Realization algorithms for systems over a principal ideal domain

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Abstract. In this paper realization algorithms for systems over a principal ideal domain are described. This is done using the Smith form or a modified Hermite form for matrices over a principal ideal domain. It is shown that Ho's algorithm and an algorithm due to Zeiger can be generalized to the ring case. Also a recursive realization algorithm, including some results concerning the partial realization problem, is presented. Applications to systems over the integers, delay differential systems and 2-D systems are discussed.

1. *INTRODUCTION*

The input-output behavior of a strictly causal linear time invariant system can be characterized by its impulse response sequence or Markov sequence $\mathcal{M} = (M_1, M_2, \ldots)$. Given the Markov sequence the input-output behavior of the system is given by

$$(1.1) \qquad y_k = \sum_{i=0}^{k-1} M_{k-i} u_i \qquad (k = 1,2,\ldots)$$

where $(u_0, u_1, \ldots)$ is the input sequence at $(y_1, y_2, \ldots)$ the output. Realization theory is concerned with the problem of finding matrices C,A,B to a given Markov sequence $\mathcal{M}$ such that the impulse response of the system in state space from

$$(1.2) \qquad \begin{aligned} x_{k+1} &= A\, x_k + Bu_k \\ y_k &= C\, x_k \end{aligned}$$

equals $\mathcal{M}$. We denote the system (1.2) simply by $\Sigma = (C,A,B)$. Thus, Σ is a realization of $\mathcal{M}$ if

$$(1.3) \qquad M_i = C\, A^{i-1}\, B \qquad (i = 1,2,\ldots)$$

and a realization algorithm constructs such a system Σ to given $\mathcal{M}$. Usually, one is particularly interested in so called canonical (or minimal) realizations (for a definition, see section 2).

If the entries of the M_i's and the matrices C,A,B are real numbers, we say that $\mathcal{M}$ is a Markov sequence over $\mathbb{R}$ and Σ is a system over $\mathbb{R}$. Realization algorithms for systems over $\mathbb{R}$ have been given by a number of authors ([3], [15],[11]).

Most of of these algorithms can be extended without any change to systems over an arbitrary field. It has been observed in [9], [7] that delay systems can be modeled as systems over the ring $\mathbb{R}[d]$ of polynomials, i.e., systems (C,A,B) in which the entries of the matrices are polynomials. Similarly, the theory of 2-D systems can be formulated in terms of systems over the ring of proper rational functions or over the ring of stable proper rational functions (see [1], [2],[17]). Therefore, it is useful to have a generalization of realization theory to systems over rings. A basis for such a theory is laid in [12], [13]. For very readable surveys of this theory we refer to [16], [8].

This paper will be concerned with explicit algorithms for the construction of canonical realizations of Markov sequences over a principal ideal

domain. In [12], Silverman's algorithm is used to compute a realization of a Markov sequence over a principal ideal domain. The realization is obtained by first computing a realization over the quotient field of the domain and then applying a suitable state space transformation.
In section 2 a more direct realization algorithm is proposed, which is related to an algorithm due to Zeiger (cf.[6]).
It is also shown that the original Zeiger algorithm and the Ho algorithm can be extended to systems over a principal ideal domain, but the algorithm described in this paper seems to be more appealing.
In section 3, a recursive algorithm similar to Rissanen's algorithm (see [1]) is described which to some extent can also be used for obtaining partial realizations.
In a final section some examples are given of application of the algorithm described in section 2.

2. *THE REALIZATION ALGORITHM*

In this paper, R denotes a principal ideal domain, with quotient field $\mathcal{Q}(R)$, unless otherwise stated. The set of $m \times n$ matrices over R will be denoted by $R^{m\times n}$. The <u>rank</u> of a matrix A will be its rank as a matrix over $\mathcal{Q}(R)$. A matrix $A \in R^{m\times n}$ will be called <u>right regular</u> if there does not exist a nonzero vector $x \in R^n$ satisfying $Ax = 0$. Equivalently A is right regular if rank $A = n$. The matrix A is called right invertible if there exists $A^+ \in R^{n\times m}$ such that $AA^+ = I$. Left regularity and left invertibility are defined similarly.
Consider a sequence $\mathcal{M} = (M_1, M_2, \ldots)$ of matrices $M_k \in R^{m\times p}$. A system $\Sigma = (C,A,B)$, where $C \in R^{m\times n}$, $A \in R^{n\times n}$, $B \in R^{n\times p}$ is called a realization of $\mathcal{M}$ if

