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Recent Developments of Lyapunov-type Inequalities

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Abstract

In this work, first we give a survey of the most basic results on Lyapunov-type inequality, and next we sketch some recent developments related to this type of inequalities.

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1 Introduction

Let $\eta(t)$ be a real-valued absolutely continuous function on $[a, b]$ with $\eta'(t)$ of integrable square and $\eta(a) = 0 = \eta(b)$. Then for $s \in (a, b)$, we have

$$\int_a^b [\eta'(t)]^2 dt \geq \frac{4\eta^2(s)}{b-a},$$

which is a very useful tool for the study of the qualitative nature of the solutions of second-order ordinary differential equations. If $\eta(t)$ is nonzero on $[a, b]$, then the equality holds only if $s = \dfrac{a+b}{2}$ and $\eta(t) = \eta(s)\left\{1 - \left|\dfrac{2t-a-b}{b-a}\right|\right\}$.

In particular with the aid of this inequality, one may show that if $q(t)$ is a real-valued function such that the second-order differential equation

$$x''(t) + q(t)x(t) = 0$$

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has a nontrivial solution having two distinct zeros on $[a, b]$, then $q^+(t) = \dfrac{q(t) + |q(t)|}{2}$ must satisfy the Lyapunov inequality

$$\int_a^b q^+(t)dt > \frac{4}{b-a}.$$

This result is due originally to Lyapunov [19]. The Lyapunov inequality and many of its generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations and also in time scales.

In this work, first we will give a survey of the most basic results on Lyapunov-type inequalities, and next we will sketch some recent developments related to this type of inequalities.

In a celebrated paper of 1893, the Russian mathematician Lyapunov [19] proved the following remarkable result.

Theorem 1.1. *If $x(t)$ is a nontrivial solution of*

$$x''(t) + q(t)x(t) = 0 \tag{1.1}$$

with $x(a) = 0 = x(b)$, where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and $x(t) \neq 0$ for $t \in (a, b)$, then the so-called Lyapunov inequality

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a} \tag{1.2}$$

holds.

As it was first noticed by Wintner [31] and subsequently by several other authors, an application of Sturm's comparison theorem allows the replacement of $|q(t)|$ in (1.2) by $q^+(t)$, where $q^+(t) = \max\{0, q(t)\}$ is the nonnegative part of $q(t)$.

We are obligated to mention here that although the inequality (1.2) is known as the classical Lyapunov inequality, it was pointed out by Cheng [5] that Lyapunov neither stated nor proved the Theorem 1.1 but rather in [19], he only claimed the following:

Theorem 1.2. *Let $q(t)$ be a nontrivial, continuous and nonnegative function with period w and let*

$$\int_0^w q(s)ds \leq \frac{4}{w}. \tag{1.3}$$

Then the roots of the characteristic equation corresponding to Hill's equation

$$x''(t) + q(t)x(t) = 0, \quad -\infty < t < \infty \tag{1.4}$$

are purely imaginary with modulus one.

By Floquet theory, this means that all solutions of (1.4) are bounded on $(-\infty, \infty)$, i.e., the equation is stable. Lyapunov's proof requires the calculation of series expansions of the coefficients in the characteristic equation

$$s^2 - 2As + 1 = 0, \quad A = \frac{f(w) + g'(w)}{2}, \tag{1.5}$$

where f and g are solutions of the equation (1.4) which satisfy the initial conditions $f(0) = 1$, $f'(0) = 0$ and $g(0) = 0$, $g'(0) = 1$, respectively.

As also observed by Cheng, in an attempt to obtain alternative proof of Theorem 1.2, Borg [2] proved the following result of Beurling [1].

Theorem 1.3. *If $x(t)$ is a nontrivial solution of the boundary value problem*

$$x''(t) + q(t)x(t) = 0, \tag{1.6}$$

$$x(a) = x(b) = 0, \tag{1.7}$$

$$x(t) > 0, \quad a < t < b, \tag{1.8}$$

where q is a real-valued continuous function on $[a, b]$, then

$$\int_a^b \left| \frac{x''(s)}{x(s)} \right| ds > \frac{4}{b-a}. \tag{1.9}$$

Here we outline Borg's proof since it is particularly simple. He starts with the inequalities

$$\begin{aligned} \int_a^b \left| \frac{x''(s)}{x(s)} \right| ds &> \frac{1}{\|x\|_\infty} \int_a^b |x''(s)|\, ds \\ &> \frac{1}{\|x\|_\infty} \int_c^d |x''(s)|\, ds \\ &> |x'(d) - x'(c)| \end{aligned} \tag{1.10}$$

for arbitrary $a \le c < d \le b$. Now let $\|x\|_\infty = x(t_1)$. By Rolle's theorem, we can choose $a < c < t_1$ and $t_1 < d < b$ such that

$$x'(c) = \frac{\|x\|_\infty}{t_1 - a}, \quad -x'(d) = \frac{\|x\|_\infty}{b - t_1}. \tag{1.11}$$

Combining (1.11) with (1.10), we obtain

$$\int_a^b \left| \frac{x''(s)}{x(s)} \right| ds > \frac{1}{t_1 - a} + \frac{1}{b - t_1} > \frac{4}{b-a}. \tag{1.12}$$

The last inequality, i.e., $\dfrac{4}{b-a}$ is simply obtained by minimization of the right-hand side of inequality (1.12).

This seems to be the first proof of the "Lyapunov inequality" to appear in the literature. Borg went on to use (1.9) to give a new and much shorter proof of the Lyapunov stability theorem.

