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# Realization of Vassiliev Invariants by Unknotting Number One Knots

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**Abstract.** We show that for any natural number $n$ and any knot $K$, there are infinitely many unknotting number one knots, all of whose Vassiliev invariants of order less than or equal to $n$ coincide with those of $K$.

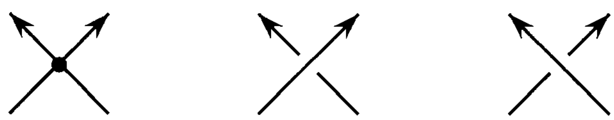
## 1. Introduction.

In 1990, V. A. Vassiliev [21] defined a sequence of knot invariants and J. S. Birman and X.-S. Lin [3] succeeded in giving an axiomatic description for Vassiliev invariants.

Our definition of Vassiliev invariants follows the Birman-Lin's axioms in [3] or D. Bar-Natan [1]. Whenever we have a knot invariant $v$ which takes value in some abelian group, we can extend it to an invariant of singular knots by the Vassiliev skein relation:

$$v(K_D) = v(K_+) - v(V_-)\,.$$

Here *a singular knot* is an immersion of a circle in $R^3$ whose only singularities are transversal double points and $K_D$, $K_+$ and $K_-$ denote the diagrams of singular knots which are identical except near one point as is shown in Fig. 1.1. An invariant $v$ is called *a Vassiliev invariant of order n* and is denoted by $v_n$, if $n$ is the smallest integer such that $v$ vanishes on all singular knots with more than $n$ double points.



$K_D$ $\quad$ $K_+$ $\quad$ $K_-$

FIGURE 1.1.

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The set of all Vassiliev invariants is at least as powerful as all of quantum group invariants. However, for any knot $K$ and for any positive integer $n$, some examples of knots have been constructed, all of whose Vassiliev invariants of order at most $n$ coincide with those of $K$ ([4][7][10][14]). Our purpose is to construct such examples of knots whose unknotting numbers are equal to one by using local moves called $C_n$-moves. Namely in this paper we show the following results.

THEOREM 1.1. *Let $n$ be a natural number and $K$ an oriented knot in $S^3$. Then there are infinitely many unknotting number one knots $J_m$ $(m = 1, 2, \cdots)$ such that $K$ and $J_m$ are $C_{n+1}$-equivalent.*

LEMMA 1.2. *Let $K$ and $J$ be $C_{n+1}$-equivalent oriented knots. Then $v(K) = v(J)$ for any Vassiliev invariant $v$ of order less than or equal to $n$,*

We will define $C_n$-moves and the $C_n$-equivalence in the next section. The following theorem is an immediate consequence of Theorem 1.1 and Lemma 1.2.

MAIN THEOREM. *Let $n$ be a natural number and $K$ an oriented knot in $S^3$. Then there are infinitely many unknotting number one knots $J_m$ $(m = 1, 2, \cdots,)$ such that $v(J_m) = v(K)$ for any Vassiliev invariant $v$ of order less than or equal to $n$.*

REMARK. A $C_n$-move is originally defined by K. Habiro in [5]. Habiro [6] showed that two oriented knots have the same Vassiliev invariants of order less than or equal to $n$ if and only if they are $C_{n+1}$-equivalent by using the clasper theory. Lemma 1.2 is the 'if' part of Habiro's result and we give a simple proof of Lemma 1.2 in the next section. Our results are obtained not by using the clasper theory, only by using the argument of knot diagrams. We do not use the 'only if' part, the difficult half, of Habiro's result. Our proof of Theorem 1.1 is elementary and constructive. After finishing the first version of this paper the first author showed a simple proof of Main theorem in [11]. However the proof essentially uses the difficult half of Habiro's result. See also [22] and [13].

## 2. Band description of local moves.

We use a concept 'band description of knots' defined in [19] for the proof of Theorem 1.1. Note that the prototypes of band description appear in [17], [23] and [24]. In particular in [24] it is shown that any knot can be expressed as a band sum of a trivial knot and some Borromean rings. The concept of band description is a development of this fact.

A *tangle* $T$ is a disjoint union of properly embedded arcs in the unit 3-ball $B^3$. A tangle $T$ is *trivial* if there exists a properly embedded disk in $B^3$ contaning $T$. A *local move* is a pair of trivial tangles $(T_1, T_2)$ with $\partial T_1 = \partial T_2$ such that for each component $t$ of $T_1$ there exists a component $u$ of $T_2$ with $\partial t = \partial u$.

Let $(T_1, T_2)$ be a local move, $t_1$ a component of $T_1$ and $t_2$ a component of $T_2$ such that $\partial t_1 = \partial t_2$. Let $N_1$ and $N_2$ be regular neighbourhoods of $t_1$ and $t_2$ respectively such that $N_1 \cap \partial B^3 = N_2 \cap \partial B^3$. Let $\alpha$ be a disjoint union of properly embedded arcs in $B^2 \times [0, 1]$ as

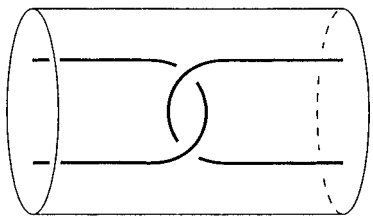

FIGURE 2.1.

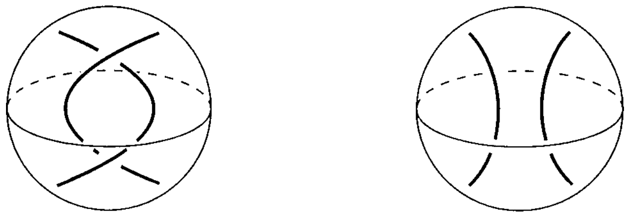

FIGURE 2.2.

illustrated in Fig. 2.1. Let $\psi_i : B^2 \times [0,1] \to N_i$ be homeomorphisms with $\psi_i(B^2 \times \{0,1\}) = N_i \cap \partial B^3$ for $i = 1, 2$. Suppose that $\psi_1(\partial\alpha) = \psi_2(\partial\alpha)$ and $\psi_1(\alpha)$ and $\psi_2(\alpha)$ are ambient isotopic in $B^3$ relative to $\partial B^3$. Then we say that a local move $((T_1 - t_1) \cup \psi_1(\alpha), (T_2 - t_2) \cup \psi_2(\alpha))$ is a *double of* $(T_1, T_2)$ *with respect to the components* $t_1$ *and* $t_2$.