(2.1) $$M_k = CA^{k-1}B \qquad (k = 1,2,\ldots)$$

In this case $\mathcal{M}$ is called the <u>Markov sequence</u> of Σ. The number n is called the <u>dimension</u> of the realization.
Given a system $\Sigma = (C,A,B)$ we define for $k = 1,2,\ldots$

(2.2) $$Q(\Sigma,k) := [B, AB, \ldots, A^{k-1}B]$$

(2.3) $$P(\Sigma,k) := [C', A'C', \ldots, (A')^{k-1}C']'$$

A system Σ is called <u>reachable</u> if $Q(\Sigma,n)$ is right invertible and <u>observable</u> if $P(\Sigma,n)$ is right regular. A reachable and observable realization is called canonical.

In order to construct such a canonical realization we form the infinite Hankel matrix

$$(2.4) \qquad H := \begin{bmatrix} M_1 & M_2 & M_3 & \cdots \\ M_2 & M_3 & \cdots & \\ M_3 & \cdots & & \\ \vdots & & & \end{bmatrix}$$

In addition, we consider Hankel blocks

$$(2.5) \qquad H_{\ell k} := \begin{bmatrix} M_1 & \cdots & M_k \\ \vdots & & \vdots \\ M_\ell & \cdots & M_{\ell+k-1} \end{bmatrix}$$

We define $\operatorname{rank} H = \sup\limits_{\ell,k} \operatorname{rank} H_{\ell k}$. The following result is instrumental.

(2.6) THEOREM. *Suppose that for a certain pair of integers* ℓ, k *we have* $\operatorname{rank} H_{\ell k} = \operatorname{rank} H =: n$. *If matrices* $P \in R^{\ell m \times n}$, $Q \in R^{n \times kp}$, $Q_k \in R^{n \times p}$ *satisfy*

(i) $H_{\ell,k+1} = P[Q, Q_k]$

(ii) Q *is right invertible*

(iii) P *is right regular*

then there exists a unique realization $\Sigma = (C,A,B)$ *of* M *such that* $P = P(\Sigma,\ell)$, $[Q,Q_k] = Q(\Sigma,k+1)$, *viz.*

$$(2.7) \qquad C = P_0,\ A = [Q_1,\ldots,Q_{k-1},Q_k]Q^+,\ B = Q_0$$

where P_0 *is the matrix consisting of the first* m *rows of* P, $Q_i \in R^{n \times p}$ *is defined by the block decomposition* $Q = [Q_0, Q_1, \ldots, Q_{k-1}]$ *and* Q^+ *is a right inverse of* Q.

PROOF Considering M as the Markov sequence of a system over $\mathcal{Q}(R)$, we find a canonical $\mathcal{Q}(R)$-realization $\bar{\Sigma} = (\bar{C},\bar{A},\bar{B})$ of M of dimension n. Then we have

$$(2.8) \qquad PQ = H_{\ell k} = \bar{P}\bar{Q}$$

where $\bar{P} := P(\bar{\Sigma},\ell)$, $\bar{Q} := Q(\bar{\Sigma},k)$. Let $\bar{P}^+$ be a left inverse (over $\mathcal{Q}(R)$) of $\bar{P}$ and $\bar{Q}^+$ a right inverse of $\bar{Q}$. Then we have

$$\bar{P}^+ PQ\bar{Q}^+ = I.$$

Thus, if we define $S := Q\bar{Q}^+ \in \mathcal{Q}(R)^{n\times n}$, then S is invertible and $S^{-1} = \bar{P}^+P$. The system $\Sigma = (C,A,B)$ defined by $A := S\bar{A}S^{-1}$, $B = S\bar{B}$, $C := \bar{C}S^{-1}$ is also a realization of M over $\mathcal{Q}(R)$. Equation (2.8) implies

$$Q = S\bar{Q} \quad , \quad P = \bar{P}S^{-1}$$

Hence, $P = P(\Sigma,\ell)$, $Q = Q(\Sigma,k)$. But then, we must have $C = P_0 \in R^{m\times n}$, $B = Q_0 \in R^{n\times p}$. In addition,

$$H_{\ell,k+1} = P[Q,Q_k] = P\ Q(\Sigma,k+1)$$

and consequently, $Q_k = A^kB$. It follows that

$$[Q_1,\dots,Q_k] = [AQ_0,\dots,AQ_{k-1}] = AQ$$

and hence (2.7), which implies $A \in R^{n\times n}$. That Σ is also canonical over R follows easily from (ii) and (iii) and the Cayley-Hamilton theorem. □