Another generalization of the classical Lyapunov inequality (1.2) is due to Hartman [15]: Let $m(t) > 0$ be continuous on $[a, b]$ and x be a nontrivial solution of (1.6)–(1.7). Then

$$\int_a^b m(s)q^+(s)ds > \gamma(b-a), \tag{1.13}$$

where

$$\gamma = \inf_{t\in[a,b]} \frac{m(t)}{(t-a)(b-t)}.$$

Evidently this inequality is just (1.2) when $m(t) = 1$. Also in a nonstability application of (1.2), Hartman and Wintner and other authors have used it to give estimates of the number of zeros of a solution of (1.6)–(1.7).

For authors who contributed the Lyapunov-type inequalities, we also refer to Cheng [4, 5], Dahiya and Singh [7], Elbert [9], Eliason [10–12], Hartman [15], Kwong [17], Lee et al. [18], Pachpatte [21–23], Panigrahi [24], Parhi and Panigrahi [25, 26] and Reid [27].

2 Generalizations

Since the appearance of Lyapunov's fundamental paper [19], various proofs and generalizations or improvements have appeared in the literature. For example, Hartman [15, Chap. XI] has generalized the classical Lyapunov inequality for the linear differential equation

$$(r(t)x'(t))' + q(t)x(t) = 0, \quad r(t) > 0 \tag{2.1}$$

as follows.

Theorem 2.1. *If $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros of a nontrivial solution of equation* (2.1)*), then*

$$\int_a^b q^+(s)ds > \frac{4}{\int_a^b r^{-1}(s)ds}, \tag{2.2}$$

where $q^+(t) = \max\{0, q(t)\}$ is the nonnegative part of $q(t)$.

Thus, the inequality (1.2) is strengthened to

$$\int_a^b q^+(s)ds > \frac{4}{b-a} \tag{2.3}$$

for the equation (1.1) by Theorem 2.1. The inequality (2.3) is the best possible in the sense that if the constant 4 in (2.3) is replaced by any larger constant, then there exists an example of (1.1) for which (2.3) no longer holds (see [15, p. 345], [17]). However, stronger results were obtained in Brown and Hinton [3] and Kwong [17]. In [17], it is shown that

$$\int_a^c q^+(s)ds > \frac{1}{c-a} \tag{2.4}$$

and

$$\int_c^b q^+(s)ds > \frac{1}{b-c}, \tag{2.5}$$

where $c \in (a,b)$ such that $x'(c) = 0$. Hence

$$\int_a^b q^+(s)ds > \frac{1}{c-a} + \frac{1}{b-c} = \frac{b-a}{(c-a)(b-c)} \geq \frac{4}{b-a}. \tag{2.6}$$

In [3, Corollary 4.1], the authors obtained

$$\left|\int_a^b q(s)ds\right| > \frac{4}{b-a}, \tag{2.7}$$

from which (1.2) can be obtained.

The Lyapunov inequality has been extended in many directions and its half-linear differential equation

$$\left(r(t)\left|x'(t)\right|^{\lambda-2}x'(t)\right)' + q(t)\left|x(t)\right|^{\lambda-2}x(t) = 0,\; r(t) > 0 \text{ and } \lambda > 1 \tag{2.8}$$

extension can be found in Došlý and Řehák's recent book [8, p. 190] as follows.

Theorem 2.2. *Let $a, b \in \mathbb{R}$ with $a < b$ be consecutive zeros of a nontrivial solution of equation* (2.8)*. Then*

$$\int_a^b q^+(s)ds > \frac{2^\lambda}{\left(\int_a^b r^{1/(1-\lambda)}(s)ds\right)^{\lambda-1}}, \tag{2.9}$$

where $q^+(t) = \max\{0, q(t)\}$ is the nonnegative part of $q(t)$.

In 1997, Pachpatte [22] has generalized the Lyapunov inequality for differential equations of the form

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + p(t)x'(t) + q(t)x(t) + f(t,x(t)) = 0, \tag{2.10}$$

$$\left(r(t)\left|x(t)\right|^{\beta}\left|x'(t)\right|^{\gamma-2}x'(t)\right)' + p(t)x'(t) + q(t)x(t) + f(t,x(t)) = 0, \tag{2.11}$$

as follows.

Theorem 2.3. *Let x be a solution of equation* (2.10) *with $x(a) = 0 = x(b)$ and $x(t) \neq 0$ for $t \in (a, b)$. Let $|x(t)|$ be maximized at a point $c \in (a, b)$. Then*

$$\begin{aligned} 1 \ \leq\ & \frac{1}{2^{\alpha+1}} \left(\int_a^b r^{-1/\alpha}(s) ds \right)^{\alpha} \times \\ & \times \left(\frac{1}{M^{\alpha-1}} \int_a^b \left| q(s) - \frac{p'(s)}{2} \right| ds + \frac{1}{M^{\alpha}} \int_a^b w(s, M) ds \right), \end{aligned} \tag{2.12}$$

where $M = \max\{|x(t)| : a \leq t \leq b\}$, $|f(t, x(t))| \leq w(t, |x(t)|)$ and $\alpha \geq 1$.

