

REALIZING ROTATION VECTORS FOR TORUS HOMEOMORPHISMS

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ABSTRACT. We consider the rotation set $\rho(F)$ for a lift F of a homeomorphism $f: T^2 \rightarrow T^2$, which is homotopic to the identity. Our main result is that if a vector v lies in the interior of $\rho(F)$ and has both coordinates rational, then there is a periodic point $x \in T^2$ with the property that

$$\frac{F^q(x_0) - x_0}{q} = v$$

where $x_0 \in R^2$ is any lift of x and q is the least period of x .

In this article we consider the rotation set $\rho(F)$ as defined in [MZ], for a lift F of a homeomorphism $f: T^2 \rightarrow T^2$, which is homotopic to the identity. Our main result is that if a vector v lies in the interior of $\rho(F)$ and has both coordinates rational, then there is a periodic point $x \in T^2$ with the property that

$$\frac{F^q(x_0) - x_0}{q} = v$$

where $x_0 \in R^2$ is any lift of x and q is the least period of x . This should be compared with the well-known fact that if a homeomorphism of the circle has rational rotation number p/q then it has a periodic point (with rotation number p/q).

R. MacKay and J. Llibre [ML] have proved a similar result using the ideas our Proposition (2.4) below. They require the stronger hypothesis that v is in the interior of the convex hull of vectors in $\rho(F)$ which represent periodic orbits of f .

1. BACKGROUND AND DEFINITIONS

Suppose $f: T^2 \rightarrow T^2$ is a homeomorphism homotopic to the identity map, and let $F: R^2 \rightarrow R^2$ be a lift.

(1.1) Definition. Let $\rho(F)$ denote the set of accumulation points of the subset of R^2

$$\left\{ \frac{F^n(x) - x}{n} \mid x \in R^2, n \in Z^+ \right\},$$

Received by the editors August 23, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 58F22.

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0002-9947/89 \$1.00 + \$.25 per page

thus $\nu \in \rho(F)$ if there are sequences $x_i \in R^2$ and $n_i \in \mathbb{Z}^+$ with $\lim n_i = \infty$ such that

$$\lim_{i \rightarrow \infty} \frac{F^{n_i}(x_i) - x_i}{n_i} = \nu.$$

In [MZ] the rotation set is defined for a map homotopic to the identity (rather than a homeomorphism) $f: T^n \rightarrow T^n$. However, we shall be concerned only with homeomorphisms of T^2 . In [MZ] it is shown that for homeomorphisms of T^2 , $\rho(F)$ is convex.

We now briefly review the elementary theory of attractor-repeller pairs and chain recurrence developed by Charles Conley in [C]. In the following $f: X \rightarrow X$ will denote a homeomorphism of a compact metric space X .

(1.2) Definition. An ε -chain for f is a sequence x_1, x_2, \dots, x_n of points in X such that

$$d(f(x_i), x_{i+1}) < \varepsilon \quad \text{for } 1 \leq i \leq n-1.$$

If $x_1 = x_n$ it is called a periodic ε -chain.

A point $x \in X$ is called *chain recurrent* if for every $\varepsilon > 0$ there is an n (depending on ε) and an ε -chain x_1, x_2, \dots, x_n with $x_1 = x_n = x$. The set \mathbf{R} of chain recurrent points is called the *chain recurrent set* of f .

It is easily seen that \mathbf{R} is compact and invariant under f .

If $A \subset X$ is a compact subset and there is an open neighborhood U of A such that $f(\text{cl}(U)) \subset U$ and $\bigcap_{n \geq 0} f^n(\text{cl}(U)) = A$, then A is called an *attractor* and U is its isolating neighborhood. It is easy to see that if $V = X - \text{cl}(U)$ and $A^* = \bigcap_{n \geq 0} f^{-n}(\text{cl}(V))$, then A^* is an attractor for f^{-1} with isolating neighborhood V . The set A^* is called the *repeller* dual to A . It is clear that A^* is independent of the choice of isolating neighborhood U for A . Obviously $f(A) = A$ and $f(A^*) = A^*$.

If we define a relation \sim on \mathbf{R} by $x \sim y$ if for every $\varepsilon > 0$ there is an ε -chain from x to y and another from y to x , then it is clear that \sim is an equivalence relation.