(2.9) REMARK. Obviously, theorem (2.6) remains valid if R is any integral domain. □

The following result states that for sufficiently large k a factorization of the form

$$H_{\ell,k+1} = P[Q,Q_k]$$

is always possible, once the factorization

$$H_{\ell k} = PQ$$

is given.

(2.10) THEOREM. *Let* $P \in R^{\ell m\times n}$, $Q \in R^{n\times kp}$ *satisfy the conditions* (ii) *and* (iii) *of theorem* (2.6) *and assume that* rank $H_{\ell k}$ = rank $H \le k$. *If*

$$H_{\ell k} = PQ$$

then there exists a unique $Q_k \in R^{m\times n}$ *such that*

$$H_{\ell,k+1} = P[Q,Q_k]$$

PROOF. There exists a realization of dimension $\le k$. (see [13]). By the Cayley-Hamilton theorem the sequence M satisfies a recurrence relation of the form

$$M_{k+j} = \alpha_1 M_j + \ldots + \alpha_k M_{j+k-1} \quad (j = 1,2,\ldots)$$

If we write $W := [\alpha_1 I,\ldots,\alpha_k I]' \in R^{kp\times p}$, then it follows that

$$H_{\ell,k+1} = [H_{\ell k}, H_{\ell k}W] = P[Q,QW]$$

Hence we may choose $Q_k = QW$. The uniqueness of Q_k follows from the right regularity of P. □

(2.11) REMARK. Also this result is valid for more general rings than principal ideal domains. Obviously, it suffices that the Markov parameters satisfy a recurrence relation of order $\le k$. This is for example the case for integrally closed rings (see [13]). □

Now the question arises of how to compute a factorization of $H_{\ell,k+1}$. One way of doing this depends on the Smith canonical form. We start by factorizing $H_{\ell k}$ as follows. There exist invertible matrices U and V and an $n \times n$ diagonal matrix D such that (see [10])

$$H_{\ell k} = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V \tag{2.12}$$

(Some of the zero matrices in (2.15) may be empty). The matrix D is regular (i.e. right and left regular). If we define

$$P := U \begin{bmatrix} D \\ 0 \end{bmatrix}, \quad Q := [I,0]V, \quad Q^+ := V^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix}$$

we see that P is right regular and $QQ^+ = I$, so that Q is right invertible.

In addition $H_{\ell k} = PQ$. Now if we decompose $H_{\ell,k+1}$ as

$$H_{\ell,k+1} = [H_{\ell k}, S],$$

it follows from theorem (2.10) that there exists a matrix Q_k such that $S = PQ_k$, hence

$$U^{-1}S = \begin{bmatrix} DQ_k \\ 0 \end{bmatrix}$$

i.e., the first n rows of $U^{-1}S$ are divisible by the corresponding diagonal element of D, and the remaining rows are zero. Thus, we are able to determine Q_k.

(2.13) REMARK. If these conditions on S are not satisfied, this implies that M does not have a realization of dimension less than k+1. □

The computation of a Smith form might be rather elaborate. Therefore, it is useful to point out that it is also possible to compute a realization of M using a slight modification of the Hermite form of $H_{\ell k}$.

(2.14) THEOREM. *If* H *is a* $p \times q$ *matrix over* R *of rank* n, *there exists a* $p \times p$ *permutation matrix* Π, *a* $q \times q$ *invertible matrix* V *and a* $p \times n$ *matrix* F *satisfying*

$$F_{ij} = 0 \quad (i < j) \quad , \quad F_{ii} \neq 0$$

such that

$$H = \Pi[F,0]V$$

The proof of this result is analogous to the proof for the ordinary Hermite form (see [10]) and will be omitted.