Theorem 2.4. *Let x be a solution of equation* (2.11) *with $x(a) = 0 = x(b)$ and $x(t) \neq 0$ for $t \in (a, b)$. Let $|x(t)|$ be maximized at a point $c \in (a, b)$. Then*

$$\begin{aligned} 1 \ \leq\ & \left(\int_a^b r^{-1/(\gamma-1)}(s) ds \right)^{\gamma-1} \times \\ & \times \left(\frac{1}{M^{\beta+\gamma-2}} \int_a^b \left| q(s) - \frac{p'(s)}{2} \right| ds + \frac{1}{M^{\beta+\gamma-1}} \int_a^b w(s, M) ds \right), \end{aligned} \tag{2.13}$$

where $M = \max\{|x(t)| : a \leq t \leq b\}$, $|f(t, x(t))| \leq w(t, |x(t)|)$, $\alpha \geq 1$, $\beta \geq 0$, $\gamma \geq 2$ and $\gamma > \beta$.

In 1999, Parhi and Panigrahi [25] established an inequality similar to (1.2) for third-order differential equations of the form

$$x'''(t) + q(t)x(t) = 0. \tag{2.14}$$

Their results are as follows.

Theorem 2.5. *Let $x(a) = x(b) = 0$. If there exists $d \in (a, b)$ such that $x''(d) = 0$, then*

$$\int_a^b |q(s)| \, ds > \frac{4}{(b-a)^2}. \tag{2.15}$$

Theorem 2.6. *If $x''(t) \neq 0$, $t \in (a, b)$ and $x(t)$ has three consecutive zeros $a < b < a'$, then*

$$\int_a^{a'} |q(s)| \, ds > \frac{4}{(a'-a)^2}. \tag{2.16}$$

In 2003, Yang [32] generalized Parhi and Panigrahi' [25] above results to certain higher-order differential equations. His results are as follows.

Theorem 2.7. *Let $n \in N$ and $q(t) \in C([a, b])$. If there exists $d \in (a, b)$ such that $x^{(2n)}(d) = 0$, where $x(t)$ is a solution of the differential equation*

$$x^{(2n+1)}(t) + q(t)x(t) = 0 \tag{2.17}$$

satisfying

$$x^{(i)}(a) = x^{(i)}(b) = 0, \quad i = 0, 1, \ldots, n-1, \quad x(t) \neq 0, \quad t \in (a, b), \tag{2.18}$$

then

$$\int_a^b |q(s)|\, ds > \frac{n! 2^{n+1}}{(b-a)^{2n}}. \tag{2.19}$$

Theorem 2.8. *Let $n \in \mathbb{N}$, $n \geq 2$ and $q(t) \in C([a, b])$. If the differential equation*

$$x^{(n)}(t) + q(t)x(t) = 0 \tag{2.20}$$

has a solution $x(t)$ satisfying the boundary value problem

$$x(a) = x(t_2) = \ldots = x(t_{n-1}) = x(b) = 0, \tag{2.21}$$

where $a < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$ and $x(t) \neq 0$, $t \in (t_k, t_{k+1})$, $k = 1, 2, \ldots, n-1$, then

$$\int_a^b |q(s)|\, ds > \frac{(n-2)! n^n}{(n-1)^{n-2}(b-a)^{n-1}}. \tag{2.22}$$

Theorem 2.9. *Consider the differential equation*

$$x^{(2n)}(t) + q(t)x(t) = 0 \tag{2.23}$$

and suppose a solution $x(t)$ of (2.23) *satisfies the boundary value conditions*

$$x(a) = x'(a) = \ldots = x^{(n-1)}(a) = 0, \tag{2.24}$$

$$x(b) = x'(b) = \ldots = x^{(n-1)}(b) = 0, \tag{2.25}$$

$x(t) \neq 0$, $t \in (a, b)$, $q(t) \in C([a, b])$. Then

$$\int_a^b |q(s)|\, (s-a)^{2n-1}(b-s)^{2n-1} ds \geq (2n-1)[(n-1)!]^2 (b-a)^{2n-1}, \tag{2.26}$$

especially,

$$\int_a^b |q(s)|\, ds > \frac{4^{2n-1}(2n-1)[(n-1)!]^2}{(b-a)^{2n-1}}. \tag{2.27}$$

Theorem 2.10. *Let us consider the boundary value problem*

$$x^{(2n)}(t) + q(t)x(t) = 0, \tag{2.28}$$

$$x^{(2i)}(a) = x^{(2i)}(b) = 0, \quad i = 0, 1, \ldots, n-1. \tag{2.29}$$

If $x(t)$ is a solution of (2.28) *satisfying $x(t) \neq 0$, $t \in (a, b)$, then*

$$\int_a^b |q(s)|\, ds > \frac{2^n}{(b-a)^n}. \tag{2.30}$$

Although there is an extensive literature on Lyapunov-type inequalities for different classes of differential equations, there is not much done for linear and nonlinear systems. In 2003, Guseinov and Kaymakçalan [13] obtained Lyapunov-type inequalities for the linear Hamiltonian system

$$\begin{aligned} x'(t) &= a(t)x(t) + b(t)u(t), \\ u'(t) &= -c(t)x(t) - a(t)u(t), \end{aligned} \qquad t \in \mathbb{R} \tag{2.31}$$

and the discrete Hamiltonian system

$$\begin{aligned} \Delta x(t) &= a(t)x(t+1) + b(t)u(t), \\ \Delta u(t) &= -c(t)x(t+1) - a(t)u(t), \end{aligned} \qquad t \in \mathbb{Z}, \tag{2.32}$$

where $1 - a(t) \neq 0$ and $b(t) \geq 0$ for all $t \in \mathbb{Z}$. Their results are as follows.

