

# Recent advances on Failure and Recovery in Networks of Networks

Louis M. Shekhtman<sup>a,1,\*</sup>, Michael M. Danziger<sup>a</sup>, Shlomo Havlin<sup>a</sup>

<sup>a</sup> *Department of Physics , Ramat Gan, Israel*

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## Abstract

Until recently, network science has focused on the properties of single isolated networks that do not interact or depend on other networks. However it has now been recognized that many real-networks, such as power grids, transportation systems, and communication infrastructures interact and depend on other networks. Here we will present a review of the framework developed in recent years for studying the vulnerability and recovery of networks composed of interdependent networks. In interdependent networks, when nodes in one network fail, they cause dependent nodes in other networks to also fail. This is also the case when some nodes, like for example certain people, play a role in two networks, i.e. in a multiplex. Dependency relations may act recursively and can lead to cascades of failures concluding in sudden fragmentation of the system. We review the analytical solutions for the critical threshold and the giant component of a network of  $n$  interdependent networks. The general theory and behavior of interdependent networks has many novel features that are not present in classical network theory. Interdependent networks embedded in space are significantly more vulnerable compared to non-embedded networks. In particular, small localized attacks may lead to cascading failures and catastrophic consequences. Finally, when recovery of components is possible, global spontaneous recovery of the networks and hysteresis phenomena occur. The theory developed for this process points to an optimal repairing strategy for a network of networks. Understanding **realistic** effects present in networks of networks is required in order to **move towards determining** system vulnerability.

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\*Corresponding author

*Email address:* lsheks@gmail.com (Louis M. Shekhtman)

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## 1. Introduction

Classical graph studies involved simple random graphs (Erdős-Rényi networks) or regular lattices, however once more data became available about real-world complex systems, researchers quickly discovered that real networks have far more complex structures. First of all, many real networks have some nodes that act as hubs with far more connections than other nodes [1, 2, 3]. Beyond this, many studies have found other non-random structures such as the small-world structure [4], community structure [5, 6], clustering [4], degree-degree correlations [7, 8], and unique spatial structures [9] in networks. Understanding the topological structure of real-world networks has provided insights into fields as diverse as epidemiology [10, 11, 12, 13, 14], climate [15, 16], economics [17, 18], sociology [19], infrastructure [20], traffic [21], physiological networks [22], and brain networks [23, 24].

One of the most important properties of networks is their robustness to failures or in other words what fraction of nodes remain connected after some other subset of nodes is removed? To answer this question it is often useful to use percolation theory which tells us what fraction of nodes are in the largest component,  $P_\infty$  [25, 26]. Defining functionality based on the size of the largest component makes sense in many contexts. For example, in the case of a communications network it is usually pertinent to ask what fraction of nodes are able to communicate? If  $P_\infty \approx 1$  then the network is functional and most nodes can easily communicate. However if  $P_\infty \approx 0$  then very few nodes can communicate and the network is essentially non-functional. The term *giant connected component* is used when  $P_\infty$  is a non-zero fraction of an infinite system.

The formal framework of percolation theory in the context of networks involves varying  $1 - p$ , the fraction of nodes removed at random and calculating the corresponding size of the largest component,  $P_\infty(p)$ . In general, for single isolated networks  $P_\infty(p)$  undergoes a second-order, con-

tinuous phase transition [27] as  $p$  decreases. The point where the transition occurs is typically referred to as  $p_c$ . For  $p > p_c$  we have  $P_\infty(p) > 0$ , but for  $p < p_c$ ,  $P_\infty(p) = 0$ . For Erdős-Rényi networks it was found that  $p_c = 1 / \langle k \rangle$  where  $\langle k \rangle$  is the mean degree of the network [28, 29, 30]. In contrast, for scale-free networks where the degree distribution follows  $p(k) \sim k^{-\lambda}$ , it was found that for  $\lambda < 3$ ,  $p_c = 0$  [31], indicating that only when essentially all nodes are removed does the giant component reach zero.

Most real networks do not operate in isolation, but are instead merely one system in a network of networks [32, 33, 34, 35, 36, 37, 38]. One type of important relationship between networks is interdependence [39, 40]. This occurs in infrastructure where power grids may depend on communications systems and in many biological systems where functionality requires numerous organs and metabolic pathways to work together. Another example occurs in sociology where an individual may participate in multiple social networks [41, 42, 43]. We show an example of the complexity of interdependence in modern infrastructure in Fig. 1. **Here we will review some of our recent results on interdependent networks and point readers to other articles where they can learn more about the subject.**

In interdependent networks there are two types of links, the usual connectivity links that are also present in single networks, as well as a new type of links called *dependency links* [39, 40, 45, 46, 47, 48, 49]. These dependency links imply that the node at one end of the link relies on the node at the other end of the link to function. Thus if the node on one end of a dependency link fails, then the node on the other end will also fail.