If we apply this theorem to $H = H_{\ell k}$ we obtain

$$H_{\ell k} = \Pi[F,0]V$$

Then we define $P := \Pi F$, $Q := [I, 0]$ and we have the desired factorization. The matrix Q_k has to be determined from the equation $PQ_k = S$, i.e., $FQ_k = \Pi^{-1}S$. However, since $[I_n, 0]F$ is a regular matrix, Q_k is uniquely determined by the $n \times n$ equation:

$$[I_n, 0]FQ_k = [I_n, 0]\Pi^{-1}S$$

and this equation is easy to solve because of the triangular character of $[I_n 0]F$. It follows from theorem (2.10) that a solution exists and satisfies the equation $FQ_k = \Pi^{-1}S$, provided rank $H \leq k$.

(2.15) REMARK. The algorithm given, is closely related to Zeiger's algorithm (cf.[6]). In this algorithm for systems over a field, the factorization $H_{\ell k} = PQ$ with Q right invertible and P left invertible yields the realization $C = P_1$, $A = P^+ (\sigma H)_{\ell k} Q^+$, $B = Q_1$ where $(\sigma H)_{\ell k}$ is the Hankel block of the shifted Markov sequence and P^+ is a left inverse of P. In the case of a system over a ring this algorithm is not directly applicable since it is usually not possible to factorize $H_{\ell k}$ such that P is left invertible and Q is right invertible (see remark (2.17)). However, if one is willing to perform calculations in the quotient field $\mathcal{Q}(R)$, then one can use Zeiger's algorithm, since it follows from theorem (2.6) and (2.10) that the resulting (C,A,B) is a realization over R. □

(2.16) REMARK. The method of computing a factorization using the Smith form (2.15) is obviously related to Ho's algorithm. The proper generalization of Ho's algorithm to the ring case is the following: starting from the factorization

$$UH_{\ell k}V = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where U and V are invertible and D is a regular diagonal matrix, we construct $\Sigma = (C,A,B)$ from

$$C = [I_m, 0]H_{\ell k}V \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

$$DA = [I_n, 0]U(\sigma H)_{\ell k} V \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

$$DB = [I_n, 0]\, UH_{\ell k} \begin{bmatrix} I_p \\ 0 \end{bmatrix}.$$

Then (C,A,B) is the realization of $\mathcal{M}$ corresponding to the factorization $H_{\ell k} = PQ$, where

$$P = U^{-1}\begin{bmatrix} D \\ 0 \end{bmatrix}, \quad Q = [I,0]V^{-1}.$$

The solvability of the equations for A and B again follows from theorem (2.6) and (2.10).
The algorithm proposed in this section, in particular if the modified Hermite form is used, is simpler than the algorithms mentioned in remark (2.15) and (2.16). In the algorithm given in remark (2.15) it is necessary to do calculations in $\mathcal{Q}(R)$ and inverses of both P and Q have to be calculated. For the algorithm mentioned in remark (2.16) it is necessary to compute the Smith form which is more elaborate than the Hermite form. (It is not necessary, however, that the diagonal elements in the Smith form satisfy the usual divisibility condition).

(2.17) REMARK. A realization $\Sigma = (C,A,B)$ is called split if both (A,B) and (A',C') are reachable (see [16]). If a Markov sequence M admits a split realization Σ then every canonical realization $\bar{\Sigma}$ of M is split, since it follows from the realization isomorphism theorem that $P(\bar{\Sigma},n) = P(\Sigma,n)T$ for some invertible matrix T. Obviously the realization given in theorem (2.6) is split iff P is left invertible. Therefore, if we construct P and Q using (2.12), the realization is split iff the invariant factors of $H_{\ell k}$ are invertible. Thus we recover the result of Sontag ([16, theorem 4.8]). □

3. *A RECURSIVE REALIZATION ALGORITHM*

In practical situations, the total Markov sequence is not always immediately available. For this reason it is useful to have partial realization algorithms, where finite Markov sequences are processed and where the computational results are updated as soon as new data is available. For systems over fields such partial realization algorithms are known (see [4], [5], [11]).
However, for systems over rings the problem of finding minimal partial realization algorithms is still unsolved. To some extent, the following theorem gives a result on partial realization.
(3.1) THEOREM. *Let* $M = (M_1,\dots,M_N)$ *be a finite sequence with* $M_k \in R^{m\times p}$. *Let* k *and* ℓ *be positive integers such that* $k + \ell = N$. *Suppose that we have the factorization*

$$H_{\ell,k+1} = P[Q,Q_k] \tag{3.2}$$

where P *is right regular and* Q *is right invertible with right inverse* Q^+.
If .