Theorem 2.11. *Assume that $b(t) \geq 0$ for all $t \in \mathbb{R}$ and assume* (2.31) *has a real solution $(x(t), u(t))$ such that $x(a) = x(b) = 0$ and x is not identically zero on $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. Then the Lyapunov inequality*

$$\int_a^b |a(s)|\, ds + \left(\int_a^b b(s) ds \right)^{1/2} \left(\int_a^b c^+(s) ds \right)^{1/2} \geq 2 \tag{2.33}$$

holds, where $c^+(t) = \max\{0, c(t)\}$ is the nonnegative part of $c(t)$.

Theorem 2.12. *Let $n, m \in \mathbb{Z}$ with $n \leq m - 2$. Assume* (2.32) *has a real solution $(x(t), u(t))$ such that $x(n) = x(m) = 0$ and x is not identically zero on $[n, m]$. Then the inequality*

$$\sum_{t=n}^{m-2} |a(t)| + \left(\sum_{t=n}^{m-1} b(t) \right)^{1/2} \left(\sum_{t=n}^{m-2} c^+(t) \right)^{1/2} \geq 2 \tag{2.34}$$

holds.

Theorem 2.13. *Suppose $1 - a(t) > 0$ and $b(t) > 0$ for all $t \in \mathbb{Z}$. Let $n, m \in \mathbb{Z}$ with $n \leq m - 2$. Assume* (2.32) *has a real solution $(x(t), u(t))$ such that $x(n) = 0$ and $x(m-1)x(m) < 0$. Then the inequality*

$$\sum_{t=n}^{m-2} |a(t)| + \left(\sum_{t=n}^{m-2} b(t) \right)^{1/2} \left(\sum_{t=n}^{m-2} c^+(t) \right)^{1/2} > 1 \tag{2.35}$$

holds.

Theorem 2.14. *Suppose $1 - a(t) > 0$ and $b(t) > 0$ for all $t \in \mathbb{Z}$. Let $n, m \in \mathbb{Z}$ with $n \leq m - 1$. Assume* (2.32) *has a real solution $(x(t), u(t))$ such that $x(n-1)x(n) < 0$, $x(m) = 0$. Then the inequality*

$$\sum_{t=n}^{m-2} |a(t)| + \left(\sum_{t=n}^{m-1} b(t) \right)^{1/2} \left(\sum_{t=n-1}^{m-2} c^+(t) \right)^{1/2} > 1 \tag{2.36}$$

holds.

Theorem 2.15. *Suppose* $1 - a(t) > 0$*,* $b(t) > 0$ *and* $c(t) > 0$ *for all* $t \in \mathbb{Z}$*. Let* $n, m \in \mathbb{Z}$ *with* $n \leq m - 1$*. Assume* (2.32) *has a real solution* $(x(t), u(t))$ *such that* $x(n-1)x(n) < 0$ *and* $x(m-1)x(m) < 0$*, and* $x(t) \neq 0$ *for all* $t \in [n, m-1]$*. Then the inequality*

$$\sum_{t=n-1}^{m-2} |a(t)| + \left(\sum_{t=n-1}^{m-1} b(t)\right)^{1/2} \left(\sum_{t=n-1}^{m-2} c(t)\right)^{1/2} > 1 \tag{2.37}$$

holds.

Theorem 2.16. *Suppose* $1 - a(t) > 0$*,* $b(t) > 0$ *and* $c(t) > 0$ *for all* $t \in \mathbb{Z}$*. Let* $n, m \in \mathbb{Z}$ *with* $n \leq m - 2$*. Assume* (2.32) *has a real solution* $(x(t), u(t))$ *such that* x *has generalized zeros at* n *and* m*, and* x *is not identically zero on* $[n, m]$*. Then the inequality*

$$\sum_{t=n-1}^{m-2} |a(t)| + \left(\sum_{t=n-1}^{m-1} b(t)\right)^{1/2} \left(\sum_{t=n-1}^{m-2} c(t)\right)^{1/2} > 1 \tag{2.38}$$

holds.

In 2005, Jiang and Zhou [16] obtained Lyapunov-type inequalities for the linear Hamiltonian system on time scales

$$\begin{aligned} x^{\Delta}(t) &= a(t)x(\sigma(t)) + b(t)u(t), \\ u^{\Delta}(t) &= -c(t)x(\sigma(t)) - a(t)u(t), \end{aligned} \qquad t \in \mathbb{T}, \tag{2.39}$$

where $1 - \mu(t)a(t) \neq 0$ and $b(t) \geq 0$ for all $t \in \mathbb{T}$. Their results are as follows.

Theorem 2.17. *Let* $a, b \in \mathbb{T}^{\kappa}$ *with* $\sigma(a) < b$*. Assume that* (2.39) *has a real solution* $(x(t), u(t))$ *such that* $x(\sigma(a)) = 0 = x(\sigma(b))$ *and* x *is not identically zero on* $[\sigma(a), b]$*. Then the inequality*

$$\int_{\sigma(a)}^{b} |a(t)|\,\Delta t + \left(\int_{\sigma(a)}^{\sigma(b)} b(t)\Delta t\right)^{1/2} \left(\int_{\sigma(a)}^{b} c^{+}(t)\Delta t\right)^{1/2} \geq 2 \tag{2.40}$$

holds.

Theorem 2.18. *Suppose* $1 - \mu(t)a(t) > 0$ *and* $b(t) > 0$ *for all* $t \in \mathbb{T}$*. Let* $a, b \in \mathbb{T}^{\kappa}$ *with* $\sigma(a) < b$*. Assume that* (2.39) *has a real solution* $(x(t), u(t))$ *such that* $x(\sigma(a)) = 0$ *and* $x(b)x(\sigma(b)) < 0$*. Then the inequality*

$$\int_{\sigma(a)}^{b} |a(t)|\,\Delta t + \left(\int_{\sigma(a)}^{b} b(t)\Delta t\right)^{1/2} \left(\int_{\sigma(a)}^{b} c^{+}(t)\Delta t\right)^{1/2} > 1 \tag{2.41}$$

holds.

