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TOPICAL REVIEW

Recent progress in the boundary control method

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Abstract
The review covers the period 1997–2007 of development of the boundary control method, which is an approach to inverse problems based on their relations to control theory (Belishev 1986). The method solves the problems on unknown manifolds: given inverse data of a dynamical system associated with a manifold it recovers the manifold, the operator governing the system and the states of the system defined on the manifold. The main subject of the review is the extension of the boundary control method to the inverse problems of electrodynamics, elasticity theory, impedance tomography, problems on graphs as well as some new relations of the method to functional analysis and topology.

To the centenary of the birth of Mark Grigor'evich Krein

Contents

1. Introduction

1.1. About the review

The boundary control method (BCm) (Belishev 1986) is an approach to inverse problems based on their relations to control theory. This review covers the period 1997–2007 of development of the method. As the previous review [15], this one deals mainly (but not only) with problems on Riemannian manifolds, the manifolds being not given but subject to reconstruction. However, the list of problems is extended: besides the scalar (acoustic) ones, it includes inverse problems of electrodynamics, elasticity theory, electric impedance tomography and problems on graphs. Such an extension of the method, as well as its new relations to functional analysis and topology, is the main subject of the paper.

1.2. Philosophy

The boundary control method follows the general principles of system theory (see [71, p 256]), which we expose in a slight abuse of notation and terms.

P1. If a dynamical system α is tested via causal experiments and its (causal) input/output map R is determined from the equations of motion, then R depends only on a subsystem α_0 of α which is completely controllable (reachable). The other part of α has no effects on R and may be chosen completely arbitrarily without altering R.

P2. If any two systems, α' and α'', are completely controllable, and have the same input/output map R then they differ only in the representation of their state space.

P3. If a realization α of R is completely controllable, then it is essentially uniquely determined by R, since the representation of states can never be inferred from input/output experiments.

Turn to the problem of determination of a Riemannian manifold from its boundary inverse data. If two manifolds are isometric then, identifying properly their boundaries, we get two manifolds with a common boundary, which possess identical boundary data. Such manifolds are called equivalent: they are indistinguishable for the external observer extracting information about the manifold from the boundary measurements (see *P2*). Therefore, these data do not determine the manifold uniquely and the formulation of the problem has to be clarified. The relevant setup is given in the form of two questions.

(1) Does the coincidence of the inverse data imply the equivalence of the manifolds?

(2) Given the data of an unknown manifold, how to construct a manifold possessing these data?

For certain classes of data, the BCm gives an affirmative answer to the first question and provides a procedure constructing a representative of the class of equivalent manifolds. The procedure consists of three steps. Let α be a system associated with a manifold Ω, $\mathcal{F}$ a space of inputs (controls), $\mathcal{H}$ a space of states, $W : \mathcal{F} \ni f \mapsto u^f \in \mathcal{H}$ an input/state map, $\mathcal{U} = W\mathcal{F}$ a set of reachable states, $R : \mathcal{F} \to \mathcal{F}$ an input/output map playing the role of the inverse data.

Step 1 (coordinatization). To each point $x_0 \in \Omega$ we attach an object $\tilde{x}_0$ (a 'coordinate' of x_0) properly constructed from the states $u^f = Wf \in \mathcal{H}$. A set $\tilde{\Omega} = \{\tilde{x}_0 \mid x_0 \in \Omega\}$ is endowed with a metric $\tilde{d}$ in such a way that (i) $\tilde{d}$ is defined in intrinsic terms of the reachable set $\mathcal{U}$ and is determined by the map W and (ii) $\tilde{d}$ turns $\tilde{\Omega}$ into an isometric copy of the original Ω. Loosely speaking, the copy $\tilde{\Omega}$ is the manifold Ω coded (coordinatized) via W.

Step 2 (constructing a model). The data R determine the inner product $(u^{f'}, u^{f''})_{\mathcal{H}}$ for any $f', f'' \in \mathcal{F}$, which is one of the key points of the BCm. By this, R determines an (auxiliary) space $\tilde{\mathcal{H}}$ and a map $\tilde{W} : \mathcal{F} \to \tilde{\mathcal{H}}$ such that the correspondence $Wf \mapsto \tilde{W}f$ is an isometry. The pair $\{\tilde{\mathcal{H}}, \tilde{W}\}$ plays the role of a copy (model) of $\{\mathcal{H}, W\}$. The external observer with knowledge of R can construct the model without leaving the space $\mathcal{F}$. Also note that the model corresponds not to the system α as a whole but its controllable part (reachable set) $\mathcal{U}$ (see *P1*).

Step 3 (reproducing $\tilde{\Omega}$). Repeating the construction of $\tilde{x}_0$ from the states $u^f = Wf \in \mathcal{H}$ but replacing the original states by the model ones $\tilde{u}^f = \tilde{W}f \in \tilde{\mathcal{H}}$, we reproduce a sample of the copy $\tilde{\Omega}$ and endow it with the relevant metric. Since the correspondence model/system is an isometry, the sample is isometric to the copy $\tilde{\Omega}$ and we can identify the sample with the copy (see *P2*). By construction, the recovered $\tilde{\Omega}$ is isometric to the original Ω and, identifying properly $\partial\tilde{\Omega}$ to $\partial\Omega$, we get two manifolds with a common boundary. By construction, the data $\tilde{R}$ of $\tilde{\Omega}$ coincide with the given R. Hence, the required representative is provided.

Dealing with problems on manifolds, the BCm supplements the principles *P1–P3* with canonical realizations of systems associated with manifolds.

In operator theory, 'to solve an inverse problem' is usually understood as 'given certain data (spectrum, characteristic function, etc) to construct a canonical realization (model) belonging to a certain class of operators and possessing these data'. The BCm can be positioned as an approach representing this viewpoint in inverse problems of mathematical physics: the method constructs *canonical models* relevant to problems on manifolds.

1.3. Coordinatization

The key point of the program outlined above (steps 1–3) is a proper choice of the 'coordinates' $\tilde{x}_0$ on Ω. There are a few known recipes for systems governed by hyperbolic and elliptic equations.

(i) $\tilde{x}_0$ *is a Dirac measure* δ_{x_0}. If the system α is controllable from the boundary[1] then, for each $x_0 \in \Omega$, one can represent $\delta_{x_0} = \lim_{j\to\infty} u^{f_j}$, where $u^{f_j} = Wf_j$ are the states

[1] This property is of crucial character: it names the BC-method.

(waves) produced by the boundary controls $f_j \in \mathcal{F}$, the sequence $\{f_j\}$ being universal, i.e., available for any Ω. Informally, δ_{x_0} is interpreted as a wave produced by a boundary control and focused on the point x_0. Solving inverse problems, we determine a model $\{\tilde{\mathcal{H}}, \tilde{W}\}$ from R and then construct the copy $\tilde{\Omega} = \left\{\tilde{\delta}_{x_0} \middle| x_0 \in \Omega\right\}$, $\tilde{\delta}_{x_0} = \lim_{j\to\infty} \tilde{u}^{f_j}$, $\tilde{u}^{f_j} = \tilde{W} f_j$. Coordinatization by the focused waves was originated in the first paper on the BCm [8]; a procedure solving the problems on graphs is in fact a version of this trick (see section 5).

(ii) $\tilde{x}_0$ *is a pair of semigeodesic coordinates* (γ_0, τ_0) *of the point* x_0. The set of these coordinates Θ (a pattern of the manifold Ω) can be recovered from R through the geometric optics formulae (see sections 2 and 3) and endowed with a metric, which turns the pattern into an isometric copy of Ω. Such a coordinatization was introduced in [12] and applied to reconstruction of manifolds in [47, 50] (see also [15]).

(iii) $\tilde{x}_0$ *is a boundary distance function* $\tau_{x_0} : \partial\Omega \to \overline{\mathbf{R}}_+$, $\tau_{x_0}(\cdot) := \mathrm{dist}(\cdot, x_0)$. For a certain class of metrics and manifolds, the set $\tilde{\Omega} = \left\{\tau_{x_0} \middle| x_0 \in \Omega\right\}$ endowed with $C(\partial\Omega)$-metric is an isometric copy of Ω (Kurylev [70]). The determination $R \Rightarrow \tau_{x_0}$ is derived from a procedure proposed in [49] for solving the dynamical inverse problem.

(iv) $\tilde{x}_0$ *is a family of increasing subspaces (a nest), which consists of waves produced by the infinitesimal source supported at* x_0. Such a source ('wave cap') is built up of waves $u^f = Wf$. The set of nests $\tilde{\Omega}$ is endowed with a travel time metric, which is also determined by the operator W. By this, $\tilde{\Omega}$ can be reproduced through any model $\{\tilde{\mathcal{H}}, \tilde{W}\}$. Emphasizing its nature, we call $\tilde{\Omega}$ a wave copy of the manifold Ω. Coordinatization by nests is the most promoted and promising variant of the BCm for hyperbolic problems: it has an operator background, is relevant to a wide class of dynamical systems with finite speed of wave propagation, and provides time-optimal results in dynamical inverse problems. The wave copy was proposed in [10][2] (see also [30]); in this review, we present its acoustic and electromagnetic versions.

(v) In the 2-dim problem of determining Ω from the elliptic Dirichlet-to-Neumann map, $\tilde{x}_0$ is a Dirac measure δ_{x_0} interpreted as a multiplicative functional of a proper algebra of states, whereas the set $\tilde{\Omega}$ of such functionals (a spectrum of the algebra) is homeomorphic to Ω through the Gelfand transform. The DN-map determines the algebra (up to isometry), its spectrum $\tilde{\Omega}$ and, hence, determines Ω up to homeomorphism [24].

We relate further progress in the BCm to the application of known and search for new variants of coordinatization.

1.4. Content

Section 2: acoustics. In this section, we deal mainly with hyperbolic dynamical systems whose states are scalar functions. The notion of the wave copy is introduced and applied to the problem of determining a Riemannian manifold from its dynamical or spectral inverse boundary data[3]. Some allied problems (in particular, the linkage between the spectral and dynamical data, and the data continuation) are considered. We reveal the operator background of the wave copy, discuss the relations of the BCm to general system theory, and touch on some open problems. The results on numerical testing of the BCm-algorithms are demonstrated.

[2] In fact, [10] is the first paper where the reconstruction of a manifold by the BCm is realized. The procedure recovers $\Omega \subset \mathbb{R}^n$ but needs no change to recover a Riemannian manifold.

[3] So, we solve a solved problem (see [15]) but apply a new version of the BCm available for further extension to more complicated systems.

Section 3: electrodynamics. The Maxwell system on a Riemannian manifold is considered. It is shown that electromagnetic waves are also well suited for constructing the wave copy. The construction provides time-optimal determination of the manifold from the dynamical electromagnetic boundary data. The results on recovery of the velocity $c = (\epsilon\mu)^{-\frac{1}{2}}$ in $\Omega \subset \mathbb{R}^3$ are presented. In addition, a sampling algorithm recovering c provided $\mu = 1$ is described.

Section 4: elasticity. A feature of dynamical systems of elasticity theory is the presence of wave modes propagating with different velocities. The interaction between such modes complicates the structure of reachable sets and the character of controllability. As a result, the application of the BCm encounters serious difficulties. Complete results are obtained for the 1-dim two-velocity system (a beam): we present a procedure recovering the parameters of the beam at its controllable part and provide the characterization of the dynamical inverse data. The results on the 3-dim Lamé system are very far from being complete: the time-optimal determination of the Lamé parameters is one of the most required problems in applications (geophysics, engineering, etc) but also one of the most difficult inverse problems. We deal with a simplified version of the Lamé system and recover the velocities of the shear and pressure waves in the subdomains controllable from the boundary. Inverse problems of elasticity theory is one of most important directions for further development of the BCm.

Section 5: impedance tomography of manifolds. Elliptic problems are a new area of application of the BCm. We present a new approach to the determination of a 2-dim manifold from its Dirichlet-to-Neumann map[4]. The procedure is based on the coordinatization described above in section 1.3(v). Our approach provides explicit formulae expressing the Betti numbers of 2-dim and 3-dim manifolds through their DN-maps. Extending the DN-operator to differential forms and introducing a relevant analog of the classical Hilbert transform, we generalize the formulae to the n-dim case and reveal some relations of the DN-operator and the Hilbert transform to algebraic topology (V A Sharafutdinov). The formulae are based on the Hodge–Morrey–Friedrichs decompositions of the harmonic field spaces.

Section 6: problems on graphs. These problems are also new for the BCm. We show that the spectral data of a tree composed of a finite number of strings of variable density, determine the tree up to a spatial isometry on the plane. A procedure recovering the tree exploits the coordinatization $\tilde{x}_0 = \delta_{x_0}$. However, here we do not construct the Dirac measures as focused waves, but identify them through the inverse data, in the spirit of duality 'controllability–observability' well known in system theory.

Reducing the volume of the review, we demonstrate the proofs of a few key propositions only, referring the reader to the original papers. By the same reasoning, we deal with the C^∞-smooth case and with data given on the whole boundary[5].

In the text, the following abbreviations are in use: BCm—the boundary control method; IP—inverse problem; RM—Riemannian manifold; sgc—semigeodesic coordinates, AF—amplitude formula. Everywhere 'smooth' means C^∞-smooth. With the exception of sections 2.3.7 and 5.1, we deal with real-valued functions and spaces. The reader is appealed to not ignore the Comments: certain of our results (in particular, concerning the partial and finitely smooth data) and open problems are placed there.

[4] Uniqueness of determination was shown by Lassas and Uhlmann in [80].

[5] Actually, determination from data given on a part of the boundary is a folklore of the BCm: all its variants, including the first one [8], are available for this case (see [15]).

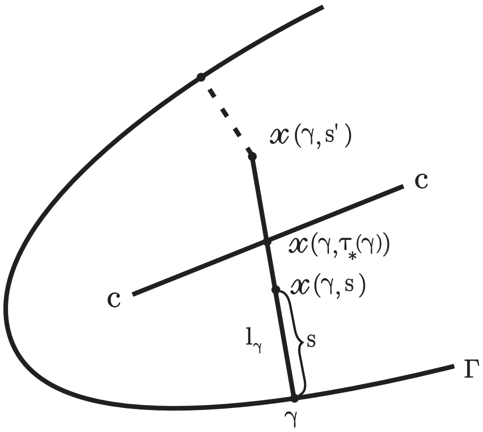

Figure 1. Cut locus.

2. Acoustics

2.1. Geometry

2.1.1. Eikonal, cut locus, sgc. Let Ω be a smooth compact Riemannian manifold (RM), $\Gamma := \partial\Omega \in C^\infty$, $\dim \Omega = n \geqslant 2$; d and g the distance and metric tensor in Ω (we write also (Ω, d) and/or (Ω, g)). For a subset $A \subset \Omega$, we denote its metric neighborhoods by

$$\Omega^r[A] := \{x \in \Omega \mid d(x, A) < r\} \qquad r > 0, \quad \Omega^0[A] := A.$$

For $A = \Gamma$, we set $\Omega^r := \Omega^r[\Gamma]$.

A function $\tau(\cdot) := d(\cdot, \Gamma)$ on Ω is called an *eikonal*. By the definitions, we have $\Omega^r = \{x \in \Omega \mid \tau(x) < r\}$, $r > 0$; the level sets of the eikonal

$$\Gamma^s := \{x \in \Omega \mid \tau(x) = s\}, \qquad s \geqslant 0$$

are the hypersurfaces equidistant to Γ. Later, in dynamics, the value

$$T_* := \max_\Omega \tau(\cdot) = \inf\{r > 0 \mid \Omega^r = \Omega\}$$

is interpreted as the time needed for waves moving from Γ with unit speed to fill Ω.

Recall the definition of a separation set (*cut locus*) of Ω w.r.t. Γ (see, e.g., [64]). Let $l_\gamma[0, s]$ be a segment of length s of a geodesic l_γ emanating from $\gamma \in \Gamma$ orthogonally to Γ; let $x(\gamma, s)$ be its second endpoint. The value $\tau_*(\gamma)$ is said to be a critical length if $\tau(x(\gamma, s)) = s$ for $0 \leqslant s \leqslant \tau_*(\gamma)$ (i.e., $l_\gamma[0, s]$ minimizes the distance between $x(\gamma, s)$ and Γ) and $\tau(x(\gamma, s)) < s$ for $s > \tau_*(\gamma)$ (i.e., $l_\gamma[0, s]$ does not minimize the distance; see figure 1, where $s < \tau_*(\gamma) < s'$). The function $\tau_*(\cdot)$ is continuous on Γ.

The point $x(\gamma, \tau_*(\gamma))$ is a separation point on l_γ. A set of separation points $c := \bigcup_{\gamma\in\Gamma} x(\gamma, \tau_*(\gamma))$ is called a cut locus (of Ω w.r.t. Γ). The cut locus is a closed set of zero volume. We denote $T_c := d(c, \Gamma)$.

For $x \in \Omega\backslash c$, there is a unique point $\gamma(x) \in \Gamma$ nearest to x and a pair $(\gamma(x), \tau(x))$ constitutes the *semigeodesic coordinates* (sgc) of x. A set

$$\Theta := \overline{\{(\gamma(x), \tau(x)) \mid x \in \Omega\backslash c\}} = \{(\gamma, s) \mid \gamma \in \Gamma, 0 \leqslant s \leqslant \tau_*(\gamma)\} \subset \Gamma \times [0, T_*]$$

is called a *pattern* of Ω; its subset $\Theta^T := \Theta \cap \{\Gamma \times [0, T]\}$ is regarded as the pattern of the submanifold Ω^T (see figure 2). The patterns are the subgraphs of the functions $\tau_*(\cdot)$

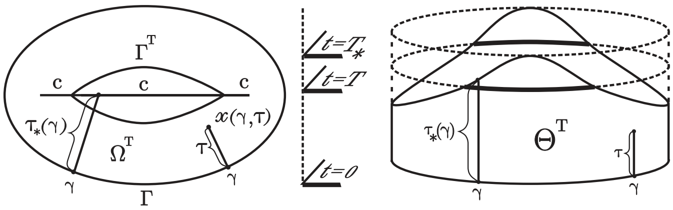

Figure 2. Manifold and its pattern.

and $\tau_*^T(\cdot) := \min\{\tau_*(\cdot), T\}$. Note in addition that $\Gamma^T \backslash c$ is a smooth (maybe disconnected) $(n-1)$-dimensional submanifold in Ω. In the following, dealing with a fixed $T > 0$, we assume Γ^T to be a piecewise smooth hypersurface. This condition provides the integration by parts in Ω^T.

2.1.2. Caps. For $\gamma \in \Gamma$, $s > 0$ and a (small) $\varepsilon > 0$, let $\sigma_\varepsilon(\gamma) := \{\gamma' \in \Gamma \mid d(\gamma', \gamma) < \varepsilon\}$ be a patch of the boundary. Introduce a family of closed subdomains (*caps*)

$$\omega_\gamma^{s,\varepsilon} := \overline{\Omega}^s[\sigma_\varepsilon(\gamma)] \cap \{\overline{\Omega}^s \backslash \Omega^{s-\varepsilon}\} = \{x \in \Omega \mid d(x, \sigma_\varepsilon(\gamma)) \leqslant s, s - \varepsilon \leqslant \tau(x) \leqslant s\}, \qquad \gamma \in \Gamma, s > 0, 0 < \varepsilon < s \tag{2.1}$$

and set $\omega_\gamma^s = \lim_{\varepsilon \to 0} \omega_\gamma^{s,\varepsilon} := \bigcap_{0<\varepsilon<s} \omega_\gamma^{s,\varepsilon}$.

Lemma 1. *The relation*

$$\omega_\gamma^s = \begin{cases} x(\gamma, s), & 0 < s \leqslant \tau_*(\gamma) \\ \{\emptyset\}, & s > \tau_*(\gamma) \end{cases} \tag{2.2}$$

holds.

Proof. See the appendix. So, the cap $\omega_\gamma^{s,\varepsilon}$ shrinks to the point $x(\gamma, s)$ if $(\gamma, s) \in \Theta$, and terminates (disappears for small enough ε) if $(\gamma, s) \notin \Theta$ (in figure 3 the cap $\omega_\gamma^{s,\varepsilon}$ is shadowed, $\tau_*(\gamma) > s > \tau_*(\gamma')$).[6] By this, the metric neighborhoods of the cup behave as follows:

$$\lim_{\varepsilon \to 0} \Omega^r\left[\omega_\gamma^{s,\varepsilon}\right] = \Omega^r[x(\gamma, s)], \qquad (\gamma, s) \in \Theta, \quad r \geqslant 0. \tag{2.3}$$

If a point $x \in c$ is represented as $x = x(\gamma'', s) = x(\gamma''', s)$ (see figure 3), then the passage to the limit (2.3) produces one and the same family of neighborhoods

$$\lim_{\varepsilon \to 0} \Omega^r\left[\omega_{\gamma''}^{s,\varepsilon}\right] = \lim_{\varepsilon \to 0} \Omega^r\left[\omega_{\gamma'''}^{s,\varepsilon}\right] = \Omega^r[x], \qquad r \geqslant 0 \tag{2.4}$$

and, hence, the limit does not depend on the representation of x. □

2.1.3. Manifold (Ω^T, d^T): metric copy. Fix $T > 0$, let (Ω^T, d^T) be the submanifold of Ω endowed with the intrinsic distance

$$d^T(x', x'') := \inf_{l \subset \Omega^T} (l) \int_{x'}^{x''} |\mathrm{d}x|.$$

In the following, the metric neighborhoods $\Omega^r[A]$, $A \subset \Omega^T$ are understood in the sense of the metric d^T. The caps $\omega_\gamma^{s,\varepsilon}$, $0 < s < T$ lie in Ω^T. For $T \leqslant T_*$, we have $\partial\Omega^T = \Gamma \cup \Gamma^T$; if $T > T_*$ then $\Gamma^T = \{\emptyset\}$ and $(\Omega^T, d^T) = (\Omega, d)$.