Two local moves $(T_1, T_2)$ and $(U_1, U_2)$ are *equivalent*, denoted by $(T_1, T_2) \cong (U_1, U_2)$, if there is an orientation preserving self-homeomorphism $\psi : B^3 \to B^3$ such that $\psi(T_i)$ and $U_i$ are ambient isotopic in $B^3$ relative to $\partial B^3$ for $i = 1, 2$. Let $K_1$ and $K_2$ be oriented knots in the oriented three-sphere $S^3$. We say that $K_1$ and $K_2$ are *related by a local move* $(T_1, T_2)$ if there is an orientation preserving embedding $h : B^3 \to S^3$ such that $K_i \cap h(B^3) = h(T_i)$ for $i = 1, 2$ and $K_1 - h(B^3) = K_2 - h(B^3)$ together with orientations. If $K_1$ and $K_2$ are related by a local move $(T_1, T_2)$ and $(T_1, T_2) \cong (U_1, U_2)$, then $K_1$ and $K_2$ are related by $(U_1, U_2)$.

A $C_1$*-move* is a local move as illustrated in Fig. 2.2. A double of a $C_k$-move is called a $C_{k+1}$*-move*. Note that any doubles of equivalent local moves with respect to the corresponding components are equivalent. Therefore we have that for each natural number $n$ there are only finitely many $C_n$-moves up to equivalence. Two knots $K_1$ and $K_2$ are $C_n$*-equivalent* if $K_1$ and $K_2$ are related by a finite sequence of $C_n$-moves and ambient isotopies.

We note that our definition of $C_k$-move follows that in [5], and is different from the one in [6]. However by an easy induction on $k$ it is shown that these two definitions are equivalent.

A local move $(T_1, T_2)$ is *Brunnian* if for each pair of components $t_1$ and $t_2$ of $T_1$ and $T_2$ respectively with $\partial t_1 = \partial t_2$, $T_1 - t_1$ is ambient isotopic of $T_2 - t_2$ in $B^3$ relative to $\partial B^3$.

LEMMA 2.1. *A $C_n$-move is Brunnian.*

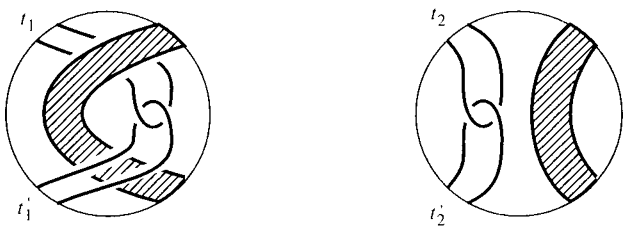


FIGURE 2.3.

PFOOF. It is easy to see that a double of a Brunnian local move is Brunnian. Since a $C_1$-move is Brunnian the result follows. □

Now we define the similarity of knots ([10][18]) to prove Lemma 1.2. We say that a knot $K$ is *n-similar* to a knot $L$ if the following occurs: There exists a diagram $D(K)$ of $K$ and a collection $\mathcal{A} = \{A_1, A_2, \cdots, A_n\}$ of $n$ pairwise disjoint, nonempty sets of crossings of $D(K)$ such that for any nonempty, not necessarily proper subcollection $\mathcal{A}'$ of $\mathcal{A}$, the diagram which is obtained from $D(K)$ by switching all the crossings in $\cup\mathcal{A}'$ is a diagram of $L$. The first author showed the following in [10].

LEMMA 2.2 ([10]). *If a knot $K$ is $(n+1)$-similar to $L$, then the Vassiliev invariants of order less than or equal to $n$ of $K$ coincide with those of $L$.*

PROOF OF LEMMA 1.2. It is sufficient to show the case that $K$ and $J$ are related by a $C_{n+1}$-move $(T_1, T_2)$. Let $h : B^3 \to S^3$ be the orientation preserving embedding such that $K \cap h(B^3) = h(T_1)$, $J \cap h(B^3) = h(T_2)$ and $K - h(B^3) = J - h(B^3)$. Let $t_1, t_2, \cdots, t_{n+2}$ be the components of $T_1$ and $u_1, u_2, \cdots, u_{n+2}$ the components of $T_2$ such that $\partial t_{n+2} = \partial u_{n+2}$. Let $D$ be a properly embedded disk in $B^3$ containing $T_1$. By the Brunnian property of $(T_1, T_2)$ we may suppose without loss of generality that $t_1 \cup \cdots \cup t_{n+1} = u_1 \cup \cdots \cup u_{n+1}$. Up to ambient isotopy in $S^3$ we may take regular projections of $K$ and $J$ respectively such that they differ only on the disk that is an injective image of $h(D)$. Let $A_i$ be the set of crossing points of $h(u_i)$ and $h(u_{n+2})$ at which $h(u_i)$ goes over $h(u_{n+2})$ in the regular projection of $J$. Then the sets $A_1, \cdots, A_{n+1}$ show that $J$ is $(n+1)$-similar to $K$. Then we have the result by Lemma 2.2. See also [16] for related results. □

We say that a $C_n$-move $(n \geq 2)$ as illustrated in Fig. 2.3 is *special* where each of the shaded regions represents $n-2$ times iteratedly doubled arcs. The arcs $t_1, t_1', t_2$ and $t_2'$ are called *specified arcs* of this special $C_n$-move.

