$\mathscr{Z}$-REALCOMPACTIFICATIONS AND NORMAL BASES

R. A. ALO and H. L. SHAPIRO

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In a recent paper (see [2]), Orrin Frink introduced a method to provide Hausdorff compactifications for Tychonoff or completely regular T_1 spaces X. His method utilized the notion of a *normal base*. A normal base $\mathscr{Z}$ for the closed sets of a space X is a base which is a disjunctive ring of sets, disjoint members of which may be separated by disjoint complements of members of $\mathscr{Z}$.

Frink showed that if X has a normal base, then the Wallman space, $\omega(\mathscr{Z})$, consisting of the $\mathscr{Z}$-ultrafilters is a Hausdorff compactification of X. This also showed that X must be a Tychonoff space. In this note we use the notion of $\mathscr{Z}$-ultrafilters in a *countably productive* normal base $\mathscr{Z}$ to introduce a new space $\eta(\mathscr{Z})$ consisting of all those $\mathscr{Z}$-ultrafilters with the countable intersection property.

Every normal base $\mathscr{Z}$ of X corresponds to a normal base $\mathscr{Z}^*$ in $\eta(\mathscr{Z})$ (and also in $\omega(\mathscr{Z})$). We show that every collection of $\mathscr{Z}^*$-ultrafilters with the countable intersection property is fixed, that is the intersection of all the members of the collection is non empty. In light of this fact, we say that $\eta(\mathscr{Z})$ is *$\mathscr{Z}^*$-real-compact*. We also show that $\eta(\mathscr{Z})$ is contained in the Q-closure of X in $\omega(\mathscr{Z})$. Finally if $\mathscr{Z}$ is the collection of all zero-sets then $\eta(\mathscr{Z})$ is precisely the Hewitt real compactification of X. We have attempted to show that every realcompactification Y of a space X can be obtained as a space $\eta(\mathscr{Z})$. This remains an open question.

Many examples exist of normal bases which are countably productive. One of the most important is the collection of all zero-sets of a completely regular T_1 space. Gillman and Jerison in [3] have shown that this family is countably productive and also that it satisfies the requirements for a normal base. Thus every Tychonoff space has a countably productive normal base.

DEFINITIONS. A base $\mathscr{Z}$ for the closed sets of a T_1 space X is said to be *disjunctive* if given any closed set F and any point x not in F there is a closed set A of $\mathscr{Z}$ that contains x and is disjoint from F. The base is said to be *normal* if any two disjoint members A and B of $\mathscr{Z}$ are subsets respectively of disjoint complements C' and D' of members of $\mathscr{Z}$.

A family $\mathscr{Z}$ of subsets of a set X is a *ring* of sets if it is closed under

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finite unions and intersections. We say that $\mathscr{Z}$ is *countably productive* if it is closed under countable intersections. We say that $\mathscr{Z}$ has the *countable intersection property* if every countable collection of subsets of $\mathscr{Z}$ has non empty intersection.

A base $\mathscr{Z}$ for the closed sets of a T_1 space X is a *normal base* if it is a normal disjunctive ring of sets.

A proper subset of a normal base $\mathscr{Z}$ is called a *$\mathscr{Z}$-filter* if it is closed under finite intersections and contains every superset in $\mathscr{Z}$ of each of its members. We also assume that no $\mathscr{Z}$-filter contains the empty set. A *$\mathscr{Z}$-ultrafilter* is a maximal $\mathscr{Z}$-filter.

If $\mathscr{Z}$ is a base for the closed sets in X we say that X is *$\mathscr{Z}$-realcompact* if every $\mathscr{Z}$-ultrafilter with the countable intersection property has a non empty intersection.

If $\mathscr{Z}$ is any distinguished family of subsets of a space X, we will represent the family of complements, $X-Z$, for Z in $\mathscr{Z}$ by $\mathfrak{C}\mathscr{Z}$. In particular, if $\mathscr{Z}$ is a normal base of closed sets in a Tychonoff space X then $\mathfrak{C}\mathscr{Z}$ is a base for the open sets.

Before stating our main results we will give three lemmas that will be needed. The proof of Lemma 1 can be found in [1].

