Realization and stabilization of 2-D systems

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by

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0. General introduction

During recent years several state space models concerning discrete 2-D systems (systems with two time parameters) have appeared in the litterature. These are used for example in image processing. To these models are attached the names of Attasi [1], Fornasini-Marchesini [2], Givone-Roesser [3], the first two models being special cases of the third. This is shown in [4].
In this paper it is shown that all these models are special cases of a new model which is a straightforward generalization of the 1-D case.

1. Problem introduction

A 2-D I/O (Input/Output) system is characterised by the following convolution equation.

(1.1) $$y_{kh} = \sum_{i,j=0}^{\infty} F_{k-i,h-j}\, u_{ij} \qquad \begin{array}{l} k = 0,1,\ldots \\ h = 0,1,\ldots \end{array}$$

where $y_{kh} \in \mathbb{R}^{p}$, $u_{ij} \in \mathbb{R}^{m}$, $F_{ln} \in \mathbb{R}^{p\times m}$

for $p \in \mathbb{N}$, $m \in \mathbb{N}$ fixed.

If $F_{ln} = 0$ when $l < 0$ or $n < 0$, the system is said to be causal.

Next we introduce some notation.

(1.2) $\mathbb{R}[s,z]$ denotes the set of polynomials in the variables s and z with real coëfficients.

(1.3) $\mathbb{R}^{p\times m}[s,z]$ denotes the set of $p\times m$ matrices with entries in $\mathbb{R}[s,z]$.

(1.4) $\mathbb{R}(s,z)$ denotes the set of rational functions in s and z.

(1.5) $\mathbb{R}^{p\times m}(s,z)$ denotes the set of $p\times m$ matrices with entries in $\mathbb{R}(s,z)$. The elements of $\mathbb{R}[s,z]$ can also be considered as polynomials in z with coëfficients in $\mathbb{R}[s]$, thus $\mathbb{R}[s,z] = \mathbb{R}[s][z]$.

Analogously, $\mathbb{R}^{p\times m}[s,z] = \mathbb{R}[s]^{p\times m}[z]$.
A polynomial $q \in R[s,z]$ seen as an element of $\mathbb{R}[s][z]$ will be notated as $\bar{q}$.

Analogously for P and $\bar{P}$ where $P \in \mathbb{R}^{p\times m}[s,z]$ and $\bar{P} \in \mathbb{R}^{p\times m}[s][z]$.

Let $T \in \mathbb{R}^{p\times m}(s,z)$, T can be written in the form $P/_{q} = \bar{P}/_{\bar{q}}$ where P, $\bar{P}$, q, $\bar{q}$ are as above.

Definition

(1.6) $T \in \mathbb{R}^{p\times m}(s,z)$ is called proper if for $T = \bar{P}/_{\bar{q}}$

1° degree of $\bar{q}(z)$ is not less than the degree of $\bar{P}(z)$
2° degree of the coëfficient of the highest power in z in $\bar{q}(z)$ is not less than the degree of all other coëfficients of $\bar{q}(z)$ and the entries of $\bar{P}(z)$.

(1.7) $T \in \mathbb{R}^{p\times m}(s,z)$ is strictly proper if "not less" is replaced by "greater" in the above definition.

Let $q(s,z) \in \mathbb{R}[s,z]$, suppose the degree of $q(s,z)$ in s is m and the degree of $q(s,z)$ in z is n.
Then for q to be the denominator of a proper T it is necessary and sufficient that the coëfficient of the monomial $z^n s^m$ is not equal to zero.
This coëfficient can w.l.o.g. taken to be unity.

<u>Examples</u> $\frac{1}{z+s}$ and $\frac{zs + s^2}{z^2 s + s}$ are not proper.

$$\frac{z + 3s}{z^2 s + zs + 1} \text{ is proper.}$$

Consider the formal power series representation of $T(s,z)$

(1.8) $T(s,z) = \sum_{k,h=0}^{\infty} L_{kh} z^{-k} s^{-h}$

Now define an I/O system by taking $F_{kh} = L_{kh}$ in (1.1) for all k,h.

Then we have:

(1.9) $T(s,z)$ is proper iff the associated I/O system is causal.

In the next we are going to construct a state space realization of a proper $T(s,z)$, which is an undefined object up to now.
For that purpose we need some theorems on linear systems over a commutative ring, as can be found in [5], [6], [7], [8].

2. Linear systems over commutative rings

Let R be a commutative ring.