The structure of the network of networks is based on networks having dependency links between them. Possible structures are shown in Fig. 2 and include treelike structures, a loop, and a random-regular configuration where each networks has the same number of dependency relations.

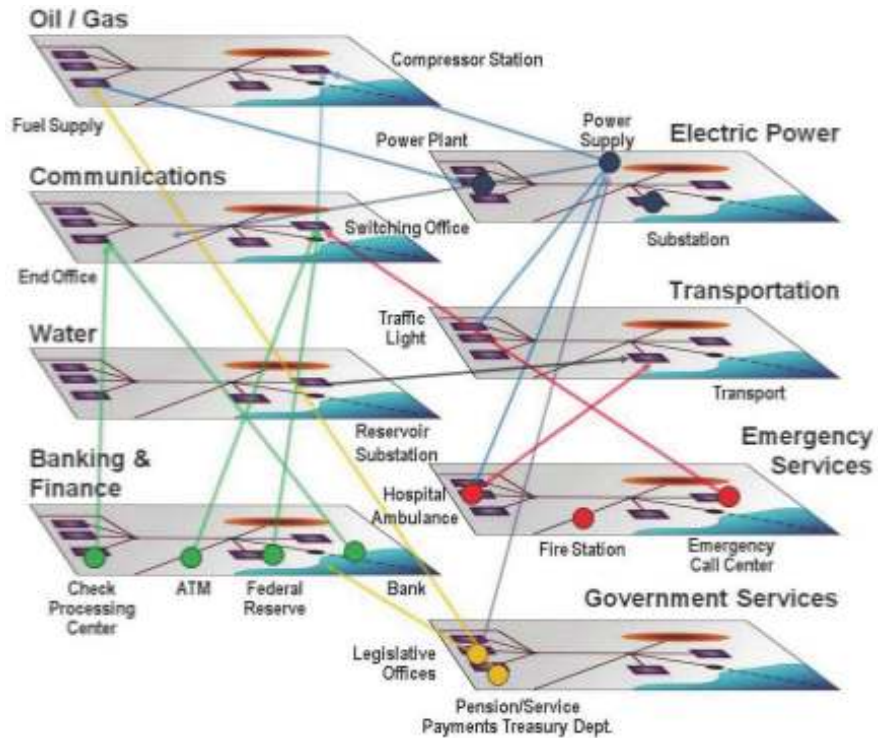


Figure 1: Modern infrastructure involves many dependency relations as shown in the Fig. from [44].

## 2. Robustness of interdependent networks

Just as earlier studies addressed the robustness of single networks using percolation methods [51, 52, 53, 54], similar methods, as well as other different methods, have been developed for interdependent networks [39, 40, 55]. When a fraction of nodes are removed from a network of interdependent networks, they cause dependent nodes in the other networks to fail [39]. The failure of these nodes will cause more nodes to become disconnected from the giant component, which will then cause other dependent nodes to fail. This process continues until a steady state is reached. Recent work examined the nature of this cascade and found that at criticality the length of the cascade has unique scaling properties [56]. Further, this cascade can be linked to other cascades which cause blackouts in power grids [20, 57]. In real space, the cascade has characteristic spatial-temporal scales which can be modeled by a characteristic dependency length [58]. Due to the cascade, the removal of a single node can cause the system to collapse entirely,

