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REALIZABILITY OF THE ADAMS-NOVIKOV SPECTRAL SEQUENCE FOR FORMAL A-MODULES

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ABSTRACT. We show that the formal A-module Adams-Novikov spectral sequence of Ravenel does not naturally arise from a filtration on a map of spectra by examining the case $A = \mathbb{Z}[i]$. We also prove that when A is the ring of integers in a nontrivial extension of $\mathbb{Q}_p$, the map $(L, W) \to (L_A, W_A)$ of Hopf algebroids, classifying formal groups and formal A-modules respectively, does not arise from compatible maps of E_∞-ring spectra $(MU, MU \wedge MU) \to (R, S)$.

1. INTRODUCTION

The recent development of highly structured categories of spectra has led to questions about what kinds of algebro-geometric procedures can be imported into homotopy theory. In particular, it often leads to the hope that there might be "algebraic extensions" of the sphere spectrum that play the role of the ring of integers in a number field. For example, the notion of a Galois extension of a ring spectrum is defined in [6]. Unfortunately, the sphere spectrum has no Galois extensions unless one inverts a set of integers, roughly because Galois extensions cannot have ramified primes.

Additionally, Schwänzl, Vogt, and Waldhausen have shown that there exists no A_∞-ring spectrum R such that $H\mathbb{Z} \wedge R \simeq H\mathbb{Z}[i]$ by making use of a calculation in topological Hochschild homology [7]. (Assuming R is connective, it would necessarily have the homotopy type of $\mathbb{S} \vee \mathbb{S}$, with some A_∞-structure imposed.)

This might suggest that another approach would be in order, based on different algebraic properties of the sphere spectrum. One such approach is through the Adams-Novikov spectral sequence. This spectral sequence computes the stable homotopy groups of the sphere and has an E_2-term given by Ext groups of the Hopf algebroid (MU_*, MU_*MU). Due to the work of Quillen [4], it is possible to identify this E_2-term as Ext groups of the Hopf algebroid (L, W) representing formal groups and strict isomorphisms between them.

Ravenel, in his article [5], defined the Adams-Novikov spectral sequence for formal A-modules. For a ring A, a formal A-module is a formal group together with an action of the ring A by endomorphisms of formal groups. There is a Hopf algebroid (L_A, W_A) representing formal A-modules and strict isomorphisms between them. A formal group is a formal $\mathbb{Z}$-module. These Hopf algebroids are functorial in A.

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If A is the ring of integers in an algebraic extension of $\mathbb{Q}$, we get a map of Ext-groups

$$\mathrm{Ext}^{**}_{(L,W)}(L,L) \to \mathrm{Ext}^{**}_{(L_A,W_A)}(L_A,L_A).$$

The domain of this map is the E_2-term of the Adams-Novikov spectral sequence for the homotopy groups of the sphere, while the range is, by definition, the Adams-Novikov E_2-term for formal A-modules. It is natural to ask whether this map of spectral sequences arises as a filtration of a map $\mathbb{S} \to \mathbb{S}_A$ from the sphere spectrum to an algebraic extension of $\mathbb{S}$.

Unfortunately, the answer is no in general. If $A = \mathbb{Z}[i]$, we will indicate as an example a computation of the first few terms of the 2-primary formal $\mathbb{Z}[i]$-module Adams-Novikov spectral sequence in section 3, together with the map from the ordinary Adams-Novikov sequence. This map would violate the nontrivial extension in the 3-stem. (A rough calculation seems to indicate that for an extension field totally ramified at an odd prime, the existence of the Toda differential does not immediately give rise to a contradiction.)

The extension in the 3-stem is detected by the ordinary Adams spectral sequence, suggesting that there is some incompatibility with the Steenrod algebra. By making this incompatibility precise, we find that the following general result holds.

Theorem 1.1. *Let A be the ring of integers in a finite extension field of $\mathbb{Q}$. There is no diagram of E_∞-ring spectra*

$$\begin{array}{ccccc} MU & \xrightarrow{\eta_L} & MU \wedge MU & \xleftarrow{\eta_R} & MU \\ \downarrow & & \downarrow & & \downarrow \\ R & \xrightarrow{\eta_L} & S & \xleftarrow{\eta_R} & R \end{array}$$

realizing the diagram

$$\begin{array}{ccccc} L & \xrightarrow{\eta_L} & W & \xleftarrow{\eta_R} & L \\ \downarrow & & \downarrow & & \downarrow \\ L_A & \xrightarrow{\eta_L} & W_A & \xleftarrow{\eta_R} & L_A \end{array}$$

on homotopy groups unless $A = \mathbb{Z}$.