Definition

(2.1) A system Σ is (A,B,C,D) where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$ for some integers n,m,p. n is called the rank of the system.
If $m = p = 1$ we call the system scalar.
We will use an interpretation in terms of discrete-time dynamics.

$$x_{k+1} = Ax_k + Bu_k \qquad x_0 = 0$$

$$y_k = Cx_k + Du_k \qquad k = 0,1,2,\ldots$$

Usually $x_k \in R^n$ will be called the state, $u_k \in R^m$ is called the input and $y_k \in R^p$ is called the output.
The I/O map f_Σ: $(u_0,u_1,\ldots) \to (y_0,y_1,\ldots)$

is completely determined by $(F_0,F_1,F_2,\ldots)$ where

(2.2) $F_0 = D$, $F_i = CA^{i-1}B \quad i = 1,2,\ldots$ see also (1.1)
In fact every I/O map (linear, shift invariant and causal defined in the usual way) is given by a sequence $(F_0,F_1,F_2,\ldots)$
Now let there be given an I/O map f_Σ characterised by $(F_0,F_1,F_2,\ldots)$.
We say that the system (A,B,C,D) realizes f_Σ if (2.2) holds.

Because the Cayley-Hamilton theorem is valid over a commutative ring we have:

Theorem [9] ch 10.11.

(2.3) An I/O map f_Σ is realizable iff it is recurrent. Where recurrency of $(F_0,F_1,F_2,\ldots)$ is defined as:

$$F_{n+k} = \sum_{i=1}^{n-1} \alpha_i F_{i+k} \text{ for all } k \geq 0.$$

where $\alpha_i \in R \quad i = 1,\ldots,n-1$
and some integer n.

The formal power series associated with $(F_0,F_1\ldots)$ is defined by:

$$W(z) = \sum_{i=0}^{\infty} F_i z^{-i}$$

Theorem [5]

(2.4) $(F_0,F_1,F_2\ldots)$ is realizable iff the associated formal power series $W(z)$ is rational.

In the case R is field another necessary and sufficient condition is:

$(F_0,F_1,F_2\ldots)$ is realizable iff the Hankel matrix

(2.5)
$$\begin{bmatrix} F_1 & F_2 & F_3 & \cdots \\ F_2 & F_3 & F_4 & \cdots \\ F_3 & F_4 & F_5 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix} \quad \text{has finite rank}$$

The smallest integer n such that all minors of order greater than n are zero will be called the rank of the Hankel matrix.

The definitions of reachability and observability are the same as in the case where R is a field.

We have:

(2.6) A realization is reachable iff the columns of B, AB, ..., $A^{n-1}B$ span R^n.

(2.7) A realization is observable iff $CA^i x = 0 \quad i = 0,1,\ldots,n-1$ implies $x=0$

Definition

(2.8) A realization is minimal if n is minimal.

Contrary to the case where R is field we have if R is a ring:

Minimality does not imply reachability and observability [5].

However if R is a principal ideal domain (P.I.D) we have:

Theorem [6]

(2.9) If the Hankel-matrix associated with an I/O map characterised by the sequence $(F_0,F_1,F_2\ldots)$ has finite rank n then there exists a reachable and observable realization which has itself rank n.

This theorem is proved by introducing the quotient field K of R and then proving that there is a minimal realization over K which is in fact a realization over the P.I.D. R.

The ring which will be of central importance here is the ring of proper rational functions in one variable s.

(2.10)
$$R_g = \{\frac{a(s)}{b(s)} \mid \text{degree } b \geq \text{degree } a\}$$

This ring is actually a P.I.D. as can easily be proved [5].

3. The realization procedure

Let $T(s,z) \in \mathbb{R}^{p\times m}(s,z)$ and $T = \bar{P}/\bar{q}$ where $\bar{P} \in \mathbb{R}^{p\times m}[s][z]$ and $\bar{q} \in \mathbb{R}[s][z]$.

Suppose T is proper and let $W(z) = \sum_{i=0}^{\infty} F_i z^{-i}$ be its associated formal power series where F_i are matrices whose entries are proper rational functions in s.