LEMMA 2.3. *Any $C_n$-move $(n \geq 2)$ is equivalent to a special $C_n$-move.*

PROOF. We will prove by an induction on $n$. Is is clear that the result holds for $n = 2$. Let $(T_1, T_2)$ be a $C_{n+1}$-move. Then $(T_1, T_2)$ is a double of a $C_n$-move $(U_1, U_2)$. By the hypothesis of the induction, we may suppose that $(U_1, U_2)$ is a special $C_n$-move. If $(T_1, T_2)$ is

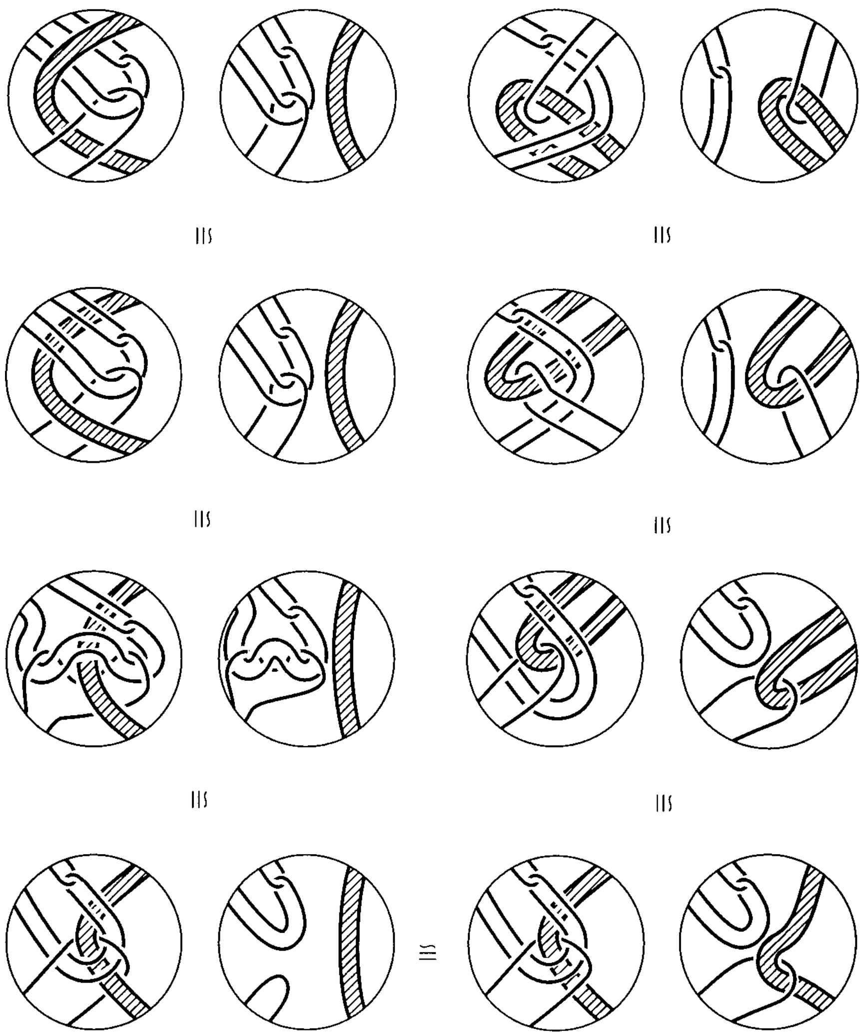

FIGURE 2.4.

a double of $(U_1, U_2)$ with respect to the components that are not specified arcs, then $(T_1, T_2)$ itself is a special $C_{n+1}$-move. Suppose that $(T_1, T_2)$ is a double of $(U_1, U_2)$ with respect to specified arcs. Then by the deformation illustrated in Fig. 2.4 we have that $(T_1, T_2)$ is equivalent to a special $C_{n+1}$-move. □

COROLLARY 2.4. *$C_{n+1}$-equivalence implies $C_n$-equivalence.*

PROOF. It is easy to see that a special $C_{n+1}$-move is realized by twice applications of a $C_n$-move as is shown in Fig. 2.5. Thus we have the result. □

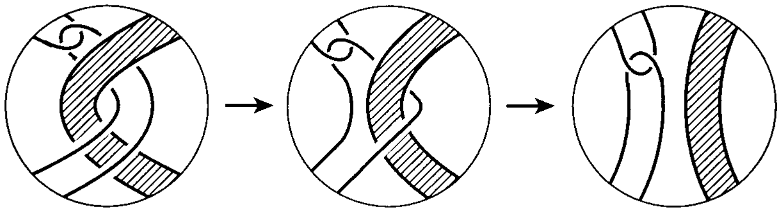

FIGURE 2.5.

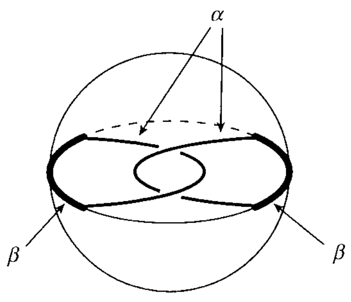


FIGURE 2.6.

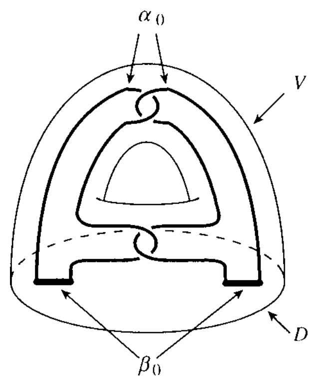


FIGURE 2.7.

A $C_1$-*link model* is a pair $(\alpha, \beta)$ where $\alpha$ is a disjoint union of $k+1$ properly embedded arcs in $B^3$ and $\beta$ is a disjoint union of arcs on $\partial B^3$ with $\partial\alpha = \partial\beta$ as illustrated in Fig. 2.6.

Suppose that a $C_k$-link model $(\alpha, \beta)$ has been defined where $\alpha$ is a disjoint union of $k+1$ properly embedded arcs in $B^3$ and $\beta$ is a disjoint union of $k+1$ arcs on $\partial B^3$ with $\partial\alpha = \partial\beta$ such that $\alpha \cup \beta$ is a disjoint union of $k+1$ circles. Let $\gamma$ be a component of $\alpha \cup \beta$ and $N$ a regular neighbourhood of $\gamma$ in $B^3$. Let $V$ be an oriented solid torus, $D$ a disk in $\partial V$, $\alpha_0$ properly embedded arcs in $V$ and $\beta_0$ arcs on $D$ as illustrated in Fig. 2.7.