LEMMA 1. *If $\mathscr{Z}$ is a normal base for X and if $\mathscr{F}$ is a $\mathscr{Z}$-filter on X, then $\mathscr{F}$ is a $\mathscr{Z}$-ultrafilter if and only if for each Z in $\mathscr{Z}$ either Z is in $\mathscr{F}$ or there is an A in $\mathscr{F}$ such that A is included in the complement of Z.*

LEMMA 2. *Let $\mathscr{Z}$ be a countably productive base for the closed sets of X and let $\mathscr{F}$ be a $\mathscr{Z}$-ultrafilter with the countable intersection property. If $(A_n)_{n\in N}$ is a sequence of sets in $\mathscr{F}$, then the intersection A of the sets A_n is in $\mathscr{F}$.*

PROOF. We first note that A is in $\mathscr{Z}$ since $\mathscr{Z}$ is countably productive. If F is in $\mathscr{F}$ then $F \cap A$ is non empty since $\mathscr{F}$ has the countable intersection property. It follows that $\mathscr{F} \cup \{A\}$ generates a $\mathscr{Z}$-filter $\mathscr{B}$ containing $\mathscr{F}$. Consequently A is in $\mathscr{F}$ since $\mathscr{F}$ must equal $\mathscr{B}$.

Lemma 1 characterizes $\mathscr{Z}$-ultrafilters in a manner which directly relates the filter with the definition of the basic open sets in Frink's compactifications (see[2]). To investigate $\mathscr{Z}$-realcompact spaces we need a similar result for the filters on the trace of a normal base with a subspace.

LEMMA 3. *Let A be a subspace of a space X, let $\mathscr{Z}(X)$ be a base for the closed sets of X, let $\mathscr{Z}(A)$ be the trace on A of $\mathscr{Z}(X)$, and let $cl_X Z$ be in $\mathscr{Z}(X)$ for all Z in $\mathscr{Z}(A)$. If $\mathscr{F}$ is a $\mathscr{Z}(X)$-ultrafilter on X and if $\{F \cap A : F \text{ is in } \mathscr{F}\}$ is a base for a $\mathscr{Z}(A)$-filter $\mathscr{F}(A)$, then $\mathscr{F}(A)$ is a $\mathscr{Z}(A)$-ultrafilter.*

PROOF. By hypothesis $\mathscr{F}(A)$ is a $\mathscr{Z}(A)$ filter so there is a $\mathscr{Z}(A)$-ultrafilter $\mathscr{G}$ that contains $\mathscr{F}(A)$. Let $\mathscr{H}$ be the set of all B in $\mathscr{Z}(X)$ such

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that B contains $cl_X G$ for some G in $\mathscr{G}$. It is easy to see that $\mathscr{H}$ is a $\mathscr{Z}(X)$-filter. Since F contains $cl_X(F \cap A)$, if F is in $\mathscr{F}$ then $F \cap A$ is in $\mathscr{F}(A)$ and F is in $\mathscr{H}$. Thus the $\mathscr{Z}(X)$-ultrafilter $\mathscr{F}$ must be equivalent to $\mathscr{H}$.

If G is in the $\mathscr{Z}(A)$-ultrafilter $\mathscr{G}$ then G is in $\mathscr{Z}(A)$ and $cl_X G$ is in $\mathscr{Z}(X)$. Thus $cl_X G$ is in $\mathscr{H} = \mathscr{F}$ and $G = cl_A G = cl_X G \cap A$ is in the $\mathscr{Z}(A)$-filter $\mathscr{F}(A)$. It follows that $\mathscr{G}$ is equivalent to $\mathscr{F}(A)$ and that $\mathscr{F}(A)$ is a $\mathscr{Z}(A)$-ultrafilter. This completes the proof of the lemma.

We now consider X to be always a Tychonoff space with a countably productive normal base $\mathscr{Z}$ for the closed subsets of X. For this base $\mathscr{Z}$, the space $\eta(\mathscr{Z})$ is obtained in the following manner. The points of $\eta(\mathscr{Z})$ are the $\mathscr{Z}$-ultrafilters of X with the countable intersection property. For each Z in $\mathscr{Z}$ we define the set Z^* to be the family of all $\mathscr{Z}$-ultrafilters with the countable intersection property having Z as a member. The collection $\mathscr{Z}^*$ of sets Z^* for Z in $\mathscr{Z}$ is taken as a base for the closed subsets of $\eta(\mathscr{Z})$. The space $\eta(\mathscr{Z})$ is a $\mathscr{Z}^*$-realcompact Hausdorff space. In particular if $\mathscr{Z}$ is the collection of zero-sets of X then $\eta(\mathscr{Z})$ is the Hewitt realcompactification νX (see Gillman and Jerison [3]).