It follows from this that there exists no E_∞ "algebraic sphere" $\mathbb{S}_A$ such that $\pi_*(\mathbb{S}_A \wedge MU) \cong L_A$ as an L-algebra. Given such an $\mathbb{S}_A$, an explicit computation with the Künneth spectral sequence for

$$\mathbb{S}_A \wedge MU \wedge MU \simeq (\mathbb{S}_A \wedge MU) \underset{MU}{\wedge} (MU \wedge MU),$$

together with unit maps arising from the weak equivalence

$$(\mathbb{S}_A \wedge MU) \underset{\mathbb{S}_A}{\wedge} (\mathbb{S}_A \wedge MU) \to \mathbb{S}_A \wedge MU \wedge MU,$$

would show that the pair $(\mathbb{S}_A \wedge MU, \mathbb{S}_A \wedge MU \wedge MU)$ violates Theorem 1.1.

As a partial converse, if we further assume that (R, S) forms a Hopf algebroid of spectra (having comultiplication and augmentation maps that satisfy appropriate diagrams, in addition to the given left and right units), the algebraic sphere $\mathbb{S}_A$ could be recovered by the cobar construction $C(R, S, R)$. (This follows because the natural map $R \wedge MU \to S$ is a weak equivalence, and the smash product with MU can be moved inside the cobar construction.)

The proof of Theorem 1.1 proceeds by calculating what the analog of the dual Steenrod algebra would be. This proof occupies section 4. In fact, the theorem holds locally at *any* prime of A whose decomposition group is nontrivial; this is a phenomenon associated to any extension of the local field, rather than merely to the ramified primes.

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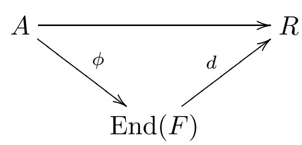
2. The Lazard ring for formal A-modules

In this section, we will make explicit the form of the Lazard ring for formal A-modules for specific choices of A. The formulas from this section are taken from Hazewinkel [3], section IV.21.

If A is a ring, a formal A-module is a formal group law F over an A-algebra R with a map $\phi : A \to \mathrm{End}(F)$ such that the diagram



commutes, where d is the differential at 0. For $a \in A$, the endomorphism $\phi(a)$ of F is a power series $[a](X) \in R[[X]]$.

We now restrict to a particular case. Let K be a finite extension of $\mathbb{Q}_p$ of degree n, with ring of integers A. Choose a uniformizer π of A. As ideals of A, $(p) = (\pi^e)$, where e is the ramification index. The residue field $A/(\pi)$ has order $q = p^f$. These satisfy $e \cdot f = n$.

For a positive integer n, define $\nu(n) = 1$ if n is not a prime power and ℓ if n is a power of some prime ℓ. Finally, let $C_n(X,Y) = \nu(n)^{-1}((X+Y)^n - X^n - Y^n)$.

Define L_A to be the Lazard ring for formal A-modules, and F_A the universal formal A-module over L_A. As a ring,

$$L_A \cong A[Y_1, Y_2, Y_3, \ldots].$$

The coefficients can be identified. Modulo $Y_1, \ldots, Y_{n-2}$ and terms of degree $n+1$ and higher in X and Y, we find the following:

$$F_A(X,Y) \equiv \begin{cases} X + Y + Y_{n-1}\nu(n)C_n(X,Y) & \text{if } n \neq q^m, \\ X + Y + \pi^{-1}Y_{n-1}\nu(n)C_n(X,Y) & \text{if } n = q^m \end{cases} \tag{1}$$

([3], IV.21.4.8).

By restriction, a formal A-module is also a formal group law, and this corresponds to a map $L \to L_A$. If F is the universal formal group law over $\mathbb{Z}$, we know that $L \cong \mathbb{Z}[X_1, X_2, \ldots]$. Modulo $X_1, \ldots, X_{n-2}$ and terms of degree $n+1$ and higher,

$$F(X,Y) \equiv X + Y + X_{n-1}C_n(X,Y).$$

Therefore, the map $L \to L_A$ can be expressed as follows:

$$X_n \mapsto \begin{cases} Y_n, & n \neq p^m - 1, \\ pY_n, & n = p^m - 1, n \neq q^k - 1, \\ \left(\frac{p}{\pi}\right) Y_n, & n = q^k - 1. \end{cases} \tag{2}$$

The ring L_A is part of a Hopf algebroid (L_A, W_A) representing formal A-modules and strict isomorphisms between them. There are left and right unit maps $\eta_L, \eta_R : L_A \to W_A$. As a module over $\eta_L(L_A)$,

$$W_A \cong L_A[b_1, b_2, \ldots].$$

We choose the coefficients b_i to be the coefficients of the universal strict isomorphism

$$f(x) = x + \sum b_i\, x^{i+1}$$

from $f^{-1} \circ F_A \circ f$ to F_A.