(3.3) $\quad \operatorname{rank} H_{\ell+1,k} = \operatorname{rank} H_{\ell,k} =: n$

and $k \geq n$, *then there exists a unique partial realization* $\Sigma = (C,A,B)$ *satisfying* $[Q,Q_k] = Q(\Sigma,k+1)$, $P = P(\Sigma,\ell)$, *viz*

$$C = P_0,\ A = [Q_1,\ldots,Q_k]Q^+,\ B = Q_0$$

where $P_0 \in R^{m\times n}$ *consists of the first* m *rows of* P *and* $Q_i \in R^{n\times p}$ *is defined by the block decomposition* $Q = [Q_0,Q_1,\ldots,Q_{k-1}]$.

PROOF. Defining $S \in R^{\ell m\times p}$ by the decomposition $H_{\ell,k+1} = [H_{\ell k},S]$, we conclude from (3.2) that $S = H_{\ell k}W$, where $W := Q^+Q_k$. If we decompose W by $W' = [W_1',\ldots,W_k']$ where $W_i \in R^{p\times p}$, then the Markov parameters satisfy the following recurrence relation

(3.4) $\quad M_{k+j} = M_jW_1 + M_{j+1}W_2 + \ldots + M_{k+j-1}W_k$

for $j = 1,\ldots,\ell$. Now, let us define M_i for $i > N$ by this recurrence relation. Then the result will follow from Theorem 2.6 if we know that $\operatorname{rank} H = \operatorname{rank} H_{\ell k} =: n$. According to [15] it suffices to show that

(3.5) $\quad \operatorname{rank} H_{\ell+1,k+j} = n$

for $j = 1,2,\ldots$. For $j = 0$ this equality follows from (3.3). For $j \geq 0$ we have, by (3.4):

$$H_{\ell+1,k+j+1} = [H_{\ell+1,k+j},\ H_{\ell+1,k+j}\,\tilde{W}_j],$$

where $\tilde{W}_j := [0,\ldots,0,W_1,\ldots,W_k]' \in R^{(k+j)p\times p}$

This equation implies (3.5). $\square$

Let us suppose that we are given an infinite sequence $\mathcal{M} = (M_1,M_2,\ldots)$, and that we want to compute a partial realization of $(M_1,\ldots,M_N)$ where N is a given positive integer.

The algorithm is based on recursive construction of the modified Hermite form, $\Pi_{\ell k}$, $V_{\ell k}$, $T_{\ell k}$, $F_{\ell k}$ of $H_{\ell k}$, that is,

$$\Pi_{\ell k}H_{\ell k}V_{\ell k} = T_{\ell k} = [F_{\ell k},\ 0]$$

where rank $F_{\ell k} = n$. We start constructing a modified Hermite form of $H_{11} = M_1$, $(\ell = 1, k = 1)$ (see(2.14)). Thus we obtain matrices $\Pi_{11}, V_{11}, T_{11}, F_{11}$ such that

$$\Pi_{11} H_{11} V_{11} = T_{11} = [F_{11}, 0]$$

and F_{11} is right regular and lower triangular. If $M_1 = 0$ then F_{11} is the empty matrix. We proceed recursively as in case α or case β depending upon the following properties (for general ℓ, k),

$$P\colon\ n \le k,\ n + p \le km,\ V_{\ell k} = \begin{bmatrix} U_{\ell k} & W_k \\ 0 & I_p \end{bmatrix}$$

for suitable matrices $U_{\ell k}$, $W_{\ell k}$ *and* $I_p \in R^{p\times p}$

<u>Case α:</u> *Property P is satisfied*: We add a block row to $H_{\ell k}$ and write

$$\begin{bmatrix} \Pi_{\ell k} & 0 \\ 0 & I_m \end{bmatrix} H_{\ell+1,k} V_{\ell k} = \begin{bmatrix} F_{\ell k} & 0 \\ S_1 & S_2 \end{bmatrix}.$$

then, if $S_2 = 0$ we obtain a partial realization of $(M_1, \dots, M_{k+\ell})$ as follows: Define

$$P := \Pi_{\ell k}^{-1} F_{\ell k},\quad Q := [I_n, 0] U_{\ell k}^{-1},\ Q_{k-1} := Q W_{\ell k}$$