Theorem 2.19. *Suppose* $1-\mu(t)a(t)>0$ *and* $b(t)>0$ *for all* $t\in\mathbb{T}$*. Let* $a,b\in\mathbb{T}^\kappa$ *with* $a<b$*. Assume that* (2.39) *has a real solution* $(x(t),u(t))$ *such that* $x(a)x(\sigma(a))<0$ *and* $x(\sigma(b))=0$*. Then the inequality*

$$\int_{\sigma(a)}^{b}|a(t)|\,\Delta t+\left(\int_{\sigma(a)}^{\sigma(b)}b(t)\Delta t\right)^{1/2}\left(\int_a^b c^+(t)\Delta t\right)^{1/2}>1 \tag{2.42}$$

holds.

Theorem 2.20. *Suppose* $1-\mu(t)a(t)>0$*,* $b(t)>0$ *and* $c(t)>0$ *for all* $t\in\mathbb{T}$*. Let* $a,b\in\mathbb{T}^\kappa$ *with* $a<b$*. Assume that* (2.39) *has a real solution* $(x(t),u(t))$ *such that* $x(a)x(\sigma(a))<0$ *and* $x(b)x(\sigma(b))<0$*. Then the inequality*

$$\int_a^b|a(t)|\,\Delta t+\left(\int_a^{\sigma(b)}b(t)\Delta t\right)^{1/2}\left(\int_a^b c(t)\Delta t\right)^{1/2}>1 \tag{2.43}$$

holds.

Theorem 2.21. *Suppose* $1-\mu(t)a(t)>0$*,* $b(t)>0$ *and* $c(t)>0$ *for all* $t\in\mathbb{T}$*. Let* $a,b\in\mathbb{T}^\kappa$ *with* $\sigma(a)<b$*. Assume that* (2.39) *has a real solution* $(x(t),u(t))$ *with generalized zeros in* $\sigma(a)$ *and* $\sigma(b)$ *and* x *is not identically zero on* $[\sigma(a),b]$*. Then the inequality*

$$\int_a^{\sigma(b)}|a(t)|\,\Delta t+\left(\int_a^{\sigma(b)}b(t)\Delta t\right)^{1/2}\left(\int_a^{\sigma(b)}c(t)\Delta t\right)^{1/2}>1 \tag{2.44}$$

holds.

In 2006, Napoli and Pinasco [20] generalized the Lyapunov inequality for quasilinear systems as follows.

Theorem 2.22. *Let* $1<p,q<\infty$*,* $f(x)>0$*,* $g(x)>0$*, and the nonnegative parameters* α *and* β *satisfy* $\dfrac{\alpha}{p}+\dfrac{\beta}{q}=1$*. If the system*

$$\left.\begin{aligned}-\left(|u'(x)|^{p-2}u'(x)\right)'&=f(x)\,|u(x)|^{\alpha-2}u(x)\,|v(x)|^{\beta}\\ -\left(|v'(x)|^{q-2}v'(x)\right)'&=g(x)\,|u(x)|^{\alpha}\,|v(x)|^{\beta-2}v(x)\end{aligned}\right\} \tag{2.45}$$

has a real nontrivial solution $(u(x),v(x))$ *such that* $u(a)=u(b)=v(a)=v(b)=0$*, where* $a,b\in\mathbb{R}$ *with* $a<b$ *are consecutive zeros and* u *and* v *are not identically zero on* $[a,b]$*, then*

$$(b-a)^{\alpha+\beta-1}\left(\int_a^b f(x)dx\right)^{\frac{\alpha}{p}}\left(\int_a^b g(x)dx\right)^{\frac{\beta}{q}}\geq 2^{\alpha+\beta}. \tag{2.46}$$

In 2007, Tiryaki, Ünal and Çakmak [28] obtained Lyapunov-type inequalities for the nonlinear system

$$\begin{aligned} x'(t) &= \alpha_1(t)x(t) + \beta_1(t)\,|u(t)|^{\gamma-2}\,u(t), \\ u'(t) &= -\beta_2(t)\,|x(t)|^{\beta-2}\,x(t) - \alpha_1(t)u(t), \end{aligned} \qquad t \in \mathbb{R}, \tag{2.47}$$

where $\gamma > 1$, $\beta > 1$ are real constants, and $\beta_1(t) > 0$ for $t \in [t_0, \infty)$. Their results are as follows.

Theorem 2.23. *Assume that* (2.47) *has a real solution* $(x(t), u(t))$ *such that* $x(a) = x(b) = 0$ *and* x *is not identically zero on* $[a, b]$*, where* $a, b \in \mathbb{R}$ *with* $a < b$*. Then the inequality*

$$\int_a^b |\alpha_1(t)|\,dt + M^{\frac{\beta}{\alpha}-1}\left(\int_a^b \beta_1(t)dt\right)^{1/\gamma}\left(\int_a^b \beta_2^+(t)dt\right)^{1/\alpha} \geq 2 \tag{2.48}$$

holds, where $\dfrac{1}{\alpha} + \dfrac{1}{\gamma} = 1$ *and* $M = \max\limits_{a<t<b} |x(t)|$.

