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REALIZING SULEIMANOVA SPECTRA VIA PERMUTATIVE MATRICES*

PIETRO PAPARELLA†

Abstract. A *permutative matrix* is a square matrix such that every row is a permutation of the first row. A constructive version of a result attributed to Suleĭmanova is given via permutative matrices. A well-known result is strenghthened by showing that all realizable spectra containing at most four elements can be realized by a permutative matrix or by a direct sum of permutative matrices. The paper concludes by posing a problem.

Key words. Suleĭmanova spectrum, Permutative matrix, Real nonnegative inverse eigenvalue problem.

AMS subject classifications. 15A18, 15A29, 15B99.

1. Introduction. Introduced by Suleĭmanova in [13], the longstanding *real nonnegative inverse eigenvalue problem* (RNIEP) is to determine necessary and sufficient conditions on a set $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ so that σ is the spectrum an n-by-n entrywise nonnegative matrix.

If A is an n-by-n nonnegative matrix with spectrum σ, then σ said to be *realizable* and the matrix A is called a *realizing matrix* for σ. It is well-known that if σ is realizable, then

$$s_k(\sigma) := \sum_{i=1}^{n} \lambda_i^k \geq 0, \ \forall\ k \in \mathbb{N} \tag{1.1}$$

$$\rho(\sigma) := \max_{1 \leq i \leq n} |\lambda_i| \in \sigma. \tag{1.2}$$

For additional background and results, see, e.g., [2, 9] and references therein.

A set $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ is called a *Suleĭmanova spectrum* if $s_1(\sigma) \geq 0$ and σ contains exactly one positive element. Suleĭmanova [13] announced (and loosely proved) that every such spectrum is realizable. Fiedler [3] showed that every Suleĭmanova spectrum is *symmetrically realizable* (i.e., realizable by a symmetric nonnegative matrix), however, his proof is by induction and does not explicitly yield a realizing

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†Division of Engineering and Mathematics, University of Washington Bothell, Bothell, WA 98011-8246, USA (pietrop@uw.edu).

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matrix for all orders. In [6], Johnson and Paparella provide a constructive version of Fiedler's result for Hadamard orders.

Friedland [4] and Perfect [10] proved Suleĭmanova's result via companion matrices (for other proofs, see references in [4]). In particular, the coefficients $c_0, c_1, \ldots, c_{n-1}$ of the polynomial $p(t) := \prod_{k=1}^{n}(t-\lambda_k) = t^n + \sum_{k=0}^{n-1} c_k t^k$ are nonpositive so that the companion matrix of p is nonnegative. As noted in [11, p. 1380], the construction of the companion matrix of p requires evaluating the elementary symmetric functions at $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a computation with $\mathcal{O}(2^n)$ complexity.

The computation of a realizing matrix for a realizable spectrum is of obvious interest for numerical purposes, but for many known theoretical results, a realizing matrix is not readily available. Indeed, according to Chu:

> Very few of these theoretical results are ready for implementation to actually compute [the realizing] matrix. The most constructive result we have seen is the sufficient condition studied by Soules [12]. But the condition there is still limited because the construction depends on the specification of the Perron vector – in particular, the components of the Perron eigenvector need to satisfy certain inequalities in order for the construction to work. [1, p. 18].

In this work, we provide a constructive version of Suleĭmanova's result via *permutative matrices*. The paper is organized as follows: Section 2 contains notation and definitions; Section 3 contains the main results; in Section 4, we show that if $\sigma = \{\lambda_1, \ldots, \lambda_n\}$, $n \leq 4$, satisfies (1.1) and (1.2), then σ is realizable by a permutative matrix or by a direct sum of permutative matrices; and we conclude by posing a problem in Section 5.

2. Notation. The set of m-by-n matrices with entries from a field $\mathbb{F}$ (in this paper, $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$) is denoted by $M_{m,n}(\mathbb{F})$ (when $m = n$, $M_{n,n}(\mathbb{F})$ is abbreviated to $M_n(\mathbb{F})$). For $A = [a_{ij}] \in M_n(\mathbb{C})$, $\sigma(A)$ denotes the *spectrum* of A.

The set of n-by-1 column vectors is identified with the set of all n-tuples with entries in $\mathbb{F}$ and thus denoted by $\mathbb{F}^n$. Given $x \in \mathbb{F}^n$, x_i denotes the i^{th} entry of x.

For the following, the size of each object will be clear from the context in which it appears:

- I denotes the identity matrix;
- e denotes the all-ones vector; and
- J denotes the all-ones matrix, i.e., $J = ee^\top$.