[6] This illustration is taken from [30].

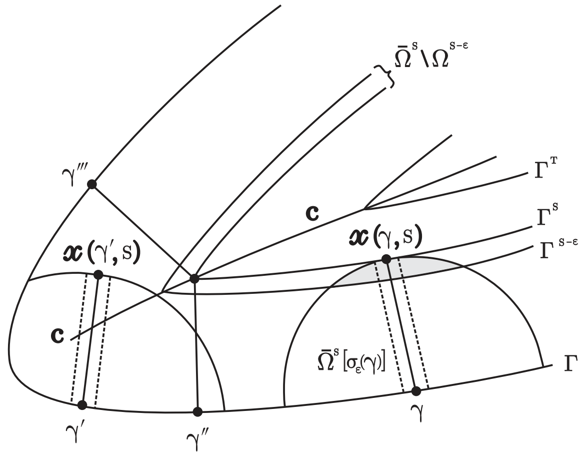

Figure 3. Cap.

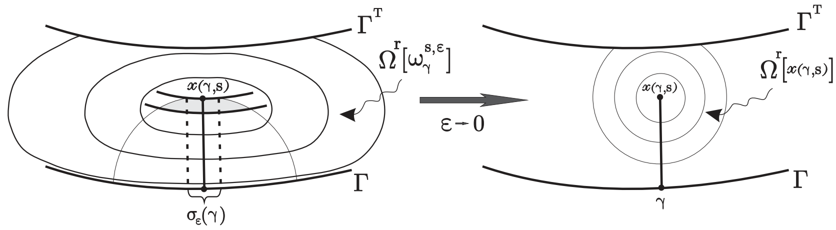

Figure 4. Centering.

For a point $x \in \Omega^T$, a family of the neighborhoods $\check{x} := \{\Omega^r[x]\}_{r\geqslant 0}$ is said to be a *metric nest* centered at x. The passage to the limit (2.3), which we call *a centering*, produces the nest $\check{x}(\gamma, s)$ from the cap $\omega_\gamma^{s,\varepsilon}$ (see figure 4). One more passage to the limit (in the evident sense) extends (2.3) to the nests centered at the boundary points:

$$\check{\gamma} = \lim_{s\to 0} \check{x}(\gamma, s). \tag{2.5}$$

Introduce a set of metric nests $\check{\Omega}^T := \{\check{x} \mid x \in \Omega^T\}$ and endow it with a function $\check{d}^T : \check{\Omega}^T \times \check{\Omega}^T \to \overline{\mathbb{R}}_+$,

$$\check{d}^T(\check{x}', \check{x}'') := 2\inf\{r > 0 \mid \Omega^r[x'] \cap \Omega^r[x''] \neq \{\emptyset\}\}, \tag{2.6}$$

where x' and x'' are the centers of $\check{x}'$ and $\check{x}''$. In view of the evident equality $\check{d}^T(\check{x}', \check{x}'') = d^T(x', x'')$, function $\check{d}^T$ is a metric, whereas a metric space $(\check{\Omega}^T, \check{d}^T)$ is isometric to (Ω^T, d^T). Identifying $\check{\gamma} \equiv \gamma$, we get two isometric manifolds $(\check{\Omega}^T, \check{d}^T)$ and (Ω^T, d^T) with the common boundary Γ. We call $(\check{\Omega}^T, \check{d}^T)$ a *metric copy* of the original manifold (Ω^T, d^T) (see figure 5).

2.2. Dynamics

2.2.1. System α^T. Propagation of acoustic (scalar) waves in the manifold Ω^T, initiated by boundary sources acting on Γ, is described by a dynamical system α^T of the form

$$u_{tt} - \Delta u = 0 \qquad \text{in} \quad (\text{int}\,\Omega^T) \times (0, T) \tag{2.7}$$

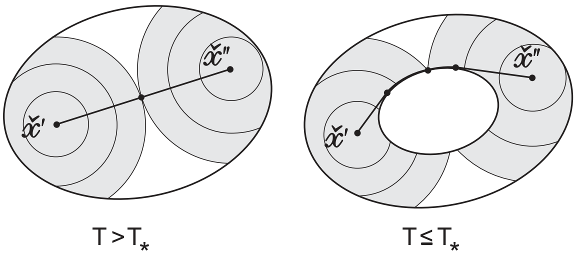

Figure 5. Metric copy.

$$u = 0 \qquad \text{in} \quad \{(x,t) \mid x \in \Omega^T, 0 \leqslant t \leqslant \tau(x)\} \tag{2.8}$$

$$u = f \qquad \text{on} \quad \Gamma \times [0,T] \tag{2.9}$$

where Δ is the Beltrami–Laplace operator, $\operatorname{int}\Omega^T := \Omega^T \backslash \Gamma$, f is a source (Dirichlet *boundary control*), $u = u^f(x,t)$ is a solution (*wave*). By hyperbolicity of the wave equation (2.7), problem (2.7)–(2.9) is well posed, the solution satisfying $\operatorname{supp} u^f(\cdot,t) \subset \overline{\Omega}^t$, $0 \leqslant t \leqslant T$.

An *outer space* of the system (space of controls) is $\mathcal{F}^T := L_2([0,T]; L_2(\Gamma))$. An *inner space* (space of states) is $\mathcal{H}^T := L_2(\Omega^T)$; the waves $u^f(\cdot,t)$ are time-dependent elements of $\mathcal{H}^T$. The input/state map is realized by a *control operator* $W^T : \mathcal{F}^T \to \mathcal{H}^T$

$$W^T f := u^f(\cdot,T),$$

which is a continuous operator[7].

The input/output map is a *response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\operatorname{Dom} R^T = \{f \in H^1([0,T]; H^1(\Gamma)) \mid f|_{t=0} = 0\}$ ($H^\alpha(\cdots)$—the Sobolev classes),

$$R^T f := \left.\frac{\partial u^f}{\partial \nu}\right|_{\Gamma\times[0,T]},$$

where $\nu = \nu(\gamma)$ is an outward normal at $\gamma \in \Gamma$.

One more intrinsic operator of the system α^T is a so-called *continued response operator* associated with the problem

$$u_{tt} - \Delta u = 0 \qquad \text{in} \quad \{(x,t) \mid x \in \operatorname{int}\Omega^T, 0 < t < 2T - \tau(x)\} \tag{2.10}$$

$$u = 0 \qquad \text{in} \quad \{(x,t) \mid x \in \Omega^T, 0 \leqslant t \leqslant \tau(x)\} \tag{2.11}$$

$$u = f \qquad \text{on} \quad \Gamma \times [0,2T] \tag{2.12}$$

which is a natural extension of problem (2.7)–(2.9) by hyperbolicity. The operator is $R^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$, $\operatorname{Dom} R^{2T} = \{f \in H^1([0,2T]; H^1(\Gamma)) \mid f|_{t=0} = 0\}$, $R^{2T} f := \left.\frac{\partial u^f}{\partial \nu}\right|_{\Gamma\times[0,2T]}$; the term 'continued' is motivated by the evident relation

$$(R^{2T} f)|_{\Gamma\times[0,T]} = R^T(f|_{\Gamma\times[0,T]}), \qquad f \in \operatorname{Dom} R^{2T}. \tag{2.13}$$

The operator R^{2T} is determined by the submanifold Ω^T; later it plays the role of data in the dynamical IP. More about this operator in section 2.3.11.

[7] The continuity of W^T is proven in [78]. The referee of this review has informed us that there is a flaw in the proof. At the moment we are not ready to comment on this remark. However, the continuity just simplifies the considerations but plays no crucial role: for instance, the analog of W^T in electrodynamics is not continuous (see section 3).

The central object of the BCm is a *connecting operator* $C^T : \mathcal{F}^T \to \mathcal{F}^T, C^T := (W^T)^* W^T$. By definition, we have

$$(C^T f', f'')_{\mathcal{F}^T} = (W^T f', W^T f'')_{\mathcal{H}^T} = (u^{f'}(\cdot, T), u^{f''}(\cdot, T))_{\mathcal{H}^T}, \tag{2.14}$$

i.e., C^T connects the Hilbert metrics of the outer and inner spaces. The significant fact is that the connecting operator is determined by the continued response operator through an explicit formula

$$C^T = \tfrac{1}{2}(S^T)^* R^{2T} J^{2T} S^T, \tag{2.15}$$

where the map $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ extends the controls from $\Gamma \times [0, T]$ to $\Gamma \times [0, 2T]$ as odd functions of t w.r.t. $t = T$; $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$, $(J^{2T} f)(\cdot, t) := \int_0^t f(\cdot, s)\, ds$ (see [15, 22]).

2.2.2. System $\beta^{T,r}$. The sources distributed in Ω initiate the waves described by the system $\beta^{T,r}$ of the form

$$w_{tt} - \Delta w = h \qquad \text{in} \quad (\operatorname{int} \Omega^T) \times (0, r) \tag{2.16}$$

$$w|_{t=0} = w_t|_{t=0} = 0 \qquad \text{in} \quad \overline{\Omega}^T \tag{2.17}$$

$$w = 0 \qquad \text{on} \quad (\Gamma \cup \Gamma^T) \times [0, r] \tag{2.18}$$

where $h = h(x, t)$ is a *volume control*, $w = w^h(x, t)$ is a wave.

An outer space of the system is $\mathcal{G}^r := L_2([0, r]; \mathcal{H}^T)$; an inner space is $\mathcal{H}^T$. Since Γ^T can be non-smooth, the meaning of the condition (2.18) on Γ^T as well as the definition of w^f have to be clarified. By doing so, introduce the operator $-\Delta_0^T : \mathcal{H}^T \to \mathcal{H}^T$, $\operatorname{Dom}(-\Delta_0^T) = \{y \in C^\infty(\Omega^T) | y|_\Gamma = 0, \operatorname{supp} y \subset \Omega^T\}$ (so that y vanishes near Γ^T), $-\Delta_0^T y = -\Delta y$. The operator $-\Delta_0^T$ is positive definite, let $-\Delta^T$ be its extension by Friedrichs (see, e.g., [57]). Then we define by Duhamel

$$w^h(\cdot, t) := \int_0^t (-\Delta^T)^{-\frac{1}{2}} \sin[(t - s)(-\Delta^T)^{\frac{1}{2}}] h(\cdot, s)\, ds, \qquad t > 0. \tag{2.19}$$

The control operator of the system $\beta^{T,r}$ is $W^r_{\text{vol}} : \mathcal{G}^r \to \mathcal{H}^T$, $W^r_{\text{vol}} h := w^h(\cdot, r)$. By (2.19), W^r_{vol} is continuous.

2.2.3. Controllability. Here we consider a property of the systems α^T and $\beta^{T,r}$ playing a key role in the BCm. For open subsets $\sigma \subset \Gamma, \omega \subset \Omega^T$ and parameters $s \in [0, T], r \geqslant 0$, let

$$\mathcal{F}^{T,s}[\sigma] := \operatorname{clos}\{f \in \mathcal{F}^T \mid \operatorname{supp} f \subset [T - s, T], \operatorname{supp} f(\cdot, t) \subset \sigma, \forall t\},$$
$$\mathcal{F}^T[\sigma] := \mathcal{F}^{T,T}[\sigma], \qquad \mathcal{G}^r[\omega] := \operatorname{clos}\{h \in \mathcal{G}^r \mid \operatorname{supp} h(\cdot, t) \subset \omega, \forall t\}$$

be the subspaces of controls acting from σ and ω respectively; also, simplifying the notation, we omit 'σ' in the case of $\sigma = \Gamma$. Note that the controls $f \in \mathcal{F}^{T,s}[\sigma]$ act with delay $T - s$, the action time being equal to s. By hyperbolicity of problems (2.7)–(2.9) and (2.16)–(2.18), for $f \in \mathcal{F}^{T,s}[\sigma]$ and $h \in \mathcal{G}^r[\omega]$ the relations

$$\operatorname{supp} u^f(\cdot, t) \subset \overline{\Omega}^t[\sigma], \qquad \operatorname{supp} w^h(\cdot, t) \subset \overline{\Omega}^t[\omega] \tag{2.20}$$

hold and show that the waves propagate with a speed $\leqslant 1$.[8]

The sets of waves

$$\mathcal{U}^s[\sigma] := W^T \mathcal{F}^{T,s}[\sigma], \qquad \mathcal{U}^r[\omega] := W^r_{\text{vol}} \mathcal{G}^r[\omega]$$

[8] Recall that the neighborhoods are understood in the sense of the intrinsic metric d^T.

are said to be *reachable* from σ and ω (at times $t = s$ and $t = r$). Once again, simplifying the notation, we omit σ in the case of $\sigma = \Gamma$: $\mathcal{U}^s := \mathcal{U}^s[\Gamma]$.

For a subset $A \subset \Omega^T$, denote

$$\mathcal{H}^r[A] := \operatorname{clos}\{y \in \mathcal{H}^T \mid \operatorname{supp} y \subset \Omega^r[A]\}, \qquad r \geqslant 0$$

and $\mathcal{H}^r := \mathcal{H}^r[\Gamma] = L_2(\Omega^r)$. By virtue of (1.20), the embeddings $\mathcal{U}^s[\sigma] \subset \mathcal{H}^s[\sigma]$ and $\mathcal{U}^r[\omega] \subset \mathcal{H}^r[\omega]$ hold. A significant fact is that these embeddings are dense: for any σ, ω and $s > 0, r > 0$ one has

$$\operatorname{clos} \mathcal{U}^s[\sigma] = \mathcal{H}^s[\sigma], \qquad \operatorname{clos} \mathcal{U}^r[\omega] = \mathcal{H}^r[\omega]. \tag{2.21}$$

In control theory, relations (2.21) are interpreted as *local approximate controllability* of the systems α^T and $\beta^{T,r}$ in the subdomains filled with waves; the name 'BC method' is derived from the first relation (*boundary controllability*). The proof of (2.21) relies on the fundamental Holmgren–John–Tataru unique continuation theorem for the wave equation [95] (see [15] for detail).

2.2.4. Laplacian on waves. For a fixed $T > 0$, a trajectory $\{u^f(\cdot, t) \mid 0 \leqslant t \leqslant T\}$ of the system α^T does not leave the reachable set $\mathcal{U}^T \subset \mathcal{H}^T$. Because of this, the system possesses one more intrinsic operator L_0^T, which acts in the subspace $\operatorname{clos} \mathcal{U}^T$ and is introduced through its graph

$$\operatorname{graph} L_0^T := \left\{\{W^T f, -W^T f_{tt}\} \middle| f \in C_0^\infty((0,T), C^\infty(\Gamma))\right\} \tag{2.22}$$

(see [22]). Since

$$\begin{aligned} L_0^T W^T f &= -W^T f_{tt} = -u^{f_{tt}}(\cdot, T) = -u^f_{tt}(\cdot, T) \\ &= \langle \text{see } (2.7) \rangle = -\Delta u^f(\cdot, T) = -\Delta W^T f \end{aligned}$$

and $\operatorname{Dom} L_0^T \subset \operatorname{Dom}\left(-\Delta_0^T\right)$, we have $L_0^T \subset -\Delta_0^T$, so that L_0^T can be interpreted as *Laplacian on waves* filling Ω^T from the boundary. Since the class of smooth controls defining the graph of L_0^T is dense in $\mathcal{F}^T$, the set $\operatorname{Dom} L_0^T$ is dense in $\operatorname{clos} \mathcal{U}^T$ and, hence, dense in $\mathcal{H}^T$ by (2.21) (for $\sigma = \Gamma$). Therefore, L_0^T is a densely defined positive definite operator in $\mathcal{H}^T$. As such, L_0^T can be extended by Friedrichs; we denote the extension by L^T.

Theorem 1. *For any $T > 0$, the equality $L^T = -\Delta^T$ holds.*

Proof. See in the appendix. □

Thus, a certain canonical procedure (Friedrichs extension) turns the Laplacian on waves into the 'standard' Laplacian in Ω^T with zero Dirichlet conditions on $\Gamma \cup \Gamma^T$. As a result, we can replace $-\Delta^T$ in (2.19) by L^T and represent

$$W^r_{\mathrm{vol}} h := \int_0^r (L^T)^{-\frac{1}{2}} \sin[(r-s)(L^T)^{\frac{1}{2}}] h(\cdot, s)\, \mathrm{d}s. \tag{2.23}$$

Introduce an operation which we call a *dynamical extension* of subspaces: for a subspace $\mathcal{A} \subset \mathcal{H}^T$ define

$$\mathcal{E}^r \mathcal{A} := \operatorname{clos} W^r_{\mathrm{vol}} L_2([0,r]; \mathcal{A}),\ r > 0; \qquad \mathcal{E}^0 := \mathrm{id}. \tag{2.24}$$

By the volume controllability, we have

$$\mathcal{E}^r \mathcal{H}^0[\omega] = \operatorname{clos} \mathcal{U}^r[\omega] = \langle \text{see } (2.21) \rangle = \mathcal{H}^r[\omega]. \tag{2.25}$$

The Laplacian on waves are determined by the operator W^T (see (2.22)). Hence, by (2.23), the operator W^T_{vol} and the operation $\mathcal{E}^r$ are also determined by the boundary control operator W^T, which is very important for the IPs.

2.2.5. *Wave copy.* Here we construct what we call a wave copy of the manifold (Ω^T, d^T). This object is built up of waves produced by boundary controls and, as such, is determined by the operator W^T. The construction is realized in a few steps.

(i) (*wave cap*) Fix $\gamma \in \Gamma, s \in (0,T)$ and $\varepsilon \in (0,s)$; a subspace of the reachable space $\operatorname{clos}\mathcal{U}^T$ of the form

$$\begin{aligned}\mathcal{U}_\gamma^{s,\varepsilon} &:= \operatorname{clos}\mathcal{U}^s[\sigma_\varepsilon(\gamma)] \cap \{\operatorname{clos}\mathcal{U}^s \ominus \operatorname{clos}\mathcal{U}^{s-\varepsilon}\}\\ &= \operatorname{clos} W^T\mathcal{F}^{T,s}[\sigma_\varepsilon(\gamma)] \cap \{\operatorname{clos} W^T\mathcal{F}^{T,s} \ominus \operatorname{clos} W^T\mathcal{F}^{s-\varepsilon}\}\end{aligned} \tag{2.26}$$

is called a *wave cap* (compare with (2.1)).
We assign the given pair (γ, s) to a set $\tilde{\Theta}^T$ if $\mathcal{U}_\gamma^{s,\varepsilon} \neq \{0\}$ for all $\varepsilon \in (0,s)$; otherwise, if $\mathcal{U}_\gamma^{s,\varepsilon}$ terminates as $\varepsilon \to 0$,[9] this pair is ignored in the following (compare with (2.2)). We call $\tilde{\Theta}^T$ a *wave pattern* of the submanifold Ω^T.

(ii) (*nests*) For $(\gamma, s) \in \tilde{\Theta}^T$, construct a *nest of subspaces*

$$\tilde{x}(\gamma,s) := \{\mathcal{U}^r[\tilde{x}(\gamma,s)]\}_{r\geqslant 0}, \qquad \mathcal{U}^r[\tilde{x}(\gamma,s)] := \lim_{\varepsilon\to 0} \mathcal{E}^r\mathcal{U}_\gamma^{s,\varepsilon}, \tag{2.27}$$

where the limit is understood in the sense of the strong operator convergence of the orthogonal projections on the corresponding subspaces (compare with (2.3) and the definition of $\check{x}$ in section 2.1.3).

Supplement $\tilde{\Theta}^T$ with the pairs $(\gamma, 0)$ and construct the set of 'boundary' nests $\tilde{\Gamma} := \{\tilde{\gamma} \mid \gamma \in \Gamma\}, \tilde{\gamma} := \lim_{s\to 0}\tilde{x}(\gamma, s)$ (compare with (2.5)).

(iii) (*the copy*) Collect the set of all nests $\tilde{\Omega}^T := \{\tilde{x}(\gamma,s) \mid (\gamma,s) \in \tilde{\Theta}^T\}$ and endow it with a function $\tilde{d}^T : \tilde{\Omega}^T \times \tilde{\Omega}^T \to \overline{\mathbb{R}}_+$,

$$\tilde{d}^T(\tilde{x}', \tilde{x}'') := 2\inf\{r > 0 \mid \mathcal{U}^r[\tilde{x}'] \cap \mathcal{U}^r[\tilde{x}''] \neq \{0\}\} \tag{2.28}$$

(compare with (2.6)). Identify $\tilde{\Gamma}$ to Γ by $\tilde{\gamma} \equiv \gamma$.

We name the pair $(\tilde{\Omega}^T, \tilde{d}^T)$ a wave copy of the submanifold (Ω^T, d^T) and stress once again that the wave copy is determined by the boundary control operator W^T. From the standpoint of physics, the wave copy represents the submanifold in the form of collection of infinitesimal sources, which interact with each other through the waves they produce, the collection being endowed with the travel time metric.