Then we write $H_{\ell k} = [H_{\ell,k-1}, S]$ and we have

$$[H_{\ell,k-1}, S] \begin{bmatrix} U_{\ell k} & W_{\ell k} \\ 0 & I_p \end{bmatrix} = P[I_n, 0, 0]$$

where $[I_n, 0, 0] \in R^{n\times(n+(km-n-p)+p)}$. It follows that

$$H_{\ell,k-1} U_{\ell k} = P[I_n, 0]$$

and hence $H_{\ell,k-1} = PQ$ and

$$H_{\ell,k-1} W_{\ell,k} + S = 0$$

and hence $S = PQ_{k-1}$. Consequently, we have the relation (3.2) with k replaced by k - 1. Also, it is clear that P is right regular and Q is right invertible.
By P we have $k \geq n$ and (3.3) follows from the equation $S_2 = 0$. Thus we may apply theorem (3.1).
If $\ell + k \geq N$, the algorithm has terminated. If not, we notice that property P is still satisfied with ℓ replaced by $\ell + 1$ and we proceed with case α. If $S_2 \neq 0$ we determine the Hermite form of S_2 and therewith the Hermite form of $H_{\ell+1,k}$. Then we check again whether P is satisfied (with ℓ replaced by $\ell + 1$).

<u>Case β</u> *Property P is not satisfied.* We add a block column to $H_{\ell k}$ and write

$$\Pi_{\ell k}\, H_{\ell,k+1} \begin{bmatrix} V_{\ell k} & 0 \\ 0 & I_p \end{bmatrix} = [F_{\ell k}, 0, S]$$

We try to find a matrix W such that

$$\Pi\, H_{\ell,k+1} \begin{bmatrix} V_{\ell k} & W \\ 0 & I_p \end{bmatrix} = [F_{\ell k}, 0, 0]$$

The existence of such a W can be investigated by performing column operations on the matrix $[F_{\ell k}, 0, S]$. Due to the special form of $F_{\ell k}$, this investigation is very simple and explicit conditions for the existence of W can be given:

(1) The i^{th} row of S is divisible by $(F_{\ell k})_{ii}$.

(2) If the appropriate multiple of the i^{th} column is subtracted from the columns of S (so as to make the i^{th} row zero) for $i = 1,\ldots,n$, then the resulting columns have to be zero.

If we are able to construct W, then we check whether $k \geq n$. If so, we are in case α. If not, or if W does not exist we are again in case β. In the latter case, we of course have to update the Hermite form.
We show that the procedure terminates, provided H has finite rank. First we note that for a fixed value of ℓ, we cannot have infinitely often that case β holds. For k increases at every step and we must have $k \geq n$ after a number of steps, because $n \leq \ell m$. Also, condition (1) of case β cannot be violated infinitely often, since at every step the ideal in R generated by $(F_{\ell k})_{ii}$ will strictly increase unless

condition (1) is satisfied. Furthermore, condition (2) will certainly be satisfied if n=rank H and every time (2) is not satisfied, n will increase. Similarly, in case α, $S_2 = 0$ will hold if n = rank H and otherwise n will increase. This shows the finiteness of the algorithm. The algorithm given here is not a true algorithm for partial realization, since one needs a infinite sequence of Markov parameters in order to complete the algorithm. Of course, one can always extend a finite sequence such that the resulting sequence has a Hankel matrix of finite rank. However, it is not at all obvious how to extend the Markov sequence in such a way that the rank will be minimal (compare [12, 3A]).

4. *APPLICATIONS AND EXAMPLES*

A. If R is a field, then Theorem (2.6) yields a slight modification of Zeiger's algorithm. The modification seems to be computationally attractive, since for the realization only a right inverse of Q is needed. It is not necessary to compute a left inverse of P.