There is a natural embedding φ of X into $\eta(\mathscr{Z})$ where $\varphi(x)$ is the $\mathscr{Z}$-ultrafilter with the countable intersection property consisting of all $\mathscr{Z}$-sets that contain x. The mapping φ is a homeomorphism of X onto the dense subset $\varphi(X)$ of $\eta(\mathscr{Z})$.

In an equivalent manner, we could define a base for the open sets of $\eta(\mathscr{Z})$. Let U^* be the collection of all $\mathscr{Z}$-ultrafilters with the countable intersection property that have some subset of U as a member, where $X-U$ is in $\mathscr{Z}$. This is just the dual of the definition of Z^*, that is, $\eta(\mathscr{Z})-U^* = (X-U)^*$ where $X-U$ is in $\mathscr{Z}$.

THEOREM 1. *Let X be a Tychonoff space with a countably productive base $\mathscr{Z}$ and let φ be the natural embedding of X into $\eta(\mathscr{Z})$. If U, V, and $(U_n)_{n=1}^\infty$ are members of $\mathfrak{C}\mathscr{Z}$ and Z, $(Z_n)_{n=1}^\infty$ are members of $\mathscr{Z}$, then the following properties hold.*

1. *If $U \subset V$ then $U^* \subset V^*$.*
2. $(\bigcup_{n=1}^\infty U_n)^* = \bigcup_{n=1}^\infty U_n^*$ *and* $(\bigcap_{n=1}^\infty U_n)^* = \bigcap_{n=1}^\infty U_n^*$.
3. $U^* \cap \varphi(X) = \varphi(U)$ *and* $Z^* \cap \varphi(X) = \varphi(Z)$.
4. $cl_{\eta(\mathscr{Z})}\varphi(Z) = Z^*$.
5. $cl_{\eta(\mathscr{Z})}\varphi(\bigcap_{n=1}^\infty Z_n) = \bigcap_{n=1}^\infty cl_{\eta(\mathscr{Z})}\varphi(Z_n)$ *or equivalently* $(\bigcap_{n=1}^\infty Z_n)^* = \bigcap_{n=1}^\infty Z_n^*$.

The map φ of X onto the subset $\varphi(X)$ of $\eta(\mathscr{Z})$ is a homeomorphism.

PROOF. If U, V are in $\mathfrak{C}\mathscr{Z}$ and if $U \subset V$ then A a subset of U implies that A is a subset of V and therefore $U^* \subset V^*$. If $(U_n)_{n=1}^\infty$ is any sequence of members of $\mathfrak{C}\mathscr{Z}$ then $\bigcup_{n=1}^\infty U_n^* \subset (\bigcup_{n=1}^\infty U_n)^*$ and $(\bigcap_{n=1}^\infty U_n)^* \subset \bigcap_{n=1}^\infty U_n^*$.

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To complete the proof of (2) we use the fact that for $X-U$ in $\mathscr{Z}$, $\eta(\mathscr{Z})-U^* = (X-U)^*$. If α is in $(\bigcup_{n=1}^{\infty} U_n)^*$ then there is an A contained in $\bigcup_{n=1}^{\infty} U_n$ such that A is a member of α. If α is not in $\bigcup_{n=1}^{\infty} U_n^*$ then $X-U_n$ is in α for each n. Then

$$A \cap (X - \bigcup_{n=1}^{\infty} U_n) = A \cap (\bigcap_{n=1}^{\infty} (X-U_n)) = \emptyset$$

for α is in $(\bigcup_{n=1}^{\infty} U_n)^*$ which is a contradiction since α has the countable intersection property. Thus

$$(\bigcup_{n=1}^{\infty} U_n)^* = \bigcup_{n=1}^{\infty} U_n^*.$$

Using this, DeMorgan's laws, and $\eta(\mathscr{Z})-U^* = (X-U)^*$ for $X-U$ in $\mathscr{Z}$, we have

$$(\bigcap_{n=1}^{\infty} U_n)^* = \bigcap_{n=1}^{\infty} U_n^*$$

and (2) is shown.