To obtain a minimal realization of W(z) we apply theorem (2.9) to the I/O map f_Σ characterised by $(F_0,F_1,\ldots)$, the P.I.D. being R_g which gives us matrices:

$$D(s),\ C(s),\ A(s),\ B(s)$$

all of whose entries are elements of R_g, with dimensions $p \times m$, $p \times n$, $n \times n$, $n \times m$.
We have:

(3.1) $$T(s,z) = D(s) + C(s)[zI - A(s)]^{-1}\ B(s)$$

The dynamical interpretation is given by the following equations:

(3.2) $$\bar{x}_{k+1}(s) = A(s)\ \bar{x}_k(s) + B(s)\ \bar{u}_k(s) \qquad \bar{x}_0(s) = 0$$

$$\bar{y}_k(s) = C(s)\ \bar{x}_k(s) + D(s)\ \bar{u}_k(s) \text{ with appropriate dimensions.}$$

where $\bar{x}_k(s)$ is a formal power series for each $k = 0,1,\ldots$

$$\bar{x}_k(s) = \sum_{i=0}^{\infty} x_{k\,i}\, s^{-i} \text{ analogously for } \bar{u}_k(s) \text{ and } \bar{y}_k(s).$$

This minimal realization is called the <u>first level realization</u> of T(s,z).

Observe that we do not require $\bar{x}_k(s)$, $\bar{u}_k(s)$, $\bar{y}_k(s)$ to be rational. The product $A(s)\ \bar{x}_k(s)$ is well defined because rational functions are also formal power series with the usual definition of product.

The matrices D(s),C(s), A(s), B(s) are uniquely determined up to isomorphism [5].

The realization $(\bar{D}(s), \bar{C}(s), \bar{A}(s), \bar{B}(s))$ is isomorphic to $(D(s),C(s),A(s),B(s))$ if there exists an invertible matrix S(s), S(s) and $S^{-1}(s)$ both having entries in the P.I.D. R_g, such that

(3.3) $$\bar{D}(s) = D(s),\ \bar{C}(s) = C(s)\ S^{-1}(s)$$
$$\bar{A}(s) = S(s)\ A(s)\ S^{-1}(s),\ \bar{B}(s) = S(s)B(s).$$

The matrices D(s), C(s), A(s), B(s) can be seen as 1-D transfer matrices themselves.
Realizing each of them we obtain realizations

(3.4)
$$\begin{array}{llll} DD & DC & DA & DB \text{ for } D(s) \\ CD & CC & CA & CB \text{ for } C(s) \\ AD & AC & AA & AB \text{ for } A(s) \\ BD & BC & BA & BB \text{ for } B(s) \end{array}$$

(all of them are single matrices, not products)

who constitute minimal realizations such that:

$$A(s) = AD + AC[sI - AA]^{-1}\ AB$$

and analogously for D(s), C(s) and B(s). The matrix S(s) (3.2) can of course be given an analogous dynamical interpretation.

(3.4) will be called the <u>second level realization</u> of T(s,z).

The interpretation of the second level realization is the following:

$$\left|\begin{array}{l} b_{k,h+1} = BAb_{kh} + BBu_{kh},\ d_{k,h+1} = DAd_{kh} + DBu_{kh}. \\ \begin{bmatrix} x_{k+1,h} \\ a_{k,h+1} \end{bmatrix} = \begin{bmatrix} AD & AC \\ AB & AA \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + BCb_{kh} + BDu_{kh} \\ c_{k,h+1} = CA\ c_{kh} + CBx_{kh} \\ y_{kh} = CDx_{kh} + CCc_{kh} + DCd_{kh} + DDu_{kh} \end{array}\right.$$

where the vectors have suitable dimensions and all initial conditions are equal to zero.
In (3.5) x_{kh}, d_{kh}, c_{kh}, a_{kh}, b_{kh} are local state variables
Furthermore we have:

$$\bar{x}_k(s) = \sum_{h=0}^{\infty} x_{kh} s^{-h}$$

$$\bar{u}_k(s) = \sum_{h=0}^{\infty} u_{kh} s^{-h}$$

$$\bar{y}_k(s) = \sum_{h=0}^{\infty} y_{kh} s^{-h}$$

see(3.2)

A flow diagram for (3.2) and (3.5) revealing the first and second level realization is:

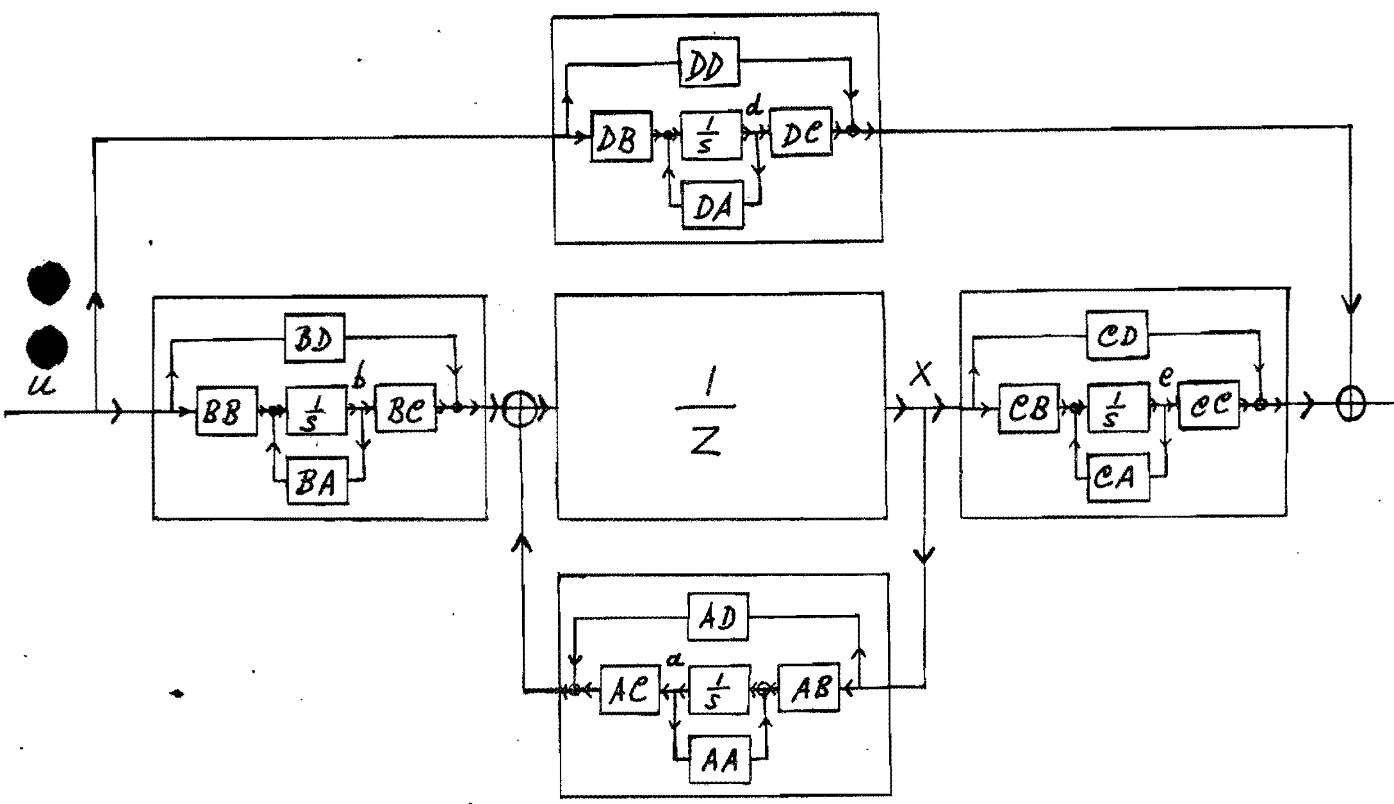

fig. 1.

We will show that the models of [1],[2],[3] are special cases of the above constructed model.
To prove this it is enough to show that the model of [3] is a special case of our model since the models in [1] and[2] are special cases of the model of [3], compare [4].

With notation as in [3] the model considered there is:

$$\begin{bmatrix} R_{k+1,h} \\ S_{k,h+1} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} R_{kh} \\ S_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh}$$

$$y_{kh} = C_1 R_{kh} + C_2 S_{kh}$$

We now have:

Theorem

(3.7) The model in (3.6) can be written in the form (3.2) and (3.5).
The corresponding matrices are:

$$D(s) = C_2[sI - A_4]^{-1} B_2 \quad , \quad C(s) = C_2[sI - A_4]^{-1} A_3 + C_1$$

$$A(s) = A_1 + A_2[sI - A_4]^{-1} A_3 \, , \; B(s) = A_2[sI - A_4]^{-1} B_2 + B_1$$

and

$DD = 0$	$DC = C_2$	$DA = A_4$	$DB = B_2$
$CD = C_1$	$CC = C_2$	$CA = A_4$	$CB = A_3$
$AD = A_1$	$AC = A_2$	$AA = A_4$	$AB = A_3$
$BD = B_1$	$BC = A_2$	$BA = A_4$	$BB = B_2$

proof: Introducing formal power series in two variables z and s (Or Z-transform in two variables)

$$y(s,z) = \sum_{k,h=0}^{\infty} y_{kh}\, z^{-k} s^{-h}$$ and assuming zero initial conditions

we obtain:

$$y(s,z) = [C_1, C_2] \begin{bmatrix} zI - A_1 & A_2 \\ -A_3 & sI-A_4 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(s,z)$$