The equivalence classes in $\mathbf{R}(f)$ for the equivalence relation \sim above are called the *chain transitive components* of $\mathbf{R}(f)$.

(1.3) Definition. A complete Lyapounov function for $f: X \rightarrow X$ is a continuous function $g: X \rightarrow R$ satisfying:

- (1) If $x \notin \mathbf{R}(f)$, then $g(f(x)) < g(x)$.
- (2) If $x, y \in \mathbf{R}(f)$, then $g(x) = g(y)$ iff $x \sim y$ (i.e., x and y are in the same chain transitive component).
- (3) $g(\mathbf{R}(f))$ is a compact nowhere dense subset of R .

By analogy with the smooth setting, elements of $g(\mathbf{R}(f))$ are called *critical values* of g .

A theorem of C. Conley [C] asserts that a complete Lyapounov function exists for any flow or homeomorphism of a compact space. The proof in [C] is given for flows; for an exposition in the case of homeomorphisms see [F2].

In general the number of chain transitive components for a homeomorphism can be infinite (even uncountable). However, if we specify a fixed $\delta > 0$ and work with δ -chains we can decompose $R(f)$ into a finite number of pieces.

(1.4) Definition. For a fixed $\delta > 0$ we say that $x, y \in \mathbf{R}(f)$ are δ -equivalent if there is a δ -chain from x to y and one from y to x . This is an equivalence relation and the equivalence classes will be called δ -transitive components of $R(f)$. We will say a compact f -invariant set $\Lambda \subset \mathbf{R}(f)$ is δ -transitive if for every $x, y \in \Lambda$, x is δ -equivalent to y .

(1.5) Lemma. Given $\delta > 0$ and a homeomorphism $f: X \rightarrow X$ of a compact space, then there are finitely many δ -transitive components.

Proof. A δ -transitive component is a union of chain transitive components. Two chain transitive components which are in different δ -transitive components must be at least distance δ apart. Hence if there were infinitely many δ -transitive components, there would be infinitely many subsets each at least distance δ from the others. This is impossible since X is compact. \square

(1.6) Theorem. Given $\delta > 0$ and a homeomorphism of a compact space $f: X \rightarrow X$, there is a complete Lyapounov function $g: X \rightarrow \mathbf{R}$ for f , and regular values for g , $c_0 < c_1 < c_2 < \dots < c_n$ such that if $\Lambda_i = \mathbf{R}(f) \cap g^{-1}([c_{i-1}, c_i])$, then $\{\Lambda_i\}$, $1 \leq i \leq n$, are the δ -transitive components of f .

Proof. Let $\Lambda_1, \dots, \Lambda_n$ be the δ -transitive components for f . We order them in such a way that if $i < j$ there is no δ -chain from Λ_i to Λ_j . This is possible since there can be no “cycle” of Λ_i ’s with each one having a δ -chain to the next and the last having a δ -chain to the first.

Let U_i denote the set of all $z \in X$ such that there is a δ -chain from Λ_i to z . U_i is an open set. Moreover, $f(\text{cl}(U_i)) \subset U_i$, because if $z \in \text{cl}(U_i)$, there is $z_0 \in U_i$ such that $d(f(z), f(z_0)) < \delta$ and consequently a δ -chain from x to z_0 gives a δ -chain $x = x_1, x_2, \dots, x_k, z_0, f(z)$ from x to $f(z)$.

Thus if $A_i = \bigcap_{n \geq 0} f^n(\text{cl } U_i)$ and $A_i^* = \bigcap_{n \geq 0} f^{-n}(X - U_i)$, then A_i, A_i^* are an attractor repeller pair and $\Lambda_i \subset A_i$. A result of Conley (see Lemma (1.7) of [F2] for a proof) asserts there is a continuous function $g_i: X \rightarrow [0, 1]$ such that $A_i = g_i^{-1}(0)$, $A_i^* = g_i^{-1}(1)$ and $g_i(f(x)) < g_i(x)$ for all $x \in X - (A_i \cup A_i^*)$. If $i < j$, then $\Lambda_j \subset A_i^*$ so $g_i(\Lambda_j) = \{1\}$.