B. In [12,2C], an example of a Markov sequence over $R = \mathbb{Z}$ is given:

$$M_1 = \begin{bmatrix} 2 & -2 \\ 2 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \quad M_3 = M_4 = \ldots = 0 .$$

Let us compute a realization for this sequence. It is easily seen that rank H_{22} = rank H = 2. We compute the Hermite decomposition of H_{22}:

$$H_{22} = \begin{bmatrix} 2 & -2 & 2 & 2 \\ 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} .$$

Hence, we obtain

$$Q = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & -1 \end{bmatrix}, \quad Q^+ = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix} .$$

The matrix Q_2 is determined from the equation $PQ_2 = S := [M_3', M_4']' = 0$. Hence $Q_2 = 0$. Consequently, we find the following realization:

$$C = P_0 = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}, \; A = Q_1, Q_2\, Q^+ = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \; B = Q_0 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

C. As has been pointed out in [9], [7], delay-differential systems can be modeled as systems over the ring $R = \mathbb{R}[d]$. For instance, if we introduce the delay operator d by $dy(t) = y(t-1)$ in the system of equations (see [7, section 7]).

$$\text{(4.1)} \qquad \begin{aligned} y_1''(t) + y_1'(t-1) &= 2u_1'(t-2) - 6u_2(t) \\ y_2''(t) + y_2'(t-1) &= -2u_1(t-3) - 2u_2(t) + 4u_2(t-1) \end{aligned}$$

we obtain $y = Wu$, where

$$W = \frac{1}{s^2 + ds} \begin{bmatrix} 2d^2 s & -6 \\ -2d^3 s & -2s + 4d \end{bmatrix}$$

and s denotes the differentiation operation: $sy = y'$. (We assume zero initial conditions.) We want to obtain a representation of the equations (4.1) in the form

$$\text{(4.2)} \qquad \begin{aligned} \dot{x}(t) &= A(d)x + B(d)u , \\ y(t) &= C(d)x . \end{aligned}$$

To this end, we consider W a rational matrix over $\mathbb{R}(d)$ and we expand in powers of s^{-1}:

$$W = M_1(d)s^{-1} + M_2(d)s^{-2} + \dots .$$

Then the matrices A,B,C in (4.2) have to satisfy $CA^kB = M_{k+1}$ $(k = 0,1,\dots)$, i.e. (C,A,B) has to be a realization of the Markov sequence $(M_1,M_2,\dots)$. In this particular example we have

$$M_1 = \begin{bmatrix} 2d^2 & 0 \\ -2d^3 & 2 \end{bmatrix}, \; M_2 = \begin{bmatrix} -2d^3 & -6 \\ 2d^4 & 6d \end{bmatrix}, \; M_3 = \begin{bmatrix} 2d^4 & 6d \\ -2d^5 & -6d^2 \end{bmatrix}.$$

We compute a Hermite form of H_{22}:

$$\begin{bmatrix} 2d^2 & 0 & -2d^3 & -6 \\ -2d^3 & -2 & 2d^4 & 6d \\ -2d^3 & -6 & 2d^4 & 6d \\ 2d^4 & 6d & -2d^5 & -6d^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ d & -2 & 0 & 0 \\ d & -6 & 0 & 0 \\ -d^2 & 6d & 0 & 0 \end{bmatrix} \begin{bmatrix} -2d^2 & 0 & 2d^3 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that

$$Q = \begin{bmatrix} -2d^2 & 0 & 2d^3 & 6 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad Q^+ = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1/6 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & 0 \\ d & -2 \\ d & -6 \\ -d^2 & 6d \end{bmatrix}.$$

The matrix Q_2 is easily obtained from $PQ_2 = S := [M_3', M_4']'$ which yields

$$Q_2 = \begin{bmatrix} -2d^4 & -6d \\ 0 & 0 \end{bmatrix}.$$

Notice, that it is not necessary to know M_4 explicitly, since Q_2 is uniquely determined by the equation $P_0Q_2 = M_3$.
Thus we find the following realization:

$$C = \begin{bmatrix} -1 & 0 \\ d & -2 \end{bmatrix}, \quad A = \begin{bmatrix} -d & 6 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -2d^2 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the equations (4.2) we obtain:

$$\dot{x}_1(t) = -x_1(t-1) + 6x_2(t) - 2u_1(t-2) ,$$

$$\dot{x}_2(t) = u_2(t) ,$$

$$y_1(t) = -x_1(t) ,$$

$$y_2(t) = x_1(t-1) - 2x_2(t) .$$

Notice that P is actually left invertible, because its diagonal elements are invertible. It follows that we have a split realization.