Now: $$\begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -A_3 & sI-A_4 \end{bmatrix}^{-1} \begin{bmatrix} zI-A_1-A_2[sI-A_4]^{-1}A_3 & -A_2[sI-A_4]^{-1} \\ 0 & I \end{bmatrix}^{-1}$$

and then by calculating both inverses in the right-hand side we obtain:

$y(s,z) = T(s,z)\, u(s,z)$ where

$$T(s,z) = C_2[sI-A_4]^{-1} B_2 + [C_1+C_2[sI-A_4]^{-1}A_3][zI-A_1-A_2[sI-A_4]^{-1}A_3]^{-1} \cdot$$

$$\cdot\, [B_1+A_2[sI-A_4]^{-1}B_2],$$ proving the theorem.

Starting with a transfer matrix our procedure will give dynamical equations of relatively small order.
The procedure of [4] for scalar transfer functions to obtain a Givone-Roesser model will usually result in large matrices.
Our second level realization gives more matrices but they are "smaller".

This can be shown as follows:

Writing (3.5) in Givone-Roesser form the corresponding matrices and vectors are:

(3.9) $$R_{kh} = x_{kh},\ A_1 = AD,\ A_2 = [AC, BC, 0, 0],\ B_1 = BD$$

$$S_{kh} = \begin{bmatrix} a_{kh} \\ b_{kh} \\ c_{kh} \\ d_{kh} \end{bmatrix},\ A_3 = \begin{bmatrix} AB \\ 0 \\ CB \\ 0 \end{bmatrix},\ A_4 = \begin{bmatrix} AA & 0 & 0 & 0 \\ 0 & BA & 0 & 0 \\ 0 & 0 & CA & 0 \\ 0 & 0 & 0 & DA \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ BB \\ 0 \\ DB \end{bmatrix}$$

$$C_1 = CD\ ,\ C_2 = [0, 0, CC, DC]$$

<u>example</u>

Consider a scalar proper transfer function

$$T(s,z) = \frac{\sum_{i=0}^{n} a_i(s)\ z^i}{\sum_{j=0}^{n} b_j(z)\ z^j}$$

properness implies that $b_n(s) \neq 0$ and that the degree of $b_n(s)$ is not less than the degree of any other coëfficient.

$$T(s,z) = \frac{\sum_{i=0}^{n} \alpha_i(s)\ z^i}{\sum_{j=0}^{n} \beta_i(s)\ z^j}$$

where

$$\alpha_i(s) = \frac{a_i(s)}{b_n(s)} \in R_g \text{ and } \beta_j(s) = \frac{b_j(s)}{b_n(s)} \in R_g$$

To simplify the example we will assume $a_n(s) = 0$.

The first level realization gives:

$$D(s) = 0 \text{ because } a_n(s) = 0,\ C(s) = [\alpha_0(s),\ \ldots,\ \alpha_{n-1}(s)]$$

$$A(s) = \begin{bmatrix} 0 & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & \cdot \\ & & & & & 1 \\ -\beta_0(s) & & & & & -\beta_{n-1}(s) \end{bmatrix} \qquad B(s) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

The second level realization gives

CD, CC, CA, CB, AD, AC, AA, AB, BD

BC = BA = BB = 0

The first level realization was very easy because of the standard controllable form of A(s) and B(s).

The resulting state space equations are

$$\begin{bmatrix} x_{k+1,h} \\ a_{k,h+1} \end{bmatrix} = \begin{bmatrix} AD & AC \\ AB & AA \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} BD \\ 0 \end{bmatrix} u_{kh}$$

$$c_{k,h+1} = CA\ c_{kh} + CB\ x_{kh}$$

$$y_{kh} = CD\ x_{kh} + CC\ c_{kh}$$

where AD is n by n, AA is m by m, CA is m by m and m is the degree of $b_n(s)$.

Two kinds of system matrices have been obtained

$$\begin{bmatrix} AD & AC \\ AB & AA \end{bmatrix} \qquad (n+m) \text{ by } (n+m)$$

representing dynamics in two directions and [CA] m by m representing dynamics in one direction.

In [4] a (n+2m) by (n+2m) system matrix is obtained for this transfer function because the authors wanted system equations in Roessers form.

It is the authors opinion that the above equations with two kinds of dynamics are more natural because they are a straightforward generalization of the 1-D case.