The wave copy is an intrinsic object of a wide class of dynamical systems governed by hyperbolic equations in Ω. The external observer investigating Ω, probes it with waves initiated at the boundary. The observer measures the result of interaction of these waves with the internal structure of Ω at the same boundary. In perfect accordance with Thesis *P1* (see section 1), such measurements contain information about the reachable set $\mathcal{U}^T$ only, and the wave copy $(\tilde{\Omega}^T, \tilde{d}^T)$ built up of elements of this set (waves), is a relevant realization of this information. As such, *the wave copy can be reproduced through the boundary inverse data*, which is the cornerstone of our program for solving the IPs by the BCm.

Controllability (2.21) links the sets of waves to the subdomains supporting the waves in a straightforward way: roughly speaking, we can identify $\mathcal{U}^r[\omega]$ with $\mathcal{H}^r[\omega]$ and, hence, with a subdomain $\Omega^r[\omega]$. Correspondingly, a wave cap $\mathcal{U}_\gamma^{s,\varepsilon} = L_2\big(\omega_\gamma^{s,\varepsilon}\big)$ can be identified with a cap $\omega_\gamma^{s,\varepsilon}$. By this, searching the construction (i)–(iii), we see that a wave pattern $\tilde{\Theta}^T$ coincides with a pattern Θ^T, each nest $\tilde{x}(\gamma,s)$ is identical to $\check{x}(\gamma,s)$, a function $\tilde{d}^T$ is a metric and, eventually, a wave copy $(\tilde{\Omega}^T, \tilde{d}^T)$ turns out to be identical to a metric copy $(\check{\Omega}^T, \check{d}^T)$. As a result, $(\tilde{\Omega}^T, \tilde{d}^T)$ is isometric to the original (Ω^T, d^T) and has the same boundary Γ.

[9] That is, $\mathcal{U}_\gamma^{s,\varepsilon} = \{0\}$ for small enough positive ε.

The coincidence of wave and metric copies is a specific feature of dynamical systems, in which the different wave modes (if many) propagate with one and the same velocity. As we shall see, in multi-velocity systems relation (2.21) does not hold and the structure of reachable sets $\mathcal{U}^s[\sigma]$ is more complicated (if known). It is a fact that poses major difficulties for solving the corresponding IPs.

2.2.6. Visualization of waves. Here the wave copy is supplemented with an additional option: given a control f, for any $x \in \Omega$, we represent the value of the wave $u^f(x, T)$ in terms of the wave copy. It is what we call *visualization*.

For a linear subset $\mathcal{A}$ of a Hilbert space, we denote by $P_{\mathcal{A}}$ the (orthogonal) projection on clos $\mathcal{A}$. By controllability (2.21), the projection $P_{\mathcal{U}^s[\sigma]}$ cuts off functions on $\Omega^s[\sigma]$, whereas $P_{\mathcal{U}^{s,\varepsilon}_\gamma} = P_{\mathcal{U}^s[\sigma_\varepsilon(\gamma)]}\,[P_{\mathcal{U}^s} - P_{\mathcal{U}^{s-\varepsilon}}]$ cuts off on the cap $\omega^{s,\varepsilon}_\gamma$ (see (2.26)).

Let $1^T \in \mathcal{H}^T, 1^T(\cdot) = 1$ be a constant function on Ω^T, $y \in C(\Omega^T)$. With a given $x = x(\gamma, s) \in \Omega^T$ we associate the Dirac measure δ_x and represent

$$y(x) = \langle \delta_x, y\rangle = \lim_{\varepsilon\to 0} \frac{\left(y, P_{\mathcal{U}^{s,\varepsilon}_\gamma} 1^T\right)_{\mathcal{H}^T}}{\left(1^T, P_{\mathcal{U}^{s,\varepsilon}_\gamma} 1^T\right)_{\mathcal{H}^T}}. \tag{2.29}$$

If a control $f \in \mathcal{F}^T$ produces the wave $u^f(\cdot, T) \in C(\Omega^T)$, then (2.21), (2.26) and (2.29) imply

$$u^f(x, T) = \lim_{\varepsilon\to 0} \frac{(W^T f, P_{\mathcal{U}^s[\sigma_\varepsilon(\gamma)]}[P_{\mathcal{U}^s} - P_{\mathcal{U}^{s-\varepsilon}}]1^T)_{\mathcal{H}^T}}{(1^T, P_{\mathcal{U}^s[\sigma_\varepsilon(\gamma)]}[P_{\mathcal{U}^s} - P_{\mathcal{U}^{s-\varepsilon}}]1^T)_{\mathcal{H}^T}}. \tag{2.30}$$

As we shall see later, for a given f, the rhs of this representation is determined by inverse data. Owing to this, the external observer can use (2.30) to make visible the wave u^f supported in the domain unreachable by direct measurements.

One more tool of visualization is the so-called amplitude formula (AF) based on geometric optics. Let $\beta = \beta(\gamma, \tau)$ be the density of the volume in sgc: $dx = \beta\, d\Gamma\, d\tau$. For $y \in \mathcal{H}^T$, a function $\tilde{y}$ on $\Gamma \times [0, T)$ defined by

$$\tilde{y}(\gamma, s) := \begin{cases} \beta^{\frac{1}{2}}(\gamma, s) y(x(\gamma, s)), & (\gamma, s) \in \Theta^T \\ 0, & \text{otherwise} \end{cases} \tag{2.31}$$

is said to be a *picture* of y. As is easy to check, the map $I^T : y \mapsto \tilde{y}$ is an isometry from $\mathcal{H}^T$ to $L_2(\Gamma \times [0, T])$. Let $f \in \mathcal{F}^T$ be smooth and vanishing near $t = 0$, so that the wave u^f is smooth. The AF represents the picture of the wave as follows:

$$\widetilde{u^f(\cdot, T)}(\gamma, s) = \lim_{t\to T-s-0} ((W^T)^*[\mathbb{I} - P_{W^T\mathcal{F}^{T,s}}]W^T f)(\gamma, t), \qquad (\gamma, s) \in \Gamma \times [0, T) \tag{2.32}$$

where $\mathbb{I}$ is the unity operator (see [15]). As will be shown, the rhs of (2.32) is determined by the inverse data. Because of this, the external observer can use the AF for visualization of the wave pictures.

2.2.7. Comments.

- Reconstruction of RM via its spectral and dynamical data was fulfilled in [47, 50], respectively. Both of these papers use one and the same BCm-scheme of reconstruction proposed in [12]. The wave copy was in fact introduced in [10]: this paper deals with the recovery of $\Omega \subset \mathbb{R}^n$, but the procedure needs no change to recover an RM. The caps are taken from [11].

- Fixing an open $\sigma \subset \Gamma$ and using the operator $W^T_\sigma = W^T|_{\mathcal{F}^T[\sigma]} : \mathcal{F}^T[\sigma] \to \mathcal{H}^T[\sigma]$, one can construct the wave copy of the submanifold $\left(\Omega^T[\sigma], d^T_\sigma\right)$ endowed with the intrinsic distance. The AF (2.32) can also be localized on σ (see [23]). Such a locality w.r.t. T and σ makes the wave copy and the AF available for problems on noncompact manifolds.
- Simple analysis shows that C^2-smoothness of (Ω, g) is enough to justify the wave copy construction. Indeed, such smoothness is enough for the uniqueness theorem [95] providing (2.21) and for preserving the geometry of the caps, whereas nothing more is necessary for constructing the wave copy. The AF requires C^N-smoothness with a finite N determined by $\dim \Omega$ (see [48]).
- The wave copy can be constructed for dynamical systems governed by equations of more general type: for instance, by $u_{tt} - \Delta u + qu = 0$ with $q \in L_\infty(\Omega)$. The only correction required is to write properly the rhs of the Duhamel formula (2.19) for a semi-bounded operator $-\Delta^T + q$ instead of $-\Delta^T$. If q is smooth enough, the AF (2.32) is also valid.
- As is mentioned in section 1, one more isometric copy of Ω can be constructed from the distant functions $\tau_{x_0} : \partial\Omega \to \overline{\mathbb{R}}_+$, $\tau_{x_0}(\cdot) := d(\cdot, x_0)$. For a certain class of manifolds and metrics, the set $\tilde{\Omega} = \left\{\tau_{x_0} \mid x_0 \in \Omega\right\}$ endowed with $C(\partial\Omega)$-metric is an isometric copy of Ω (Kurylev [70]). The use of τ_{x_0} goes back to the procedure of solving the dynamical inverse problem 'in the large' proposed by one of the authors in [49][10]. The disadvantage of this version of the BCm is that it is workable for $T > T_*$ only and, therefore, does not provide time-optimal results.

2.3. *Inverse problems*

2.3.1. Setup. Recall that $-\Delta^T$ is the Laplacian in Ω^T with homogeneous (zero) Dirichlet boundary conditions on $\Gamma \cup \Gamma^T$ (see section 2.2.2). $-\Delta^T$ is a positive definite operator in $\mathcal{H}^T$ possessing the discrete spectrum $\left\{\lambda^T_k\right\}^\infty_{k=0}$; $0 < \lambda^T_1 < \lambda^T_2 \leqslant \lambda^T_3 \leqslant \ldots$; let $\left\{\varphi^T_k\right\}^\infty_{k=0}$ be the corresponding eigenfunctions constituting an orthonormal basis in $\mathcal{H}^T$. A set of pairs $\Sigma^T_\Gamma := \left\{\lambda^T_k; \frac{\partial\varphi^T_k}{\partial\nu}\big|_\Gamma\right\}^\infty_{k=0}$ is called *spectral data* of the operator $-\Delta^T$ (on Γ). If $T > T_*$ then $\Gamma^T = \{\emptyset\}$, $\Omega^T = \Omega$ and we omit the superscript T in the notation: $\lambda^T_k =: \lambda_k$, $\varphi^T_k =: \varphi_k$, etc.

Let $T > 0$ be fixed;

- the spectral inverse problem is to determine (Ω^T, d^T) from Σ^T_Γ;
- the dynamical inverse problem is to determine (Ω^T, d^T) from the (continued) response operator R^{2T}.

Both of the problems are understood in the sense of the setups (1) and (2) given in section 1.2. Both of them will be solved by a procedure that constructs a representative of the class of equivalent manifolds (the representative will be just the wave copy!) and, thus, determines the manifold up to isometry. The procedure is based on the concept of model.

2.3.2. Model. A pair $\{\tilde{\mathcal{H}}^T, \tilde{W}^T\}$ consisting of a Hilbert space $\tilde{\mathcal{H}}^T$ and an operator $\tilde{W}^T : \mathcal{F}^T \to \tilde{\mathcal{H}}^T$ is said to be a *model* of the system α^T if $\tilde{W}^T$ and $\tilde{\mathcal{H}}^T$ are determined by the inverse data and the map $U^T : W^T f \mapsto \tilde{W}^T f$ is an isometry from $\mathcal{U}^T = \operatorname{Ran} W^T \subset \mathcal{H}^T$ onto $\tilde{\mathcal{U}}^T := \operatorname{Ran} \tilde{W}^T \subset \tilde{\mathcal{H}}^T$.

The model is an intermediate object in solving IPs. It plays the role of an auxiliary copy of the original system which the external observer can construct from measurements at the boundary. While the genuine wave $u^f(\cdot, T) = W^T f$ is invisible to the observer, its

[10] The contributions of the authors of this paper are separated: one of them elaborates a procedure solving the problem, the second one proves a geometric lemma justifying the procedure.

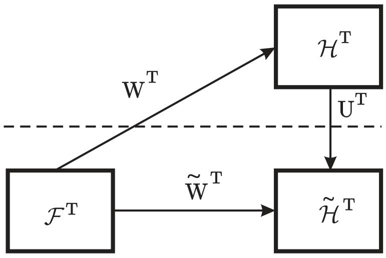

Figure 6. Model.

U^T-representation $\tilde{u}^f(\cdot, T) = \tilde{W}^T f$ can be visualized by means of the model control operator $\tilde{W}^T$ (see the diagram in figure 6, where the upper part is invisible, whereas the lower part can be extracted from inverse data).

The following fact plays a key role: given *any* model of the system α^T, one can reproduce the wave copy $(\tilde{\Omega}^T, \tilde{d}^T)$. Indeed,

(a) $\tilde{W}^T$ determines the sets $\tilde{\mathcal{U}}^s[\sigma] := U^T \mathcal{U}^s[\sigma] = \tilde{W}^T \mathcal{F}^{T,s}[\sigma]$;

(b) $\tilde{W}^T$ determines the U^T-representation of the Laplacian on waves $\tilde{L}_0^T := U^T L_0^T (U^T)^*$ through the graph

$$\text{graph } \tilde{L}_0^T = \left\{\{\tilde{W}^T f, -\tilde{W}^T f_{tt}\} \middle| f \in C_0^\infty((0,T); C^\infty(\Gamma))\right\} \tag{2.33}$$

(see (2.22)) and its extension (in $\tilde{\mathcal{H}}^T$) by Friedrichs $\tilde{L}^T = U^T L^T (U^T)^*$;

(c) the operator $\tilde{L}^T$ determines the image of the volume control operator

$$\tilde{W}^r_{\text{vol}} h := \int_0^r (\tilde{L}^T)^{-\frac{1}{2}} \sin[(r-s)(\tilde{L}^T)^{\frac{1}{2}}] h(\cdot, s)\, \mathrm{d}s$$

for $h \in L_2([0,r]; \tilde{\mathcal{H}}^T)$ (see (2.23)) and the operation

$$\tilde{\mathcal{E}}^r \tilde{\mathcal{A}} := \operatorname{clos} \tilde{W}^r_{\text{vol}} L_2\left([0,r]; \tilde{\mathcal{A}}\right)$$

extending the subspaces $\tilde{\mathcal{A}} \subset \tilde{\mathcal{H}}^T$ (see (2.24)).

Therefore, repeating the procedure (i)–(iii) of section 2.2.5 and just replacing the subspaces and operators without tildes by those with tildes, we get a U^T-representation (*a sample*) of the wave copy $(\tilde{\Omega}^T, \tilde{d}^T)$. The sample constructed in terms of the model is evidently isometric to the wave copy itself, and in the following we identify them.

So, to construct a model via given inverse data is to determine the wave copy of the original (Ω^T, d^T) and, hence, to solve the corresponding IP.

2.3.3. Solving the IPs. Solving the forward problem (2.7)–(2.9) by the Fourier method, we seek for the solution in the form of expansion

$$u^f(\cdot, t) = \sum_{k=1}^\infty c_k^f(t) \varphi_k^T(\cdot)$$

over the eigenbasis of the operator $-\Delta^T$. Standard calculations imply

$$c_k^f(t) = \int_{\Gamma \times [0,t]} \mathrm{d}\Gamma\, \mathrm{d}s \left[\frac{\sin\left[\left(\lambda_k^T\right)^{\frac{1}{2}}(t-s)\right]}{\left(\lambda_k^T\right)^{\frac{1}{2}}} \frac{\partial \varphi_k^T}{\partial \nu}(\gamma) \right] f(\gamma, s).$$

Hence, we get

$$u^f(\cdot,T) = W^T f = \sum_{k=1}^{\infty} \left(f, s_k^T\right)_{\mathcal{F}^T} \varphi_k^T(\cdot),$$

where $s_k^T = s_k^T(\gamma,t) := \left(\lambda_k^T\right)^{-\frac{1}{2}} \sin\left[\left(\lambda_k^T\right)^{\frac{1}{2}}(T-t)\right] \frac{\partial\varphi_k^T}{\partial\nu}(\gamma)$.

The *spectral model* is the pair

$$\tilde{\mathcal{H}}^T := l_2, \qquad \tilde{W}^T := \left\{\left(\cdot, s_k^T\right)_{\mathcal{F}^T}\right\}_{k=1}^{\infty}, \tag{2.34}$$

the role of isometry U^T is played by the Fourier transform $F^T : \mathcal{H}^T \to \tilde{\mathcal{H}}^T$, $F^T y := \left\{\left(y, \varphi_k^T\right)_{\mathcal{H}^T}\right\}_{k=1}^{\infty}$. So, the spectral data Σ_Γ^T determine a model and, hence, the wave copy $(\tilde{\Omega}^T, \tilde{d}^T)$. Therefore, Σ_Γ^T determines (Ω^T, d^T) up to isometry, which solves the spectral IP. Note a peculiarity of the model (2.34): the operator $\tilde{L}^T = U^T L^T (U^T)^* = F^T(-\Delta^T)(F^T)^* = \operatorname{diag}\left\{\lambda_k^T\right\}_{k=1}^{\infty}$ is given in explicit form, i.e., we do not need to determine it through the graph (2.33) and invoke the Friedrichs extension.

By (2.15), the operator R^{2T} determines the modulus of the control operator

$$| W^T | := [(W^T)^* W^T]^{\frac{1}{2}} = (C^T)^{\frac{1}{2}},$$

which enters in the polar decomposition $W^T = \Phi^T \mid W^T \mid$ with an isometry $\Phi^T : \mathcal{F}^T \to \mathcal{H}^T$. Along with the modulus, the continued response operator determines the *dynamical model*

$$\tilde{\mathcal{H}}^T := \operatorname{clos}\operatorname{Ran}(C^T)^{\frac{1}{2}}, \qquad \tilde{W}^T := (C^T)^{\frac{1}{2}}.^{11} \tag{2.35}$$

Therefore, R^{2T} determines the wave copy $(\tilde{\Omega}^T, \tilde{d}^T)$, which is isometric to the original (Ω^T, d^T). The dynamical IP is solved.

Given any model, one can visualize the wave pictures. Indeed, since U^T is an isometry, the relation $\tilde{W}^T = U^T W^T$ implies

$$(W^T)^*[\mathbb{I} - P_{W^T\mathcal{F}^{T,s}}]W^T = (\tilde{W}^T)^*[\tilde{\mathbb{I}} - P_{\tilde{W}^T\mathcal{F}^{T,s}}]\tilde{W}^T$$

and the AF (2.32) can be rewritten in the form

$$\widetilde{u^f(\cdot,T)}(\gamma,s) = \lim_{t\to T-s-0}((\tilde{W}^T)^*[\tilde{\mathbb{I}} - P_{\tilde{W}^T\mathcal{F}^{T,s}}]\tilde{W}^T f)(\gamma,t), \qquad (\gamma,s) \in \Gamma\times[0,T) \tag{2.36}$$

with the rhs determined by *any* model. The set of wave pictures is rich enough for recovering the pattern Θ^T and the metric tensor on it. The pattern endowed with the tensor is isometric to (Ω^T, d^T): see [15, 47, 50].

The waves themselves can be visualized through the spectral or dynamical model by means of the representation (2.30) written in the invariant form

$$u^f(x(\gamma,s),T) = \lim_{\varepsilon\to 0} \frac{(\tilde{W}^T f, P_{\tilde{W}^T\mathcal{F}^{T,s}[\sigma_\varepsilon(\gamma)]}[P_{\tilde{W}^T\mathcal{F}^s} - P_{\tilde{W}^T\mathcal{F}^{s-\varepsilon}}]\tilde{1}^T)_{\tilde{\mathcal{H}}^T}}{(\tilde{1}^T, P_{\tilde{W}^T\mathcal{F}^{T,s}[\sigma_\varepsilon(\gamma)]}[P_{\tilde{W}^T\mathcal{F}^s} - P_{\tilde{W}^T\mathcal{F}^{s-\varepsilon}}]\tilde{1}^T)_{\tilde{\mathcal{H}}^T}} \tag{2.37}$$

available for any model. Here $\tilde{1}^T := U^T 1^T$, and the only question is to determine the element $\tilde{1}^T \in \tilde{\mathcal{H}}^T$.

Lemma 2. *The relation*

$$(W^T)^* 1^T = (R^T)^* \varkappa^T \tag{2.38}$$

holds with $\varkappa^T = \varkappa^T(\gamma,t) := T - t$.

[11] Since $\tilde{\mathcal{H}}^T \subset \mathcal{F}^T$, the external observer can construct this model, not leaving the outer space $\mathcal{F}^T$.

Proof. See in [20, 32]. □

In the dynamical model we have $(W^T)^*1^T = (\tilde{W}^T)^*\tilde{1}^T = (C^T)^{\frac{1}{2}}\tilde{1}^T$. Hence, by (2.38), the required element $\tilde{1}^T$ can be characterized as a unique solution of an equation

$$(C^T)^{\frac{1}{2}} f = (R^T)^* \varkappa^T \qquad \text{in} \quad \mathcal{F}^T \tag{2.39}$$

belonging to clos $\mathrm{Ran}(C^T)^{\frac{1}{2}}$. Solving this equation, we get an opportunity to determine the rhs of (2.37) from dynamical inverse data.

If $T > T_*$, we have $\Gamma^T = \{\emptyset\}$, $1^T(\cdot) = 1$ in Ω and easily find

$$\tilde{1} = \{(1, \varphi_k)_{\mathcal{H}}\}_{k=1}^{\infty} = \left\{ -\frac{1}{\lambda_k} \int_{\Gamma} \frac{\partial \varphi_k}{\partial \nu}\, \mathrm{d}\Gamma \right\}_{k=1}^{\infty}$$

(the data Σ_Γ^T do not depend on T), which enables one to determine the rhs of (2.37) through the spectral model.