If (L, W) is the Hopf algebroid representing formal groups and strict isomorphisms, we know that

$$W \cong L[b_1, b_2, \ldots]$$

as a module over $\eta_L(L)$. The restriction map $L \to L_A$ extends to a map $(L, W) \to (L_A, W_A)$ of Hopf algebroids. The map $W \to W_A$ is the extension of scalars map $L[b_i] \to L_A[b_i]$.

The ring W_A has a quotient ring $A \otimes_{L_A} W_A$, where the map $L_A \to A$ classifies the additive group law. This ring, which is isomorphic to $A[b_1, b_2, \ldots]$, classifies the universal strict isomorphism whose range is the additive formal group law. We now wish to determine the image of the right unit, which is equivalent to determining the domain formal group law.

Proposition 2.1. *Modulo $\eta_R(Y_1), \ldots, \eta_R(Y_{n-1})$, the image of $\eta_R(Y_n)$ in the ring $A[b_1, b_2, \ldots]$ is $-b_n$ if $n \neq q^m - 1$ for any m, and $-\pi b_n$ if $n = q^m - 1$.*

Proof. If $f(x) = x + \sum b_i x^{i+1}$ is the universal strict isomorphism, then f is a map from G to the additive formal group law, where $G(x, y) = f^{-1}(f(x) + f(y))$. By equation (1), when we reduce modulo $\eta_R(Y_1), \ldots, \eta_R(Y_{n-2})$ and terms of degree $n + 1$ and higher, we find that

$$f^{-1}(f(x) + f(y)) \equiv \begin{cases} X + Y + \eta_R(Y_{n-1})\nu(n)C_n(X, Y) & \text{if } n \neq q^m, \\ X + Y + \eta_R(Y_{n-1})\pi^{-1}\nu(n)C_n(X, Y) & \text{if } n = q^m. \end{cases}$$

Applying f to both sides, we find that

$$f(x) + f(y) \equiv \begin{cases} f\Big(X + Y + \eta_R(Y_{n-1})\nu(n)C_n(X, Y)\Big) & \text{if } n \neq q^m, \\ f\Big(X + Y + \eta_R(Y_{n-1})\pi^{-1}\nu(n)C_n(X, Y)\Big) & \text{if } n = q^m. \end{cases}$$

Taking terms of degree n gives the formula

$$b_{n-1}X^n + b_{n-1}Y^n \equiv \begin{cases} \eta_R(Y_{n-1})\nu(n)C_n(X, Y) + b_{n-1}(X + Y)^n & \text{if } n \neq q^m, \\ \eta_R(Y_{n-1})\pi^{-1}\nu(n)C_n(X, Y) + b_{n-1}(X + Y)^n & \text{if } n = q^m. \end{cases}$$

In particular, modulo $\eta_R(Y_1), \ldots, \eta_R(Y_{n-2})$, we have $\eta_R(Y_{n-1}) \equiv -b_{n-1}$ if $n \neq q^m$, and $\eta_R(Y_{n-1}) \equiv -\pi b_{n-1}$ if $n = q^m$. Re-indexing gives the statement of the proposition. □

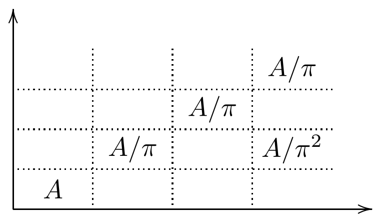
3. The formal $\mathbb{Z}_2[i]$-module Adams-Novikov sequence

In this section, we fix A to be the ring $\mathbb{Z}_2[i]$. We will compute the formal A-module Adams-Novikov spectral sequence out to dimension 3 by using the reduced bar complex

$$0 \to L_A \to \overline{W}_A \to \overline{W}_A \otimes_{L_A} \overline{W}_A \to \cdots,$$

where $\overline{W}_A$ is the kernel of the augmentation map $\epsilon : W_A \to L_A$.

Let $\pi = 1+i$ be a uniformizer for A. We can choose generators of L_A and W_A, as in section 2, such that $X_1 \mapsto \pi Y_1$ and $b_1 \mapsto b_1$. As $\eta_R X_1 = X_1 - 2b_1$, we deduce the formula $\eta_R(Y_1) = Y_1 - (2-\pi)b_1$. Similarly, $\Delta b_1 = 1 \otimes b_1 + b_1 \otimes 1$. These formulas allow us to directly compute the homology of this complex out to the 3-stem; we record the result in the following diagram. The vertical axis denotes Ext-degree, and the horizontal axis denotes total degree.