The map φ is a one-one map of X onto the subspace $\varphi(X)$ of $\eta(\mathscr{Z})$ since $\mathscr{Z}$ is a disjunctive family and X is a T_1 space. To show that φ is a homeomorphism it will be sufficient to show that (3) holds. In fact if α is in $\varphi(X) \cap U^*$ then $\alpha = \varphi(x)$ for some x in X and there is a Z in $\varphi(x)$ such that $Z \subset U$. Hence x is in $Z \subset U$ and $\varphi(x)$ is in $\varphi(U)$. Thus $\varphi(U) = U^* \cap \varphi(X)$. Since φ is one-one and onto $\varphi(X)$ this equation shows that φ is both a continuous and open map; hence φ is a homeomorphism. It follows then that $Z^* \cap \varphi(X) = \varphi(Z)$.

From (3) it follows that $cl_{\eta(\mathscr{Z})}\varphi(Z)$ is included in Z^*. Conversely if α is in Z^* and U^* is any basic open set containing α then there is an $A \subset U$ such that $A \cap Z$ is in α. Since α is a filter there is a point p in $A \cap Z$ and $\varphi(p)$ is in $U^* \cap \varphi(Z)$. Thus $cl_{\eta(\mathscr{Z})}\varphi(Z) = Z^*$.

Finally property (5) follows from DeMorgan's laws and properties (2) and (4). This completes the proof of our theorem.

We are now in a position to prove our main theorem.

THEOREM 2. *If X is a Tychonoff space with a countably productive normal base $\mathscr{Z}$ then X is homeomorphic to a dense subspace of the $\mathscr{Z}^*$-realcompact Hausdorff space $\eta(\mathscr{Z})$.*

PROOF. Let $\mathscr{F}^*$ be a $\mathscr{Z}^*$-ultrafilter on $\eta(\mathscr{Z})$ that also has the countable intersection property. By (5) of Theorem 1, the elements of $\mathscr{F}^*$ are of the form $cl_{\eta(\mathscr{Z})} Z$ for some Z in $\mathscr{Z}$ where we have identified Z with $\varphi(Z)$. Thus $\mathscr{F}^*$ is a family $cl_{\eta(\mathscr{Z})} Z_\alpha$ for α in an indexing set I. Let $\mathscr{G}$ be the collection of $cl_{\eta(\mathscr{Z})} Z_\alpha \cap X$ for $\alpha \in I$. This is precisely the collection of Z_α for $\alpha \in I$. This collection is a base for a $\mathscr{Z}$-filter $\mathscr{H}$ on X. For if Z_1 and Z_2 are in $\mathscr{G}$ then by property 5 of Theorem 1, $cl\, Z_1 \cap cl\, Z_2 = cl(Z_1 \cap Z_2)$ is in $\mathscr{F}$ and hence $Z_1 \cap Z_2$ is non empty, so $Z_1 \cap Z_2$ is in $\mathscr{G}$.

The filter $\mathscr{H}$ is precisely the $\mathscr{Z}$-filter on X generated by the family of $F \cap X$ for F in $\mathscr{F}$. By Lemma 3, $\mathscr{H}$ is a $\mathscr{Z}$-ultrafilter on X. If K is a countable subset of I then $cl(\bigcap_{n \in K} Z_n) = \bigcap_{n \in K} cl\, Z_n$ is non empty since $\mathscr{F}^*$ has the countable intersection property. Consequently $\bigcap_{n \in K} Z_n$ is non empty, $\mathscr{H}$ is in $\eta(\mathscr{Z})$, and $\mathscr{H}$ belongs to each member of $\mathscr{F}^*$. Thus $\eta(\mathscr{Z})$ is $\mathscr{Z}^*$-realcompact.

