external potential (that is, $\phi_e = \Re(w)$). Here (x, y) is the robot's current position and (x_f, y_f) its goal position. Thus $\phi_{u_i} = \phi_e$ for all i by definition. Notice that w has no extrema. The advantage of this method is that the external field will always have magnitude U and that $\phi_{u_i} = \phi_e$ for all times. The uniform field used, is rotated with time, however, it is always directed parallel to a straight line from the robot to its goal. This type of motion is analog to the Pursuit problem [19], thus the time variation of the field and robot paths will be continuous, and allows for implementation of moving goals in a natural manner.

3 Dynamic Systems

In this section, path planning in static environments is generalized to path planning in dynamic environments. The same general methods for defining objects in space as outlined previously will be used, but with some simple changes in order to incorporate translation (section 3.1), expansion and contraction (section 3.2), and rotation of objects (section 3.3). Section 3.4 describes a method for dealing with objects that come in contact and thereby closes existing paths.

3.1 Translation of Objects

In this section we show how to find the velocity field of an object moving with a velocity $\mathbf{V}$ in a uniform flow $\mathbf{U}$. We first define an inertial reference frame X-Y. Second, we define a coordinate system x-y which is fixed on the moving object, and moves with a velocity $\mathbf{V} = [V_X \;\; V_Y]$ with respect to the reference frame X-Y. Since the object is moving we need to ensure that the von Neumann boundary condition is satisfied on the surface of the object in the *x-y coordinate system* in order to ensure that a path does not pass through the object. When this is achieved, we can transform the field back to the X-Y system. The uniform flow past the object, as seen in the x-y coordinate system is:

$$\mathbf{U_o} = \mathbf{U} - \mathbf{V}\ . \qquad (13)$$

Notice that $\mathbf{U_o}$ is also a uniform field, but has different magnitude and direction than $\mathbf{U}$. Thus we can find the velocity field around this object using any of the methods outlined in section 2.1. In particular, assume that the velocity at time t at a point $(x(t), y(t))$ is $v = [v_x \;\; v_y]$ in the x-y coordinate system. In the X-Y coordinate system, this point's coordinates are given by:

$$\begin{aligned} X(t) &= X(0) + x(t) + V_X t\ , \\ Y(t) &= Y(0) + y(t) + V_Y t\ , \end{aligned} \qquad (14)$$

where $X(0), Y(0)$ is the coordinates in the X-Y coordinate system of the origin of the x-y coordinate system. The velocity $\mathbf{u}$ at this point in the X-Y system is then simply

$$\mathbf{u} = v + \mathbf{V}\ . \qquad (15)$$

As a simple example, we can find the velocity field when a circular cylinder with velocity $\mathbf{V}$ moves in a fluid with velocity $\mathbf{U}$. Using equation (8) we find

$$v_x - iv_y = ||\mathbf{U} - \mathbf{V}|| \left(e^{-i\alpha} - \frac{r^2 e^{i\alpha}}{z - z_0}\right)\ , \qquad (16)$$

where r is the radius of the cylinder, α is the angle that $\mathbf{U} - \mathbf{V}$ makes with respect to the X-axis and $z_0 = X_0 + iY_0$ is the center of the cylinder. The velocity field is then simply found by combining equations (15) and (16).

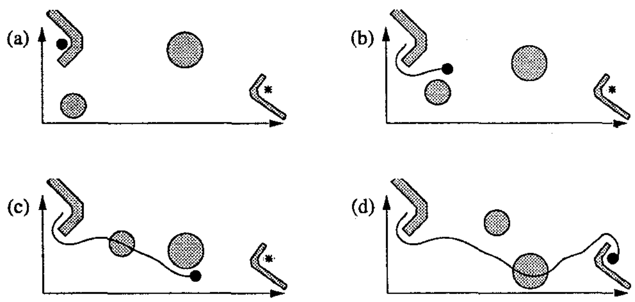

Figure 3: Robot avoiding moving circular cylinders and static walls.

Figure 3 shows a circular robot starting at the left whose goal is marked by a '*'. A moving circular cylinder starts at the bottom left, and moves to the left and up, and an other moving cylinder starts at the upper middle and moves straight down. The two L-shaped objects are static and was defined using the panel method with 18 panels each. The line in the drawing represents the robot's path. As can be

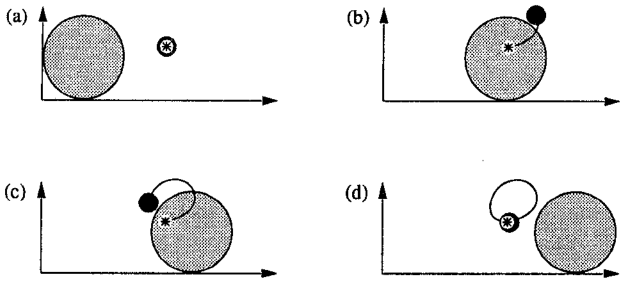

Figure 4: Robot avoiding circular cylinder crossing the robot's goal '*'.