Stability

Let $T(s,z) \in \mathbb{R}^{p\times m}(s,z)$ be the transfer matrix of a causal I/O system (1.1).

The system (1.1) is said to be BIBO (bounded input-bounded output) stable if: $\forall\, M > 0\ \exists\, N > 0$ such that $\forall i,j\ \|u_{ij}\| \le M \Rightarrow \|y_{kh}\| \le N,\ \forall\, k,h$ where $\|\ \|$ denotes the Euclidean norm.

Theorem

4.1) The I/O system is BIBO stable iff $\sum_{k,h=0}^{\infty} \|F_{kh}\| < \infty$ for a proof see [15].

Theorem (Shanks)

4.2) The I/O system is BIBO stable if $q(s,z) \neq 0$ for $|z| \ge 1$, $|s| \ge 1$.
where $q(s,z)$ is the least common multiple of all denominators of the entries of $T(s,z)$.
For a proof see [12], this proof is for the scalar case but the matrix case is completely analogous.

Theorem (Huang)

4.3) $q(s,z) \neq 0$ for $|z| \le 1$, $|s| \le 1$ iff

1^{o} $q(s,0) \neq 0$ for $|s| \le 1$

2^{o} $q(s,z) \neq 0$ for $|z| \le 1$, $|s| = 1$

for a proof see [11], [13]

By considering $q(\frac{1}{s},\frac{1}{z})$ where $q(s,z) = \sum_{j=0}^{n} b_j(s)\, z^j$ and multiplying with appropriate powers of s and z and using Huangs theorem we have:

Theorem

4.4) $q(s,z) \neq 0$ for $|s| \ge 1$, $|z| \ge 1$ iff

1^{o} $b_n(s) \neq 0$ for $|s| \ge 1$

2^{o} $q(s,z) \neq 0$ for $|z| \ge 1$, $|s| = 1$

Therefor for BIBO stability it is necessary that $b_n(s)$ is stable ($b_n(s) \neq 0$, $|s| \ge 1$).

This motivates us to introduce a subring R_σ of R_g

$$R_\sigma = \{\frac{a(s)}{b(s)} \mid b(s) \neq 0,\ |s| \ge 1\}$$

R_σ is also a P.I.D. [5]

Before introducing stabilizability of 2-D systems we state the following:

Theorem [5]

(4.5) Let R be a P.I.D., $A \in R^{n\times n}$, $B \in R^{n\times m}$ and A,B reachable. Then for every $p_1 \dots p_n \in R$ there exists $K \in R^{m\times n}$ such that $\det[zI-A+BK] = (z-p_1)(z-p_2)\dots(z-p_n)$.

5. Feedback, pole-placement and stabilization

Consider now a proper transfermatrix $T(s,z) \in R^{p\times m}(s,z)$.

Let $T(s,z) = \bar{P}(z)/\bar{q}(z)$ and $\bar{q}(z) = \sum_{j=0}^{n} b_j(s)\, z^j$.

We will assume $b_n(s) \neq 0$ for $|s| \geq 1$

Then deviding all coëfficients of all powers of z by $b_n(s)$ $T(s,z)$ can be considered as a $p \times m$ matrix whose elements are proper rational functions in z with coefficients in R_σ.

Now let $D(s) \in R_\sigma^{p\times m}$, $C(s) \in R_\sigma^{p\times n}$, $A(s) \in R_\sigma^{n\times n}$, $B(s) \in R_\sigma^{n\times m}$ be a first level minimal realization of $T(s,z)$ with dynamics:

$$\bar{x}_{k+1}(s) = A(s)\,\bar{x}_k(s) + B(s)\,\bar{u}_k(s) \qquad \bar{x}_0(s) = 0$$

$$\bar{y}_k(s) = C(s)\,\bar{x}_k(s) + D(s)\,\bar{u}_k(s) \text{ with appropriate dimensions}$$

Choose $p_1 \ldots p_n \in R_\sigma$ such that $(z-p_1)\ldots(z-p_n)$ is stable.
By theorem (4.5) there exists $K(s) \in R^{m\ n}$ such that:

$$\det[zI-A(s)+B(s)K(s)] = (z-p_1)\ldots(z-p_n).$$

We can thus stabilize the 2-D system by a feedback law.

$$\bar{u}_k(s) = -K(s)\,\bar{x}_k(s)$$

We can even take $p_1 \ldots p_n$ to be constants which is very remarkable.