D. Also 2-D systems can be modeled using systems over the principal ideal domain R of proper rational functions (see [1], [2], [17]). The realization algorithm given in this paper enables us to obtain a first level realization (see [1]), which can be described by the following equations

$$x_{k+1}(s) = A(s)x_k(s) + B(s)u_k(s)$$

$$y_k(s) = C(s)x_k(s) + D(s)u_k(s)$$

where $A(s)$, $B(s)$, $C(s)$, $D(s)$ are matrices over R. For stable 2-D systems it is more appropriate to work with the principal ideal domain

$$R_\sigma := \{r(s) \in \mathbb{R}(s) \mid r(s) \text{ is proper and has no poles for } |s| \geq 1\}.$$

For a proof that R_σ is a principal ideal domain see [16].

A direct computation of Smith or Hermite forms over R_σ seems to be rather complicated. The problem can be simplified considerably, however, by applying the following ring isomorphism to R_σ: For $r \in R_\sigma$ we define $\bar{r}(s)$ by

$$\bar{r}(s) := r(1/s) \ .$$

The set $\bar{R}_\sigma$ of rational functions, thus obtained, is characterized by

$$\bar{R}_\sigma = \{\bar{r}(s) \in \mathbb{R}(s) \mid \bar{r}(s) \text{ has no pole for } |s| \leq 1\} \ .$$

This set is also a principal ideal domain (see [16]). Now let $H(s)$ be a matrix over R_σ of which we want to compute the R_σ-Smith form. Let $\bar{H}(s)$ be the matrix obtained by applying the maps $r(s) \to \bar{r}(s)$ to each entry of $H(s)$. Then $\bar{H}(s)$ is a matrix over $\bar{R}_\sigma$. Let $h(s)$ be the least common multiple of the denominators of the entries of $\bar{H}(s)$ and let $\tilde{H}(s) := h(s)\bar{H}(s)$. Then $\tilde{H}(s)$ is a polynomial matrix. Using the standard procedure for computing Smith forms over $\mathbb{R}[s]$ we compute unimodular matrices $\tilde{U}(s)$ and $\tilde{V}(s)$ such that

$$\tilde{H}(s) = \tilde{U}(s)\tilde{D}(s)\tilde{V}(s) \ .$$

If we define $\bar{D}(s) := \tilde{D}(s)/h(s)$, then

$$\bar{H}(s) = \tilde{U}(s)\bar{D}(s)\tilde{V}(s)$$

is the Smith form decomposition of $\bar{H}(s)$ over $\bar{R}_\sigma$. Note that $\mathbb{R}[s] \subseteq \bar{R}_\sigma$, so that the unimodular polynomial matrices $\tilde{U}$ and $\tilde{V}$ are also invertible matrices over $\bar{R}_\sigma$. (Actually, this formula is the MacMillan form decomposition of the rational matrix $\bar{H}(s)$ over $\mathbb{R}[s]$. The fact that $\bar{H}(s)$ is a matrix over $\bar{R}_\sigma$ implies that $\bar{D}(s)$ is a matrix over $\bar{R}_\sigma$.)
Finally we replace s by $1/s$, i.e., we define

$$U(s) := \tilde{U}(1/s) \ , \ D(s) := \bar{D}(1/s) \ , \ V(s) := \tilde{V}(1/s)$$

and we obtain the Smith form over R_σ:

$$H(s) = U(s)D(s)V(s) \ .$$

REMARK. Notice that the matrices U and V only have poles and zeroes at $s = 0$.□

Completely similarly, one can reduce the computation of the Hermite form over R_σ to the computation over $\mathbb{R}[s]$.

REMARK. In the above applications all the rings under consideration are in fact Euclidean domains (see [14]). This fact can be exploited in the calculations for the Smith form or the modified Hermite form (see [10]).

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