- There is one more variant of the dynamical model. Let $\tilde{\mathcal{F}}^T$ be the completion of the outer space $\mathcal{F}^T$ w.r.t. the norm $\|f\|_{\tilde{\mathcal{F}}^T} := (C^T f, f)_{\mathcal{F}^T}^{\frac{1}{2}}$ (a space of generalized controls introduced in [12]), $\tilde{W}^T$ the embedding $\mathcal{F}^T$ to $\tilde{\mathcal{F}}^T$. As is easy to see, the pair $\{\tilde{\mathcal{F}}^T, \tilde{W}^T\}$ constitutes a model. Such a model possesses some interesting properties [20]; however, $\tilde{\mathcal{F}}^T$ is not a distributional space.
- Let $T > 0$ and an open $\sigma \subset \Gamma$ be fixed, the operator $R_\sigma^{2T} : \mathcal{F}^{2T}[\sigma] \to \mathcal{F}^{2T}[\sigma]$ defined on $\mathcal{F}^{2T}[\sigma] \cap \mathrm{Dom}\, R^{2T}$ by $R_\sigma^{2T} := [R^{2T} \cdot]|_{\sigma \times [0,2T]}$ given. Repeating the construction of the wave copy and applying the technique [10], one can show that this operator determines the submanifold $(\Omega^T[\sigma], g)$ up to isometry. Such a locality w.r.t. T and σ, corresponding to the general principles $P1$–$P3$, is one of main advantages of the BCm.

  Returning to the spectral IP, assume that the derivatives $\frac{\partial \varphi_k^T}{\partial \nu}$ are given on σ only. A simple consequence of our approach is that the data $\left\{\lambda_k^T; \frac{\partial \varphi_k^T}{\partial \nu}\Big|_\sigma\right\}_{k=0}^{\infty}$ determine (Ω^T, d^T) up to isometry.
- Equation (2.39) is one of the BCm-versions of the classical Gelfand–Levitan–Krein–Marchenko equations (see [9, 20]). Apropos of this, there is the conjecture that the Faddeev–Newton equations of the inverse scattering problem can also be interpreted in BCm-terms. We plan to extend the BCm on this problem. A curious fact is that the character of controllability of the corresponding dynamical system differs from that of the system on a compact manifold: a natural analog of (2.21) does not hold [55, 56]. The possible lack of controllability leads to interesting physical effects (the existence of so-called reversing waves): see [56].

2.3.4. A look at wave copy. Here we interpret the wave copy construction in terms of functional analysis.

A set $\mathcal{L} = \{A_\alpha\}$ of subspaces L_α of a Hilbert space H is said to be a (closed) *lattice* if $A_\alpha, A_{\alpha'}, A_j \in \mathcal{L}$ and $A_j \to A$ [12] implies $H \ominus A_\alpha$, $A_\alpha \cap A_{\alpha'}$, $A \in \mathcal{L}$. For a set of subspaces $b = \{B_\alpha\}$, we denote by $\mathcal{L}\langle b\rangle$ the minimal lattice containing b.

A set of subspaces $x = \{X_\alpha\}$ is called a *nest* if it is totally subordinated w.r.t. embedding: for any $X_\alpha, X_{\alpha'}$, either $X_\alpha \subset X_{\alpha'}$ or $X_{\alpha'} \supset X_\alpha$ holds [59, 63]. We write $x \subset \mathcal{L}$ if all X_α belong to the lattice $\mathcal{L}$ and denote by $\mathcal{X}[\mathcal{L}] := \{x \mid x \subset \mathcal{L}\}$ a set of the nests belonging to $\mathcal{L}$.

[12] In the sense of the strong operator convergence of the projections on A_j.

A positive definite operator L in H determines an operation $\mathcal{E}^r$ that extends subspaces by the rule

$$\begin{aligned}&\mathcal{E}^r A := \operatorname{clos}\left\{\int_0^r L^{-\frac{1}{2}}\sin[(r-s)L^{\frac{1}{2}}]h(\cdot,s)\,\mathrm{d}s \,\middle|\, h\in L_2([0,r];A)\right\}, \qquad r>0,\\ &\mathcal{E}^0 := \mathrm{id}.\end{aligned} \tag{2.40}$$

We call $\mathcal{E}^r$ a dynamical extension. A lattice $\mathcal{L}$ is said to be *dynamical* if $A \in \mathcal{L}$ implies $\mathcal{E}^r A \in \mathcal{L}, r \geqslant 0$.

A nest $x \subset \mathcal{L}$ is said to be *dynamical* if it can be parametrized as

$$x = \{X^r\}_{r\geqslant 0}, \qquad X^0 = \bigcap_{r>0} X^r, \qquad X^{r+s} = \mathcal{E}^r X^s, \qquad s>0, \qquad r\geqslant 0.$$

The set of dynamical nests is partially subordinated: we write $x \prec x'$ if $X^r \subset X'^r, r \geqslant 0$. A dynamical nest x is said to be *elementary* if $x \prec x'$ for any x' comparable with x. We denote by $\mathcal{X}_{\mathrm{e}}[\mathcal{L}]$ the set of elementary nests of a dynamical lattice $\mathcal{L}$, and endow it with a function $\delta : \mathcal{X}_{\mathrm{e}}[\mathcal{L}] \times \mathcal{X}_{\mathrm{e}}[\mathcal{L}] \to \overline{\mathbb{R}}_+$,

$$\delta(x,x') := \inf\{r \geqslant 0 \mid X^r \not\subset X'^r\} \leqslant \infty. \tag{2.41}$$

Return to the dynamical system α^T and its wave copy. The boundary control operator W^T determines the space $H := \overline{\mathcal{U}}^T = \operatorname{clos} W^T\mathcal{F}^T$, the set of its subspaces $a^T_\Gamma := \{\overline{\mathcal{U}}^s[\sigma] = \operatorname{clos} W^T\mathcal{F}^{T,s}[\sigma] \mid 0 \leqslant s \leqslant T, \sigma \subset \Gamma\}$, and the operator $L := L^T$ (Laplacian on waves). Along with L, the extension $\mathcal{E}^r$ is well defined by (2.40). Hence, we can constitute the dynamical lattice $\mathcal{L}\langle a^T_\Gamma\rangle$, the set of elementary nests $\mathcal{X}_{\mathrm{e}}\big[\mathcal{L}\langle a^T_\Gamma\rangle\big]$ and endow this set with the function δ by (2.41) (compare with (2.28)). A simple analysis shows that the pair $\big(\mathcal{X}_{\mathrm{e}}\big[\mathcal{L}\langle a^T_\Gamma\rangle\big], \delta\big)$ is identical to the wave copy $(\tilde\Omega^T, \tilde d^T)$. The copy is isometric to the original (we write $(\tilde\Omega^T, \tilde d^T) \approx (\Omega^T, d^T)$) and, hence, we arrive at

$$(\Omega^T, d^T) \approx \big(\mathcal{X}_{\mathrm{e}}[\mathcal{L}\langle a^T_\Gamma\rangle], \delta\big). \tag{2.42}$$

There are two reasons motivating such an interpretation of the wave copy. First, owing to its invariant operator nature, the rhs of (2.42) is well defined for a wide class of dynamical systems: roughly speaking, it always exists. Second, the rhs is determined by the control operator W^T and, hence, can be reproduced through any model. Thus, the pair $\big(\mathcal{X}_{\mathrm{e}}\big[\mathcal{L}\langle a^T_\Gamma\rangle\big], \delta\big)$ is a relevant (to IPs) canonical realization of the original manifold (Ω^T, d^T).

- In this connection, actually, an original and its realization are different things, which do not need to be identical. Indeed, 'equality' (2.42) is derived from controllability (2.21) and, hence, is a specificity of a class of 1-velocity systems. What does $\big(\mathcal{X}_{\mathrm{e}}\big[\mathcal{L}\langle a^T_\Gamma\rangle\big], \delta\big)$ look like in the case of more complicated (in particular, multi-velocity) systems, for which (2.21) does not hold and, as a result, (2.42) is not valid? How is such an object related to the geometry of Ω and parameters of the associated dynamical system? These questions constitute one of most interesting directions of further development for the BCm.

In addition, let us provide an abstract analog of representation (2.29). Assume that $G \subset H$ is a subspace generating the dynamical lattice $\mathcal{L}$, i.e., $\operatorname{clos}\{P_A\psi \mid \psi \in G, A \subset \mathcal{L}\} = H$ and let $\{\psi_k\}_{k=1}^N$, $N \leqslant \infty$ be an orthonormal basis in G. For $y \in H$, define an l_2^N-valued function $y(\cdot) : \mathcal{X}_e[\mathcal{L}] \to l_2^N$,

$$y(x) := \left\{\lim_{r\to 0}\frac{(y, P_{X^r}\psi_k)_H}{(\psi_k, P_{X^r}\psi_k)_H}\right\}_{k=1}^N, \tag{2.43}$$

where $X^r \in x$. The map $U : y \mapsto y(\cdot)$ realizes elements of H as functions on $\mathcal{X}_{\mathrm{e}}[\mathcal{L}]$, whereas operator ULU^{-1} turns out to be a functional model of L.

2.3.5. Visualization as triangular factorization. Returning to the definition of the pictures (2.31), let us identify the space of pictures $L_2(\Gamma\times[0,T])$ to the outer space $\mathcal{F}^T$ and consider the *picture operator* I^T as a map from $\mathcal{H}^T$ to $\mathcal{F}^T$. Let $Y^T:\mathcal{F}^T\to\mathcal{F}^T$ be the time inversion: $(Y^Tf)(\cdot,t):=f(\cdot,T-t), 0\leqslant t\leqslant T$. An operator $Y^TI^T:\mathcal{H}^T\to\mathcal{F}^T$ is isometric; the relation

$$Y^TI^T\mathcal{H}^s\subset\mathcal{F}^{T,s},\qquad 0\leqslant s\leqslant T \tag{2.44}$$

easily follows from (2.31).

A map $V^T:\mathcal{F}^T\to\mathcal{F}^T$, $V^T:=Y^TI^TW^T$ is called a *visualizing operator* [15]. The first of relations (2.20), along with (2.44), easily imply

$$V^T\mathcal{F}^{T,s}\subset\mathcal{F}^{T,s},\qquad 0\leqslant s\leqslant T$$

i.e., V^T is *triangular* w.r.t. the family $\{\mathcal{F}^{T,s}\}_{0\leqslant s\leqslant T}$ (see [59, 63]).

Since I^T and Y^T are isometries, one has

$$|V^T|:=[(V^T)^*V^T]^{\frac12}=[(W^T)^*W^T]^{\frac12}=[C^T]^{\frac12}=|\tilde W^T|$$

so that $|V^T|$ is determined by any model of the system α^T. Writing (2.32) and (2.36) in the form

$$(V^Tf)(\gamma,s)=\lim_{t\to T-s-0}(|V^T|[\mathbb{I}-P_{|V^T|\mathcal{F}^{T,s}}]|V^T|f)(\gamma,t),\qquad(\gamma,s)\in\Gamma\times[0,T)$$

we conclude that the AF recovers a triangular operator V^T from its modulus $|V^T|$[13]. In other words, the AF provides the *triangular factorization* of the connecting operator: $C^T=(V^T)^*V^T$. Such a factorization is not unique and the factor V^T can be characterized in terms of the *operator diagonal* (see [18, 52]).

2.3.6. Relations between data: variational principle. Given R^{2T}, we can recover the wave copy of (Ω^T,d^T) and, solving the forward spectral problem on the copy, determine Σ_Γ^T. Conversely, Σ_Γ^T determines the wave copy and, solving the forward dynamical problem on it, we can find R^{2T}. So, both side determinations $R^{2T}\Leftrightarrow(\Omega^T,d^T)\Leftrightarrow\Sigma_\Gamma^T$ hold but the question arises whether one can relate the dynamical and spectral data in a straightforward way, i.e., avoiding the reconstruction of the manifold. The model provides such an option.

A dynamical system α_*^T of the form

$$v_{tt}-\Delta v=0\qquad\text{in}\quad\{(x,t)\mid x\in\Omega^T,\tau(x)<t<T\} \tag{2.45}$$

$$v|_{t=T}=0,\ v_t|_{t=T}=y\qquad\text{in}\quad\overline{\Omega}^T \tag{2.46}$$

$$v=0\qquad\text{on}\quad\Gamma\times[0,T] \tag{2.47}$$

is called dual to the system α^T (2.7)–(2.9); let $v=v^y(x,t)$ be its solution for $y\in\mathcal{H}^T$. With the system α_*^T one associates an observation operator $O^T:\mathcal{H}^T\to\mathcal{F}^T$, $O^Ty:=\frac{\partial v^y}{\partial\nu}\big|_{\Gamma\times[0,T]}$. The term 'dual' is motivated by the relation $O^T=(W^T)^*$ (e.g., see [15]). Taking $y=\varphi_k^T$ (see section 2.3.1), one has $v^{\varphi_k^T}(\cdot,t)=\left(\lambda_k^T\right)^{-\frac12}\sin\left[\left(\lambda_k^T\right)^{\frac12}(t-T)\right]\varphi_k^T(\cdot)$; this implies

$$\left.\frac{\partial\varphi_k^T}{\partial\nu}\right|_\Gamma=\left[\frac{\mathrm{d}}{\mathrm{d}t}\left.\frac{\partial v^{\varphi_k^T}}{\partial\nu}\right|_\Gamma\right]\bigg|_{t=T}=\left[\frac{\mathrm{d}}{\mathrm{d}t}(W^T)^*\varphi_k^T\right]\bigg|_{t=T}. \tag{2.48}$$

Given R^{2T}, one can recover Σ_Γ^T by the following procedure:

[13] This relates the AF to optical holography.

(i) Using the dynamical model (2.35), determine the operator $\tilde{L}_0^T$ (see (2.33)) and its extension by Friedrichs $\tilde{L}^T$ in $\tilde{\mathcal{H}}^T$. Find its spectrum and choose an orthonormal basis $\{\tilde{\varphi}_k^T\}_{k=0}^{\infty} : \tilde{L}_k^T \tilde{\varphi}_k^T = \lambda_k^T \tilde{\varphi}_k^T$.

(ii) For $\varphi_k^T := (U^T)^* \tilde{\varphi}_k^T$, by isometry, we have $(W^T)^* \varphi_k^T = (\tilde{W}^T)^* \tilde{\varphi}_k^T$ and (2.48) implies

$$\left.\frac{\partial \varphi_k^T}{\partial \nu}\right|_{\Gamma} = \left.\left[\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{W}^T)^* \tilde{\varphi}_k^T\right]\right|_{t=T}. \tag{2.49}$$

Collecting $\left\{\lambda_k^T; \left.\frac{\partial \varphi_k^T}{\partial \nu}\right|_{\Gamma}\right\}_{k=0}^{\infty} = \Sigma_{\Gamma}^T$, we get the spectral data.

One more way is to extract Σ_{Γ}^T from C^T by means of the *variational principle*, which is equivalent to the spectral analysis (i). In accordance with the principle, we have

$$\lambda_1^T = \min_{\|y\|^2_{\mathcal{H}^T}=1} (-\Delta^T y, y)_{\mathcal{H}^T}, \qquad -\Delta^T \varphi_1^T = \lambda_1^T \varphi_1^T,$$

$$\lambda_2^T = \min_{\substack{\|y\|^2_{\mathcal{H}^T}=1 \\ (y,\varphi_1^T)_{\mathcal{H}^T}=0}} (-\Delta^T y, y)_{\mathcal{H}^T}, \qquad -\Delta^T \varphi_2^T = \lambda_2^T \varphi_2^T,$$

$$\ldots$$

where $y \in \mathrm{Dom}(-\Delta)^T$ and φ_k^T are the minimizers. Boundary controllability (2.21) enables one to reduce the search for minima to the reachable set $\{u^f(\cdot, T) \mid f \in \mathcal{F}_\infty^T := C_0^\infty((0,T); C^\infty(\Gamma))\} = W^T \mathcal{F}_\infty^T$ dense in $\mathcal{H}^T$, whereas for $y = u^f(\cdot, T)$ we can represent

$$\begin{aligned}
(-\Delta^T y, y)_{\mathcal{H}^T} &= (-\Delta^T u^f(\cdot, T), u^f(\cdot, T))_{\mathcal{H}^T} = \left(-u_{tt}^f(\cdot, T), u^f(\cdot, T)\right)_{\mathcal{H}^T} \\
&= -(u^{f_{tt}}(\cdot, T), u^f(\cdot, T))_{\mathcal{H}^T} = \langle \text{see (2.14)} \rangle = -(C^T f_{tt}, f)_{\mathcal{F}^T}, \\
\|y\|^2_{\mathcal{H}^T} &= \left(u^f(\cdot, T), u^f(\cdot, T)\right)_{\mathcal{H}^T} = \left(C^T f, f\right)_{\mathcal{F}^T}, \\
(y, \varphi_k^T)_{\mathcal{H}^T} &= (W^T f, \varphi_k^T)_{\mathcal{H}^T} = \left(f, (W^T)^* \varphi_k^T\right)_{\mathcal{F}^T} = \left(f, (\tilde{W}^T)^* \tilde{\varphi}_k^T\right)_{\mathcal{F}^T},
\end{aligned}$$

where $\tilde{\varphi}_k^T = U^T \varphi_k^T$. Thereafter, we can realize the variational principle through the dynamical model, not leaving the outer space $\mathcal{F}^T$.

Step 1. Find

$$\lambda_1^T = -\min_{f \in \mathcal{F}_\infty^T, (C^T f, f)_{\mathcal{F}^T}=1} (C^T f_{tt}, f)_{\mathcal{F}^T}.$$

Choose a sequence $f_p^{(1)}$ satisfying the minimization conditions and providing $-\lim_{p\to\infty} \left(C^T f_p^{(1)}{}_{tt}, f_p^{(1)}\right)_{\mathcal{F}^T} = \lambda_1^T$. Since $u^{f_p^{(1)}}(\cdot, T) = W^T f_p^{(1)} \to \varphi_1^T$ in $\mathcal{H}^T$, by isometry 'system $\leftrightarrow$ model', the sequence $\tilde{W}^T f_p^{(1)}$ converges to $\tilde{\varphi}_1^T = U^T \varphi_1^T$ in $\tilde{\mathcal{H}}^T = \mathcal{F}^T$ and we can find $\tilde{\varphi}_1^T = \lim_{p\to\infty} (C^T)^{\frac{1}{2}} f_p^{(1)}$. Then, by (2.49) we determine

$$\left.\frac{\partial \varphi_1^T}{\partial \nu}\right|_{\Gamma} = \left.\left[\frac{\mathrm{d}}{\mathrm{d}t}(C^T)^{\frac{1}{2}} \tilde{\varphi}_1^T\right]\right|_{t=T} = \left.\left[\frac{\mathrm{d}}{\mathrm{d}t} \lim_{p\to\infty} C^T f_p^{(1)}\right]\right|_{t=T}.$$

So, the pair $\left\{\lambda_1^T; \left.\frac{\partial \varphi_1^T}{\partial \nu}\right|_{\Gamma}\right\}$ is obtained.

Step 2. Find

$$\lambda_2^T = -\min_{\substack{f \in \mathcal{F}_\infty^T, (C^T f, f)_{\mathcal{F}^T}=1 \\ ((C^T)^{\frac{1}{2}} f, \tilde{\varphi}_1^T)_{\mathcal{F}^T}=0}} (C^T f_{tt}, f)_{\mathcal{F}^T}$$

and choose $f_p^{(2)}$ such that $-\lim_{p\to\infty}\left(C^T f_p^{(2)}{}_{tt}, f_p^{(2)}\right)_{\mathcal{F}^T} = \lambda_2^T$. Find $\tilde{\varphi}_2^T = \lim_{p\to\infty}(C^T)^{\frac{1}{2}} f_p^{(2)}$, determine

$$\frac{\partial\varphi_2^T}{\partial\nu}\bigg|_\Gamma = \left[\frac{\mathrm{d}}{\mathrm{d}t}\lim_{p\to\infty} C^T f_p^{(2)}\right]\bigg|_{t=T}$$

and get $\left\{\lambda_2^T; \frac{\partial\varphi_2^T}{\partial\nu}\big|_\Gamma\right\}$.

Continuing in the evident way, we get $\left\{\lambda_k^T; \frac{\partial\varphi_k^T}{\partial\nu}\big|_\Gamma\right\}_{k=0}^\infty = \Sigma_\Gamma^T$.

- Determination $R^{2T} \Rightarrow \Sigma_\Gamma^T$ through model and variational principle was proposed in [14] for solving the dynamical IP of the heat conductivity; in [25] the same trick is applied to the wave equation. This trick is also workable in the case of local data: as is easy to see, the same procedure (steps 1 and 2, …) enables one to determine Σ_σ^T from R_σ^{2T}.

2.3.7. Data continuation. Consider the question: given R^{2T} for a fixed $T > 0$, can one determine $R^{T'}$ for $T' > 2T$ in a straightforward way, not recovering (Ω^T, d^T)? In other words, one needs to continue the dynamical inverse data from $[0, 2T]$ to a bigger interval $[0, T']$, not solving the inverse (and, thereafter, the corresponding forward) problem.