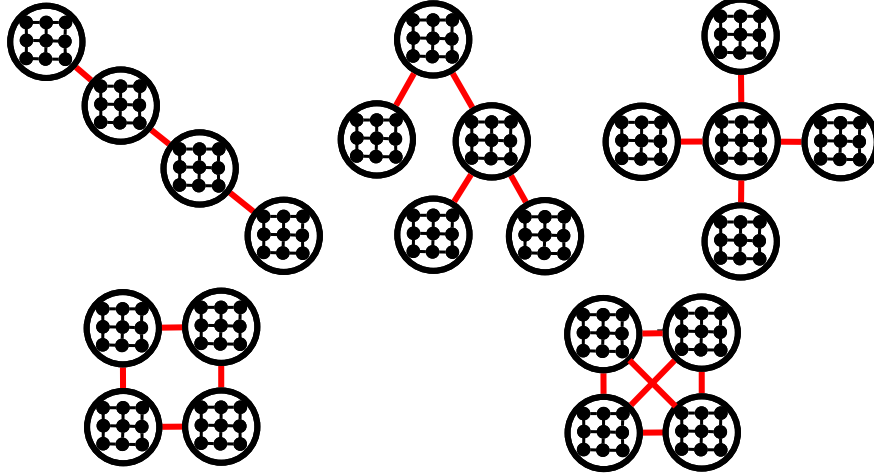


Figure 2: Networks of networks can assume many different structures defined based on which networks have dependency links between them. The top structures are all treelike structures whereas the bottom structures involve loops. Fig. after [50].

i.e. the transition is abrupt and first-order [40, 59]. **This discovery has significant implications to real world infrastructure as the discontinuity implies that a single additional failure can cause the total collapse of entire systems as was the case in the Northeastern United States Blackout in 2003 [60].**

The initial work on the robustness of interdependent networks studied a pair of interdependent Erdős-Rényi networks [39]. In that work, Buldyrev et al. [39] developed a theory based on percolation where the giant component was iteratively reduced due to the cascades. They then carried out numerical simulations to determine the size of the giant component at each step of the cascade, as well as the the giant component at the steady state, as a function of the fraction of initially removed nodes. We recall that for single Erdős-Rényi networks the equation governing the size of the giant component is  $P_\infty = p(1 - e^{-kP_\infty})$ , where  $k$  is the average degree of the network. In the case where  $1 - p$  nodes are removed from one network in a pair of interdependent Erdős-Rényi networks it was found [39, 61]

$$P_\infty = p(1 - e^{-kP_\infty})^2. \quad (1)$$

In a series of papers by Gao et al. [62, 63, 61, 64] this framework was extended for various types of

networks of networks that involve more than two networks. Simple analytic formulas were found for several cases. First for the case of  $n$  interdependent Erdős-Rényi networks whose dependencies (fully) form a tree (top structures in Fig. 2), Gao et al. [61] found that the size of the giant component is given by

$$P_\infty = p(1 - e^{-kP_\infty})^n. \quad (2)$$

Even though networks are interdependent, it need not be the case that every node in the network has a dependent node in the other network. For example, a communications system could have its own autonomous power supply and other systems could also feature autonomous nodes. To account for this, researchers defined  $q$  as the fraction of nodes in one network that have a dependent node in the second network [40]. The idea of partial dependency also leads to more structures that can be solved analytically, such as a loop of interdependent networks where each pair of networks has a fraction  $q$  nodes interdependent. Gao et al. [61] found that for a loop of interdependent Erdős-Rényi networks,

$$P_\infty = p(1 - e^{-kP_\infty})(1 - q + qP_\infty). \quad (3)$$

Note that when  $q = 0$  (no interdependence) this result reduces to the classical result for a single Erdős-Rényi network.

Another case that can be expressed analytically is the case where each network depends on exactly  $m$  other networks, i.e. a random-regular network of networks (see the bottom structures in Fig. 2). Here too Gao et al. [62, 64] was able to solve this case for Erdős-Rényi networks and obtained

$$P_\infty = \frac{p}{2^m} (1 - e^{-kP_\infty}) [1 - q + \sqrt{(1 - q)^2 + 4qP_\infty}]^m. \quad (4)$$

Note here that when  $m = 0$  or  $q = 0$ , Eq. (4) reduces to the result for single networks.

Other analytic results have been obtained by Bianconi et al. [65, 66], Baxter et al. [59, 67], and Cellai et al. [68].

### 3. Interdependent networks with realistic features

Other recent works, which are concisely reviewed below, have considered interdependent networks that have more realistic features.

#### 3.1. Realistic network structures

First, several papers have examined interdependent networks that have more realistic degree distributions. For the case of interdependent scale-free networks, it was found that a broader degree distribution actually makes the networks *less* robust, in contradiction of the case for single networks [39, 69]. Other studies have examined the influence of clustering [70, 71] and assortativity [72, 73] on interdependent networks.