Theorem 2.24. *Assume that* (2.47) *with* $\alpha_1(t) = 0$ *has a real solution* $(x(t), u(t))$ *such that* $x(a) = x(b) = 0$ *and* x *is not identically zero on* $[a, b]$*, where* $a, b \in \mathbb{R}$ *with* $a < b$*. Then there exists* $\tau \in (a, b)$ *such that the inequalities*

$$M^{\beta-\alpha}\left(\int_a^\tau \beta_1(t)dt\right)^{\alpha-1}\left(\int_a^\tau \beta_2^+(t)dt\right) \geq 1 \tag{2.49}$$

$$M^{\beta-\alpha}\left(\int_\tau^b \beta_1(t)dt\right)^{\alpha-1}\left(\int_\tau^b \beta_2^+(t)dt\right) \geq 1 \tag{2.50}$$

and

$$M^{\beta-\alpha}\left(\int_a^b \beta_1(t)dt\right)^{\alpha-1}\left(\int_a^b \beta_2^+(t)dt\right) \geq 2^\alpha \tag{2.51}$$

hold, where $\dfrac{1}{\alpha} + \dfrac{1}{\gamma} = 1$ *and* $M = \max\limits_{a<t<b} |x(t)|$.

In 2007, Guseinov and Zafer [14] obtained a Lyapunov-type inequality for the linear impulsive Hamiltonian system

$$\begin{aligned} x'(t) &= a(t)x(t) + b(t)u(t), \\ u'(t) &= -c(t)x(t) - a(t)u(t), \\ x(\tau_i+) = \alpha_i x(\tau_i-),\ & u(\tau_i+) = \alpha_i u(\tau_i-) - \beta_i x(\tau_i-), \end{aligned} \qquad t \neq \tau_i, \tag{2.52}$$

where $t \in \mathbb{R}$ and $i \in \mathbb{Z}$. Their result is as follows.

Theorem 2.25. *Let* $a, b, c \in PC[t_1, t_2]$, $b(t) > 0$, *and* $\alpha_i \neq 0$ *for all* $i \in \mathbb{Z}$. *Suppose that* (2.52) *has a real solution* $(x(t), u(t))$ *such that* $x(t_1+) = x(t_2-) = 0$ *and* $x(t) \neq 0$ *on* (t_1, t_2). *Then the inequality*

$$\int_{t_1}^{t_2} |a(s)|\, ds + \left(\int_{t_1}^{t_2} b(s) ds\right)^{1/2} \left\{\int_{t_1}^{t_2} c^+(s) ds + \sum_{t_1 \leq \tau_i < t_2} \left(\frac{\beta_i}{\alpha_i}\right)^+\right\}^{1/2} > 2 \qquad (2.53)$$

holds.

In 2008, Ünal, Çakmak and Tiryaki [29] obtained Lyapunov-type inequalities for the discrete nonlinear systems of the form

$$\begin{aligned} \Delta x(t) &= \alpha_1(t) x(t+1) + \beta_1(t) |u(t)|^{\gamma-2} u(t), \\ \Delta u(t) &= -\beta_2(t) |x(t+1)|^{\beta-2} x(t+1) - \alpha_1(t) u(t), \end{aligned} \qquad t \in \mathbb{Z}, \qquad (2.54)$$

where $\gamma > 1$ and $\beta > 1$ are constants, $\beta_1(t) > 0$ and $1 - \alpha_1(t) \neq 0$ for all $t \in \mathbb{Z}$. Their results are as follows.

Theorem 2.26. *Suppose* $\beta_1(t) > 0$ *for all* $t \in \mathbb{Z}$. *Let* $n, m \in \mathbb{Z}$ *with* $n \leq m - 2$. *Assume* (2.54) *has a real solution* $(x(t), u(t))$ *such that* $x(n) = x(m) = 0$ *and* x *is not identically zero on* $[n, m]$. *Then the inequality*

$$\sum_{t=n}^{m-2} |\alpha_1(t)| + M^{\frac{\beta}{\alpha}-1} \left(\sum_{t=n}^{m-1} \beta_1(t)\right)^{1/\gamma} \left(\sum_{t=n}^{m-2} \beta_2^+(t)\right)^{1/\alpha} \geq 2 \qquad (2.55)$$

holds, where α *is the conjugate number to* γ, *i.e.,* $\dfrac{1}{\gamma} + \dfrac{1}{\alpha} = 1$, $M = |x(\tau)| = \max\limits_{n+1 \leq t \leq m-1} |x(t)|$.

Theorem 2.27. *Suppose* $1 - \alpha_1(t) > 0$ *and* $\beta_1(t) > 0$ *for all* $t \in \mathbb{Z}$. *Let* $n, m \in \mathbb{Z}$ *with* $n \leq m - 2$. *Assume* (2.54) *has a real solution* $(x(t), u(t))$ *such that* $x(n) = 0$ *and* $x(m-1)x(m) < 0$. *Then the inequality*

$$\sum_{t=n}^{m-2} |\alpha_1(t)| + M^{\frac{\beta}{\alpha}-1} \left(\sum_{t=n}^{m-2} \beta_1(t)\right)^{1/\gamma} \left(\sum_{t=n}^{m-2} \beta_2^+(t)\right)^{1/\alpha} > 1 \qquad (2.56)$$

holds, where α *is the conjugate number to* γ, *i.e.,* $\dfrac{1}{\gamma} + \dfrac{1}{\alpha} = 1$, $M = |x(\tau)| = \max\limits_{n+1 \leq t \leq m-1} |x(t)|$.