One of possible continuation procedures is the following:

(i) given R^{2T}, recover the spectral data Σ_Γ^T (see the previous section);

(ii) represent the solution u^f of the system $\alpha^{T'}$ by Fourier (see the beginning of section 2.3.3). For a control $f \in C_0^\infty((0, T'); C^\infty(\Gamma))$ provided $\operatorname{supp} f \subset [0, 2T]$, the Fourier representation easily implies

$$(R^{T'} f)(\cdot, t)$$
$$= \begin{cases} (R^{2T} f)(\cdot, t), & 0 \leqslant t < 2T \\ \displaystyle\sum_{k=1}^\infty \left[\int_{\Gamma\times[0,t]} \mathrm{d}\Gamma\,\mathrm{d}s \frac{\sin\left[\left(\lambda_k^T\right)^{\frac{1}{2}}(t-s)\right]}{\left(\lambda_k^T\right)^{\frac{1}{2}}} \frac{\partial\varphi_k^T}{\partial\nu}(\gamma) f(\gamma, s)\right] \frac{\partial\varphi_k^T}{\partial\nu}(\cdot), & t \geqslant 2T \end{cases} \tag{2.50}$$

(the integral is in fact taken over $\Gamma \times [0, 2T]$; the series converges since the integral rapidly decreases as $k \to \infty$ by the choice of f)

(iii) using a shift w.r.t. time and choosing a suitable partition of unity on $[0, T']$, one can take off the restriction to $\operatorname{supp} f$ and extend the representation to controls $f \in C^\infty((0, T']; C^\infty(\Gamma))$ (see [22] for details). Such an extension determines $R^{T'}$.

In the case $T < T_*$, the continuation $R^{2T} \Rightarrow R^{T'}$ constructed above corresponds to the zero Dirichlet condition on Γ^T. Surely, there are infinitely many other continuations but our one is singled out and realizable through any model owing to invariancy of the Friedrichs extension. If $T > T_*$, then the continuation is unique.

- Invoking the spectral data Σ_Γ^T for the continuation, we use compactness of Ω. A more general construction based on the model [22] is available for noncompact manifolds. The same construction provides the continuation of the response operator R_σ^{2T} for any $T > 0$ and open $\sigma \subset \Gamma$.
- For the versions of the continuation procedure [22] for the Maxwell and Lamé dynamical systems, see in [45, 51]. Moreover, the procedure is realizable on a general operator level [20]; it is related to the classical problem of continuation of positive Hermitian operator functions, the role of the positivity by Bochner–Krein being played by the condition $C^T \geqslant \mathbb{O}$.

- One more BCm-procedure extending the response operator is proposed in [75]. It admits the presence of the dissipative term βu_t in the wave equation. However, this version is workable for $T > T_*$ only and under additional global conditions on controllability of the manifold.

2.3.8. Other types of equations. As was mentioned above, for the first time the determination $R^{2T} \Rightarrow \Sigma_\Gamma$ through the model and variational principle has been applied to the heat conductivity IP [14]. Here we describe briefly the application of the same trick to a system governed by the Schrödinger equation [2, 4, 5].

On a compact RM Ω with the boundary Γ consider the system

$$\begin{aligned} &\mathrm{i}u_t - \Delta u = 0 && \text{in} \quad (\operatorname{int}\Omega) \times (0, T) \\ &u|_{t=0} = 0 && \text{in} \quad \Omega \\ &\frac{\partial u}{\partial \nu} = f && \text{in} \quad \Gamma \times [0, T] \end{aligned}$$

with a (complex valued) Neumann boundary control. Its control operator $W^T : \mathcal{F}^T \to \mathcal{H} = L_2(\Omega)$, $W^T f = u^f(\cdot, T)$ and response operator $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $R^T f = u^f|_{\Gamma\times[0,T]}$ are well defined on a class $\mathcal{M}^T$ of smooth controls vanishing near $\Gamma \times \{t = 0\}$. The following two facts are basic for the BCm:

(i) for any $T > 0$, the system is approximately controllable: the reachable set $W^T\mathcal{M}^T$ is dense in $\mathcal{H}$;

(ii) the *connecting form* $c^T[f', f''] = (W^T f', W^T f'')_{\mathcal{H}}$ is represented via the response operator: $c^T[f', f''] = \mathrm{i}([R^T - (R^T)^*]f', f'')_{\mathcal{F}^T}$ (S A Avdonin).

These facts enable one to construct the appropriate dynamical model (2.35) of the system and determine the spectral data Σ_Γ from R^T, for instance, by the use of the variational principle. As a result, for any fixed $T > 0$ the operator R^T determines (Ω, d) up to isometry.

- The results of Triggiani and Yao [96] on the exact H^1-controllability of the Schrödinger system provide stronger convergence of the minimizing sequences in the variational principle (see [2]).
- A peculiarity of the heat and Schrödinger systems is the global character of their dynamical data: owing to infiniteness of the domain of influence, for any $T > 0$ the response operators R^T contain the information about Ω as a whole. By the same reasoning, (Ω, g) is determined up to isometry by the operator R^T_σ given for an open $\sigma \subset \Gamma$.
- Perhaps, the area of application of the variational principle is not exhausted yet. For instance, if the system governed by the plate equation $u_{tt} + \Delta^2 u = 0$ is approximately controllable, then the relevant dynamical and spectral data determine (Ω, g) up to isometry.

2.3.9. Not self-adjoint case. A version of the BCm for the wave equation $u_{tt} - \Delta u + b\cdot\nabla u + qu = 0$ (b is a vector field and q is a function in $\Omega \subset \mathbb{R}^n$) is based on the following observation. Denote $L := -\Delta + b\cdot\nabla + q$, let $L_\sharp := -\Delta - b\cdot\nabla + [\operatorname{div} b] + q$ be the conjugate operator. Let

$$\begin{aligned} &u_{tt} + Lu = 0 && \text{in} \quad \Omega \times (0, T) && \qquad u_{tt} + L_\sharp u = 0 && \text{in} \quad \Omega \times (0, T) \\ &u|_{t=0} = u_t|_{t=0} = 0 && \text{in} \quad \bar\Omega && \qquad u|_{t=0} = u_t|_{t=0} = 0 && \text{in} \quad \bar\Omega \\ &u = f && \text{on} \quad \Gamma \times [0, T] && \qquad u = f && \text{on} \quad \Gamma \times [0, T] \end{aligned}$$

be the conjugate systems, $u = u^f(x,t)$ and $u = u^f_\sharp(x,t)$ the solutions, $W^T : f \mapsto u^f(\cdot,T)$ and $W^T_\sharp : f \mapsto u^f_\sharp(\cdot,T)$ the control operators, $R^T : f \mapsto \frac{\partial u^f}{\partial \nu}\big|_{\Gamma\times[0,T]}$ and $R^T_\sharp : f \mapsto \frac{\partial u^f_\sharp}{\partial \nu}\big|_{\Gamma\times[0,T]}$ the response operators.[14] The *cross-connecting* operator $C^T = \left(W^T_\sharp\right)^* W^T$ can be represented in the form (2.15) and, hence, the wave products $(u^{f'}(\cdot,T), u^{f''}_\sharp(\cdot,T))_{\mathcal{H}^T} = (C^T f', f'')_{\mathcal{F}^T}$ are determined by the response operator R^{2T} (S A Avdonin [1, 16]).

As each version of the BCm, the variant available for the not self-adjoint case invokes the results on controllability. Fix $x \in \Omega$, let l^θ_x be a geodesic (straight ray) emanating from x in the direction $\theta \in S^{n-1}$. Let t^θ_x be the time needed for the point starting from x and moving along l^θ_x with unit speed, to reach the boundary Γ. Denote $T_x := \max_{\theta\in S^{n-1}} \min\left\{t^\theta_x, t^{-\theta}_x\right\}$ and introduce a subdomain $B^T := \{x \in \Omega \mid T_x < T\}$ that we call *Bardos's zone*. The relation $\{u^f(\cdot,T)|_{B^T} \mid f \in \mathcal{F}^T\} = L_2(B^T)$ is derived from the well-known result of Bardos, Lebeau and Roach [7] (see [6]) and may be interpreted as exact controllability of the wave tails [16]. Our contribution to the dynamical IP is the following [17, 16].

Theorem 2. *Assume that $L = -\Delta + b\cdot\nabla$; let T and $T' > T$ be such that $\Omega^T \subset B^{T'}$. Then the response operator $R^{2T'}$ determines $b|_{\Omega^T}$ uniquely.*

This result is supplemented with a procedure recovering the field b by means of the *amplitude integral*, which is a promoted version of the AF. If L is of general form $-\Delta + b\cdot\nabla + q$, the procedure recovers L up to a natural nonuniqueness and constructs a representative of the class of equivalent operators possessing the given $R^{2T'}$, the representative being singled out by the condition $b\perp\nabla\tau$. Theorem 2 remains valid for RMs.

However, with the exception of the case $b = \nabla \ln\rho$, which can be reduced to the self-adjoint case (see [69]), time-optimal determination $R^{2T} \Rightarrow b|_{\Omega^T}$ is not established so far. The problem can be interpreted as a question on uniqueness of a special triangular factorization for the cross-connecting operator C^T (see [18]) and we hope for development of this approach.

- One of the best uniqueness results on determination of the low-order terms belongs to V Isakov: see [66], theorem 8.3.1.
- In [74] the complex not self-adjoint operator of more general form is determined from the frequency-domain data, provided the BLR-condition of controllability.
- In [61], a modification of the BCm is proposed for determination of the complex self-adjoint operator $L = -\Delta + A + q$ containing the first-order ('magnetic') terms A in $\Omega \subset \mathbb{R}^n$. This approach provides the uniqueness of determination (up to a natural equivalence) from R^{2T}_σ given on a part of the boundary, provided T is large enough. However, in our opinion, the same results in a stronger form (on a RM, for a fixed arbitrary $T > 0$) can be obtained by the use of standard BCm tools. Moreover, visualization by the AF could give a canonical representative of the class of equivalent operators L. This is done for $A = 0$ [15] and we see no obstructions to extending this technique to the case under consideration. Perhaps the author of [61] does not fully recognize the opportunities of the 'classical' BCm.

2.3.10. On recovery of dissipation. Here we discuss an important problem, required in a lot of applications, which is so far open: all attempts to solve it by the BCm have given no result during more than 10 years. The problem is the time-optimal determination of the *dissipative term* in the wave equation.

[14] R^T and $R^T_\sharp$ are connected through a simple explicit formula: see [1, 16].

In a bounded $\Omega \subset \mathbb{R}^n$, consider the system α_β^T of the form

$$\begin{aligned} &u_{tt} + \beta u_t - \Delta u = 0 && \text{in} \quad \Omega^T \times (0, T) \\ &u = 0 && \text{in} \quad \{(x, t) \mid x \in \Omega^T, 0 \leqslant t \leqslant \tau(x)\} \\ &u = f && \text{on} \quad \Gamma \times [0, T] \end{aligned}$$

(τ is the Euclidean eikonal) with a smooth enough function $\beta \geqslant 0$; let $u = u^f(x, t)$ be its solution, $W_\beta^T : f \mapsto u^f(\cdot, T)$ the control operator. The continued response operator $R_\beta^{2T} : f \mapsto \frac{\partial u^f}{\partial \nu}\big|_{\Gamma\times[0,2T]}$ associated with the appropriate extended problem (see (2.10)–(2.12)) is determined by $\beta|_{\Omega^T}$. The dynamical IP is to determine $\beta|_{\Omega^T}$ from R_β^{2T} given for a fixed $T > 0$. For $T < T_*$, the problem is open.

Probably, the primary source of difficulties is the fact that to recover the term $\beta\frac{\partial}{\partial t}$ is to recover a *nonlocal operator*. As can be shown, for $T < T_*$ the operator W_β^T is injective. In this case, this term can be written in the form $\beta\frac{\partial}{\partial t} = \beta Q^T$, where $Q^T := W_\beta^T \frac{\partial}{\partial t}\left(W_\beta^T\right)^{-1} : u^f(\cdot, T) \mapsto u_t^f(\cdot, T)$ is a velocity operator well defined on the reachable set. As is easy to show, this operator is nonlocal: it does not preserve support of the wave.

The relation

$$\left(\frac{1}{2}(S^T)^* R_\beta^{2T} J^{2T} S^T f', f''\right)_{\mathcal{F}^T} = \int_\Omega u^{f'}(x, T) u^{f''}(x, T)\,\mathrm{d}x + \int_\Omega \mathrm{d}x\, \beta(x) \int_0^T u^{f'}(x, t) u^{f''}(x, t)\,\mathrm{d}t \qquad (2.51)$$

(see [13]) is a straightforward analog of (2.15). The rhs determines a Hilbert metric on waves, the metric being stronger than the standard L_2-metric. Hence, completing the reachable set $\mathcal{U}_\beta^T = W_\beta^T \mathcal{F}^T$ w.r.t. the corresponding norm, we obtain a new Hilbert space $\mathcal{H}_\beta^T \subset \mathcal{H}^T$.

Considering $\mathcal{H}_\beta^T$ as an inner space of system α_β^T and understanding properly the operators W_β^T, $\left(W_\beta^T\right)^*$, $P_{\mathcal{U}_\beta^T}$, we can compose the rhs of the AF (2.32).[15] The intriguing question is: what does such an AF show and whose pictures will be seen? The reason of interest is that it is the pictures, which the external observer can visualize through the dynamical model by (2.36). One more question: if one repeats step by step the procedure (i)–(iii) of section 2.2.5 constructing the wave copy, what end product will be obtained as a result? Is it an RM? In other words, possessing R_β^{2T} and applying the standard BCm-devices, what will the external observer see?

In the 1-dim case the answer is known: applying the BCm-procedure to R_β^{2T}, we get a system α_q^T governed by the equation $u_{tt} - u_{xx} + qu = 0$ with a potential $q = q(x)$ such that its response operator R_q^{2T} coincides with R_β^{2T} [13]. Systems α_q^T and α_β^T are indistinguishable for the external observer. Perhaps, in the multidimensional case the observer will get an RM with a new metric, which simulates the dissipation effects[16].

Thus, an edifying fact is that the canonical realization of a system obtained by the BCm tools (wave copy, AF, etc) does not need to be identical to the original system. Therefore, we have to take care of the transform 'realization $\mapsto$ original', which is not a trivial task. For instance, even in the 1-dim case we do not know whether there exists a formula or an efficient procedure determining β from q.

[15] However, we have no efficient description of $P_{\mathcal{U}_\beta^T}$: it is not known how to project in the metric (2.51).

[16] Such a simulation occurs in the case of the transport equation (V A Sharafutdinov).

- The best-known result on uniqueness of determination of β belongs to V Isakov (see [66], theorem 8.3.1). In [75] the dissipation is recovered by a version of the BCm provided the BLR condition on Ω, so that the time of determination is far from the optimal one. The same condition is used in the paper [62], where an approach incorporating some ideas of the BCm is proposed. A promising peculiarity of the approach is that it recovers *time-dependent* coefficients, what is mostly 'terra incognita' in hyperbolic IPs. However, the approach is rather complicated and needs to be carefully checked.

2.3.11. BCm and linear system theory. The paper [20] is an attempt to appreciate the operator background of the BCm and inscribe the method in general system theory [71].

The object defining an abstract *dynamical system with boundary control* (DSBC) is a *space of boundary values* (SBV) that is a collection $\{\mathcal{H}, \mathcal{B}, L, D, N\}$ of the Hilbert spaces $\mathcal{H}, \mathcal{B}$ and the operators $L : \mathcal{H} \to \mathcal{H}$, $D : \mathcal{H} \to \mathcal{B}$, $N : \mathcal{H} \to \mathcal{B}$ connected through the Green formula

$$(Lu, v)_{\mathcal{H}} - (u, Lv)_{\mathcal{H}} = (Du, Nv)_{\mathcal{B}} - (Nu, Dv)_{\mathcal{B}}$$

for $u, v \in \operatorname{Dom} L$. DSBC is a dynamical system α^T of the form

$$\begin{array}{lll} u_{tt} + Lu = 0 & \text{in} \quad \mathcal{H}, & 0 < t < T \\ u|_{t=0} = u_t|_{t=0} = 0 & \text{in} \quad \mathcal{H} & \\ D[u(t)] = f(t) & \text{in} \quad \mathcal{B}, & 0 \leqslant t \leqslant T, \end{array}$$

where $f \in \mathcal{F}^T := L_2([0, T]; \mathcal{B})$ is a boundary control, $u = u^f(t)$ is a solution. The control operator $W^T : \mathcal{F}^T \to \mathcal{H}$ acts by the rule $W^T f = u^f(T)$; the response operator is $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $R^T f = N[u^f(\cdot)]$; the connecting operator is $C^T = (W^T)^* W^T$. Recall that the continued response operator R^{2T} of the system on RM is defined via the extended system (2.10)–(2.13), which has no natural abstract analog. Nevertheless, R^{2T} can be introduced for an abstract DSBC: it is defined by

$$R^{2T} := -2Y^{2T} S_0^T C^T \frac{\mathrm{d}}{\mathrm{d}t} \big(S_0^T\big)^* - Y^{2T} S_0^T (R^T)^* (S_-^T) + S_+^T R^T \big(S_0^T\big)^*,$$

where $S_0^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ extends the controls from $[0, T]$ to $[0, 2T]$ by zero, $S_\pm^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ are the extensions by evenness/oddness w.r.t. $t = T$, Y^{2T} changes t for $2T - t$. Thus, R^{2T} is an intrinsic object of the DSBC α^T determined by W^T and R^T, whereas the relation (2.15) turns out to be an operator identity.

The IP is to recover the system α^T from the given R^{2T}. 'To recover' means to construct a system $\tilde{\alpha}^T$ (a realization) associated with an SBV $\{\tilde{\mathcal{H}}^T, \mathcal{B}, \tilde{L}^T, \tilde{D}^T, \tilde{N}^T\}$ provided $\operatorname{clos} \operatorname{Ran} \tilde{W}^T = \tilde{\mathcal{H}}^T$,[17] such that $\tilde{R}^{2T} = R^{2T}$.

The main result of [20] is the characterization of the inverse data: after rigorously specifying the class of systems, we prove that an operator R^{2T} is the continued response operator of a DSBC α^T iff the operator $\frac{1}{2}(S_-^T)^* R^{2T} J^{2T} S_-^T$ is positive definite in $\mathcal{F}^T$.

A realization $\tilde{\alpha}^T$ that possesses the given R^{2T} is not unique: it is determined up to an isometry $U^T : \tilde{\mathcal{H}}^T \to \tilde{\mathcal{H}}'^T$, $\tilde{W}'^T = U^T \tilde{W}^T$. We introduce some special realizations related to the triangular factorization of the connecting operator.

[17] This is a natural minimality condition.

As an application, we obtain a *conditional existence theorem* for the problem of reconstruction of (Ω^T, g) (for $T < T_c$) via R^{2T}. The assumptions of the theorem ensure the realizability of the procedure recovering the visualizing operator V^T from R^{2T}. This operator is constructed in the form of an operator (amplitude) integral [15]. Actually, the theorem is not too rich in content, because to check its assumptions is in fact to recover V^T by the BCm. However, it contains a sharp and checkable necessary condition: a certain operator integral determined by the data must converge to a unitary operator.

The BCm provides the multidimensional analogs of the classical Gelfand–Levitan–Krein–Marchenko equations introduced in [9]. The paper [20] presents an abstract version of these analogs.

2.3.12. Numerical testing and convergence of algorithms. Numerical algorithms for solving 2-dim spectral and dynamical IPs for the wave equation $\rho u_{tt} - \Delta u = 0$, which recover a variable density $\rho > 0$, were elaborated and tested by V Yu Gotlib and S A Ivanov in 1997–1999 (see [39, 40]). Most successful results (Gotlib [39]) are obtained by the algorithm realizing the amplitude formula (2.36). They are shown in figure 7: the left pictures are the tests, the right ones are the reconstructions. The first line demonstrates the recovery of $\rho(x^1, x^2)$ depending linearly on x^1. The second line corresponds to the density of the background value 1 perturbed by four smooth positive Gaussian-type inclusions: $\rho = 1 + \sum_{i=1}^4 \delta\rho_i, \delta\rho_i \approx 0.2$. The third line corresponds to the most complicated profile: there are four inclusions $\delta\rho_{1,2} \leqslant 0, \delta\rho_{3,4} \geqslant 0, | \delta\rho_i | \approx 0.2$. The algorithm works well near the boundary, whereas the zone near the cut locus turns out to be unreachable that should be expected by physical reasons: the focusing and crossing rays effects complicate the reconstruction. Unfortunately, this work has been terminated by the early death of Vadim Gotlib who was an excellent mathematician and specialist in numerical analysis.

Recent promising results on numerical testing of the BCm-algorithm elaborated by S I Kabanikhin and M A Shishlenin are presented in [69]. The algorithm recovers a density ρ in the 2-dim acoustic equation of the form $u_{tt} - \Delta u + \nabla \ln \rho \cdot \nabla u = 0$ from R^{2T}.

In the case of the wave equation with density, analysis of the algorithm based on the AF shows the following type of convergence. Let $x_0 \in \Omega$ be a point lying out of the cut locus (w.r.t. the metric $\rho|\mathrm{d}x|^2$). Then, for arbitrarily given $\varepsilon > 0$, the algorithm produces a sequence of numbers $\left\{\rho_j^\varepsilon(x_0)\right\}_{j=1}^\infty$ such that $\left|\rho(x_0) - \lim_{j\to\infty} \rho_j^\varepsilon(x_0)\right| < \varepsilon$. However, the estimates of j ensuring the last inequality are not obtained yet (if they exist). The same type of convergence occurs in recovering (Ω^T, d^T) by the algorithm, which determines the components of the metric tensor in sgc through the AF [15].