Let $\psi : V \to N$ be an orientation preserving homeomorphism such that $\psi(D) = N \cap \partial B^3$ and $\psi(\alpha_0 \cup \beta_0)$ bounds disjoint disks in $B^3$. We further assume for a technical reason that $\psi(\beta_0)$ does not contain $\gamma \cap \beta$. Then we call the pair $((\alpha - \gamma) \cup \psi(\alpha_0),\ (\beta - \gamma) \cup \psi(\beta_0))$ a $C_{k+1}$-*link model.* We also say that the pair $((\alpha - \gamma) \cup \psi(\alpha_0),\ (\beta - \gamma) \cup \psi(\beta_0))$ is a *double* of $(\alpha, \beta)$ with respect to the component $\gamma$. A *special* $C_2$-*link model* is illustrated in Fig. 2.8. The components $\gamma_1$ and $\gamma_2$ in Fig. 2.8 are called the *specified components* of this special $C_2$-link model. A double of a special $C_n$-link model with respect to a component $\gamma$ that is not a specified component is called a *special* $C_{n+1}$-*link model.* And the specified components of this special $C_{n+1}$-link model are the same as those of the special $C_n$-link model. A *link model* is a $C_n$-link model for some $n$.

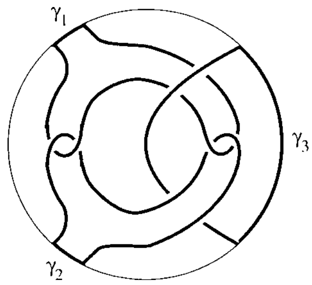


FIGURE 2.8.

Let $(\alpha_1, \beta_1), \cdots, (\alpha_\ell, \beta_\ell)$ be link models. Let $K$ be an oriented knot. Let $\psi_i : B^3 \to S^3$ be an orientation preserving embedding for $i = 1, \cdots, \ell$ and $b_1, \cdots, b_m$ mutually disjoint disks embedded in $S^3$. Suppose that they satisfy the following conditions;

(1) $\psi_i(B^3) \cap \psi_j(B^3) = \emptyset$ if $i \neq j$,

(2) $\psi_i(B^3) \cap K = \emptyset$ for each $i$,

(3) $b_i \cap K = \partial b_i \cap K$ is an arc for each $i$,

(4) $b_i \cap \bigcup_{j=1}^{\ell} \psi_j(B^3) = \partial b_i \cap \bigcup_{j=1}^{\ell} \psi_j(B^3)$ is a component of $\psi_k(\beta_k)$ for some $k$ for each $i$,

(5) $\bigcup_{i=1}^{m} b_i \cap \bigcup_{i=1}^{\ell} \psi_i(B^3) = \bigcup_{i=1}^{\ell} \psi_i(\beta_i)$.

Let $J$ be an oriented knot defined by $J = K \cup (\bigcup_{i=1}^{m} \partial b_i) \cup (\bigcup_{i=1}^{\ell} \psi_i(\alpha_i)) - \bigcup_{i=1}^{m} \mathrm{int}(\partial b_i \cap K) - \bigcup_{i=1}^{\ell} \psi_i(\mathrm{int}\,\beta_i)$ where the orientation of $J$ coincides with that of $K$ on $K - \bigcup_{i=1}^{m} b_i$. We denote $J$ by $J = \Omega(K; b_1, \cdots, b_m; (\alpha_1, \beta_1), \cdots, (\alpha_\ell, \beta_\ell); \psi_1, \cdots, \psi_\ell)$. Then we say that $J$ is a *band sum* of $K$ and link models $(\alpha_1, \beta_1), \cdots, (\alpha_\ell, \beta_\ell)$. We call each $b_i$ a *band*. Each image $\psi_i(B^3)$ is called a *link ball*.

Let $(T_1, T_2)$ be a local move. Then $(T_2, T_1)$ is also a local move. We call $(T_2, T_1)$ the *inverse* of $(T_1, T_2)$. It is easy to see that the inverse of the $C_1$-move is equivalent to itself. Then it follows inductively that the inverse of a $C_n$-move is equivalent to a $C_n$-move (but possibly not equivalent to itself).

LEMMA 2.5. *Let $n$ be a natural number greater than one. Let $K$ and $J$ be $C_n$-equivalent knots. Then $J$ is a band sum of $K$ and some special $C_n$-link models.*

PROOF. By Lemma 2.3, it is assumed that a $C_n$-move is special. We consider the sequence $K = K_0 \to K_1 \to \cdots \to K_\ell = J$, where $K_i$ and $K_{i+1}$ are related by a special $C_n$-move. We will prove by an induction on $\ell$. Let $(\alpha, \beta)$ be a link model of a $C_n$-move. Let $\beta'$ be a disjoint union of properly embedded arcs in $B^3$ that is a slight push off of $\beta$. Then we can show inductively on $n$ that the local move $(\alpha, \beta')$ is equivalent to a $C_n$-move. Conversely we can show that a $C_n$-move is equivalent to $(\alpha, \beta')$ for some link model $(\alpha, \beta)$. In particular a special $C_n$-move corresponds to a special $C_n$-link model. See for example Fig. 2.9.

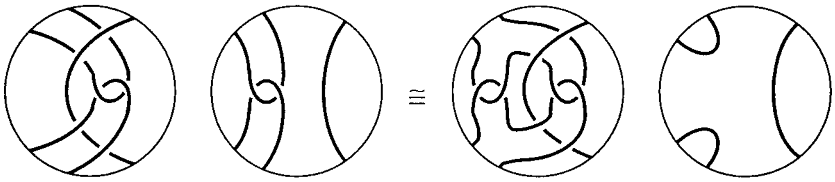

FIGURE 2.9.