Theorem 2.28. *Suppose* $1 - \alpha_1(t) > 0$ *and* $\beta_1(t) > 0$ *for all* $t \in \mathbb{Z}$. *Let* $n, m \in \mathbb{Z}$ *with* $n \leq m - 1$. *Assume* (2.54) *has a real solution* $(x(t), u(t))$ *such that* $x(n-1)x(n) < 0$ *and* $x(m) = 0$. *Then the inequality*

$$\sum_{t=n}^{m-2} |\alpha_1(t)| + M_1^{\frac{\beta}{\alpha}-1} \left(\sum_{t=n}^{m-1} \beta_1(t)\right)^{1/\gamma} \left(\sum_{t=n-1}^{m-2} \beta_2^+(t)\right)^{1/\alpha} > 1 \qquad (2.57)$$

holds, where α is the conjugate number to γ and $M_1 = |x(\tau)| = \max_{n \le t \le m-1} |x(t)|$.

Theorem 2.29. *Suppose $1 - \alpha_1(t) > 0$, $\beta_1(t) > 0$ and $\beta_2(t) > 0$ for all $t \in \mathbb{Z}$. Let $n, m \in \mathbb{Z}$ with $n \le m - 1$. Assume* (2.54) *has a real solution $(x(t), u(t))$ such that $x(n-1)x(n) < 0$ and $x(m-1)x(m) < 0$, and $x(t) \neq 0$ for all $t \in [n, m-1]$. Then the inequality*

$$\sum_{t=n-1}^{m-2} |\alpha_1(t)| + M_2^{\frac{\gamma}{\alpha}-1} \left(\sum_{t=n-1}^{m-1} \beta_1(t) \right)^{1/\alpha} \left(\sum_{t=n-1}^{m-2} \beta_2(t) \right)^{1/\beta} > 1 \tag{2.58}$$

holds, where α is the conjugate number to β and $M_2 = |u(\tau_0)| = \max_{n-1 \le \tau \le m_0 - 1} |u(\tau)|$.

Theorem 2.30. *Suppose $1 - \alpha_1(t) > 0$, $\beta_1(t) > 0$ and $\beta_2(t) > 0$ for all $t \in \mathbb{Z}$. Let $n, m \in \mathbb{Z}$ with $n \le m - 2$. Assume* (2.54) *with $\frac{1}{\gamma} + \frac{1}{\beta} = 1$ has a real solution $(x(t), u(t))$ such that x has generalized zeros at n and m, and x is not identically zero on $[n, m]$. Then the inequality*

$$\sum_{t=n-1}^{m-2} |\alpha_1(t)| + \left(\sum_{t=n-1}^{m-1} \beta_1(t) \right)^{1/\gamma} \left(\sum_{t=n-1}^{m-2} \beta_2(t) \right)^{1/\beta} > 1$$

holds.

In 2008, Ünal and Çakmak [30] obtained Lyapunov-type inequalities for nonlinear system on time scales

$$\begin{aligned} x^{\Delta}(t) &= \alpha_1(t)x(\sigma(t)) + \beta_1(t) \left| u(t) \right|^{\gamma-2} u(t), \\ u^{\Delta}(t) &= -\beta_2(t) \left| x(\sigma(t)) \right|^{\alpha-2} x(\sigma(t)) - \alpha_1(t)u(t), \end{aligned} \tag{2.59}$$

where $1 - \mu(t)\alpha_1(t) \neq 0$ and $\beta_1(t) > 0$, $\alpha > 1$ is constant and α is the conjugate number of γ, i.e., $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$. Their results are as follows.

Theorem 2.31. *Suppose $\beta_1(t) > 0$ for all $t \in \mathbb{T}$. Let $a, b \in \mathbb{T}^{\kappa}$ with $\sigma(a) < b$. Assume that* (2.59) *has a real solution $(x(t), u(t))$ such that $x(\sigma(a)) = 0 = x(\sigma(b))$ and x is not identically zero on $[\sigma(a), b]$. Then the inequality*

$$\int_{\sigma(a)}^{b} |\alpha_1(t)| \, \Delta t + \left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t) \Delta t \right)^{1/\gamma} \left(\int_{\sigma(a)}^{b} \beta_2^{+}(t) \Delta t \right)^{1/\alpha} \ge 2 \tag{2.60}$$

holds.

Theorem 2.32. *Suppose* $1 - \mu(t)\alpha_1(t) > 0$ *and* $\beta_1(t) > 0$ *for all* $t \in \mathbb{T}$. *Let* $a, b \in \mathbb{T}^\kappa$ *with* $\sigma(a) < b$. *Assume that* (2.59) *has a real solution* $(x(t), u(t))$ *such that* $x(\sigma(a)) = 0$ *and* $x(b)x(\sigma(b)) < 0$. *Then the inequality*

$$\int_{\sigma(a)}^{b} |\alpha_1(t)|\,\Delta t + \left(\int_{\sigma(a)}^{b} \beta_1(t)\Delta t\right)^{1/\gamma} \left(\int_{\sigma(a)}^{b} \beta_2^+(t)\Delta t\right)^{1/\alpha} > 1 \tag{2.61}$$

holds.

Theorem 2.33. *Suppose* $1 - \mu(t)\alpha_1(t) > 0$ *and* $\beta_1(t) > 0$ *for all* $t \in \mathbb{T}$. *Let* $a, b \in \mathbb{T}^\kappa$ *with* $a < b$. *Assume that* (2.59) *has a real solution* $(x(t), u(t))$ *such that* $x(a)x(\sigma(a)) < 0$ *and* $x(\sigma(b)) = 0$. *Then the inequality*

$$\int_{\sigma(a)}^{b} |\alpha_1(t)|\,\Delta t + \left(\int_{\sigma(a)}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma} \left(\int_{a}^{b} \beta_2^+(t)\Delta t\right)^{1/\alpha} > 1 \tag{2.62}$$

holds.