Another open problem is the stability of the time-optimal reconstruction, the stability being understood as a continuity of the map $R^{2T} \to (\Omega^T, d^T)$. The known results [94] concern the simple metrics and are not time-optimal. Our hopes for time-optimal results are connected with an approach based on the Carleman estimates [90, 91]. One more possible way is to investigate, at a general operator level, the stability of the triangular factorization $C^T = (V^T)^* V^T$ (see section 2.3.5). The principal difficulty is that in multidimensional problems C^T is not an isomorphism.

2.3.13. Kinematic IP. A new approach to the 2-dim kinematic IP exploiting some ideas of the BCm is proposed by L N Pestov [87]. Using a relevant variant of the Gelfand–Levitan–Krein–Marchenko equations and a microlocal version of the classical Hilbert transform, the approach determines the (elliptic) Dirichlet-to-Neumann operator from a hodograph $d|_{\Gamma\times\Gamma}$ of the manifold provided simplicity of its metric. Thereafter, the known results on determination of the metric from the Dirichlet-to-Neumann operator are applied.

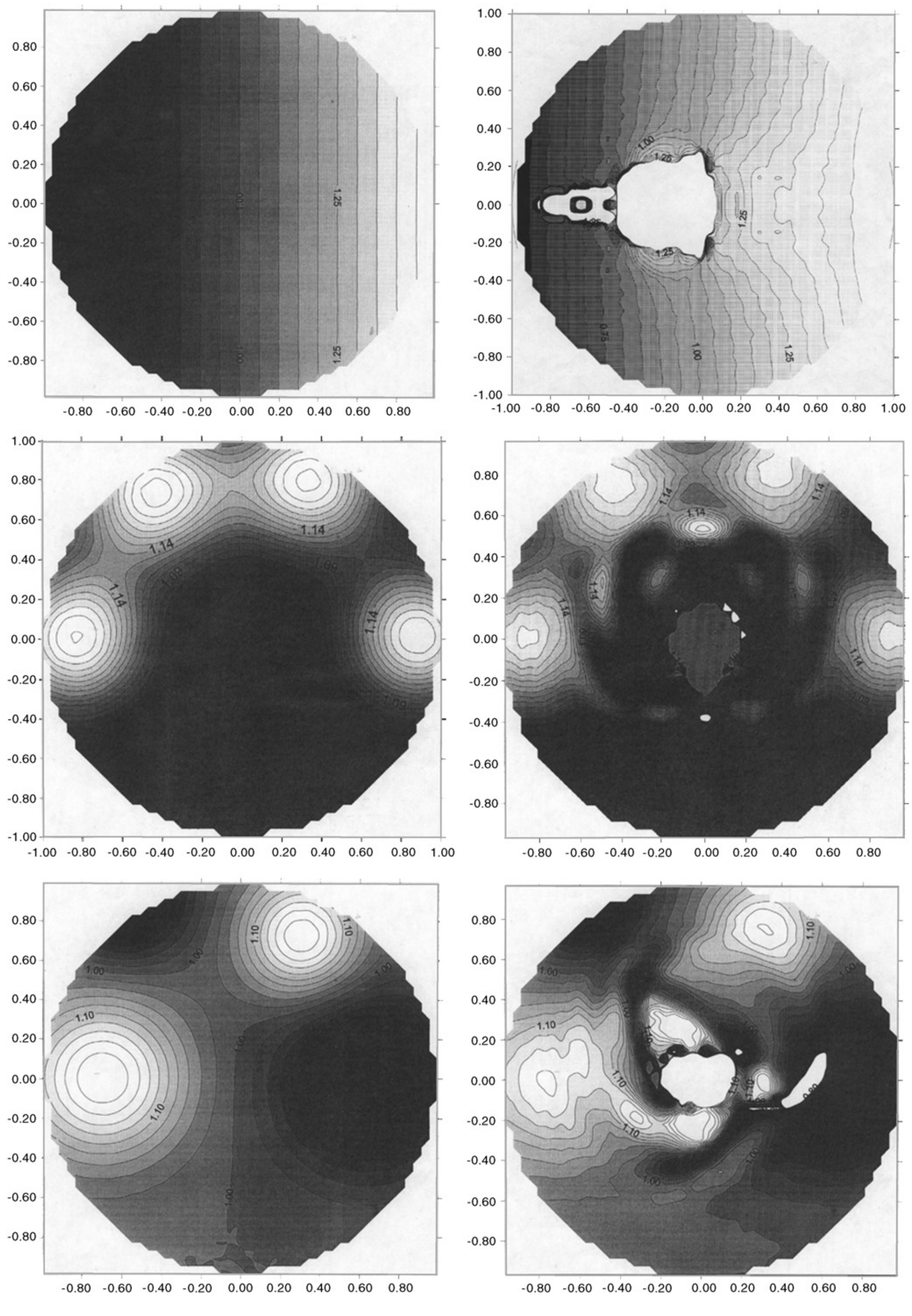

Figure 7. Numerical testing.

3. Electrodynamics

3.1. Maxwell system on RM

3.1.1. Vector analysis. Let Ω be a smooth compact orientable RM, g a metric tensor on Ω, $\dim \Omega = 3$, $\Gamma := \partial\Omega \in C^\infty$. We assume Ω oriented and denote by μ the volume 3-form in Ω. Recall certain of the definitions of the vector analysis (e.g., see [93], chapter 3.5).

With a vector field a given in Ω one associates an 1-form $a_\sharp$ by $a_\sharp(b) := g(a, b), \forall\, b$. A field $\eta^\sharp$ associated with a form η is defined by $g(\eta^\sharp, b) = \eta(b), \forall\, b$.

A scalar product of fields is defined pointwise by $a \cdot b := g(a, b)$. A vector product $a \times b$ is defined by $g(a \times b, c) = \mu(a, b, c), \forall\, c$.

We identify 0-forms and functions. A *gradient* of a function is a field $\nabla\phi := (\mathrm{d}\phi)^\sharp$. A *divergence* of a field is a function $\operatorname{div} a := \delta a_\sharp$, where $\delta = - \star\, \mathrm{d}\star$ is the codifferential, $\star$ is the Hodge operator. A *curl* of a field is a field $\operatorname{curl} a := (\star\, \mathrm{d}a_\sharp)^\sharp$. Recall the basic identities $\operatorname{div}\operatorname{curl} = 0$, $\operatorname{curl}\nabla = 0$.

The *Laplacian on function* is defined by $\Delta := \operatorname{div}\nabla$. The *Laplacian on vector field* is $\Delta := \nabla\operatorname{div} - \operatorname{curl}\operatorname{curl}$.

Let μ_Γ be the surface 2-form on $\Gamma : \mu_\Gamma(a, b) = \mu(a, b, \nu)$ for vector fields a, b on Γ[18] and the outward normal $\nu = \nu(\gamma)$. Recall the Green formulae

$$\int_\Omega \operatorname{div} au = \int_\Gamma a \cdot \nu u - \int_\Omega a \cdot \nabla u; \qquad \int_\Omega \operatorname{curl} a \cdot b = \int_\Gamma \nu \times a \cdot b + \int_\Omega a \cdot \operatorname{curl} b \tag{3.1}$$

(we omit μ and μ_Γ in integrals) and the relation

$$\nu \cdot \operatorname{curl} a = -\operatorname{div}_\Gamma \nu \times a,$$

where $\operatorname{div}_\Gamma$ is the (intrinsic) divergence on Γ.

We use a pointwise decomposition of vectors $a \in T_\gamma\Omega$ on Γ:

$$a = a_\theta + a^\nu \nu, \tag{3.2}$$

where a_θ is a tangent component of a, $a^\nu = a \cdot \nu$ and identify a_θ as the corresponding element of $T_\gamma(\Gamma)$.

3.1.2. System α^T. The *Maxwell system* on Ω is

$$u_t = \operatorname{curl} v, \qquad v_t = -\operatorname{curl} u \qquad \text{in} \quad (\operatorname{int}\Omega) \times (0, T) \tag{3.3}$$

$$u|_{t=0} = 0, \qquad v|_{t=0} = 0 \qquad \text{in} \quad \Omega \tag{3.4}$$

$$u_\theta = f \qquad \text{on} \quad \Gamma \times [0, T], \tag{3.5}$$

where f is a boundary control (a time-dependent tangent field on Γ), $u = u^f(x, t)$, $v = v^f(x, t)$ is a solution. The relations $\operatorname{div} u^f(\cdot, t) = \operatorname{div} v^f(\cdot, t) = 0, \forall\, t$ follow from (3.3) and (3.4).

Eliminating a magnetic component v and taking into account the finiteness of the domains of influence, one arrives at the system α^T of the form

$$u_{tt} - \Delta u = 0 \qquad \text{in} \quad (\operatorname{int}\Omega^T) \times (0, T) \tag{3.6}$$

$$u = 0 \qquad \text{in} \quad \{(x, t) \mid x \in \Omega^T, 0 \leqslant t \leqslant \tau(x)\} \tag{3.7}$$

$$u_\theta = f \qquad \text{on} \quad \Gamma \times [0, T] \tag{3.8}$$

[18] Here and in the following, we identify $a(\gamma) \in T_\gamma\Gamma$ and $(i_*a)(\gamma) \in T_\gamma\Omega$, where $i : \Gamma \to \Omega$ is the embedding.

(we use $\Delta = -\mathrm{curl}\,\mathrm{curl}$ on solenoidal fields), which is interpreted as an *electric subsystem* of system (3.3)–(3.5). Problem (3.6)–(3.7) is well posed; for $f \in \mathcal{M}^T := \{f \in C^\infty([0,T]; \vec{C}^\infty(\Gamma)) \mid \mathrm{supp}\, f \subset (0,T]\}$ it has a unique classical (smooth) solution $u = u^f(x,t)$.

An outer space of the system α^T is $\mathcal{F}^T := L_2([0,T]; \vec{L}_2(\Gamma))$; it contains the class $\mathcal{F}^T_+ := L_2([0,T]; \vec{H}^1(\Gamma))$ ($\vec{H}^s(\cdots)$ are the Sobolev classes).

Let $\mathcal{J} := \{y \in \vec{L}_2(\Omega) \mid \mathrm{div}\, y = 0\}$ be a *space of solenoidal vector fields* on Ω. Its subspace $\mathcal{J}^T := \mathrm{clos}\,\{y \in \mathcal{J} \mid \mathrm{supp}\, y \subset \Omega^T\}$ plays the role of an inner space of α^T.

A control operator of the system is $W^T : \mathcal{F}^T \to \mathcal{J}^T$, $\mathrm{Dom}\, W^T = \mathcal{M}^T$, $W^T := u^f(\cdot, T)$. By the results of [79], it acts continuously from $\mathcal{F}^T_+$ to $\mathcal{J}^T$ and we assume W^T extended on $\mathcal{F}^T_+$.[19]

A response operator is $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\mathrm{Dom}\, R^T = \mathcal{M}^T$, $R^T f := \nu \times \mathrm{curl}\, u^f|_{\Gamma\times[0,T]}$. A continued response operator $R^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$ is introduced through the relevant analog of system (2.10)–(2.12) (with $u = f$ replaced by $u_\theta = f$). By hyperbolicity, the operator R^{2T} is determined by the submanifold (Ω^T, g).

With the system α^T we associate a *connecting form* $c^T : \mathcal{F}^T \times \mathcal{F}^T \to \overline{\mathbf{R}}_+$, $\mathrm{Dom}\, c^T = \mathcal{F}^T_+ \times \mathcal{F}^T_+$, $c^T[f', f''] := (W^T f', W^T f'')_{\mathcal{J}^T}$. The relation

$$c^T[f', f''] = \left(\tfrac{1}{2}(S^T)^* R^{2T} J^{2T} S^T f', f''\right)_{\mathcal{J}^T} \tag{3.9}$$

holds and shows that c^T is determined by the operator R^{2T} (see [36]).

3.1.3. System $\beta^{T,r}$. Here we introduce an 'electric' analog of system (2.16)–(2.18). Begin with the operator governing its evolution.

Define an operator $-\Delta^T_0 : \mathcal{J}^T \to \mathcal{J}^T$, $\mathrm{Dom}\big(-\Delta^T_0\big) = \{y \in \vec{C}^\infty(\Omega) \cap \mathcal{J}^T \mid u_\theta|_\Gamma = 0$, $\mathrm{supp}\, y \subset \Omega^T\}$, $-\Delta^T_0 y = -\Delta y$. This operator is densely defined and positive; let $-\Delta^T$ be its extension by Friedrichs. As can be shown, for $T \leqslant T_*$ the elements of $\mathrm{Dom}(-\Delta^T)$ vanish at Γ^T, whereas on smooth elements it acts by the rule $-\Delta^T y = P_{\mathcal{J}^T}\, \mathrm{curl}\,\mathrm{curl}\, y$, where $P_{\mathcal{J}^T}$ projects square summable in Ω^T solenoidal fields on the subspace $\mathcal{J}^T$. By the presence of the projection, $-\Delta^T$ is not a local operator, in contrast to its acoustic analog of the same name. Nevertheless, as is easy to see, the additional assumption $d^T(\mathrm{supp}\, y, \Gamma^T) > 0$ ensures $\mathrm{supp}\,(-\Delta^T y) \subset \mathrm{supp}\, y$.

The (electric) system $\beta^{T,r}$ is of the form

$$w_{tt} - \Delta^T w = h \qquad \text{in} \quad \mathcal{J}^T, \qquad t \in (0,r) \tag{3.10}$$

$$w|_{t=0} = w_t|_{t=0} = 0 \qquad \text{in} \quad \mathcal{J}^T \tag{3.11}$$

where $h \in \mathcal{G}^r := L_2([0,r]; \mathcal{J}^T)$ is a volume control, $w = w^h(\cdot, t)$ is a solution defined by Duhamel[20]:

$$w^h(\cdot, t) := \int_0^t (-\Delta^T)^{-\frac{1}{2}} \sin[(t-s)(-\Delta^T)^{\frac{1}{2}}] h(\cdot, s)\, \mathrm{d}s, \qquad t > 0. \tag{3.12}$$

The control operator of the system $W^r_{\mathrm{vol}} : h \mapsto w^h(\cdot, r)$ acts continuously from $\mathcal{G}^r$ to $\mathcal{J}^T$. The following peculiarity must be taken into account: in contrast to the acoustic case, the evolution by (3.12) in general does not satisfy the finiteness of the speed of wave propagation. The reason is that, as was mentioned above, the Friedrichs extension $-\Delta^T$ is not a local operator. However, for controls h supported in Ω^T and small enough times, equation (3.10) takes the form $w_{tt} - \Delta w = h$ and such a finiteness does hold.

[19] W^T is not continuous as a map from $\mathcal{F}^T$ to $\mathcal{J}^T$ (D Tataru, private communication).

[20] Here and in the following, for the sake of simplicity, we assume $\mathrm{Ker}(-\Delta^T) = \{0\}$; the extension to the general case is evident.

3.1.4. Controllability. For open subsets $\sigma \subset \Gamma$, $\omega \subset \Omega^T$ and parameters $s \in [0, T]$, $r \geqslant 0$, let

$$\mathcal{F}^{T,s}[\sigma] := \operatorname{clos}\{f \in \mathcal{F}^T \mid \operatorname{supp} f \subset [T-s, T], \operatorname{supp} f(\cdot, t) \subset \sigma, \forall t\},$$
$$\mathcal{F}^{T,T}[\sigma] := \mathcal{F}^T[\sigma]; \qquad \mathcal{G}^r[\omega] := \operatorname{clos}\{h \in \mathcal{G}^r \mid \operatorname{supp} h(\cdot, t) \subset \omega \, \forall t\}$$

be the subspaces of controls acting from σ and ω respectively; also, simplifying the notation, we omit 'σ' in the case of $\sigma = \Gamma$. Introduce the corresponding reachable sets of the systems α^T and $\beta^{T,r}$

$$\mathcal{U}^s[\sigma] := W^T\left[\mathcal{F}^{T,s}[\sigma] \cap \mathcal{F}^T_+\right]; \qquad \mathcal{U}^r[\omega] := W^r_{\mathrm{vol}} \mathcal{G}^r[\omega].$$

For $A \subset \Omega^T$, we define the subspaces $\mathcal{J}^s[A] := \operatorname{clos}\{y \in \mathcal{J} \mid \operatorname{supp} y \subset \Omega^s[A]\}$ and recall that the neighborhoods $\Omega^s[A]$ are understood in the sense of the intrinsic distance d^T in Ω^T. Assume that ω is separated from Γ^T; as one can show, the finiteness of the speed of propagation implies

$$\mathcal{U}^s[\sigma] \subset \mathcal{J}^s[\sigma], \qquad \mathcal{U}^r[\omega] \subset \mathcal{J}^r[\omega]$$

for $0 \leqslant s \leqslant T$ and $0 \leqslant r < d^T(\omega, \Gamma^T)$.

A field $y \in J$ is called *harmonic* in ω if $\operatorname{curl} y = 0$ in ω. For two subspaces $\mathcal{A} \subset \mathcal{B} \subset \mathcal{J}^T$, we write $\mathcal{A} \approx \mathcal{B}$ if the elements of $\mathcal{B} \ominus \mathcal{A}$ are harmonic into their supports.

Theorem 3. *Let $\sigma \subset \Gamma$ and $\omega \subset \Omega^T$ be open subsets such that $\overline{\omega} \cap \Gamma^T = \{\emptyset\}$. For parameters $0 \leqslant s \leqslant T$, $r \geqslant 0$ provided $r < d^T(\omega, \Gamma^T)$, the relations*

$$\operatorname{clos} \mathcal{U}^s[\sigma] \approx \mathcal{J}^s[\sigma], \qquad \operatorname{clos} \mathcal{U}^r[\omega] \approx \mathcal{J}^r[\omega] \tag{3.13}$$

are valid.

Proof. See in the appendix. □

Note that the subspace $\overline{\mathcal{U}}^T := \operatorname{clos} \mathcal{U}^T[\Gamma] \subseteq \mathcal{J}^T$ can be referred to as a relevant (minimal) inner space of the system α^T: its trajectories do not leave this subspace. About 'unreachable' states $\mathcal{J}^T \ominus \overline{\mathcal{U}}^T$ see [35, 36].

3.1.5. Laplacian on waves. System $\beta^{T,r}_{\mathrm{rch}}$. Fix $T > 0$. By analogy with (2.22), introduce the operator $L^T_0 : \overline{\mathcal{U}}^T \to \overline{\mathcal{U}}^T$ through its graph

$$\operatorname{graph} L^T_0 := \{\{W^T f, -W^T f_{tt}\} \mid f \in C^\infty_0((0, T); \vec{C}^\infty(\Gamma))\}. \tag{3.14}$$

This operator coincides with the Laplacian $-\Delta = \operatorname{curl}\operatorname{curl}$ restricted on $W^T C^\infty_0((0, T); \vec{C}^\infty(\Gamma))$. Hence, L^T_0 is a densely defined positive operator; let L^T be its extension by Friedrichs. As can be shown, on smooth elements of its domain of definition, this extension acts as $P_{\overline{\mathcal{U}}^T} \operatorname{curl}\operatorname{curl}$.

The operator L^T determines the system $\beta^{T,r}_{\mathrm{rch}}$ of the form

$$w_{tt} - L^T w = h \qquad \text{in} \quad \overline{\mathcal{U}}^T, \quad t \in (0, r)$$
$$w|_{t=0} = w_t|_{t=0} = 0 \qquad \text{in} \quad \overline{\mathcal{U}}^T$$

with controls $h \in L_2([0, r]; \overline{\mathcal{U}}^T)$. This system evolves in the reachable subspace $\overline{\mathcal{U}}^T$, the evolution being governed by the operator

$$W^r_{\mathrm{vol\,rch}} := \int_0^r (L^T)^{-\frac{1}{2}} \sin[(r-s)(L^T)^{\frac{1}{2}}](\cdot)(s)\, \mathrm{d}s. \tag{3.15}$$

System $\beta^{T,r}_{\text{rch}}$ is evidently determined by the operator W^T and, as such, may be interpreted as the part of system $\beta^{T,r}$ reachable from the boundary Γ. In the acoustic case, these systems are identical, whereas in electrodynamics, in the case of $\mathcal{J}^T \ominus \overline{\mathcal{U}}^T \neq \{0\}$, they differ. However, the following result shows that away from Γ^T these systems evolve identically.

Lemma 3. *Let $\omega \subset \Omega^T$ be an open set such that $d^T(\omega, \Gamma^T) \geqslant r > 0$; assume $h \in L_2([0,r]; \overline{\mathcal{U}}^T)$ to be such that* $\operatorname{supp} h(\cdot, t) \subset \overline{\omega}, 0 \leqslant t \leqslant r$*. Then, for times $t \in [0,r]$ the trajectories (solutions $w^h(\cdot,t)$) of systems $\beta^{T,r}_{\text{rch}}$ and $\beta^{T,r}$ coincide and satisfy the equation* $w_{tt} - \operatorname{curl}\operatorname{curl} w = h$.

We omit the proof, based on the following simple fact: if a smooth solenoidal y satisfies $\nu \times y|_\Gamma = 0$ and $\operatorname{supp} y \cap \Gamma^T = \{\emptyset\}$, then $y \in \operatorname{Dom} L^T$ and $L^T y = -\Delta^T y = \operatorname{curl}\operatorname{curl} y$.