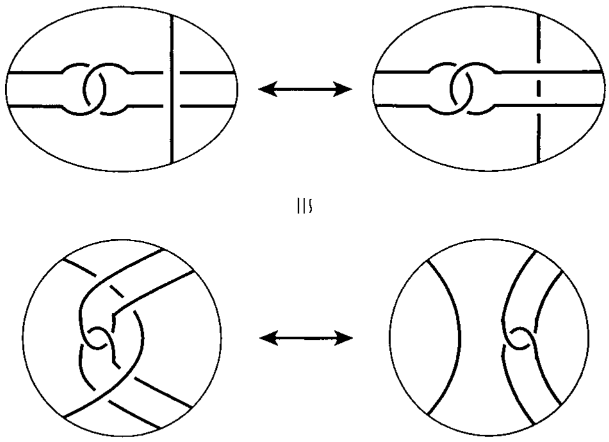

FIGURE 2.10.

Therefore we have the result in the case $\ell = 1$. By an inductive argument it is sufficient to consider the case that $J$ is a band sum of a knot $K_1$ and some special $C_n$-link models where $K$ and $K_1$ are related by a special $C_n$-move. Let $(T_1, T_2)$ be the special $C_n$-move and $h : B^3 \to S^3$ the orientation preserving embedding such that $K \cap h(B^3) = h(T_1)$, $K_1 \cap h(B^3) = h(T_2)$ and $K - h(B^3) = K_1 - h(B^3)$. We can sweep the link balls and then slide the bands out of the ball $h(B^3)$ by an ambient isotopy of $J$ which fixes $K_1$ setwisely. Note that this is possible by the triviality of the tangle $T_2$. Then we choose the link ball and bands in $h(B^3)$ so that $K_1$ is a band sum of $K$ and a special $C_n$-link model. Note that the new link ball and the bands are disjoint from the previous ones. Therefore $J$ is a band sum of $K$ and the special $C_n$-link models. □

LEMMA 2.6. *Let $(\alpha, \beta)$ be a $C_k$-link model. Let $\beta_1, \beta_2, \cdots, \beta_{k+1}$ be the components of $\beta$. We give an arbitrary orientation to each $\beta_i$. Let $K$, $J_1$ and $J_2$ be oriented knots. Suppose that $J_1 = \Omega(K; b_1, \cdots, b_{k+1}; (\alpha, \beta); \varphi)$ and $J_2 = \Omega(K; c_1, \cdots, c_{k+1}; (\alpha, \beta); \psi)$ such that $\partial b_i \supseteq \varphi(\beta_i)$ and $\partial c_i \supseteq \psi(\beta_i)$ for each $i$. For each of $b_i \cap K$ and $c_i \cap K$ we give an orientation that is coherent to the orientation of $\varphi(\beta_i)$ and $\psi(\beta_i)$ in $b_i$ and $c_i$ respectively.*

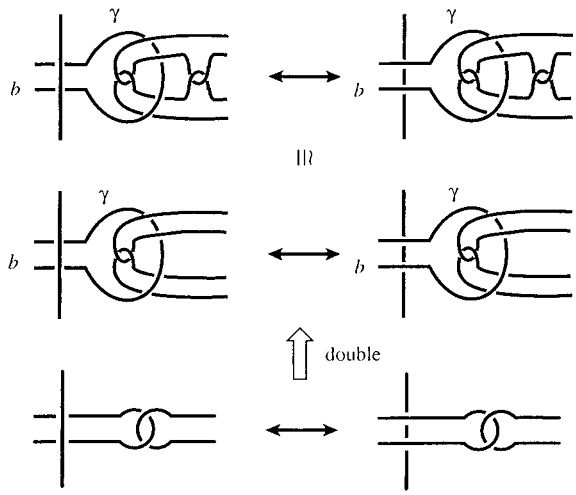


FIGURE 2.11.

*Suppose that the ordered sets of oriented arcs* $(b_1 \cap K, b_2 \cap K, \cdots, b_{k+1} \cap K)$ *and* $(c_1 \cap K, c_2 \cap K, \cdots, c_{k+1} \cap K)$ *are isotopic on the circle* $K$*. Then the knots* $J_1$ *and* $J_2$ *are* $C_{k+1}$*-equivalent.*

PROOF. First we claim that a crossing change between a band and a string is equivalent to a $C_{k+1}$-move. We will show this by an induction on $k$. When $k = 1$ the crossing change is nothing but a $C_2$-move. See Fig. 2.10.

Let $b$ be the band and $\gamma$ the component of $\alpha \cup \beta$ whose image intersects with $b$.

First suppose that $(\alpha, \beta)$ is a double of a $C_{k-1}$-link model $(\alpha', \beta')$ such that $\gamma$ is still a component of $\alpha' \cup \beta'$. Then we have that the crossing change is a 'double' of a crossing change in the case $k - 1$. See for example Fig. 2.11. Then by the hypothesis of the induction we have the result. Therefore it is sufficient to show that up to equivalence $(\alpha, \beta)$ is a double of some $C_{k-1}$-link model $(\alpha'', \beta'')$ such that $\gamma$ is still a component of $\alpha'' \cup \beta''$.

Consider the sequence of link models $(\alpha(1), \beta(1))$, $(\alpha(2), \beta(2)), \cdots, (\alpha(k), \beta(k)) = (\alpha, \beta)$ such that $(\alpha(j+1), \beta(j+1))$ is a double of $(\alpha(j), \beta(j))$ with respect to the component $\gamma(j)$ of $\alpha(j) \cup \beta(j)$ for each $1 \leq j \leq k-1$. Let $\gamma'(j+1)$ and $\gamma''(j+1)$ be the components of $\alpha(j+1) \cup \beta(j+1)$ that are not components of $\alpha(j) \cup \beta(j)$. If $\gamma'(k) \neq \gamma$ and $\gamma''(k) \neq \gamma$ then we set $(\alpha'', \beta'') = (\alpha(k-1), \beta(k-1))$ and have the conclusion. Therefore we may suppose without loss of generality that $\gamma'(k) = \gamma$. If $\{\gamma'(\ell), \gamma''(\ell)\} \cap \{\gamma(\ell), \gamma(\ell+1), \cdots, \gamma(k-1)\} = \emptyset$ for some $2 \leq \ell \leq k-1$, then by changing the order of doubling we have the conclusion. Suppose none of the cases above occur. Let $\gamma'(2), \gamma''(2)$ and $\gamma'''(2)$ be the components of $\alpha(2) \cup \beta(2)$. Then we easily have that $\gamma'(2)$ and $\gamma'''(2)$, or $\gamma''(2)$ and $\gamma'''(2)$ are still components of $\alpha(k) \cup \beta(k)$. Then by the deformation illustrated in Fig. 2.12 we have the conclusion. In Fig. 2.12 the shaded part represents iteratedly doubled arcs in the sense of link model. See for example Fig. 2.13. Thus we have shown the claim.