Now let α and β be two distinct points in $\eta(\mathscr{Z})$. By maximality of the filters α and β there are disjoint $\mathscr{Z}$-sets A and B such that A is in α but not in β and B is in β but not in α. By the normality of $\mathscr{Z}$ there are sets C and D in $\mathscr{Z}$ such that their complements are disjoint and $A \subset X - C$, $B \subset X - D$. Then α is in $(X-C)^*$ and β is in $(X-D)^*$ and $\eta(\mathscr{Z})$ is a Hausdorff space.

If U^* is any basic non empty open set in $\eta(\mathscr{Z})$ then its correspondent U is non empty. Since $U^* \cap \varphi(X) = \varphi(U)$ (see Theorem 1), it follows that $\varphi(X)$ is dense in $\eta(\mathscr{Z})$ and the theorem has been proved.

Corollary. *If $\mathscr{Z}$ is the countably productive normal base of all zero-sets of a Tychonoff space X, then $\eta(\mathscr{Z})$ is precisely the Hewitt realcompactification νX.*

If the space X has a normal base $\mathscr{Z}$ (Frink has shown that X must be a Tychonoff space) and if X is $\mathscr{Z}$-realcompact then our construction for $\eta(\mathscr{Z})$ gives precisely X.

Theorem 3. *If $\mathscr{Z}$ is a normal base on a space X and if X is $\mathscr{Z}$-realcompact then X is precisely $\eta(\mathscr{Z})$.*

Proof. If α is in $\eta(\mathscr{Z})$ then α is a $\mathscr{Z}$-ultrafilter with the countable intersection property. Since X is $\mathscr{Z}$-realcompact there is an x in X such that x belongs to every member of α. But then the ultrafilter α is included in the filter $\varphi(x)$ and thus α must be $\varphi(x)$. Hence $\varphi(X) = \eta(\mathscr{Z})$.

We have not yet determined whether or not $\eta(\mathscr{Z})$ is realcompact in the usual sense. S. Mrowka has pointed out to the authors that $\eta(\mathscr{Z})$ is contained in the Q-closure of X in $\omega(\mathscr{Z})$, the Hausdorff compactification introduced by Frink in [2]. If X is a subspace of a space Y, then a point *p is in the Q-closure* of X in Y if there does not exist a real-valued continuous function that is zero at p and positive on S.

Frink's compactification $\omega(\mathscr{Z})$ for a normal base $\mathscr{Z}$ of a space X is obtained by taking Z^* to be the family of all $\mathscr{Z}$-ultrafilters that have Z as a member. The collection of sets Z^* are taken as a base for the closed sets of $\omega(\mathscr{Z})$. Then X is shown to be a dense subspace of the compact Hausdorff space $\omega(\mathscr{Z})$.

The following Lemma will be needed to give our result.

LEMMA 4. *If $\mathscr{Z}$ is a normal base on a space X, then $\mathscr{Z}^*$ is a normal base on $\omega(\mathscr{Z})$.*

PROOF. That $\mathscr{Z}^*$ is a ring follows immediately (see Lemma 1 of [1]). If Z^* is any basic closed set of $\omega(\mathscr{Z})$ and if $\mathscr{F}$ is any point not in Z^* then by Lemma 1 there is an A in $\mathscr{F}$ such that A is included in $X-Z$. Hence $\mathscr{F}$ is in A^* and $A^* \cap Z^*$ is empty since $A \cap Z$ is empty. Thus $\mathscr{Z}^*$ is disjunctive.

If Z_1^* and Z_2^* are two disjoint $\mathscr{Z}^*$ sets then $Z_1 \cap Z_2$ is empty. By the normality of $\mathscr{Z}$ there are $\mathscr{Z}$-sets F_1 and F_2 whose complements are disjoint and such that $Z_1 \subset X-F_1$ and $Z_2 \subset X-F_2$. It follows that $Z_1^* \subset X-F_1^*$ and $Z_2^* \subset X-F_2^*$.

THEOREM 4. *If $\mathscr{Z}$ is a countably productive normal base on X then $\eta(\mathscr{Z})$ is a subset of the Q-closure of X in $\omega(\mathscr{Z})$.*

PROOF. Suppose that $\mathscr{F}$ is a point in $\omega(\mathscr{Z})$ that is not in the Q-closure of X in $\omega(\mathscr{Z})$. Then there is a real-valued continuous function f on $\omega(\mathscr{Z})$ that is zero at $\mathscr{F}$ and positive on X.