Let $h(x) = \sum_{i=1}^n 2^i g_i(x)$ and note that $h(f(x)) \leq h(x)$ for all $x \in X$. For $x \in \mathbf{R}(f) = \bigcup \Lambda_i$, $h(x)$ is an even integer between 0 and 2^{n+1} . Also note if $x, y \in \mathbf{R}(f)$, then $h(x) = h(y)$ if and only if $g_i(x) = g_i(y)$ for all i . Hence if $x \in \Lambda_i, y \in \Lambda_j, i < j$, then $h(x) \neq h(y)$ since $g_i(x) \neq g_i(y)$. Now if $g_0: X \rightarrow [0, 1]$ is a complete Lyapounov function, then $g(x) = g_0(x) + h(x)$ is the desired function. \square

2. THE δ -TRANSITIVE CASE

We begin with a sequence of results leading to our main theorem. Assume throughout that $f: T^2 \rightarrow T^2$ is a homeomorphism homotopic to the identity and $F: R^2 \rightarrow R^2$ is a lift, i.e., if $\pi: R^2 \rightarrow T^2$ is the covering projection then $\pi \circ F = f \circ \pi$.

(2.1) Lemma. *If F has no fixed points, then there is an $\varepsilon > 0$ such that no periodic ε -chain for F exists.*

Proof. This result and its proof are quite similar to (2.1) of [F1] and (2.2) of [F2]. Let

$$\delta = \min_{x \in R^2} |F(x) - x|.$$

Note this minimum is assumed since it suffices to consider only x in a compact fundamental domain for π . Hence $\delta > 0$.

A result of Oxtoby [Ox] says that there is a $\gamma > 0$ such that for any finite set of pairs $\{(x_i, y_i)\}$ of elements in R^2 with $\|x_i - y_i\| < \gamma$ there is a pairwise disjoint set of piecewise linear arcs α_i from x_i to y_i with the diameter of each $< \delta$.

Let $\varepsilon = \gamma$; we will show there is no periodic ε -chain for F . Suppose to the contrary that $z_1 = z, z_2, z_3, \dots, z_n = z$ is a periodic ε -chain. Letting $y_i = z_i, x_i = F(z_{i-1})$, we see that there are pairwise disjoint arcs α_i from $F(z_{i-1})$ to z_i , with diameter $< \delta$. By isotoping in a neighborhood of these arcs we can produce a perturbation G of F satisfying

- (1) $\|F(x) - G(x)\| < \delta$ for all $x \in R^2$, and
- (2) $G(z_{i-1}) = z_i$.

Now G has a periodic point, namely z . Hence by results of [Br or Fa] G has a fixed point p . Thus $\|F(p) - p\| \leq \|F(p) - G(p)\| + \|G(p) - p\| < \delta$ which is a contradiction. \square

(2.2) Lemma. *Suppose Λ is a δ -transitive compact invariant subset of $\mathbf{R}(f)$ for a homeomorphism $f: T^2 \rightarrow T^2$ and F is a lift of f . There is a constant $K > 0$, such that for any $x_0, y_0 \in \Lambda, x \in \pi^{-1}(x_0)$ there is a δ -chain for F from x to a point $y \in \pi^{-1}(y_0)$ with $\|y - x\| < K$.*

Proof. Fix $\omega \in \pi^{-1}(\Lambda)$ and let Q_n denote the set of $z \in \Lambda$ such that there is a δ -chain for f from $\pi(\omega)$ to z of length less than n . Q_n is open by definition and $\Lambda = \bigcup_{n \geq 1} Q_n$ so compactness of Λ implies $Q_N = \Lambda$ for some $N > 0$. Hence given $y_0 \in \Lambda$ there is a δ -chain from $\pi(\omega)$ to y_0 of length less than N . Lifting this to R^2 , starting at w , we obtain a δ -chain from w to some $y' \in \pi^{-1}(y_0)$. If $P = \sup \|F(\nu) - \nu\|$, then since this δ -chain from w to y' has length less than N , it follows that $\|y' - w\| < C_1 = N(P + \delta)$.