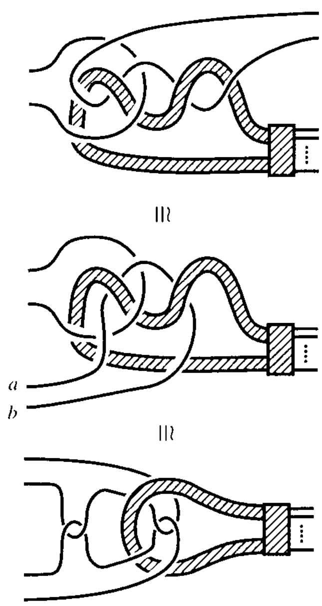


FIGURE 2.12.

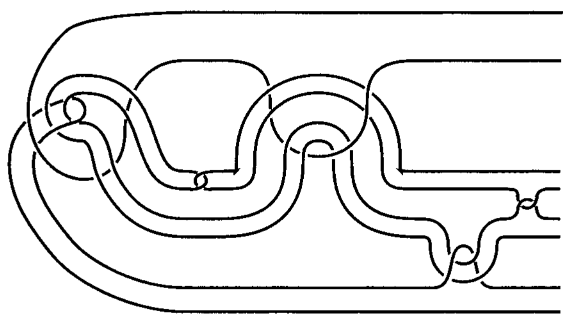

FIGURE 2.13.

Note that a full twist of a band is removable by the crossing change described above as illustrated in Fig. 2.14. Thus we have the result. □

LEMMA 2.7. *Let* $(\alpha, \beta)$ *be a special* $C_k$*-link model and* $\gamma_1, \gamma_2$ *the specified components of* $\alpha \cup \beta$*. Let* $K$ *and* $J$ *be oriented knots. Suppose that* $J = \Omega(K; b_1, \cdots, b_{k+1}; (\alpha, \beta); \varphi)$ *such that* $b_i \cap \varphi(\gamma_i) \neq \emptyset$ *for* $i = 1, 2$*. Then there is a special* $C_k$*-link model* $(\alpha', \beta')$ *with the same specified components* $\gamma_1, \gamma_2$ *and an oriented knot* $H = \Omega(K; c_1, \cdots, c_{k+1}; (\alpha', \beta'); \varphi')$ *that satisfies the following conditions*;

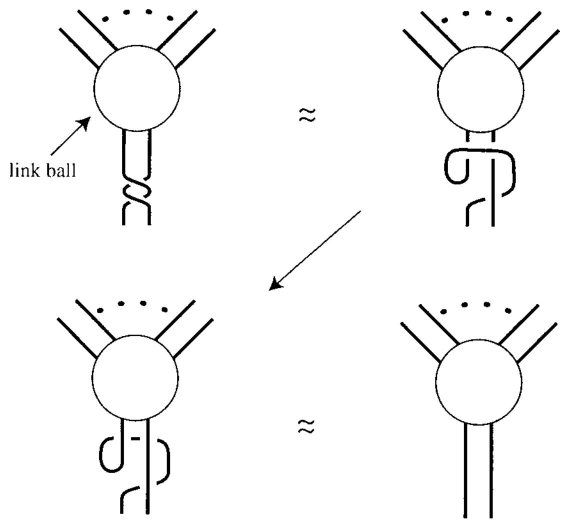


FIGURE 2.14.

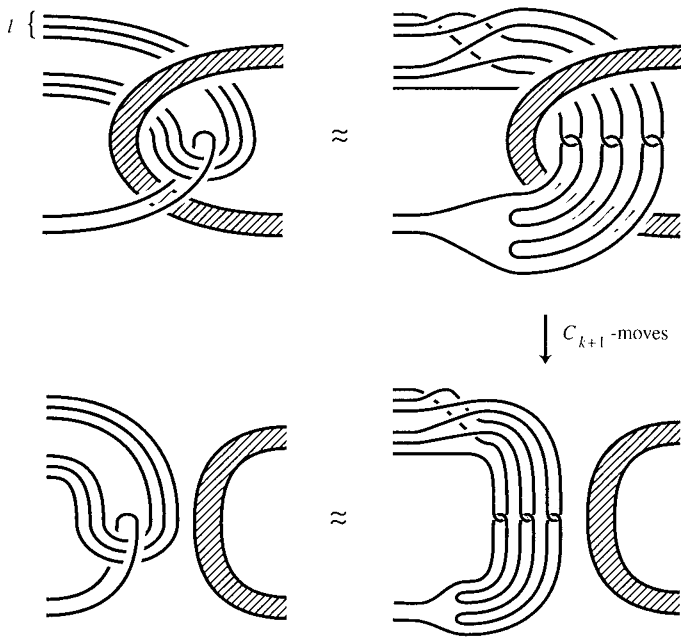


FIGURE 2.15.