Introduce an operation extending the subspaces $\mathcal{A} \subset \overline{\mathcal{U}}^T$ by the rule

$$\mathcal{E}^r \mathcal{A} := \operatorname{clos} W^r_{\text{vol rch}} L_2([0,r]; \mathcal{A}), \qquad r > 0, \quad \mathcal{E}^0 := \text{id}. \tag{3.16}$$

Note that, along with the operators L^T and W^r_{volrch}, the operation $\mathcal{E}^r$ is also determined by the boundary control operator W^T: see (3.14)–(3.16).

In addition to the assumptions of lemma 3, let us suppose that ω is homeomorphic to an $\mathbb{R}^3$-ball. In this case, the set $\{\operatorname{curl}\psi \mid \psi \in \vec{C}^\infty_0(\omega)\}$ is dense in $\mathcal{J}^0[\omega] = \operatorname{clos}\{y \in \mathcal{J} \mid \operatorname{supp} y \subset \omega\}$ (see [82, 93]), whereas each $\operatorname{curl}\psi$ is orthogonal to the fields harmonic in Ω^T. By the boundary controllability (3.13), the latter implies $\mathcal{J}^0[\omega] \subset \overline{\mathcal{U}}^T$ and, hence, the operation (3.16) is well defined on $\mathcal{J}^0[\omega]$. In the meantime, the volume controllability and lemma 3 lead to

$$\mathcal{E}^r \mathcal{J}^0[\omega] = \operatorname{clos} \mathcal{U}^r[\omega] = \langle \text{see (3.13)} \rangle \approx \mathcal{J}^r[\omega], \qquad 0 < r \leqslant d^T(\omega, \Gamma^T). \tag{3.17}$$

3.1.6. Wave copy. Let us repeat the procedure constructing the wave copy (section 2.2.5), just replacing the 'acoustic' objects by their 'electric' analogs denoted by the same symbols.

(i) For fixed $\gamma \in \Gamma$, $s \in (0,T)$ and $\varepsilon \in (0,s)$, introduce a wave cap $\mathcal{U}^{s,\varepsilon}_\gamma$ by (2.26).

Lemma 4. *For a small enough ε, the equality $\mathcal{U}^{s,\varepsilon}_\gamma = \mathcal{J}^0\big[\omega^{s,\varepsilon}_\gamma\big]$ holds.*

Proof. See in the appendix. □

Then, sending $\varepsilon \to 0$ and controlling the behavior of the caps $\mathcal{U}^{s,\varepsilon}_\gamma$, we define the 'electric' pattern $\tilde{\Theta}^T_{\text{e}}$. By lemma 4, the behavior is the same as in the acoustic case. Therefore, $\tilde{\Theta}^T_{\text{e}}$ turns out to be identical to $\tilde{\Theta}^T$ constructed in section 2.2.5 (and, hence, to the original Θ^T).

(ii) and (iii) Constructing the nests by (2.27), we get the set $\tilde{\Omega}^T_{\text{e}}$. A minor correction is required, when this set is endowed with a distance: we have to take into account the restriction $0 < r \leqslant d^T(\omega, \Gamma^T)$ in (3.17). By doing so, we say two nests $\tilde{x}' = \tilde{x}(\gamma', s')$ and $\tilde{x}'' = \tilde{x}(\gamma'', s'')$ are close (to each other) if $2\inf\{r > 0 \mid \mathcal{U}^r[\tilde{x}'] \cap \mathcal{U}^r[\tilde{x}''] \neq \{0\}\} < \min\{T - s', T - s''\}$.[21] Then, for the close nests, we define by (2.28)

$$\tilde{d}^T_{\text{e}}(\tilde{x}', \tilde{x}'') := 2\inf\{r > 0 \mid \mathcal{U}^r[\tilde{x}'] \cap \mathcal{U}^r[\tilde{x}''] \neq \{0\}\}.$$

Such a local distance determines the global intrinsic metric on $\tilde{\Omega}^T_{\text{e}}$ (we denote it by the same symbol $\tilde{d}^T_{\text{e}}$), whereas the *electric wave copy* $\big(\tilde{\Omega}^T_{\text{e}}, \tilde{d}^T_{\text{e}}\big)$ turns out to be identical to the acoustic one and, hence, isometric to the original submanifold (Ω^T, d^T).

Summarizing, we can claim that electromagnetic waves are also well suited for constructing the wave copy.

[21] This condition means that the sources of waves supported near $x(\gamma', s')$ and $x(\gamma'', s'')$ begin to interact before the waves reach the boundary Γ^T.

- Probably, $\left(\tilde{\Omega}^T_{\rm e}, \tilde{d}^T_{\rm e}\right)$ is identical to the rhs of (2.42) but it is not established. One more interesting question concerns the visualization by (2.43): what does it give? In other words, what do Maxwell's waves look like in an invariant representation?

3.2. Inverse problem

3.2.1. Dynamical model: solving the IP. Given the continued response operator R^{2T} we can recover the connecting form $c^T|_{\mathcal{M}^T\times\mathcal{M}^T}$ by (3.9). Closing this form in $\mathcal{F}^T$,[22] we determine Dom $\bar{W}^T = \text{Dom}|\bar{W}^T|$ (the bar denotes the operator closure) and, hence, the modulus $|\bar{W}^T|$ of the control operator.

The operator $|\bar{W}^T|$ determines the dynamical model

$$\tilde{\mathcal{H}}^T := \text{clos Ran}|\bar{W}^T|, \qquad \tilde{W}^T := |\bar{W}^T| \tag{3.18}$$

(see (2.35)).

Reproducing through this model the wave copy $\left(\tilde{\Omega}^T_{\rm e}, \tilde{d}^T_{\rm e}\right)$, we get an RM isometric to (Ω^T, d^T). The IP is solved and we arrive at the following theorem.

Theorem 4. *For any $T > 0$, the continued response operator R^{2T} of system (3.6)–(3.8) determines the submanifold (Ω^T, d^T) up to isometry.*

- As ever in the BCm, C^∞-smoothness is in fact not necessary: the only restriction is the applicability of uniqueness theorems providing the controllability (3.13). Since for the Maxwell system $\Omega, \Gamma, g \in C^2$ is enough [60], theorem 4 remains valid in this case.
- Let $T > 0$ and an open $\sigma \subset \Gamma$ be fixed. By hyperbolicity of the system α^T, the operator $R^{2T}_\sigma : \mathcal{F}^{2T}[\sigma] \to \mathcal{F}^{2T}[\sigma]$ defined on $\mathcal{F}^{2T}[\sigma] \cap \text{Dom}\, R^{2T}$ by $R^{2T}_\sigma := [R^{2T}\cdot]|_{\sigma\times[0,2T]}$ is determined by the submanifold $(\Omega^T[\sigma], g)$. Analyzing the wave copy construction, it is easy to show that R^{2T}_σ determines (up to isometry) a neighborhood $(\Omega^\varepsilon[\sigma], g)$ for small enough $\varepsilon > 0$. However, the time-optimal determination $R^{2T}_\sigma \Rightarrow (\Omega^T[\sigma], g)$ is yet an open problem. The difficulty (rather, of technical character) is that Maxwell's L^T is not a local operator.

3.2.2. M-transform. An 'electric' version of the amplitude formula (2.36) will be presented in the next section. Here we describe the main ingredient of this version that is the so-called M-transform.

Lifting the field of the outward normals from Γ into Ω, we define a field $\nu := -\nabla\tau(\cdot)$ in $\Omega\backslash c$[23] and represent the fields in Ω pointwise as

$$a = a_\theta + a^\nu \nu, \tag{3.19}$$

where $a^\nu := a\cdot\nu$, $a^\nu\nu$ and $a_\theta = a - a^\nu\nu$ are the longitudinal and transversal components of a vector $a \in T_x\Omega$. Such a representation induces the decomposition

$$\vec{L}_2(\Omega) = \mathcal{L}_\theta \oplus \mathcal{L}_\nu \tag{3.20}$$

on the subspaces $\mathcal{L}_\theta := \{v \in \vec{L}_2(\Omega) \mid \nu\cdot v = 0\}$ and $\mathcal{L}_\nu := \{w \in \vec{L}_2(\Omega) \mid \nu \times w = 0\}$.

Postponing the rigorous definition of the transform, let us introduce it formally. Fix $s \in [0, T_*]$, let P^s be an (orthogonal) projection in $\mathcal{J}$ onto $\mathcal{J}^s = \mathcal{J}^0[\Omega^s] = \text{clos}\,\{y \in \mathcal{J} \mid \text{supp}\, y \subset \Omega^s\}$. For a smooth $y \in \mathcal{J}$, a field $P^s y$ is supported in $\overline{\Omega}^s$ and, by $\text{div}\, y = 0$, satisfies

[22] That is, completing $\mathcal{M}^T$ w.r.t. the norm $\{\|f\|^2_{\mathcal{F}^T} + c^T[f,f]\}^{\frac{1}{2}}$, which is the graph-norm of W^T.
[23] Since vol $c = 0$, the field ν is defined almost everywhere in Ω.

$\nu \cdot P^s y = 0$ on $\Gamma^s \backslash c$. Therefore, $P^s y|_{\Gamma^{s-0}\backslash c}$ is a tangent field on $\Gamma^s\backslash c$. This field is a jump appearing on Γ^s as result of projecting y on $\mathcal{J}^s$. A *transform* $M : \mathcal{J} \to \mathcal{L}_\theta$,

$$My := P^s y|_{\Gamma^{s-0}\backslash c} \qquad \text{on} \quad \Gamma^s\backslash c, \qquad 0 \leqslant s \leqslant T_*$$

maps y to a collection of these jumps.

The rigorous definition requires some additional assumptions on behavior of the equidistant surfaces Γ^s. It is convenient to impose them in the following implicit form. For $y \in \mathcal{J} \cap \vec{C}^\infty(\Omega)$, consider the (scalar) problem

$$\Delta p = 0 \qquad \text{in} \quad \Omega^s \tag{3.21}$$

$$p = 0 \qquad \text{on} \quad \Gamma \tag{3.22}$$

$$\nu \cdot \nabla p = \nu \cdot y \qquad \text{on} \quad \Gamma^s\backslash c \tag{3.23}$$

let $p = p^s(x)$ be its solution depending on s as a parameter. We assume that for all $s \in (0, T_*]$, except for a finite set $\{s_k\} : 0 < s_1 < s_2 < \cdots < s_N = T_*$,

(α) the problem has a unique $H^{\frac{3}{2}}(\Omega^s)$-solution and $\|p^s|_{\Gamma^s\backslash c}\|_{H^1(\Gamma^s\backslash c)} \leqslant$ const uniformly w.r.t. s, with the constant determined by y;

(β) the integrals $\int_{\Gamma^\tau} |\nabla p^s|^2 \, d\Gamma$ are uniformly bounded as $\tau \to s - 0$ and $\lim_{\tau\to s-0} \int_{\Gamma^\tau} |\nu \cdot \nabla p^s - \nu \cdot y|^2 d\Gamma = 0$;

(γ) the solution p^s is differentiable w.r.t. s in $\{(x, s) \mid \tau(x) < s\}$ and the integrals $\int_{\Gamma^\tau} \left|\frac{\partial p^s}{\partial s}\right|^2 d\Gamma$ are uniformly bounded as $\tau \to s - 0$.

Note that (α) and (β) hold if Γ^s is a Lipshitz surface[24], whereas all conditions are fulfilled for $s \in (0, T_c)$.

The projection on $\mathcal{J}^s$ acts by the rule

$$P^s y = \begin{cases} y - \nabla p^s & \text{in} \quad \Omega^s \\ 0 & \text{in} \quad \Omega\backslash\overline{\Omega}^s \end{cases} \tag{3.24}$$

(e.g., see [37]); therefore $P^s y|_{\Gamma^{s-0}\backslash c} = y_\theta - (\nabla p^s)_\theta$ on $\Gamma^s\backslash c$. Hence provided (α), the transform M is well defined on $\mathcal{J} \cap \vec{C}^\infty(\Omega)$.

Denote by X^s the projection in $\mathcal{L}_\theta$ onto a subspace $\mathcal{L}^s_\theta := \{w \in \mathcal{L}_\theta \mid \operatorname{supp} w \subset \overline{\Omega}^s\}$. This projection cuts off transversal fields on Ω^s.

Lemma 5. *Under the conditions (α)–(γ), the transform M can be extended by continuity to a (partial) isometry from $\mathcal{J}$ onto $\mathcal{L}_\theta$. For the extended transform, the relation $MP^s = X^s M$, $s \in [0, T_*]$ holds.*

We omit the proof, which generalizes the proof of theorem 1 from [21], the conditions (α)–(γ) being used to justify such a generalization. A family of projections $\{P^s\}_{0\leqslant s\leqslant T_*}$ is a spectral measure in $\mathcal{J}$ and lemma 5 shows that M diagonalizes this measure.

A remarkable property of M is the locality w.r.t. s: the embedding $\operatorname{supp} y \subset \overline{\Omega}^{s'}\backslash\Omega^s$ implies $\operatorname{supp} My \subset \overline{\Omega}^{s'}\backslash\Omega^s$. Note in addition that M is in general a partial isometry: the case $\operatorname{Ker} M \neq \{0\}$ is possible if the surfaces Γ^s contain the folds.

Fix $T > 0$ and introduce a *reduced transform* $M^T : \mathcal{J}^T \to \mathcal{L}^T_\theta$, $M^T := j^*_{\mathcal{L}^T_\theta} M j_{\mathcal{J}^T}$, where $j_{\mathcal{J}^T} : \mathcal{J}^T \to \mathcal{J}$ and $j_{\mathcal{L}^T_\theta} : \mathcal{L}^T_\theta \to \mathcal{L}_\theta$ are the embedding. If $T < T_c$ then M^T is a unitary operator, whereas the fact that M^T is an isometry turns out to be equivalent to the well-known operator Riccati equation of the layer-stripping method for the family of Neumann-to-Dirichlet operators $K^s : \frac{\partial p^s}{\partial \nu}\big|_{\Gamma^s} \mapsto p^s|_{\Gamma^s}$ associated with problems (3.21)–(3.23) (see [19, 38, 46]).

[24] M Mitrea, private communication.

- In [19, 21], the transform M is supplemented with a *transform* $N : \{\nabla p \mid p \in H^1_0(\Omega)\} \to \mathcal{L}_\nu$ mapping potential vector fields to longitudinal fields. N is also defined through jumps appearing as a result of projecting on the subspaces of potential fields supported in Ω^s.

3.2.3. Visualization. The appearance of the M-transform has been motivated by the following question. Given R^{2T}, the external observer can construct the model (3.18) of the system α^T and then, by analogy with the acoustic case, compose the rhs of the amplitude formula (2.36). What will the observer see: what kind of wave pictures? Here we show that the answer is 'M-transform of the waves'.

In this section, we assume Ω and $T > 0$ to be such that $\mathcal{D}^s := \{y \in \mathcal{J}^s \mid \operatorname{curl} y = 0, \nu \times y|_\Gamma = 0\} = \{0\}$ for all $s \in [0, T]$. The motivation of this restriction[25] is the following. The proof of theorem 3 (see the appendix) enables one to clarify the first relation in (3.13): the equality $\mathcal{J}^s \ominus \operatorname{clos} \mathcal{U}^s = \mathcal{D}^s$ holds. Therefore, the condition $\mathcal{D}^s = \{0\}$ ensures $\operatorname{clos} \mathcal{U}^s = \mathcal{J}^s$ and implies the equality of the projections

$$P_{\mathcal{U}^s} = P^s, \tag{3.25}$$

which is used below for visualization.

By analogy with (2.31), for a field $v \in \mathcal{L}^T_\theta$ define its $\vec{L}_2(\Gamma)$-valued picture on $\Gamma \times [0, T)$:

$$\tilde{v}(\gamma, s) := \begin{cases} \beta^{\frac{1}{2}}(\gamma, s)[v(x(\gamma, s))]^{\mathrm{par}}, & (\gamma, s) \in \operatorname{int} \Theta^T \\ 0, & \text{otherwise} \end{cases}$$

where $[v(x(\gamma, s))]^{\mathrm{par}}$ is the result of the parallel translation of the vector $v(x(\gamma, s))$ along the geodesic l_γ from the point $x(\gamma, s)$ to the point $\gamma \in \Gamma$. Identifying a *space of pictures* $L_2([0, T] : \vec{L}_2(\Gamma))$ with the space of controls $\mathcal{F}^T$, we introduce the *picture operator* $I^T : \mathcal{L}^T_\theta \to \mathcal{F}^T$, $I^T v := \tilde{v}$ (compare with section 2.3.5).

In its original form, the geometric optics relation describing the propagation of wave discontinuities in the Maxwell system is

$$\beta^{\frac{1}{2}}(\gamma, s)[(P^s y)(x(\gamma, s - 0))]^{\mathrm{par}} = \lim_{t \to T - s - 0} ((W^T)^*[\mathbb{I} - P^s]y)(\gamma, t),$$

for a smooth $y \in \mathcal{J}^T$, where $(\gamma, s) \in \Theta^T$ (see [36]). Taking $f \in \mathcal{M}^T$, putting $y = u^f(\cdot, T)$, and recalling the definition of M-transform, we get

$$\begin{aligned} \widetilde{Mu^f(\cdot, T)}(\gamma, s) &= \lim_{t \to T - s - 0} ((W^T)^*[\mathbb{I} - P^s]W^T f)(\gamma, t) = \langle \text{see } (3.25) \rangle \\ &= \lim_{t \to T - s - 0} ((W^T)^*[\mathbf{1} - P_{\mathcal{U}^s}]W^T f)(\gamma, t) = \langle \text{see } (3.18) \rangle \\ &= \lim_{t \to T - s - 0} ((\tilde{W}^T)^*[\mathbf{1} - P_{\tilde{W}^T \mathcal{F}^{T,s}}]\tilde{W}^T f)(\gamma, t), \qquad (\gamma, s) \in \Gamma \times [0, T], \end{aligned}$$

where the rhs of the last equality is determined by R^{2T} (through the model (3.18)). Thus, the external observer can construct the operator $V^T := Y^T I^T M^T W^T : \mathcal{F}^T \to \mathcal{F}^T$ visualizing the M^T-pictures of waves[26].

Given the pictures, the observer can determine the image of the Laplacian on waves $\tilde{L}^T_0 := (I^T M^T) L^T_0 (I^T M^T)^*$ through the graph

$$\operatorname{graph} \tilde{L}^T_0 := \{\{V^T f, -V^T f_{tt}\} \mid f \in C^\infty_0((0, T); \vec{C}^\infty(\Gamma))\}.$$

[25] In fact, it is a condition of topological character—see [35, 36, 88]. Note that $\mathcal{D}^s = \{0\}$ holds for all $s < T_c$.

[26] Note that V^T provides the relevant triangular factorization of the connecting operator associated with the connecting form c^T.

In the case of $T \leqslant T_c$, the structure of $\tilde{L}_0^T$ is known [36, 46]:

$$\tilde{L}_0^T = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \tilde{L}_\Gamma^T, \tag{3.26}$$

where $\tilde{L}_\Gamma^T = \oplus \int_0^T L(t)\,\mathrm{d}t$ is a fiberwise operator acting by the rule

$$\left(\tilde{L}_\Gamma^T f\right)(\cdot, t) = L(t) f(\cdot, t) \qquad \text{in} \quad \vec{L}_2(\Gamma), t \in [0, T],$$

each $L(t)$ is a 2-order PDO; in the local coordinates γ^1, γ^2 on Γ the representation

$$L(t) = \sum_{i,j=1}^{2} \tilde{g}^{ij}(\gamma^1, \gamma^2, t)\frac{D^2}{\partial\gamma^i \partial\gamma^j} + \text{low order terms} \tag{3.27}$$

holds, where $\tilde{g}^{ij}$ is the inverse of the matrix of the metric tensor g in sgc.

So, the external observer can use representations (3.26) and (3.27) for recovering the metric tensor $\breve{g}$ on the pattern Θ^T, which turns Θ^T into an isometric copy of (Ω^T, g).

- For $T < T_c$, the reconstruction of (Ω^T, g) through visualization by the AF is proposed in [36, 46]. For $T > T_c$, this program is not yet realized: the main obstruction is the possible presence of the defect subspaces $\mathcal{D}^s \neq \{0\}$, which violates the equality (3.25) and, hence, distorts the M-transform and wave pictures.

3.3. Determination of velocity

3.3.1. Maxwell system in $\mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $\Gamma := \partial\Omega \in C^\infty$, ϵ and μ two smooth functions strictly positive in Ω. A function $c := (\epsilon\mu)^{\frac{1}{2}}$ endows Ω with a conformal (*optic*) metric $\mathrm{d}\tau^2 := c^{-2}|\mathrm{d}x|^2$ turning Ω into an RM (Ω, g), g being called an *optic metric tensor*. The Maxwell system

$$\begin{aligned}
&\epsilon u_t = \operatorname{curl} v, \quad \mu v_t = -\operatorname{curl} u && \text{in} \quad (\operatorname{int}\Omega) \times (0, T)\\
&u|_{t=0} = 0 \quad v|_{t=0} = 0 && \text{in} \quad \Omega\\
&u_\theta = f && \text{on} \quad \Gamma \times [0, T]
\end{aligned}$$

describes the propagation of electromagnetic waves in an inhomogeneous isotropic body; $c := (\epsilon\mu)^{\frac{1}{2}}$ is a (variable) *velocity of propagation*.