For each integer n, let F_n be the set of $\mathscr{G}$ in $\omega(\mathscr{Z})$ such that $f(\mathscr{G}) \geqq 1/n$. The sets F_n are closed in $\omega(\mathscr{Z})$ and $\mathscr{F}$ is not in F_n for any n. Since $\mathscr{Z}^*$ is a normal base on $\omega(\mathscr{Z})$, there is a Z_n in $\mathscr{Z}$ such that $\mathscr{F}$ is in Z_n^* and $Z_n^* \cap F_n$ is empty. Then Z_n is in $\mathscr{F}$ for each n and Z_n is included in $X-F_n$. But X is included in $\bigcup_{n=1}^{\infty} F_n$ so the intersection of the sets Z_n must be empty. Hence $\mathscr{F}$ does not have the countable intersection property and $\mathscr{F}$ is not in $\eta(\mathscr{Z})$. This completes the proof.

The Q-closure of a subspace of a space is always realcompact. It remains an open question as to whether or not $\eta(\mathscr{Z})$ is precisely the Q-closure of X in $\omega(\mathscr{Z})$.

We can now give an example of a space X with different normal bases $\mathscr{Z}_1$ and $\mathscr{Z}_2$ for which the $\mathscr{Z}_i$-realcompactifications are not equal ($i = 1, 2$). In particular let X be an uncountable discrete space. Let $\mathscr{Z}_1$ be the collection of all subsets A of X such that A or $X-A$ is at most countable and let $\mathscr{Z}_2$ be the collection of all subsets of X.

It is easy to verify that $\mathscr{Z}_1$ and $\mathscr{Z}_2$ are countably productive normal bases on X; moreover, that $\eta(\mathscr{Z}_2) = \nu X = \varphi(X)$ and that $\omega(\mathscr{Z}_2) = \beta X$ where νX is the Hewitt realcompactification and βX is the Stone-Cech compactification of X. Now $\eta(\mathscr{Z}_1)$ is not equal to $\varphi(X)$ for there is a member $\mathscr{F}$ of $\eta(\mathscr{Z}_1)$ that is not in $\varphi(X)$. In fact let $\mathscr{F}$ be the $\mathscr{Z}_1$-filter that is the collection of all subsets of X whose complement is at most countable. It is a $\mathscr{Z}_1$-ultrafilter since $X-Z$ is in $\mathscr{F}$ for any member Z of a filter containing $\mathscr{F}$ where Z is not in $\mathscr{F}$ (see Lemma 1). If $(F_n)_{n=1}^{\infty}$ is any sequence of sets in $\mathscr{F}$, then their common intersection F is non empty since the complement of F is countable and hence not equal to X. This shows that $\mathscr{F}$ has the countable intersection property. Finally $X - \{x\}$ is in $\mathscr{F}$ for each x in X and the common intersection of these sets is empty. It follows that the common intersection of the sets F in $\mathscr{F}$ must be empty. Thus $\mathscr{F}$ is not in $\varphi(X)$. Hence in this case we have that $\mathscr{Z}_1 \subset \mathscr{Z}_2$ and $\eta(\mathscr{Z}_2)$ is, homeomorphically, a proper subset of $\eta(\mathscr{Z}_1)$.

In addition $\omega(\mathscr{Z}_1)$ is not equal to βX. For if it were, then $\eta(\mathscr{Z}_1)$ would be included in the Q-closure of X in βX which in turn is included in βX, by Theorem 4. But the Q-closure of X in βX is $vX = X$. Hence $\eta(\mathscr{Z}_1)$ is included in $\nu X = X$ and since $\eta(\mathscr{Z}_1)$ contains X homeomorphically we would have that $X = \eta(\mathscr{Z}_1)$, a contradiction.

References

[1] R. A. Alo and H. L. Shapiro, 'A note on compactifications and semi-normal spaces', *J. Austral. Math. Soc.* 8 (1968), 102—108.

[2] Orrin Frink, 'Compactifications and semi-normal spaces', *Amer. J. Math.* 86 (1964), 602—607.

[3] L. Gillman and M. Jerison, *Rings of continuous functions*. (New York: Van Nostrand, 1960.)

The Carnegie Institute of Technology
The Pennsylvania State University

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