The elements in total degree 3 are generated by the cycles $\nu_A = b_1^2 - \pi Y_1 b_1$ and $\eta_A^3 = b_1 \otimes b_1 \otimes b_1$ in the bar complex. The map of Adams-Novikov sequences sends the element $\nu = b_1^2 - X_1 b_1$ to ν_A, and sends η^3 to η_A^3.

This violates the extension in the 2-primary Adams-Novikov spectral sequence. A lift of ν to $\pi_3(\mathbb{S})$ satisfies $4\nu = \eta^3$, but any lifting of ν_A would have to be π^3-torsion, and hence be killed by $\pi^4 = -4$.

4. The formal A-module Steenrod algebra

We now prove the following. Let A be the ring of integers in a finite extension of $\mathbb{Q}_p$, and let (L_A, W_A) be the Hopf algebroid associated to A.

Theorem 4.1. *There is no diagram of E_∞-ring spectra*

$$\begin{array}{ccccc} MU & \xrightarrow{\eta_L} & MU \wedge MU & \xleftarrow{\eta_R} & MU \\ \downarrow & & \downarrow & & \downarrow \\ R & \xrightarrow{\eta_L} & S & \xleftarrow{\eta_R} & R \end{array}$$

realizing the diagram

$$\begin{array}{ccccc} L & \xrightarrow{\eta_L} & W & \xleftarrow{\eta_R} & L \\ \downarrow & & \downarrow & & \downarrow \\ L_A & \xrightarrow{\eta_L} & W_A & \xleftarrow{\eta_R} & L_A \end{array}$$

on homotopy groups unless $A = \mathbb{Z}_p$.

Theorem 1.1 follows immediately by completing at any prime that is not totally split.

Proof. Suppose that we had a diagram of E_∞-ring spectra as stated. We have that R is (-1)-connected with $\pi_0 R = A$, so we can form a diagram of E_∞-ring spectra

$$\begin{array}{ccc} MU & \longrightarrow & H\mathbb{F}_p \\ \downarrow & & \downarrow \\ R & \longrightarrow & H\mathbb{F}_q, \end{array}$$

where $\mathbb{F}_q$ is the residue field of A.

We then get a map of E_∞-ring spectra

$$H\mathbb{F}_p \underset{MU}{\wedge} (MU \wedge MU) \to H\mathbb{F}_q \underset{R}{\wedge} S,$$

where the smash product is taken along the left unit. The Künneth spectral sequence of [2] shows that on homotopy groups, this map is the extension of scalars map

$$\mathbb{F}_p[b_1, b_2, \ldots] \to \mathbb{F}_q[b_1, b_2, \ldots].$$

These rings respectively classify the universal strict isomorphisms into the additive formal groups over $\mathbb{F}_p$ and $\mathbb{F}_q$.

Smashing along the right unit then gives a map of E_∞-ring spectra

$$H\mathbb{F}_p \underset{MU}{\wedge} (MU \wedge MU) \underset{MU}{\wedge} H\mathbb{F}_p \to H\mathbb{F}_q \underset{R}{\wedge} S \underset{R}{\wedge} H\mathbb{F}_q.$$

The homotopy groups of the left-hand spectrum form the dual Steenrod algebra $\mathcal{A}_*$. Write $\mathcal{B}_* = \pi_*(H\mathbb{F}_q \wedge_R S \wedge_R H\mathbb{F}_q)$.

We get by naturality a map of Künneth spectral sequences

$$\mathrm{Tor}^{L}_{**}(\mathbb{F}_p[b_i], \mathbb{F}_p) \to \mathrm{Tor}^{L_A}_{**}(\mathbb{F}_q[b_i], \mathbb{F}_q)$$

that converges to some filtration of the map $\mathcal{A}_* \to \mathcal{B}_*$.

Proposition 2.1 shows that the image of $\eta_R(Y_i)$ in $\mathbb{F}_q[b_1, b_2, \ldots]$ is congruent modulo $\eta_R(Y_1), \ldots, \eta_R(Y_{i-1})$ to $-b_i$ if $i \neq q^m - 1$, and 0 if $i = q^m - 1$. The same proposition shows that a similar result holds for X_i with q replaced by p. This determines the right module structure of $\mathbb{F}_p[b_i]$ and $\mathbb{F}_q[b_i]$.