K(s) can be given a dynamical interpretation by realizing the 1-D transfer matrix K(s) as follows:

$$u_{kh} = -KD\, x_{kh} - KC\, l_{kh}$$

$$l_{k,h+1} = KA\, l_{kh} + KB\, x_{kh}$$

with appropriate dimensions and zero initial condition.

KA is stable because $K(s) \in R_\sigma^{m\times n}$.

We will now consider more closely the reachability condition which is restrictive for applying the above procedure.
First we have:

A(s), B(s) reachable

is equivalent to:

There exists $L(s) \in R^{n\cdot m\times n}$ such that:

(5.1) $[B(s), A(s)B(s), \ldots, A^{n-1}(s)B(s)]\ L(s) = I$

Theorem

(5.2) $A(s)$, $B(s)$ is reachable iff $[B(s), A(s)B(s), \ldots, A^{n-1}(s)B(s)]$ has rank n for all $|s| \geq 1$ and for $|s| \to \infty$ or equivalently:

(5.3) rank $[B(\frac{1}{s}), \ldots, A^{n-1}(\frac{1}{s})\ B(\frac{1}{s})]$ is n for all $|s| \leq 1$ after multiplying with an appropriate power of s to obtain again reational functions in s

Proof: by (5.1) necessity is obvious.
The condition is also sufficient.
First replace s by $\frac{1}{s}$ and multiply with an appropriate power of s.
Now suppose $A(s)$, $B(s)$ is not reachable.
Then we have [10] that:
The greatest common divisor of all $n \times n$ minors of (5.3) is not invertible in the ring

$$\bar{R}_\sigma = \{\frac{a(s)}{b(s)} \mid b(s) \neq 0\ |s| \leq 1\}$$

Therefor there exists s_0, $|s_0| \leq 1$ such that all $n \times n$ minors of (5.3) are zero for $s = s_0$ thus for $s = s_0$ (5.3) has not full rank.

The case $s_0 = 0$ corresponds to the case $|s| \to \infty$ in (5.2).

Not making the stability requirements, the ring of interest is therefor R_g, we have the next:

Theorem

(5.4) $A(s)$, $B(s)$ is reachable iff AD, BD is reachable see (3.4) for AD and BD.

This theorem can be proved in the same way as theorem (5.2) by using the ring

$$\tilde{R}_\sigma = \{\frac{a(s)}{b(s)} \mid b(0) \neq 0\}$$

instead of $\bar{R}_\sigma$

There is still another characterization of the reachability of $A(s)$, $B(s)$.

In [4] modal controllability is defined as follows

(5.5) $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ and $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ are modally controllable if:

$\begin{bmatrix} Z-A_1 & -A_2 \\ -A_3 & s-A_4 \end{bmatrix}$ and $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ are left coprime with respect to (w.r.t.) $\mathbb{C}[s,z]$

where left coprimeness is defined by:
Every left common factor is necessarily unimodular.

Instead of $\mathbb{R}[s,z]$ we take here $\mathbb{C}[s,z]$, the ring of polynomials in two variables with complex coëfficients, because the field of coëfficients has to be algebraically closed. See also [4] part 1.

Suppose $A(s,z) \in \mathbb{C}^{n\times n}[s,z]$, $B(s,z) \in \mathbb{C}^{n\times m}[s,z]$.
In [4] the following is proved.

Theorem

(5.6) $A(s,z)$ and $B(s,z)$ are left comprime w.r.t. $\mathbb{C}[s,z]$ iff:

1° $A(s,z)$ and $B(s,z)$ are left comprime w.r.t. $\mathbb{C}(s)[z]$

2° $A(s,z)$ and $B(s,z)$ are left comprime w.r.t. $\mathbb{C}(z)[s]$

where $\mathbb{C}(s)$ ($\mathbb{C}(z)$) is the field of rational functions in s (z) with complex coëfficients.
Next suppose (AD, AC, AA, AB) and (BD, BC, BA, BB) are realizations of A(s) and B(s).
We then have:

Theorem

(5.7) If $\begin{bmatrix} z-AD & -AC & -BC \\ -AB & S-AA & 0 \\ 0 & 0 & s-BA \end{bmatrix}$ and $\begin{bmatrix} BD \\ 0 \\ BB \end{bmatrix}$ are

left comprime w.r.t. $\mathbb{C}[s,z]$ then A(s), B(s) is reachable.

Proof Suppose that A(s), B(s) is not reachable.
Then $[z-A(s)]$ and $B(s)$ are not left comprime or equivalently:
$[z-A(s), B(s)]$ is not right invertible
Therefor there exists

$$L(s,z) \in \mathbb{R}(s)[z] \text{ s.t. } \quad z - A(s) = L(s,z)\,\tilde{A}$$
$$B(s) = L(s,z)\,\tilde{B}$$

and $L(s,z)$ is not unimodular.

