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# RECENT DEVELOPMENT OF THE HOMOTOPY PERTURBATION METHOD

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Abstract. The homotopy perturbation method is extremely accessible to non-mathematicians and engineers. The method decomposes a complex problem under study into a series of simple problems that are easy to be solved. This note gives an elementary introduction to the basic solution procedure of the homotopy perturbation method. Particular attention is paid to constructing a suitable homotopy equation.

This special issue on "the homotopy perturbation method and its application" of Topological Methods in Nonlinear Analysis consists mainly of a collection of recently obtained results and various new interpretations of earlier conclusions pertinent to the application of the homotopy perturbation method for real-life nonlinear problems, ranging from advanced calculus to fractional calculus(see Momani and Odibat's contributions), from peridic problems to solitary problems (see J. C. Lan, Ozis, Z. L. Tao's papers), from biology to engineering applications (see L. Xu, Sadighi, Chowdhury's papers), from stochastic system to scalar images (see El-Tawil and Q. Ma's papers). The aim of this special issue is to bring to the fore the many new and exciting applications of the homotopy perturbation iteration method, thereby capturing both the interest and imagination of the wider communities in various fields, such as in mathematics, physics, information science, computational science, biologics, medicine, and others.

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The special issue is a review of the state of the art of the field of the homotopy perturbation method. In selecting presentations, efforts were made to cover the field from all its key aspects to motivate the concepts, mathematical framework, and applications. No particular order has been followed in the presentation of the special issue, both achievements and limitations are discussed. It is intended to serve as a reference and tutorial resource, as well as to create a vision for future direction of this field.

The homotopy perturbation method [11]–[16] proposed by Ji-Huan He in 1998 has been proved by many authors to be a powerful mathematical tool for various kinds of nonlinear problems [1]–[10], [19]–[23], [25]–[27], [30], it is a promising and evolving method. Besides its mathematical importance and its links to other branches of mathematics, it is widely used in all ramifications of modern sciences.

The method does not need a small parameter or linearization, the solution procedure is very simple, and only few iterations lead to high accurate solutions which are valid for the whole solution domain.

Hereby I will illustrate the general solution procedure of the method. Consider a nonlinear equation in the form

$$Lu + Nu = 0,$$

where $L$ and $N$ are linear operator and nonlinear operator, respectively. In order to use the homotopy perturbation, a suitable construction of a homotopy equation is of vital importance. Generally, a homotopy can be constructed in the form

$$Lu + p(Nu + Nu - Lu) = 0,$$

where $L$ can be a linear operator or a simple nonlinear operator, and the solution of $Lu = 0$ with possible some unknown parameter can best describe the original nonlinear system. For example, for a nonlinear oscillator we can choose $Lu = u + \omega^2 u$, where $\omega$ is the frequency of the nonlinear oscillator. We use a simple example to illustrate the solution procedure.

## 1. Mathematical model

We consider a simple mathematical model in the form

$$u'' + u^2 = 0, \quad u(0) = u(1) = 0. \tag{1.1}$$

## 2. Qualitative sketch, trial function solution

This is a boundary value problem, so we choose such an initial guess

$$u_0(t) = at(1 - t), \tag{2.1}$$

where $a$ is an unknown constant. The trial-function, equation (2.1), satisfies the boundary conditions.

## 3. Construction of a homotopy

According to the initial guess, a homotopy should be constructed

$$u'' + 2a + p(u^2 - 2a) = 0. \tag{3.1}$$

When $p = 0$, the solution of equation (3.1) is equation (2.1). When $p = 1$, it turns out to be the original one.

## 4. Solution procedure similar to that of classical perturbation method

Using $p$ as an expanding parameter as that one in classic perturbation method, we have

$$u_0'' + 2a = 0, \qquad u_0(0) = u_0(1) = 0, \tag{4.1}$$

$$u_1'' + u_0^2 - 2a = 0, \qquad u_1(0) = u_1(1) = 0. \tag{4.2}$$

Generally, we need only few items. Setting $p = 1$, we obtain the first-order approximate solution which reads

$$u(t) = u_0(t) + u_1(t) = at(1-t) + at^2 - a^2\left(\frac{1}{30}t^6 - \frac{1}{10}t^5 + \frac{1}{12}t^4\right) - \left(a - \frac{1}{60}a^2\right)t.$$

## 5. Optimal identification of the unknown parameter in the trial function

There are many approaches to identification of the unknown parameters in the obtained solution. We suggest hereby the method of weighted residuals, especially the least squares method

$$\int_0^1 R\frac{\partial R}{\partial a}\,dt = 0,$$

where $R$ is the residual $R(u(t)) = Lu + Nu$.

We can also use the parameter-expansion method[14] to achieve the above iteration scheme. We re-write equation (1.1) in the form

$$u'' + 0 + 1 \cdot u^2 = 0. \tag{5.1}$$

We seek an expansion of the form [17]

$$u = u_0 + pu_1 + p^2u_2 + \dots \tag{5.2}$$

where the ellipsis dots stand for terms proportional to powers of p greater than 2, $p$ is a bookkeeping parameter [18], $p = 1$. The constants, 0 and 1, in left-hand side of equation (5.1) can be, respectively, expanded in a similar way [14], [17] and [18]:

$$o = 2a + a_1p + a_2p^2 + \dots, \quad 1 = b_1p + b_2p^2 + \dots \tag{5.3}$$

Substituting equations (5.3) to (5.2), we have

$$(u_0 + pu_1 + p^2u_2 + \dots)'' + (2a + a_1p + a_2p^2 + \dots) \\ + (b_1p + b_2p^2 + \dots) \cdot (u_0 + pu_1 + p^2u_2 + \dots)^2 = 0,$$

and equating coefficients of like powers of $p$, we obtain same equations as illustrated in (4.1) and (4.2). For detailed solution procedure for parameter-expansion method, please refer to [24], [28] and [29].

I hope that this issue will prove to be a timely and valuable reference for researchers in this area. Special thanks go to the referees for their valuable work. I here thank Prof. Lech Górniewicz for providing us with the opportunity to produce this special issue on this promising technology. I should also thank co-guest editor of this special issue Dr. Lan Xu of Donghua University for her careful preparation of this special issue.

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