A similar argument shows that given $x_0 \in \Lambda$ there is an $x' \in \pi^{-1}(x_0)$ with a δ -chain from x' to w and $\|x' - w\| < C_2$ for some constant C_2 independent of

x_0 . Piecing these together we obtain a δ -chain from x' to y' with $\|y' - x'\| < K = C_1 + C_2$. Now given any $x \in \pi^{-1}(x_0)$ translate this δ -chain by the integer vector $x - x'$ to obtain a δ -chain from x to y , where $y = y' + (x - x')$ satisfies $\pi(y) = y_0$ and $\|y - x\| = \|y' - x'\| < K$. \square

(2.3) Definition. If $\Lambda \subset T^2$ is a compact invariant set for $f: T^2 \rightarrow T^2$, and F is a lift of f , we denote by $\rho(f, \Lambda)$, the accumulation points of the set

$$\left\{ \frac{F^n(x) - x}{n} \mid \pi(x) \in \Lambda \text{ and } n > 0 \right\}.$$

(2.4) Proposition. Suppose $\Lambda \subset T^2$ is a compact invariant subset of $\mathbf{R}(f)$ for $f: T^2 \rightarrow T^2$ and for some $\delta > 0$, Λ is δ -transitive. If 0 is in the interior of the convex hull of $\rho(F, \Lambda)$, then there is a periodic δ -chain for F .

Proof. The hypothesis guarantees that there are vectors $\nu_1, \nu_2, \nu_3, \nu_4 \in \rho(F, \Lambda)$ such that 0 is in the interior of their convex hull (see Steinitz's theorem in [HDK]). Choose neighborhoods U_i of ν_i in R^2 so small that whenever $\nu'_i \in U_i$, 0 is also in the interior of the convex hull of ν'_1, ν'_2, ν'_3 and ν'_4 . Fix $z_0 \in \Lambda$ and $z \in \pi^{-1}(z_0)$. Now by (2.2) and the fact that $\nu_1 \in \rho(F, \Lambda)$ we can find $x_i \in R^2$ and $n_i > i$ such that

- (1) $\lim_{i \rightarrow \infty} \frac{F^{n_i}(x_i) - x_i}{n_i} = \nu_1$.
- (2) There is a δ -chain from z to x_i and $\|x_i - z\| < K$.
- (3) There is a δ -chain from $F^{n_i}(x_i)$ to $z'_i \in \pi^{-1}(z_0)$ and $\|F^{n_i}(x_i) - z'_i\| < K$.

Notice that piecing together the δ -chain from z to x_i , the orbit segment from x_i to $F^{n_i}(x_i)$ and the δ -chain from $F^{n_i}(x_i)$ to z'_i we obtain a δ -chain from z to z'_i . Also (1), (2), and (3) imply

$$\lim_{n \rightarrow \infty} \frac{z'_i - z}{n_i} = \nu_1.$$

Choose i sufficiently large that

$$\frac{z'_i - z}{n_i} \in U_1$$

and set $w_1 = z'_i - z$, $m_1 = n_i$ so that there is a δ -chain from z to $z + w_1$ and $w_1/m_1 \in U_1$. Note that $\pi(z'_i) = \pi(z) = z_0$ implies w_1 is an integer vector.

Now in a similar fashion construct w_2, m_2, w_3, m_3 , and w_4, m_4 , with the analogous properties.

Since 0 is in the convex hull of $w_1/m_1, w_2/m_2, w_3/m_3, w_4/m_4$ and the vectors w_1, w_2, w_3, w_4 are integers, it is possible to solve

$$Aw_1 + Bw_2 + Cw_3 + Dw_4 = 0$$

for positive integers A, B, C, D . Any translate of a δ -chain by an integer vector is another δ -chain. Hence piecing together A translates of the δ -chain from z to $z + w$, with B translates of the δ -chain from z to $z + w_2$, C translates of the δ -chain from z to $z + w_3$, etc., we obtain a δ -chain from z to $z + Aw_1 + Bw_2 + Cw_3 + Dw_4 = z$ as desired. \square

3. THE GENERAL CASE

As before we assume $f: T^2 \rightarrow T^2$ is a homeomorphism and $F: R^2 \rightarrow R^2$ is a lift.