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Definition 2.1. For $x \in \mathbb{C}^n$ and permutation matrices $P_2, \ldots, P_n \in M_n(\mathbb{R})$, a *permutative matrix*[1] is any matrix of the form

$$\begin{bmatrix} x^\top \\ (P_2 x)^\top \\ \vdots \\ (P_n x)^\top \end{bmatrix} \in M_n(\mathbb{C}).$$

According to Definition 2.1, all one-by-one matrices are considered permutative.

3. Main results. We begin with the following lemmas.

Lemma 3.1. *For $x \in \mathbb{C}^n$, let*

$$P = P_x = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & \cdots & i & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{array} & \begin{bmatrix} x_1 & x_2 & \cdots & x_i & \cdots & x_n \\ x_2 & x_1 & \cdots & x_i & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ x_i & x_2 & \cdots & x_1 & \cdots & x_n \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ x_n & x_2 & \cdots & x_i & \cdots & x_1 \end{bmatrix} \end{array} = \begin{bmatrix} x^\top \\ (P_{\alpha_2} x)^\top \\ \vdots \\ (P_{\alpha_i} x)^\top \\ \vdots \\ (P_{\alpha_n} x)^\top \end{bmatrix},$$

where P_{α_i} is the permutation matrix corresponding to the permutation α_i defined by $\alpha_i(x) = (1\, i)$, $i = 2, \ldots, n$. Then $\sigma(P) = \{s, \delta_2, \ldots, \delta_n\}$, where $s := \sum_{i=1}^n x_i$ and $\delta_i := x_1 - x_i$, $i = 2, \ldots, n$.

Proof. Since every row sum of P is s, it follows that $Pe = se$, i.e., $s \in \sigma(P)$.

Since

$$P - \delta_i I = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & \cdots & i & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{array} & \begin{bmatrix} x_i & x_2 & \cdots & x_i & \cdots & x_n \\ x_2 & x_i & \cdots & x_i & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ x_i & x_2 & \cdots & x_i & \cdots & x_n \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ x_n & x_2 & \cdots & x_i & \cdots & x_i \end{bmatrix} \end{array},$$

it follows that the homogeneous linear system $(P - \delta_i I)\hat{x} = 0$ has a nontrivial solution (notice that the first and i^{th} rows of $P - \delta_i I$ are identical). Thus, $\delta_i \in \sigma(P)$.

[1]Terminolgy due to Charles R. Johnson.

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Moreover, if

$$v_i := \begin{array}{c} {\scriptstyle 1} \\ \vdots \\ {\scriptstyle i-1} \\ {\scriptstyle i} \\ {\scriptstyle i+1} \\ \vdots \\ {\scriptstyle n} \end{array} \begin{bmatrix} x_i \\ \vdots \\ x_i \\ x_1 - s \\ x_i \\ \vdots \\ x_i \end{bmatrix}, \ i = 2, \ldots, n$$

then

$$Pv_i = \begin{array}{c} {\scriptstyle 1} \\ \vdots \\ {\scriptstyle i-1} \\ {\scriptstyle i} \\ {\scriptstyle i+1} \\ \vdots \\ {\scriptstyle n} \end{array} \begin{bmatrix} x_i(s-x_i) + x_i(x_1 - s) \\ \vdots \\ x_i(s-x_i) + x_i(x_1 - s) \\ x_i(s-x_1) + x_1(x_1 - s) \\ x_i(s-x_i) + x_i(x_1 - s) \\ \vdots \\ x_i(s-x_i) + x_i(x_1 - s) \end{bmatrix} = (x_1 - x_i) \begin{bmatrix} x_i \\ \vdots \\ x_i \\ x_1 - s \\ x_i \\ \vdots \\ x_i \end{bmatrix} = \delta_i v_i,$$

so that (δ_i, v_i) is a right-eigenpair for P. ☐

Lemma 3.2. *If*

$$M = M_n := \begin{bmatrix} 1 & e^\top \\ e & -I \end{bmatrix} \in M_n(\mathbb{R}), \ n \geq 2,$$

then

$$M^{-1} = M_n^{-1} = \frac{1}{n} \begin{bmatrix} 1 & e^\top \\ e & J - nI \end{bmatrix}.$$

Proof. Clearly,

$$nMM^{-1} = \begin{bmatrix} 1 & e^\top \\ e & -I \end{bmatrix} \cdot \begin{bmatrix} 1 & e^\top \\ e & J - nI \end{bmatrix} = \begin{bmatrix} n & e^\top + e^\top(J - nI) \\ 0 & nI \end{bmatrix},$$

but $e^\top + e^\top(J - nI) = e^\top + (n-1)e^\top - ne^\top = 0$; dividing through by n establishes the result. ☐

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THEOREM 3.3 (Suleĭmanova [13]). *Every Suleĭmanova spectrum is realizable.*