Theorem 2.34. *Suppose* $1 - \mu(t)\alpha_1(t) > 0$, $\beta_1(t) > 0$ *and* $\beta_2(t) > 0$ *for all* $t \in \mathbb{T}$. *Let* $a, b \in \mathbb{T}^\kappa$ *with* $a < b$. *Assume that* (2.59) *has a real solution* $(x(t), u(t))$ *such that* $x(a)x(\sigma(a)) < 0$ *and* $x(b)x(\sigma(b)) < 0$. *Then the inequality*

$$\int_{a}^{b} |\alpha_1(t)|\,\Delta t + \left(\int_{a}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma} \left(\int_{a}^{b} \beta_2(t)\Delta t\right)^{1/\alpha} > 1 \tag{2.63}$$

holds.

Theorem 2.35. *Suppose* $1 - \mu(t)\alpha_1(t) > 0$, $\beta_1(t) > 0$ *and* $\beta_2(t) > 0$ *for all* $t \in \mathbb{T}$. *Let* $a, b \in \mathbb{T}^\kappa$ *with* $\sigma(a) < b$. *Assume that* (2.59) *has a real solution* $(x(t), u(t))$ *with generalized zeros in* $\sigma(a)$ *and* $\sigma(b)$ *and* x *is not identically zero on* $[\sigma(a), b]$. *Then the inequality*

$$\int_{a}^{\sigma(b)} |\alpha_1(t)|\,\Delta t + \left(\int_{a}^{\sigma(b)} \beta_1(t)\Delta t\right)^{1/\gamma} \left(\int_{a}^{\sigma(b)} \beta_2(t)\Delta t\right)^{1/\alpha} > 1 \tag{2.64}$$

holds.

In 2010, Yang and Lo [33] obtained a Lyapunov-type inequality for even-order differential equations. Their result is as follow.

Theorem 2.36. *Consider the* $2n$*-order linear differential equation*

$$(r_{2n-1}(t)(r_{2n-2}(t)(\ldots(r_2(t)(r_1(t)x')')'\ldots)')')' + q(t)x = 0, \tag{2.65}$$

where $r_k \in C^{2n-k}([a,b],(0,+\infty))$, $k = 1,2,\ldots,2n-1$, $q \in C([a,b],\mathbb{R})$. *If* $x(t)$ *is a nonzero solution of* (2.65) *satisfying*

$$x^{(k)}(a) = x^{(k)}(b) = 0, \quad k = 0,1,2,\ldots,n-1, \tag{2.66}$$

then we have

$$\int_a^b \frac{ds}{r_1(s)} \int_a^b |q(s)|\,ds > 4 \tag{2.67}$$

for $n = 1$,

$$2\int_a^b |q(s)|\,ds > \min_{c\in[a,b]} H_2(c) \tag{2.68}$$

for $n = 2$, *where*

$$H_2(c) = \frac{1}{\int_a^c \frac{ds}{r_1(s)} \int_a^c \frac{ds}{r_2(s)} \int_a^c \frac{ds}{r_3(s)}} + \frac{1}{\int_c^b \frac{ds}{r_1(s)} \int_c^b \frac{ds}{r_2(s)} \int_c^b \frac{ds}{r_3(s)}}, \tag{2.69}$$

and

$$2\left[\prod_{k=n+2}^{2n-1} \int_a^b \frac{ds}{r_k(s)}\right] \int_a^b |q(s)|\,ds > \min_{c\in[a,b]} H_n(c) \tag{2.70}$$

for $n \geq 3$, *where*

$$H_n(c) = \frac{1}{\prod_{k=1}^{n+1} \int_a^c \frac{ds}{r_k(s)}} + \frac{1}{\prod_{k=1}^{n+1} \int_c^b \frac{ds}{r_k(s)}}. \tag{2.71}$$

In 2010, Çakmak [6] obtained Lyapunov-type inequalities for certain higher-order differential equations. His results, which are improvements of the results of Yang [32] (see Theorems 2.8 and 2.10) are as follows.

Theorem 2.37. *Let* $n \in \mathbb{N}$, $n \geq 2$, $q(t) \in C([a,b])$. *If the differential equation*

$$x^{(n)} + q(t)x = 0 \tag{2.72}$$

has a solution $x(t)$ *satisfying the boundary value problem*

$$x(a) = x(t_2) = \ldots = x(t_{n-1}) = x(b) = 0, \tag{2.73}$$

where $a = t_1 < t_2 < \ldots < t_{n-1} < t_n = b$ *and* $x(t) \neq 0$ *for* $t \in (t_k, t_{k+1})$, $k = 1,2,\ldots,n-1$, *then*

$$\int_a^b |q(s)|\,ds > \frac{(n-2)!n^n}{(n-1)^{n-1}(b-a)^{n-1}}. \tag{2.74}$$

Theorem 2.38. *Consider the boundary value problem*

$$x^{(2n)} + q(t)x = 0, \tag{2.75}$$

$$x^{(2i)}(a) = x^{(2i)}(b) = 0, \quad i = 0,1,\ldots,n-1. \tag{2.76}$$

If $x(t)$ *is a solution of* (2.75) *satisfying* $x(t) \neq 0$ *for* $t \in (a,b)$, *then*

$$\int_a^b |q(s)|\,ds > \frac{2^{2n}}{(b-a)^{2n-1}}. \tag{2.77}$$

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