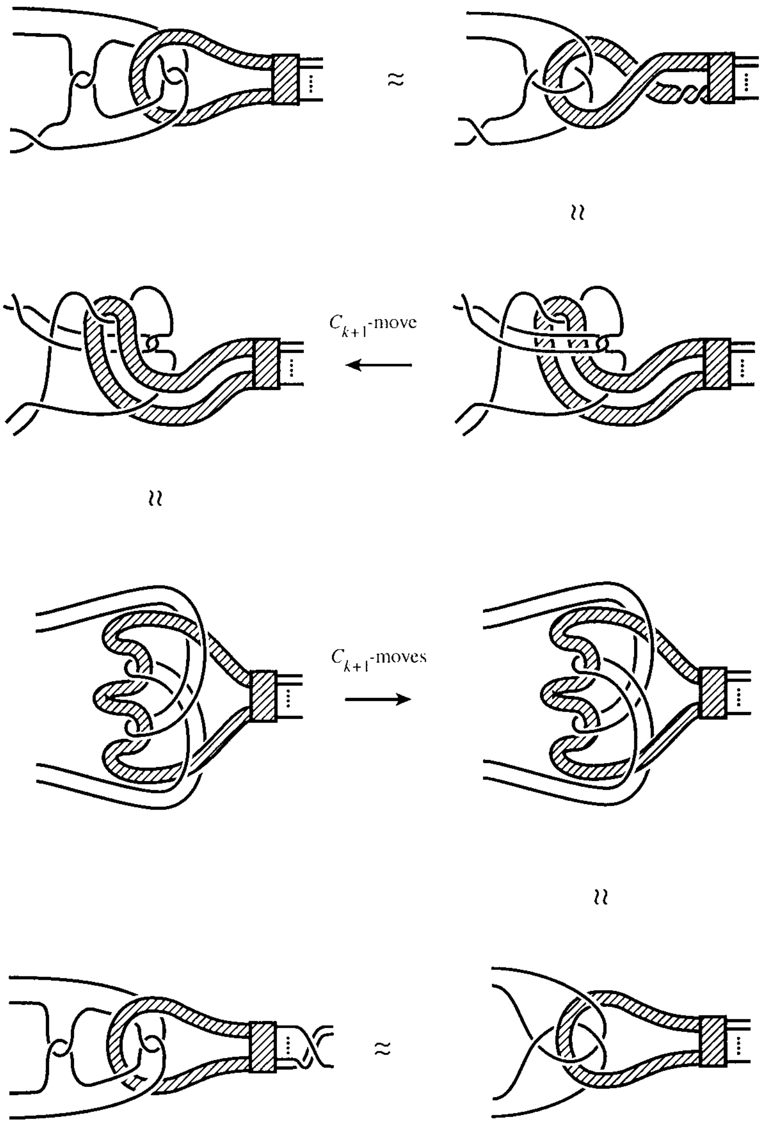


FIGURE 2.16.

(1) $K \cap b_i = K \cap c_i$ *for each* $i$,
(2) $\varphi'(B^3) = \varphi(B^3)$,
(3) $c_i \cap \varphi'(\gamma_i) = b_i \cap \varphi(\gamma_i)$ *for* $i = 1, 2$,
(4) $b_1 \cup c_1$ *is an annulus*,
(5) $b_2 \cup c_2$ *is a Möbius band*,

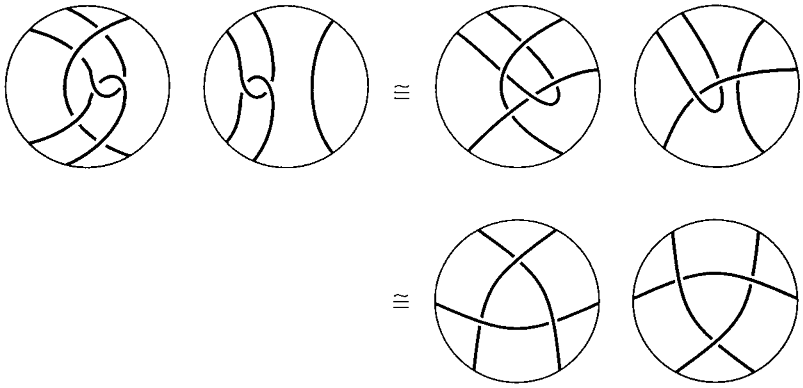


FIGURE 2.17.

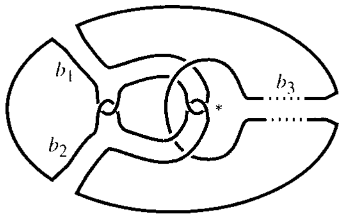


FIGURE 2.18.

(6) *$J$ and $H$ are $C_{k+1}$-equivalent.*

PROOF. First we note that the move illustrated in Fig. 2.15 is realized by $\ell$-times applications of $C_{k+1}$-moves where the shaded region represents iteratedly doubled $k$ arcs. Then by the deformation illustrated by Fig. 2.16 we have the result. □

PROOF OF THEOREM 1.1. First we note that a $C_2$-move is equivalent to a *delta move* defined in [9] as illustrated in Fig. 2.17. We note that the same move is defined in [8] independently. It is shown in [9] that knots are transformed into each other by delta moves. Then by Lemma 2.5 we have that $K$ is a band sum of a trivial knot $K_0$ and some special $C_2$-link models. Let $(\alpha, \beta)$ be a special $C_2$-link model. Suppose that $K = \Omega(K_0; b_1, \cdots, b_{3\ell}; (\alpha, \beta), \cdots, (\alpha, \beta); \varphi_1, \cdots, \varphi_\ell)$ such that $b_i \cap \varphi_j(\beta) \neq \emptyset$ if and only if $3(j-1) < i \leq 3j$. We deform $K \cap (b_1 \cup b_2 \cup b_3 \cup \varphi_1(\alpha \cup \beta))$ up to $C_3$-equivalence using Lemmas 2.6 and 2.7, the result is still denoted by the same symbols, so that the knot $K' = \Omega(K_0; b_1, b_2, b_3; (\alpha, \beta); \varphi_1)$ is just as illustrated in Fig. 2.18. Note that by a crossing change at $*$ in Fig. 2.18 we have trivial knot.

Next we deform $K \cap (b_4 \cup b_5 \cup b_6 \cup \varphi_2(\alpha \cup \beta))$ up to $C_3$-equivalence using Lemmas 2.6 and 2.7 so that they are as illustrated in Fig. 2.19.