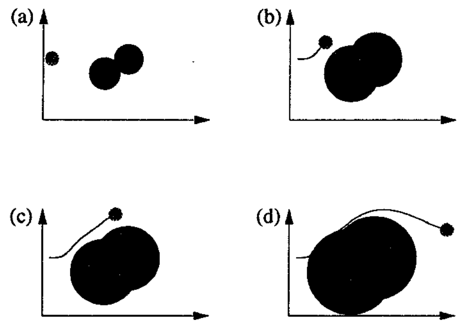

Figure 5: Robot moving over an expanding object.

seen from these figures, the robot manages to maneuver in the dynamic environment without hitting the moving objects or the static walls, while still reaching its goal. For this particular plot, each computation took about 0.05 seconds using non-optimized matlab code on a Sparc 5. It is expected that the algorithm can be speeded up considerable, as only about 10^2 operations are required for each of the circles, and only about 10^4 operation for each of the L-shaped objects, due to the simple analytical forms.

Figure 4 shows how a robot that initially is at rest at its goal, marked by a '*' (figure 4(a)), while a larger circular moving objects moves from left to right. As the circular object is approaching, the robot starts to move away from its goal in order to avoid the moving object, part (b) and (c). When the object has moved away from the robot's goal, the robot can again reach its goal position, as shown in part (d).

3.2 Expanding Objects

Objects that expand or contract are simple to model. The only requirement is that the von Neumann boundary condition is satisfied on the dynamically changing boundary. Thus, if a boundary expands/contracts with a velocity V the normal velocity of the boundary must be V. The magnitude of the normal vector is then V on the boundary and is made to drop off with the inverse of the distance to the boundary, as potentials obeying Laplace's equation drops off with the inverse of the distance. Such objects might be useful in situations where, e.g., one needs to avoid liquids that are dripping on a surface, or to incorporate time-varying safety zones.

Figure 5 shows how the robot navigates around an expanding object using this method.

3.3 Rotating Objects

Determining the velocity potential for a rotating object is more complicated. It can, however, be shown that the stream function on the boundary of an object rotating with an angular velocity ω around the origin, and a translational velocity $\mathbf{U} = [U_x \;\; iU_y] = Ue^{-i\alpha}$ in a *static* fluid, is given by (see [12] for a proof):

$$2i\psi = -Uze^{i\alpha} + U\bar{z}e^{i\alpha} + i\omega z\bar{z} \,, \tag{17}$$

where $\bar{z}$ is the complex conjugate of z.

Now, if we suppose that the domain outside a contour C in the z-plane to be mapped conformally on to the outside of the unit circle $|\xi| = 1$ in a complex ξ-plane, by the relation

$$z = f(\xi) \,, \tag{18}$$

the points at infinity in the z-plane and ξ-plane correspond. Therefore, for the liquid to be at rest at infinity, the complex potential w cannot contain positive powers of z (or ξ) when expanded in a power series in z (or ξ). Now, we define a general point on the unit circle boundary by $\sigma = e^{i\theta}$, where the stream function in equation (17) must apply. Thus, by equations (17) and (18), the stream function ψ on the boundary C becomes

$$\begin{aligned} 2i\psi &= B(\sigma) \\ &= -Uf(\sigma)e^{-i\alpha} + U\bar{f}(1/\sigma)e^{i\alpha} + \\ &\quad i\omega f(\sigma)\bar{f}(1/\sigma) \,. \end{aligned} \tag{19}$$

The function $B(\sigma)$ is called the *boundary function* and can be expanded in negative powers of σ, grouped in $B_1(\sigma)$, and in positive powers of σ, grouped in $B_2(\sigma)$. That is, $B(\sigma) = B_1(\sigma) + B_2(\sigma)$. It can further be shown, that if the *fluid* is moving with velocity $-\mathbf{U}$ and the *object* is rotating with angular velocity ω about the point z_0, the complex potential that satisfies the boundary condition (equation (17)) is then (see [12] for proof):

$$w = B_1(\xi) + Uf(\xi)e^{-i\alpha} - i\omega\bar{z}_0 z \,. \tag{20}$$

To get the velocity field for the contour C rotating about the point z_0 in a fluid flowing uniformly with velocity $Ue^{-i\alpha}$ we use equation (6).

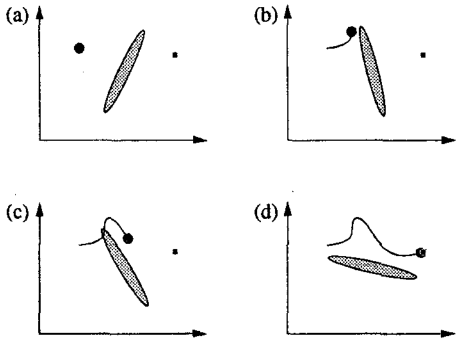

Figure 6: The robot navigates to its goal at '*', avoiding the rotating ellipse. The robot's goal is so high that the robot decides to go above the rotating ellipse.