We now have:

$$\begin{bmatrix} z-AD & -AC & -BC \\ -AB & s-AA & 0 \\ 0 & 0 & s-BA \end{bmatrix} = \begin{bmatrix} I & -AC & -BC \\ 0 & s-AA & 0 \\ 0 & 0 & s-BA \end{bmatrix} \begin{bmatrix} z-AD-AC[s-AA]^{-1}AB & 0 & 0 \\ -[s-AA]^{-1}AB & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and thus

$$\bar{A} = \begin{bmatrix} z-AD & -AC & -BC \\ -AB & s-AA & 0 \\ 0 & 0 & s-BA \end{bmatrix} = \begin{bmatrix} L(s,z) & -AC & -BC \\ 0 & s-AA & 0 \\ 0 & 0 & s-BA \end{bmatrix} \begin{bmatrix} \tilde{A} & 0 & 0 \\ -[s-AA]^{-1}AB & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} BD \\ 0 \\ BB \end{bmatrix} = \begin{bmatrix} L(s,z) & -AC & -BC \\ 0 & s-AA & 0 \\ 0 & 0 & s-BA \end{bmatrix} \begin{bmatrix} \tilde{B} \\ 0 \\ [s-BA]^{-1}BB \end{bmatrix}$$

Hence $\bar{A}$ and $\bar{B}$ are not left coprime w.r.t. $\mathbb{R}(s)[z]$ and therefor not left coprime w.r.t. $\mathbb{C}[s,z]$.

So we have that the modal controllability of

$$\begin{bmatrix} AD & AC & BC \\ AA & AA & 0 \\ 0 & 0 & BA \end{bmatrix} \text{ and } \begin{bmatrix} BD \\ 0 \\ BB \end{bmatrix} \text{ implies the reachability of:}$$

$$[AD + AC[s-AA]^{-1}AB] \text{ and } [BD + BC[s-BA]^{-1}BB]$$

We can prove a partial inverse of theorem (5.7)

<u>Theorem</u>

(5.8) If $A(s)$, $B(s)$ is reachable then

$$\begin{bmatrix} s-AD & -AC & -BC \\ -AB & s-AA & 0 \\ 0 & 0 & s-BA \end{bmatrix} \text{ and } \begin{bmatrix} BD \\ 0 \\ BB \end{bmatrix} \text{ are left coprime w.r.t. } \mathbb{C}(s)[z]$$

<u>Proof</u> Suppose $A(s)$ and $B(s)$ are reachable and thus $[z-A(s)]$ and $B(s)$ are left coprime. Therefor there exists $L(s,z)$ and $Q(s,z)$ with entries from $\mathbb{R}(s)[z]$ s.t.

(5.9) $$[z-A(s)]\, L(s,z) + B(s)\, Q(s,z) = I$$

Now we have:

$$\begin{bmatrix} z-AD & -AC & -BC \\ -AB & s-AA & 0 \\ 0 & 0 & s-BA \end{bmatrix} \begin{bmatrix} L & L\,A_2 & L\,B_2 \\ A_1L & [s-AA]^{-1}+A_1LA_2 & A_1L\,B_2 \\ -B_1Q & -B_1QA_2 & [s-BA]^{-1}-B_1QB_2 \end{bmatrix} +$$

$$\begin{bmatrix} BD \\ 0 \\ BB \end{bmatrix} [Q,\ QA_2,\ QB_2] = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

where $A_1 = [s-AA]^{-1}AB \qquad A_2 = AC[s-AA]^{-1}$

$B_1 = [s-BA]^{-1}BB \qquad B_2 = BC[s-BA]^{-1}$

$L = L(s,z) \qquad Q = Q(s,z)$ from (5.9)

which proves the theorem.

The complete inversion of theorem (5.7) is still under investigation in particular the role minimal realizations of A(s) and B(s) play.

5. Conclusions

In this paper a realization procedure has been described as an application of the theory of linear systems over commutative rings. In [14] Sontag makes a remark about this.

Under certain conditions the existence of a stabilizing feedback regulator has been proved and connections with [4] have been made.

It is the authors opinion that the algebraic methods used here will prove to be very fertile in 2-D systems theory.

In 5. the reachability condition is rather severe but in the case of a scalar transfer function which can be first level realized in standard controllable form this condition is always satisfied. Compare the example in 3.

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