(3.1) Proposition. *Suppose ν_1, ν_2, ν_3 and ν_4 are extreme points of the convex set $\rho(F)$ and 0 is in the interior of their convex hull. Then F possesses a fixed point.*

Proof. In [MZ] it is shown that since ν_i is an extreme point of $\rho(F)$ there is an ergodic Borel measure realizing ν_i and hence a nonwandering point $x_i \in T^2$ such that if $x \in \pi^{-1}(x_i)$

$$\lim_{n \rightarrow \infty} \frac{F^n(x) - x}{n} = \nu_i.$$

We will need only the fact that such an x_i exists with $x_i \in \mathbf{R}(f)$.

To show that F has a fixed point it suffices by (2.1) to show that for every $\delta > 0$ there is a periodic δ -chain for F . Given $\delta > 0$, let $\mathbf{R}(f) = \Lambda_1 \cup \Lambda_2 \cup \dots \cup \Lambda_m$ be a decomposition of the chain recurrent set into δ -transitive pieces as given in (1.6) and let $g: T^2 \rightarrow R$ be a complete Lyapounov function compatible with this decomposition. We will show that there exists a piece Λ_j of this decomposition and points $y_i \in \Lambda_j, i = 1, 2, 3, 4$, such that whenever $y \in \pi^{-1}(y_i)$,

$$\nu_i = \lim_{n \rightarrow \infty} \frac{F^n(y) - y}{n}.$$

It then follows by (2.4) that F has a δ -chain. Since this holds for all $\delta > 0$ we conclude by (2.1) that F has a fixed point.

Choose a smooth approximation $g_0: T^2 \rightarrow R$ to g and regular values c_1, c_2, \dots, c_m such that the manifolds with boundary $M_i = g_0^{-1}((-\infty, c_i])$ satisfy

- (1) $f(M_i) \subset \text{int } M_i$, and
- (2) $\Lambda_i \subset M_i - M_{i-1}$.

Let N_i be the manifold $\text{cl}(M_i - M_{i-1})$, so $T^2 = \bigcup N_i$ and $N_i \cap N_k$ consists of a finite set of circles if $k = i \pm 1$ and otherwise is empty if $i \neq k$.

These circles are the components of $g_0^{-1}(\{c_1, c_2, \dots, c_m\})$. We first observe that none of these circles is essential in T^2 . If there were such a circle, say

γ , then it would be in the boundary of M_j for some j and M_j would have to have another boundary component which is isotopic to γ . (There might also be some inessential circles in the boundary of M_j .) It follows that M_j is an essential annulus (perhaps with some disks removed) in T^2 . Let \tilde{M}_j be a component of $\pi^{-1}(M_j)$ and choose a lift F_0 of f so that $F_0(\tilde{M}_j) \subset \tilde{M}_j$. Now \tilde{M}_j is an infinite strip (perhaps with holes) which has a rational slope. It follows since $F_0(\tilde{M}_j) \subset \tilde{M}_j$ that for any $x \in R^2$, if $\lim_{n \rightarrow \infty} \frac{F^n(x) - x}{n}$ exists, then it must lie on a line with this slope, since $F_0^n(x)$ is constrained between parallel translates of \tilde{M}_j . From this and the fact that $F(x) = F_0(x) + w$ for some integer vector w , it follows that the convex hull of the vectors ν_i given in our hypothesis is a line segment. This contradicts the assumption that 0 is in the interior of the convex hull; so none of the boundary components of the N_i can be essential in T^2 .

Since each of these boundary circles is inessential, each of them bounds a unique smooth disk in T^2 . The complement of the union of these disks consists of the interior of a single one of the N_i 's, say N_j . The complement of $\text{int}(N_j)$ in T^2 consists of a finite set of disks, say D_1, D_2, \dots, D_r . Number these disks so that

$$D_i \subset M_j \quad \text{for } 1 \leq i \leq s$$

and

$$D_i \subset \text{cl}(T^2 - M_j) \quad \text{for } s < i \leq r.$$

Then

$$f(D_i) \subset \bigcup_{k=1}^s D_k \quad \text{for } 1 \leq i \leq s$$

and

$$f^{-1}(D_i) \subset \bigcup_{k=s+1}^r D_k \quad \text{if } s < i \leq r.$$

Consider now a point $x \in \pi^{-1}(x_1)$ such that

$$\nu_1 = \lim_{n \rightarrow \infty} \frac{F^n(x) - x}{n}.$$

We will show that if x_1 is not in Λ_j , there is another point $y_1 \in \Lambda_j$ so that whenever $y \in \pi^{-1}(y_1)$,

$$\nu_1 = \lim_{n \rightarrow \infty} \frac{F^n(y) - y}{n}.$$

Since the same is true for ν_2, ν_3 , and ν_4 , we will have completed the proof by the remarks above.