The sequence $(p, X_1, X_2, \ldots)$ is regular in L, so we can resolve $\mathbb{F}_p$ over L by the Koszul complex

$$\bigotimes_{i=0}^{\infty} (L \xrightarrow{X_i} L),$$

where $X_0 = p$ by convention. The tensor product is taken over L. Similarly, $(\pi, Y_1, Y_2, \ldots)$ is regular in L_A, so we have a similar Koszul resolution for $\mathbb{F}_q$ over L_A.

The following Tor calculations follow:

$$\begin{aligned} \mathrm{Tor}^{L}_{**}(\mathbb{F}_p[b_i], \mathbb{F}_p) &\cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_0, \tau_1, \tau_2, \ldots], \\ \mathrm{Tor}^{L_A}_{**}(\mathbb{F}_q[b_i], \mathbb{F}_q) &\cong \mathbb{F}_q[\xi_f, \xi_{2f} \ldots] \otimes \Lambda[\sigma_0, \sigma_f, \sigma_{2f}, \ldots]. \end{aligned}$$

The elements ξ_k live in Tor degree zero and total degree $2(p^k - 1)$. (The element ξ_k is represented by b_{p^k-1}.) The elements τ_k live in Tor degree 1 and total degree $2p^k - 1$, while the elements σ_{kf} live in Tor degree 1 and total degree $2q^k - 1$. Because the terms in the spectral sequence are generated by terms in homological degrees 0 and 1, the spectral sequence collapses.

The map of spectral sequences sends ξ_k to 0 if f does not divide k. In fact, the Tor-degree zero terms form the algebras $\mathbb{F}_p \otimes_L W \otimes_L \mathbb{F}_p$ and $\mathbb{F}_q \otimes_{L_A} W_A \otimes_{L_A} \mathbb{F}_q$ respectively. These rings classify the strict automorphisms of the additive formal group laws over $\mathbb{F}_p$ and $\mathbb{F}_q$. As such, they are the affine coordinate rings of the group schemes of p-series and q-series respectively.

The map of Tor-groups is induced by a map of Koszul resolutions; this map of resolutions is formed by tensoring together the maps

$$\begin{array}{ccc} L & \xrightarrow{X_i} & L \\ \downarrow & & \downarrow \\ L_A & \xrightarrow[Y_i]{} & L_A. \end{array}$$

The right vertical map is the natural inclusion. In order to make the diagram commute, the left vertical map must be the inclusion if $i \neq p^m - 1$, multiplication by p if $i = p^m - 1$, $i \neq q^k - 1$ for any k, and multiplication by π if $i = q^k - 1$, by equation (2).

We then carry forward the computation on Tor. We find that the image of τ_{kf} is σ_{kf} if the extension is unramified. In any other case, τ_k maps to zero.

However, there can be no such map $\mathcal{A}_* \to \mathcal{B}_*$ that respects the E_∞-structure unless the field extension is trivial. The reason is as follows.

The element ξ_f in $\mathcal{A}_*$ would have nonzero image in $\mathcal{B}_*$, and hence so does its conjugate $\chi\xi_f$. The field extension is nontrivial, so either $f > 1$ (implying that $\mathcal{B}_*$ is trivial in dimensions 2 through $2q - 3$), or the extension is totally ramified and τ_0 maps to 0. In either case, $\chi\tau_{f-1}$ maps to 0.

If $p = 2$, this is immediately a contradiction, as $\chi\tau_{f-1}$ lifts to a class in $\mathcal{A}_*$ that squares to a nonzero multiple of $\chi\xi_f$.

If $p > 2$, the map $\mathcal{A}_* \to \mathcal{B}_*$ is a map of E_∞-algebras over $\mathrm{H}\mathbb{F}_p$, and so it should respect the Dyer-Lashof operations. However, $\beta Q^{p^k} \chi\tau_{f-1} = \chi\xi_f$ in $\mathcal{A}_*$ ([1], Theorem III.2.3). The element $\chi\tau_{f-1}$ has trivial image and $\chi\xi_f$ has nontrivial image, which gives a contradiction. □

Remark 4.2. For clarity, we have chosen to assume that the object S is an E_∞-ring spectrum; it is possible to weaken this assumption. There is a folklore result (for which the author does not know of a reference in the literature) to the effect that $\chi\xi_f = \beta Q^{p^k}(\chi\tau_{f-1})$ can be identified with the p-fold Massey product $\langle \chi\tau_{f-1}, \ldots, \chi\tau_{f-1} \rangle$. The fact that the map $\mathcal{A}_* \to \mathcal{B}_*$ gives rise to a contradiction only depends on the existence of p-fold Massey products, and these are well-defined for A_p-algebras. We could therefore restrict our assumptions to S being an A_p-algebra over $R \wedge R^{op}$.

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