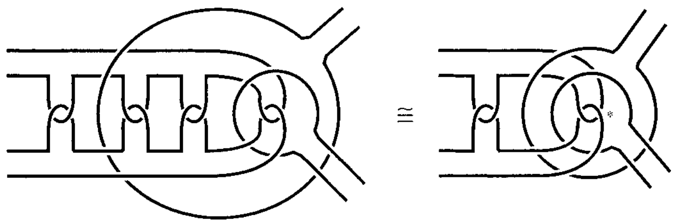

FIGURE 2.19.

We continue similar deformations and finally have a knot $K_1$ that is $C_3$-equivalent to $K$ so that the crossing change at the crossing corresponding to $*$ in Fig. 2.19 deforms $K_1$ into a trivial knot.

Since $K$ and $K_1$ are $C_3$-equivalent, we can express $K$ as a band sum of $K_1$ and some special $C_3$-link models. Then by similar deformations of $K$ up to $C_4$-equivalence we have a knot $K_2$ and a crossing $*$ whose change deforms $K_2$ into a trivial knot. Then we express $K$ as a band sum of $K_2$ and some special $C_4$-link models. We continue the process above and finally have a knot $K_{n-1}$ that is $C_{n+1}$-equivalent to $K$. Note that the unknotting number of $K_{n-1}$ is 0 or 1. By the result in [12], we have the following: There exists a $C_{n+1}$-move such that by operating this $C_{n+1}$-move for $K_{n-1}$ repeatedly, we have an infinite sequence of mutually $C_{n+1}$-equivalent knots $J_1'' = K_{n-1}, J_2'', J_3'', \cdots$, no two of whose order $n+1$ Vassiliev invariants coincide. Note that each $J_m''$ can be expressed as a band sum of $K_{n-1}$ and some special $C_{n+1}$-link models. By a similar deformation up to $C_{n+2}$-equivalence we have a knot $J_m'$ with unknotting number 0 or 1. Since $C_{n+2}$-equivalence does not change order $n+1$ Vassiliev invariants we have an infinite sequence of mutually $C_{n+1}$-equivalent knots $J_1' = K_{n-1}, J_2', J_3', \cdots$. At most one of them is a trivial knot. Therefore by removing it if it be we have the desired sequence $J_1, J_2, \cdots$. □

## References


[ 1 ] D. BAR-NATAN, On the Vassiliev invariants, Topology **34** (1995), 423–472.

[ 2 ] J. S. BIRMAN, New points of view in knot theory, Bull. Amer. Math. Soc. (N.S.) **28** (1993), 253–287.

[ 3 ] J. S. BIRMAN and X.-S. LIN, Knot polynomials and Vassiliev's invariants, Invent. Math. **111** (1993), 225–270.

[ 4 ] M. N. GUSAROV, On $n$-equivalence of knots and invariants of finite degree, *Topology of Manifold and Varieties* (ed. O. Viro), Adv. Soviet Math. **18** (1994).

[ 5 ] K. HABIRO, Master thesis of University of Tokyo (1994).

[ 6 ] K. HABIRO, Claspers and finite type invariants of links, Geom. Topol. **4** (2000), 1–83, [http://www.maths.warwick.ac.uk/gt/GTVol4/paper1.abs.html].

[ 7 ] X.-S. LIN, Finite type link invariants of 3-manifold, Topology **33** (1994), 45–71.

[ 8 ] S. V. MATVEEV, Generalized surgeries of three-dimensional manifolds and representations of homology spheres (in Russian), Mat. Zametki **42** (1987), 268–278, 345.

[ 9 ] H. MURAKAMI and Y. NAKANISHI, On a certain move generating link-homology, Math. Ann. **284** (1989), 75–89.

[10] Y. OHYAMA, Vassiliev invariants and similarity of knots, Proc. Amer. Math. Soc. **123** (1995), 287–291.

[11] Y. OHYAMA, Web diagrams and realization of vassiliev invariants by knots, J. Knot Theory Ramifications **9** (2000), 693–701.

[12] Y. OHYAMA and T. TSUKAMOTO, On Habiro's $C_n$-moves and Vassiliev invariants of order $n$, J. Knot Theory Ramifications **8** (1999), 15–23.

[13] Y. OHYAMA and H. YAMADA, Delta and clasp-pass distances and Vassiliev invariants of knots, to appear in J. Knot Theory Ramifications.

[14] T. STANFORD, Braid commutators and Vassiliev invariants, Pacific J. Math., **174** (1996), 269–276.

[15] T. STANFORD, Vassiliev invariants and knots modulo pure braid subgroup, to appear in Ann. Math.

[16] T. STANFORD, Four observations on $n$-triviality and Brunnian links, J. Knot Theory Ramifications **9** (2000), 213–219..

[17] S. SUZUKI, Local knots of 2-spheres in 4-manifolds, Proc. Japan Acad. **45** (1969), 34–38.

[18] K. TANIYAMA, On similarity of links, Gakujutu Kenkyu (issued by the school of education of Waseda University) **14** (1992), 33–36.

[19] K. TANIYAMA and A. YASUHARA, Local moves on spatial graphs and finite type invariants, to appear in Pacific J. Math.

[20] T. TSUKAMOTO, Clasp-pass moves and Vassiliev invariants of type three of knots, Proc. Amer. Math. Soc. **128** (2000), 1859–1867.

[21] V. A. VASSILIEV, Cohomology of knot space, *Theory of Singularities and its Applications* (ed. V. I. Arnold), Adv. Soviet Math. **1** (1990), Amer. Math. Soc.

[22] H. YAMADA, Delta distance and Vassiliev invariants of knots, J. Knot Theory Ramifications **9** (2000), 967–974.

[23] M. YAMAMOTO, Knots in spatial embeddings of the complete graph on four vertices, Topology Appl. **36** (1990), 291–298.

[24] A. YASUHARA, Delta-unknotting operation and adaptability of certain graphs, *Proceedings of Knots 96*, (ed. S. Suzuki), World Sci. Publ. Co. (1997), 115–121.

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