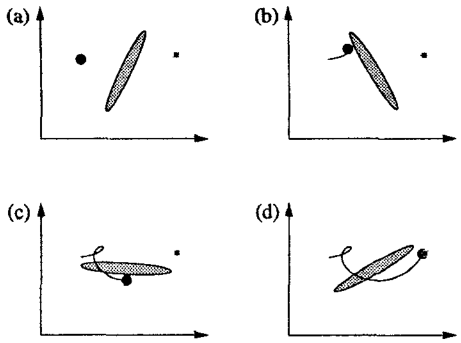

Figure 7: The robot navigates to its goal at '*', avoiding the rotating ellipse. The robot at first tries to go above the ellipse, but is forced to go below the ellipse.

The effect of a rotating object can be illustrated by two simple examples where we let a robot navigate to its goal in the presence of a rotating ellipse. Figures 6 and 7 illustrate how a circular robot navigates around a rotating ellipse, depending on its goal position. The initial position of the robot is to the left, and is the same for both cases. The initial position and the angular velocity of the ellipse is also the same in both figures. In figure 6 the robot moves in the correct general direction initially and passes above the rotating ellipse. In figure 7, however, the robot initially believes it can pass above the ellipse (part (b)), but it is unable to, and is forced to move below the ellipse (part (c)) and to its goal in part (d).

3.4 Interaction Among Obstacles

The main problem left to solve is how to deal with situations where objects move in such a manner as to close an existing path. A solution to this problem is to add circulation around the objects. By changing the tangential velocity on the boundary, we are able to move the streamlines around the object. In particular we are able to move the stagnation lines. As mentioned, all objects have one and only one stagnation line. The problem is, therefore, to make a smooth transition from two stagnation lines to one when two objects come in contact and cuts a path. This is achieved by making the surface of the objects "rotate" in opposite directions as the two objects approach each other.

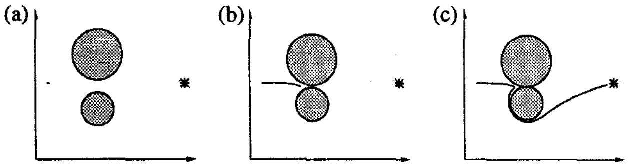

Figure 8: A point robot moves from the left to its goal at '*'.

An example of using this method is given in figure 8, where a point robot starts at the left, and the goal position is at the right marked by a '*'. The two circles move against each other. Initially the robot believes it can take the shortest path between the two objects (figure 8 (a)–(b)). However, as the robot moves closer to the two objects and the two objects are approaching each other, the circulation around each object is increased and the stagnation lines of both are moved to the center line between the two objects. Finally merging the two stagnation lines into one. At (b) the robot feels this effect and realizes it can not pass between the two objects, and takes the shorter path around the smaller circle to its goal in part (c).

4 Concluding Remarks

In this paper we introduce the use of harmonic functions for real-time path planning in dynamic environments.

This method inherits the attractive features of harmonic functions for path planning. The use of *analytical* solutions to Laplace's equation in dynamic environments makes real-time path planning possible, and modifications allow for several moving objects as well. The method requires a global model of the environment.

In static cases, a harmonic potential will have no local minima, a path will not intersect with any object and a convergence to the goal position is guaranteed if the goal position is reachable. The same features of harmonic potentials carries over to the dynamic case at every instant. That is, if there exist a path at the current time leading to the goal that does not intersect with any obstacle, the robot will move along this path. In dynamic environments it is not possible to guarantee convergence to the goal position or collision avoidance for all dynamic environments over some finite time. This path planner only considers the possible paths at each instant, thus making it possible to trap it by alternating opening and closing existing paths.

Several extensions to this work are possible, in addition to experimental evaluation. Utilization of harmonic potentials for control of robots with multiple d.o.f. should be developed. Taking the curvature of the streamlines into account, and moving away from areas of high curvature is an attractive approach. This can be seen as locally applying elastic bands on the path, similarly to [1]. Extensions to n-dimensions should also be studied. Further, moving along a fluid path is not a necessary constraint for a robot. Thus, the fluid paths may be varied so as to optimize some criteria while still guaranteeing the absence of local minima, through the introduction of intermediate goals or possibly by regularization methods.

Acknowledgment

This work was supported in part by NASA under contract JPL-959774. H.J.S.F. acknowledges the support of NFR (Norwegian Research Council) through grant 109338/410. Further, the authors would like to thank Jens Feder, Winfried S. Lohmiller and Christophe Bernard for interesting discussions and comments.

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