Proof. Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be a Suleĭmanova spectrum and assume, without loss of generality, that $\lambda_1 \geq 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If $\lambda := [\lambda_1 \; \lambda_2 \; \cdots \; \lambda_n]^\top \in \mathbb{R}^n$, then, following Lemma 3.2, the solution x of the linear system

$$\begin{cases} x_1 + x_2 + \cdots + x_n = \lambda_1 \\ x_1 - x_2 = \lambda_2 \\ \qquad\vdots \\ x_1 - x_n = \lambda_n \end{cases}$$

is given by

$$x = M^{-1}\lambda = \frac{1}{n}\begin{bmatrix} s_1(\sigma) \\ s_1(\sigma) - n\lambda_2 \\ \vdots \\ s_1(\sigma) - n\lambda_n \end{bmatrix}.$$

which is clearly nonnegative. Following Lemma 3.1, the nonnegative matrix P_x realizes σ. □

EXAMPLE 3.4. If $\sigma = \{10, -1, -2, -3\}$, then σ is realizable by

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \\ 3 & 2 & 1 & 4 \\ 4 & 2 & 3 & 1 \end{bmatrix}.$$

COROLLARY 3.5. *If $\sigma = \{\lambda_1, -\lambda_2, \ldots, -\lambda_n\}$ is a Suleĭmanova spectrum such that $s_1(\sigma) = 0$ and $\lambda_1 > 0$, then the n-by-n nonnegative matrix*

$$P := \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_i & \cdots & \lambda_n \\ \lambda_2 & 0 & \cdots & \lambda_i & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ \lambda_i & \lambda_2 & \cdots & 0 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ \lambda_n & \lambda_2 & \cdots & \lambda_i & \cdots & 0 \end{bmatrix}$$

realizes σ.

EXAMPLE 3.6. If $\sigma = \{6, -1, -2, -3\}$, then σ is realizable by

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 3 \\ 2 & 1 & 0 & 3 \\ 3 & 1 & 2 & 0 \end{bmatrix}.$$

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4. Connection to the RNIEP. It is well-known that for $1 \leq n \leq 4$, conditions (1.1) and (1.2) are also sufficient for realizability (see, e.g., [6, 7]). In this section, we strengthen this result by demonstrating that the realizing matrix can be taken to be permutative or a direct sum of permutative matrices.

THEOREM 4.1. *If* $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ *and* $1 \leq n \leq 4$*, then* σ *is realizable if and only if* σ *satisfies* (1.1) *and* (1.2)*. Futhermore, the realizing matrix can be taken to be permutative or a direct sum of permutative matrices.*

Proof. Without loss of generality, assume that $\rho(\sigma) = 1$.

The case when $n = 1$ is trivial, but it is worth mentioning that $\sigma = \{1\}$ is realized by the permutative matrix [1].

If $\sigma = \{1, \lambda\}$, $-1 \leq \lambda \leq 1$, then the permutative matrix

$$\frac{1}{2}\begin{bmatrix} 1+\lambda & 1-\lambda \\ 1-\lambda & 1+\lambda \end{bmatrix}$$

realizes σ.

As established in [6], if $\sigma = \{1, \mu, \lambda\}$, where $-1 \leq \mu, \lambda \leq 1$, then the matrix

$$\begin{bmatrix} (1+\lambda)/2 & (1-\lambda)/2 & 0 \\ (1-\lambda)/2 & (1+\lambda)/2 & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

realizes σ when $1 \geq \mu \geq \lambda \geq 0$ or $1 \geq \mu \geq 0 > \lambda$. Notice that this matrix is a direct sum of permutative matrices. If $0 > \mu \geq \lambda$, then, following Theorem 3.3, σ is realizable by a permutative matrix.

When $n = 4$, all realizable spectra can be realized by matrices of the form

$$\begin{bmatrix} a+b & a-b & 0 & 0 \\ a-b & a+b & 0 & 0 \\ 0 & 0 & c+d & c-d \\ 0 & 0 & c-d & c+d \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}$$

(for full details, see [6, pp. 10–11]). ◻

5. Concluding remarks. In [4], Fiedler posed the *symmetric nonnegative inverse eigenvalue problem* (SNIEP), which requires the realizing matrix to be symmetric. Obviously, if $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ is a solution to the SNIEP, then it is a solution to the RNIEP. In [5], Johnson, Laffey, and Loewy that showed that the RNIEP strictly

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contains the SNIEP when $n \geq 5$. It is in the spirit of this problem that we pose the following.

PROBLEM 5.1. *Can all realizable real spectra be realized by a permutative matrix or by a direct sum of permutative matrices?*

At this point there is no evidence that suggests an affirmative answer to Problem 5.1; however, a negative answer could be just as difficult: one possibility, communicated to me by R. Loewy, is to find an *extreme nonnegative matrix* [8] with a real spectrum that can not be realized by a permutative matrix, or a direct sum of permutative matrices.

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