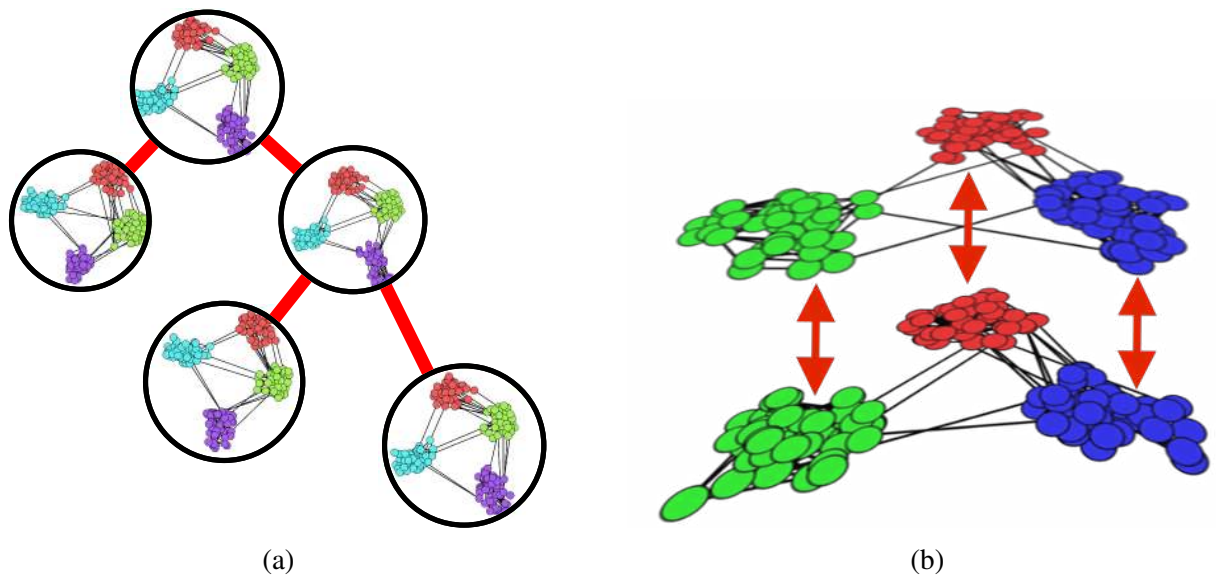


Figure 3: The model studied in [74]. **(a)** A treelike network of networks where each network has a modular structure. Dependency links are restricted such that they only connect nodes within the same communities, i.e a node in module  $m_a$  in network  $i$  will depend on a node also in module  $m_a$  in network  $j$ . **(b)** Demonstration of the dependency relations between a pair of interdependent networks. Dependency links exist between nodes of the same color in different layers. Figure after [74].

Very recent work has also considered the effects of modularity or community structure on the robustness of single and interdependent networks [75, 74]. In those two works [75, 74], the researchers considered the case of attacks on the nodes that are interconnected, i.e. have at least one

link to another community. It has been shown that interconnected nodes serve as efficient attack targets in many individual infrastructure networks, but especially in the Western U.S. power grid [76]. In [74] the authors considered the case of several interdependent networks where each network has the same number of modules of equal size. The dependency links were then restricted to be between nodes in the same community, but in different networks. It was noted that this model is realistic for infrastructure across cities, since each city has its own infrastructure and most of the interdependence is within a single city. Nonetheless, infrastructure networks are likely to be connected both within and across many cities. For example, a power grid and a communication network will likely encompass many cities, however it is likely that a power station and communication tower that are interdependent will be in the same city. A diagram of the model is shown in Fig. 3. It was found that for this model, as nodes are removed the system may undergo either one or two percolation transitions depending on the number of modules. If the system undergoes two transitions, the first transition occurs when the modules separate but are able to continue functioning individually. Only after further damage do the modules themselves also fail.

Another realistic structure for which particularly interesting effects were found is interdependent spatially embedded networks. Most critical infrastructures are embedded in space [9, 77, 78] and thus understanding how spatial embedding affects the cascade of failures is very important. **Spatially embedded networks have unique properties since most links will be of short length. For example, it is unlikely for two power stations on opposite sides of the United States will be connected, rather stations are usually connected to other nearby stations. This property, where nodes are connected to other nodes that are spatially nearby, is not present in random networks and leads to significantly different results.** To model spatial networks, many researchers used 2D lattices and noted that any other embedded network will be in the same universality class [79, 26]. One early work on interdependent spatially embedded networks considered varying  $q$ , the fraction of nodes which are interdependent. It was found analytically, that unlike random networks where a finite fraction of nodes must be interdependent in order for the transition to be first order, spatially



embedded networks undergo a first order transition for any value of  $q > 0$  [80].

Another study considered the case of  $q = 1$ , but where dependency links were restricted to be between nodes within some geographical distance,  $r$ . It was found that for low  $r$  the percolation transition was continuous, but as  $r$  increased, the transition became first order. The critical dependency length, where the transition becomes first-order occurs at  $r_c \approx 8$  [81].