Suppose now that $x_1 \in D_p$ for $1 \leq p \leq s$. There exists $q > 0$ such that $f^q(D_p) \subset D_p$ (recall that x_1 is recurrent). Hence if $D \subset R^2$ is the lift of

D_p containing x , then $F^q(D) \subset D + w$ for some integer vector w . If we set $G(z) = F^q(z) - w$, then $G(D) \subset D$ so there is a fixed point z_0 for G . Clearly

$$\nu_1 = \lim_{n \rightarrow \infty} \frac{F^n(x) - x}{n} = \lim_{n \rightarrow \infty} \frac{F^n(z_0) - z_0}{n} = \frac{w}{q}.$$

If $x_1 \in D_p$ and $s < p$, then a similar argument applied to f^{-1} leads to a fixed point z_0 of G with the same properties.

We want to find a fixed point for G which is in $\pi^{-1}(N_j)$. To do this we consider fixed points of f^q on T^2 . We will use the fact that f^q is homotopic to a map with no fixed points so the index sum of the set of fixed points in any Nielsen class for f^q is zero (see [B, Theorem 3, p. 94]). Recall that two fixed points p_1 and p_2 are in the same Nielsen class for f^q provided any lift of f^q to R^2 which pointwise fixes $\pi^{-1}(p_1)$ also pointwise fixes $\pi^{-1}(p_2)$.

We will consider points in the Nielsen class of the point $\pi(z_0)$ where z_0 is the fixed point of G mentioned above. Any such points which are not in N_j will lie in a D_i with a lift \tilde{D}_i for which $G(\tilde{D}_i) \subset \tilde{D}_i$ or with $G^{-1}(\tilde{D}_i) \subset \tilde{D}_i$. Hence the contribution to the index of the points in D_i will be $+1$. Thus the index of the set of fixed points in the Nielsen class of $\pi(z_0)$ which are not in N_j is positive (the disk D_p contributes at least one $+1$). It follows there must be a fixed point $y_1 \in N_j$ of f^q in the Nielsen class of $\pi(z_0)$. Since y_1 is in the Nielsen class of $\pi(z_0)$, if $y \in \pi^{-1}(y_1)$, then $G(y) = y$. Hence

$$\nu_1 = \lim_{n \rightarrow \infty} \frac{F^n(y) - y}{n}.$$

Also y_1 is a periodic point of f in N_j so $y_1 \in \Lambda_j$. The same argument implies the existence of $y_2, y_3, y_4 \in \Lambda_j$, so this completes the proof. \square

(3.2) Theorem. *Suppose $f: T^2 \rightarrow T^2$ is a homeomorphism homotopic to the identity and $F: R^2 \rightarrow R^2$ is a lift. If ν is a vector with rational coordinates in the interior of $\rho(F)$, then there is a point $p \in R^2$ such that $\pi(p) \in T^2$ is a periodic point for f and*

$$\nu = \lim_{n \rightarrow \infty} \frac{F^n(p) - p}{n}.$$

Proof. Suppose $\nu = (r/q, s/q)$ with the greatest common divisor of r, s , and q equal to 1. If $G(x) = F^q(x) - (r, s)$, then a fixed point p of G will satisfy $F^q(p) = p + (r, s)$ and hence be the desired point.

It is easy to check (see [MZ]) that $\rho(G) = q\rho(F) - (r, s)$. Thus since $(r/q, s/q)$ is in the interior of $\rho(F)$, it follows that 0 is in the interior of $\rho(G)$. Since $\rho(G)$ is closed and convex there exist extreme points $\nu_1, \nu_2, \nu_3, \nu_4 \in \rho(G)$ such that 0 is in their convex hull (see Steinitz's theorem in [HDK]). It now follows from (3.1) that G possesses a fixed point p . \square

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