Later work considered combining  $q$  and  $r$ , and found that as  $q$  decreases,  $r_c$  increases [82, 83]. Further work showed that when there are more than 2 interdependent networks,  $r_c$  decreases [50]. Very recent work examined a more realistic spatial multiplex structure where interdependent nodes overlap ( $r = 0$ ), but connectivity lengths are drawn from an exponential distribution [84].

### 3.2. Realistic dependency relations

Besides varying the structure of the networks themselves, it is also possible to place restrictions on the nature of the dependency links. For example, in the previous section we saw that for spatially embedded networks dependency links may be restricted to a finite length. Another realistic configuration of dependency links involves assigning high degree nodes in one network to be dependent on high degree nodes in the other network [85]. This makes sense since if a certain location is central then it is likely that both a power station and a communication tower located there will also be central. When dependency links are applied in such a way, interdependent scale-free networks become robust [86, 87, 88]. Many other studies have explored different variations of this idea [73, 89, 87, 90, 91, 92, 93, 68, 94, 95].

A recent development involved the recognition that it is often not sufficient to say whether a node is in the giant component or not, rather it also matters if the node participates in a process occurring on the network. A classic example is resistor networks. In these systems, nodes must not only be in the giant component, but also carry current in order for them to be considered functional (see Fig. 4) [96, 97, 98]. In the case of interdependent resistor networks, the nodes must both themselves carry current and have their dependent nodes carry current in order to be functional

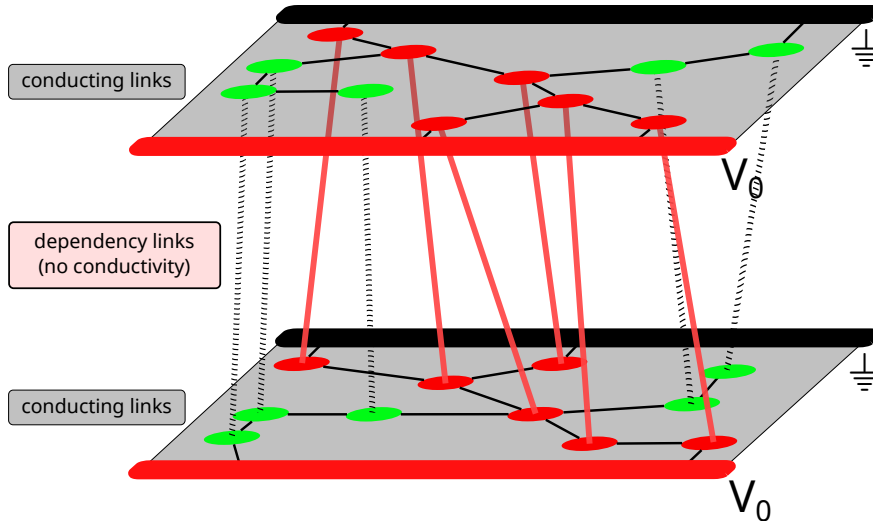


Figure 4: Two networks with process-based dependency are shown. Each network is connected to a source and a ground. In the figure, the red nodes are current-carrying nodes whereas the green nodes are dead ends. Nodes in one network are only considered functioning only if they carry current *and* the node they depend on carries current. Fig. from [99].

[99].

A study on interdependent resistor networks examined the fraction of nodes that carry current, as well as the overall conductivity of the system in the context of process based dependency [99]. Because functionality in this case relates to a microscopic physical process which can be studied in the laboratory, these results can be used to create new technologies such as a sensitive thermal or gas sensor.

### 3.3. Localized attacks on interdependent networks

A further aspect of interdependent networks where realistic features have been incorporated involves the nodes chosen to be removed. In other words, what happens if certain nodes are preferentially attacked? Initial studies examined targeted attacks on high-degree nodes in a pair of scale-free networks [100] and further work quickly expanded this to attacks on a more general network of networks [101].

More recent works have examined new types of attacks called localized attacks [102, 103]. In a study on a pair of interdependent spatially embedded networks, researchers examined what size

of localized attack, i.e. 'hole,' must be made in a network in order for the cascade to lead to the collapse of the entire system. It was found that there is a metastable regime where even though the system is robust against random failures, it is vulnerable to geographically localized attacks. Further, regardless of the actual size of the system, the size of the 'hole,' denoted  $r_h^c$ , that must be made to collapse the system, remains fixed for a given degree and dependency length. This is in marked contrast to the case of a single lattice where the size of a hole must grow with the size of the system in order to lead to collapse. This sort of attack is very realistic in the context of an Electromagnetic Pulse (EMP) detonation. **In Fig. 5 we determine the state of the system, i.e. stable, unstable, or metastable, according to system properties such as the degree of the system and the typical length of the dependency lengths. Given these two parameters for any real system, one can evaluate if the system is stable or vulnerable to either random or local failure. This information can help regulators allocate resources to make the system more resilient.**

A further question related to attacks on interdependent networks involves determining the optimal way to prevent cascades of failures and avoid abrupt collapse. Aside from several suggestions mentioned earlier such as having high degree nodes depend on other high degree nodes, having short range dependency links, and increasing the number of autonomous nodes, there have been several more explicit studies on this question. These studies have examined which nodes should be made autonomous and other methods in order to increase the robustness of interdependent networks [86, 105]

#### **4. Recovery in single networks and interdependent networks**

Despite efforts to build resilient infrastructure, there will always be situations where failures occur. In order to deal with this problem researchers have begun studying efficient methods to restore and recover interdependent networks. It was recently found [106] that spontaneous recovery can occur in single networks. This occurs when node recoveries are introduced in a system based on the Watts opinion model [107] and causes the network to fluctuate near its critical point between

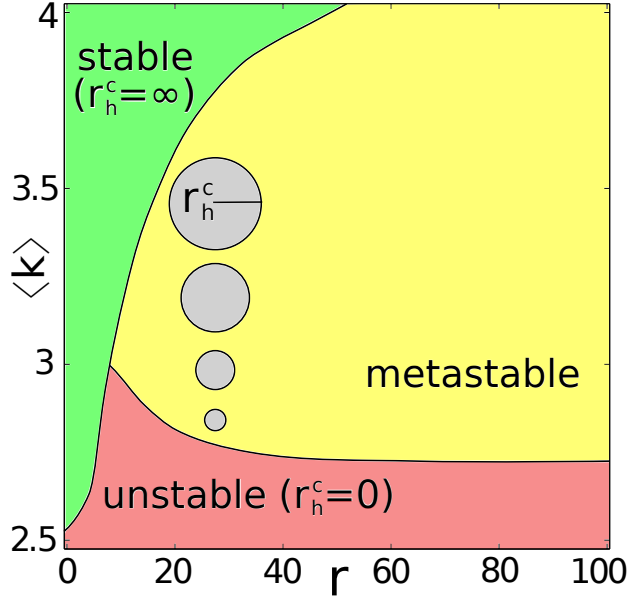


Figure 5: Here we show the phase diagram for localized attacks on interdependent spatial networks with an average degree  $\langle k \rangle$  and dependency links of length  $r$ . Depending on these two parameters, the size needed for a hole to lead to the collapse of the entire system may be either  $r_h^c = 0$  (unstable case),  $r_h^c = \infty$  (stable case) or  $r_h^c$  finite (metastable case). In the metastable case a random attack on the network will not lead to its collapse, but a localized attack, even if it's a zero fraction of the system size, will cause the entire system to fail. **The sizes of the circles represent the increase of the critical attack size,  $r_h^c$ , as  $\langle k \rangle$  increases (for fixed  $r$ ).** After [104].

active and inactive states.

A system of two interdependent networks has a much more complex phase diagram, characterized by two critical points and several spinodals (Fig. 6). These spinodals indicate the lines in parameter space, which if crossed lead the system to either fail to leave a collapsed state or recover to a functioning state, depending on the direction of the crossing. Given some level of failure in both networks, the optimal recovery strategy is to repair the fraction of nodes in each network that takes the shortest path in phase space towards the restoration spinodal [108]. Essentially this reveals how much effort to put into restoring each network in order to restore the overall functioning of the system of systems.

Several other approaches for restoration of interdependent network have also recently been proposed [109, 110].



## 5. Conclusion

There continue to be new developments in the understanding of interdependent networks. The recognition of the importance of analyzing the relations between systems is becoming more and more crucial as we move towards all sorts of smart technologies, like SMART grids, SMART cities, and the internet of things (IOT). Furthermore, interdependent networks have found significant applications outside of infrastructure in areas like finance [111, 112] and climate studies [113]. Further studies on the robustness of interdependent networks, recovery in interdependent networks, and other new aspects of interdependent networks will likely lead to more new discoveries of other rich phenomenon.

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