Isolating the electric component, we get the system α^T of the form

$$u_{tt} + \frac{1}{\epsilon}\operatorname{curl}\frac{1}{\mu}\operatorname{curl} u = 0 \qquad \text{in} \quad (\operatorname{int}\Omega^T) \times (0, T) \tag{3.28}$$

$$u = 0 \qquad \text{in} \quad \{(x, t) \mid x \in \Omega^T, \quad 0 \leqslant t \leqslant \tau(x)\} \tag{3.29}$$

$$u_\theta = f \qquad \text{on} \quad \Gamma \times [0, T] \tag{3.30}$$

where $\tau(\cdot)$ is the *optic eikonal*. The outer space of the system is $\mathcal{F}^T := L_2([0, T]; \vec{L}_2(\Gamma))$; the inner space is $\mathcal{J}_\epsilon := \{y \in L_{2,\epsilon}(\Omega; \mathbb{R}^3) \mid \operatorname{div} \epsilon y = 0\}$ with the inner product $(u, v)_{\mathcal{J}_\epsilon} := \int_\Omega u \cdot v\, \epsilon\, \mathrm{d}x$. The continued response operator R^{2T} associated with the relevant extended system acts in $\mathcal{F}^{2T}$ on $\mathcal{M}^{2T}$ by the rule $R^{2T} f = \nu \times \frac{1}{\mu}\operatorname{curl} u^f|_{\Gamma\times[0,2T]}$. By hyperbolicity of problem (3.28)–(3.30), we have $\operatorname{supp} u^f(\cdot, t) \subset \overline{\Omega}^t$, whereas the operator R^{2T} is determined by $\epsilon, \mu|_{\Omega^T}$ and, hence, contains information about $c|_{\Omega^T}$.

The dynamical IP for the system α^T is to recover $c|_{\Omega^T}$ via given R^{2T}.

3.3.2. Uniqueness. Here we show that R^{2T} determines $c|_{\Omega^T}$ uniquely. By the results of section 2.2, the operator R^{2T} determines the wave copy $(\tilde{\Omega}^T, \tilde{d}^T)$ of the submanifold (Ω^T, d^T) (d^T is the intrinsic *optic distance*) and it remains to embed the copy in $\Omega \subset \mathbb{R}^3$. The subdomain Ω^T is endowed with two metric tensors: the optic g and the Euclidean $h = c^2 g$. Let $i : (\Omega^T, d^T) \to (\tilde{\Omega}^T, \tilde{d}^T)$ be an isometry identifying the boundaries: $i(\gamma) \equiv \gamma, \gamma \in \Gamma$. The copy $\tilde{\Omega}^T$ also carries two tensors: $\tilde{g} := i^* g$ (already recovered from R^{2T})[27] and $\tilde{h} := i^* h = \tilde{c}^2 \tilde{g}$ (unknown), where $\tilde{c} := c \circ i^{-1}$. Let $K_{\tilde{g}}$ and $K_{\tilde{h}}$ be the corresponding scalar curvatures; since $\tilde{h}$ is Euclidean, we have $K_{\tilde{h}} = 0$. To represent the metric $\tilde{g}$ in the form $\tilde{c}^{-2}\tilde{h}$ is to solve the *Yamabe problem*[28]: given $\tilde{g}$, to find a multiplier $\tilde{c}^2$ such that a conformal deformation $\tilde{h} = \tilde{c}^2 \tilde{g}$ possesses a prescribed constant (in our case, zero) curvature (see [72]).

In the general case, the main tool for solving the Yamabe problem in dimension $n \geqslant 3$ is an equation

$$\frac{4(n-1)}{n-2}\Delta_{\tilde{g}}\eta - K_{\tilde{g}}\eta + K_{\tilde{h}}\eta^{\frac{n+2}{n-1}} = 0 \qquad \text{in} \quad \tilde{\Omega}^T.$$

Solving the equation, we get $\tilde{c}^2 = \eta^{\frac{4}{n-2}}$. In our case, we have $n = 3$, $K_{\tilde{h}} = 0$, $\eta = \tilde{c}^{\frac{1}{2}}$ and the equation takes the form

$$\left(\Delta_{\tilde{g}}\eta - \tfrac{1}{8}K_{\tilde{g}}\right)\tilde{c}^{\frac{1}{2}} = 0 \qquad \text{in} \quad \tilde{\Omega}^T. \tag{3.31}$$

The response operator R^{2T} determines the traces $c|_\Gamma$ and $\frac{\partial c}{\partial \nu}\big|_\Gamma = c\frac{\partial c}{\partial \nu_e}\big|_\Gamma$ (ν_e is an Euclidean normal)[29]. Hence, by the equalities $\tilde{c}^{\frac{1}{2}}\big|_\Gamma = c^{\frac{1}{2}}\big|_\Gamma$ and $\frac{\partial \tilde{c}^{\frac{1}{2}}}{\partial \nu}\big|_\Gamma = \frac{1}{2}c^{\frac{1}{2}}\frac{\partial c}{\partial \nu_e}\big|_\Gamma$, the Cauchy data of a solution $\tilde{c}^{\frac{1}{2}}$ of an elliptic equation (3.31) are given. These data determine the solution uniquely. Thus, the function $\tilde{c}$ and the tensor $\tilde{h} = \tilde{c}^2\tilde{g}$ are recovered in $\tilde{\Omega}^T$.

Knowledge of $\tilde{h}$ enables one to recover the embedding $i^{-1} : (\tilde{\Omega}^T, \tilde{d}^T) \to (\Omega^T, d^T) \subset \Omega$. Indeed, let x^1, x^2, x^3 be the Cartesian coordinates on Ω. Since x^k are harmonic in h-metric, the functions $\tilde{x}^k := x^k \circ i^{-1}$ satisfy $\Delta_{\tilde{h}}\tilde{x}^k = 0$ in $\tilde{\Omega}^T$, whereas $\tilde{x}^k|_\Gamma$ and $\frac{\partial \tilde{x}^k}{\partial \nu}\big|_\Gamma = c\frac{\partial x^k}{\partial \nu_e}\big|_\Gamma$ are known. These Cauchy data determine $\tilde{x}^k$ in $\tilde{\Omega}^T$ uniquely; hence, the map $i^{-1} : \tilde{\Omega}^T \ni m \mapsto \{\tilde{x}^k(m)\}_{k=1}^3 \in \Omega^T$ is recovered. Subsequently, we recover $c = \tilde{c} \circ i$ and solve the IP.

As is shown in [36], for times $T \leqslant T_c$, R^{2T} determines $c|_{\Omega^T}$ uniquely. Here we see that this result is valid for arbitrary $T > 0$. Once again, we emphasize the time-optimal character of the determination by the BCm.

- A reasonable question is whether R^{2T} determines $\epsilon|_{\Omega^T}$ and $\mu|_{\Omega^T}$ individually. In [45] it is shown that such a determination holds for large enough times: namely, for $T > T_*$ the operator R^{2T} determines ϵ and μ in Ω uniquely. To prove this, the authors invoke the data continuation (see section 2.3.7) and then apply the Fourier transform reducing the problem to the known frequency-domain results (P Ola, L Paivarinta and E Somersalo). However, time-optimal determination $R^{2T} \Rightarrow \epsilon, \mu|_{\Omega^T}$ is so far an open problem.

3.3.3. Sampling algorithm for $\mu = 1$. The procedure of determination $R^{2T} \Rightarrow c|_{\Omega^T}$ proposed above is hardly available for numerical realization. The reason is basically not the use of the wave copy but the necessity to solve the Cauchy problem for the elliptic equation (3.31). Here we describe a more realistic version of the BCm belonging to a class of so-called *sampling*

[27] More precisely, we have recovered a distance $\tilde{d}^T$ but this distance evidently determines the tensor.
[28] G Ya Perelman, private communication.
[29] V M Babich, private communication.

algorithms (see, e.g., [89]). The version is a straightforward analog of the acoustic variant proposed in [9] and developed in [15].

So, we return to system (3.28)–(3.30) and deal with the case $\mu = 1$ corresponding to a class of dielectrics in applications. We assume also that for all $T > 0$, except for a finite set $s_1, s_2, \ldots, s_N$, the defect subspace $\mathcal{D}^T = \mathcal{J}_\epsilon^T \ominus \operatorname{clos} \mathcal{U}^T$ is of finite dimension.

Fix a point $x_0 \in \Omega$ and a vector $l \in \mathbb{R}^3$, $|\, l \,| = 1$; denote $r_{x_0}(x) :=|\, x - x_0 \,|$. A field

$$m_{x_0}(x) := \nabla r_{x_0} \times l = \operatorname{curl} \frac{l}{r_{x_0}}, \qquad x \in \Omega \backslash \{x_0\}$$

is said to be a *mark field* of the point x_0; the following of its properties can be checked by straightforward calculations.

(1) The relation

$$\operatorname{curl}\operatorname{curl} m_{x_0} = 0 \qquad \text{in} \quad \Omega \backslash \{x_0\} \tag{3.32}$$

holds.

(2) The mark field is summable but not square summable in Ω. Therefore, if $x_0 \notin \overline{\Omega}^T$ and $x_0 \to \overline{\Omega}^T$ (i.e., $\operatorname{dist}_{\mathbb{R}^3}(x_0, \overline{\Omega}^T) \to 0$) then $\left\| m_{x_0}^T \right\|_{\mathcal{J}_\epsilon}^2 \to \infty$, where $m_{x_0}^T$ is the projection of the restriction $m_{x_0}|_{\Omega^T}$ in $L_{2,\epsilon}(\Omega^T; \mathbb{R}^3)$ onto $\mathcal{J}_\epsilon^T$. The boundary controllability (3.13) along with the condition $\dim \mathcal{D}^T < \infty$ imply

$$\left\| P_{\mathcal{U}^T} m_{x_0}^T \right\|_{\mathcal{J}_\epsilon}^2 \to \infty \qquad \text{as} \quad x_0 \to \overline{\Omega}^T, \tag{3.33}$$

where $P_{\mathcal{U}^T}$ is the projection in $\mathcal{J}_\epsilon^T$ onto $\operatorname{clos} \mathcal{U}^T$.

(3) The norm in (3.33) is determined by the operator R^{2T}. Indeed, the operator determines the connecting form by (3.9); hence, we can choose a complete system of controls $\{f_k\}_{k=1}^\infty \subset \mathcal{M}^T$, $\operatorname{clos}\operatorname{span}\{f_k\}_{k=1}^\infty = \mathcal{F}^T$ such that $c[f_j, f_k] = \delta_{jk}$. By this choice, the corresponding waves $u_k := W^T f_k$ satisfy $(u_j, u_k)_{\mathcal{J}_\epsilon} = \delta_{jk}$ and constitute an orthonormal basis in $\operatorname{clos} \mathcal{U}^T$ (that is called a *wave basis* in BCm). Representing

$$P_{\mathcal{U}^T} m_{x_0}^T = \sum_{k=1}^\infty a_k^T u_k$$

and taking into account that $x_0 \notin \overline{\Omega}^T$ and $\operatorname{supp} u_k \subset \Omega^T$, we derive

$$\begin{aligned} a_k^T &= \left(P_{\mathcal{U}^T} m_{x_0}^T, u_k\right)_{\mathcal{J}_\epsilon} = \int_\Omega m_{x_0} \cdot u^{f_k}(\cdot, T)\epsilon \,\mathrm{d}x = \langle \text{see (3.29)} \rangle \\ &= \int_\Omega \epsilon \,\mathrm{d}x \, m_{x_0} \cdot \int_0^T (T-t) u_{tt}^{f_k}(\cdot, t) \,\mathrm{d}t = \langle \text{see (3.28)} \rangle \\ &= -\int_0^T \mathrm{d}t \,(T-t) \int_\Omega m_{x_0} \cdot \operatorname{curl}\operatorname{curl} u^{f_k}(\cdot, t) \,\mathrm{d}x = \langle \text{see (3.1), (3.33)} \rangle \\ &= -\int_{\Gamma \times [0,T]} \mathrm{d}\Gamma \,\mathrm{d}t \,(T-t)[m_{x_0} \cdot (\nu \times \operatorname{curl} u^{f_k}(\cdot, t)) - \nu \times \operatorname{curl} m_{x_0} \cdot u^{f_k}(\cdot, t)] \\ &= -\int_{\Gamma \times [0,T]} \mathrm{d}\Gamma \,\mathrm{d}t \,(T-t)\big[m_{x_0} \cdot (R^T f_k)(\cdot, t) - \nu \times \operatorname{curl} m_{x_0} \cdot f_k(\cdot, t)\big]. \end{aligned} \tag{3.34}$$

Hence, the Fourier coefficients are determined by R^{2T} and we can find

$$\left\| P_{\mathcal{U}^T} m_{x_0}^T \right\|_{\mathcal{J}_\epsilon}^2 = \sum_{k=1}^\infty \left(a_k^T\right)^2. \tag{3.35}$$

Assume, for the sake of simplicity, that along with R^{2T} we are given a positive bound $\epsilon_0 \leqslant \epsilon(\cdot)$ (so that the velocity satisfies $c(\cdot) \leqslant \epsilon_0^{-\frac{1}{2}}$). Now we can recover ϵ in Ω^T by the following procedure.

Step 1. Fix a point $x_0 \in \Omega$ and choose a (small enough) $s \in (0, T)$ providing $x_0 \notin \Omega^s$. Since R^{2T} is given, R^{2s} is also known. Determine the connecting form c^s (see (3.9)) and construct a c^s-orthonormal system of controls $\{f_k\}_{k=1}^{\infty} \subset \mathcal{M}^s$ complete in $\mathcal{F}^s$.

Step 2. Find the Fourier coefficients a_k^s by (3.34). Determine the norm $\|P_{\mathcal{U}^s} m_{x_0}^s\|_{\mathcal{J}_\epsilon}^2$ by (3.35).

Step 3. Moving x_0 along a straight line connecting x_0 with Γ and, simultaneously, controlling the value of the norm, detect the position of x_0 at which the norm blows up. This corresponds to $x_0 \in \Gamma^s$, and we have $\tau(x_0) = s$. Varying the initial position of x_0 and the lines, recover the surface $\Gamma^s \subset \Omega$.

Step 4. Varying $s \leqslant T$ and repeating the previous steps, recover a family of the surfaces $\{\Gamma^s\}_{0<s\leqslant T}$. The family determines the optic eikonal τ in Ω^T by $\tau|_{\Gamma^s} = s$.

Step 5. Determine ϵ by the Jacobi equation $|\nabla\tau|^2 = c^{-2} = \epsilon$ in Ω^T outside the optic cut locus. Since the function ϵ is smooth and the cut locus is of zero volume, ϵ is determined everywhere in Ω^T.

The IP is solved. In [31] the inequality

$$\operatorname{dist}_{\mathbb{R}^3}(x_0, \Gamma^T) \leqslant \text{const} \left(\sum_{k=1}^{N} \left(a_k^T\right)^2 \right)^{-1}$$

for the partial sums of the series (3.35) is obtained. This estimate can be used for approximately determining the shape of Γ^T.

- The sampling algorithm is also of local character. Fix an open $\sigma \subset \Gamma$ and $T > 0$; assume that for all $s \in (0, T]$ the subspaces $\mathcal{J}_\epsilon^s[\sigma] \ominus \operatorname{clos} \mathcal{U}^s[\sigma]$ are finite dimensional. Then, the algorithm determines $\mu|_{\Omega^T[\sigma]}$ from the 'partial' response operator R_σ^{2T}.
- C^2-smoothness of ϵ, μ and Γ is enough to justify all the above-presented results on recovering the velocity.
- Time-optimal determination of tensorial ϵ, μ (crystal optics) is one of most difficult and challenging problems. The problem is stubborn, first of all, because of very complicated structure of the reachable sets $\mathcal{U}^s[\sigma]$. The known results [76, 77] concern an artificial case: ϵ and μ are assumed to be such that the velocity of propagation is one and the same for all wave modes (polarizations). This assumption provides the 'isotropic' controllability (3.13) and makes it possible to apply one of standard versions of the BCm (coordinatization by distance functions). However, if such a controllability holds, one could obtain stronger (time-optimal) results, for instance, by constructing the wave copy.

4. Elasticity

*4.1. 1-*dim *two-velocity system (beam)*

4.1.1. Forward and inverse problem. The initial boundary value problem under consideration is

$$\rho u_{tt} - u_{xx} + Au_x + Bu = 0, \qquad x > 0, \quad t > 0 \tag{4.1}$$

$$u|_{t=0} = u_t|_{t=0} = 0, \qquad x \geqslant 0 \tag{4.2}$$

$$u|_{x=0} = f, \qquad t \geqslant 0 \tag{4.3}$$

where $\rho = \{\rho_i \delta_{ij}\}_{i,j=1}^2$ is a diagonal 2×2-matrix, $\rho_{1,2} = \text{const}$, $0 < \rho_1 < \rho_2$; $A = A(x)$, $B = B(x)$ are smooth 2×2-matrix-functions satisfying the self-adjointness conditions[30]

$$A^\# = -A, \qquad \frac{\mathrm{d}A}{\mathrm{d}x} = B^\# - B, \qquad x \geqslant 0; \tag{4.4}$$

an $\mathbb{R}^2$-valued function $f = \text{col}\{f_1(t), f_2(t)\} \in L_2^{\text{loc}}([0,\infty); \mathbb{R}^2)$ is a *boundary control*, $u^f = \text{col}\{u_1^f(x,t), u_2^f(x,t)\}$ is a solution (*wave*). The components $u_{1,2}^f$ are interpreted as the wave modes propagating along the semi-axis (*a beam*) with different velocities $c_1 = \rho_1^{-\frac{1}{2}}$ and $c_2 = \rho_2^{-\frac{1}{2}} < c_1$. If $A \not\equiv 0$ and/or B is not diagonal, the modes interact with each other. By hyperbolicity of the problem, we have

$$\operatorname{supp} u^f(\cdot, t) \subset \overline{\Omega}_1^t, \qquad t > 0 \tag{4.5}$$

where $\Omega_i^s := (0, c_i s)$, $i = 1, 2$.

The problem

$$\rho u_{tt} - u_{xx} + A u_x + B u = 0 \qquad \text{in} \quad \Omega_1^T \times (0, T) \tag{4.6}$$

$$u = 0 \qquad \text{in} \quad \left\{(x,t) \mid x \in \overline{\Omega}_1^T, \quad 0 \leqslant t \leqslant x/c_1\right\} \tag{4.7}$$

$$u|_{x=0} = f, \qquad 0 \leqslant t \leqslant T \tag{4.8}$$

is well posed and we refer to it as a dynamical system α^T. Its outer space $\mathcal{F}^T := L_2([0,T]; \mathbb{R}^2)$ contains a family of subspaces $\mathcal{F}^{T,s} := \{f \in \mathcal{F}^T \mid \operatorname{supp} f \subset [T-s, T]\}$ formed by delayed controls. Introduce a space $\mathcal{H} := L_{2,\rho}([0,\infty); \mathbb{R}^2)$ with the inner product $(y, w)_{\mathcal{H}} := \int_0^\infty [\rho y(x)] \cdot w(x)\, \mathrm{d}x$ and a family of its subspaces $\mathcal{H}^\xi := \{y \in \mathcal{H} \mid \operatorname{supp} y \subset [0, \xi]\}$, $\xi \geqslant 0$. The inner space of the system α^T is $\mathcal{H}^{c_1 T}$; by (4.5), the waves $u^f(\cdot, t)$ are time-dependent elements of $\mathcal{H}^{c_1 T}$. The control operator $W^T : \mathcal{F}^T \to \mathcal{H}^{c_1 T}$ acts by the rule $W^T f := u^f(\cdot, T)$ and maps $\mathcal{F}^T$ onto $\operatorname{Ran} W^T$ isomorphically [42].

The problem

$$\begin{aligned}
&\rho u_{tt} - u_{xx} + A u_x + B u = 0 && \text{in} \quad \left\{(x,t) \mid x \in \Omega_1^T, 0 < t < 2T - x/c_1\right\} \\
&u = 0 && \text{in} \quad \left\{(x,t) \mid x \in \overline{\Omega}_1^T, 0 \leqslant t \leqslant x/c_1\right\} \\
&u|_{x=0} = f, && 0 \leqslant t \leqslant 2T,
\end{aligned}$$

is well posed and can be regarded as a natural extension of problem (4.6)–(4.8) by hyperbolicity. This extension defines a continued response operator R^{2T} of the system α^T acting in $\mathcal{F}^{2T}$ on $\operatorname{Dom} R^{2T} = \{f \in H^1([0,2T]; \mathbb{R}^2) \mid f(0) = 0\}$ by the rule $R^{2T} : f \mapsto -\frac{\partial u^f}{\partial x}\big|_{[0,2T]}$. The representation

$$(R^{2T} f)(t) = -\rho^{\frac{1}{2}} \frac{\mathrm{d}f}{\mathrm{d}t}(t) + \int_0^t r(t-s) f(s)\, \mathrm{d}s, \qquad 0 \leqslant t \leqslant 2T \tag{4.9}$$

holds, where $r = \{r_{ij}(t)\}_{i,j=1}^2$, $0 \leqslant t \leqslant 2T$ is a smooth matrix-function satisfying $r = r^\#$ and called a reply function of the system α^T (see [42]).

The connecting operator $C^T : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$, $C^T := (W^T)^* W^T$ can be represented through the response operator by the general formula

$$C^T = \tfrac{1}{2} (S^T)^* R^{2T} J^{2T} S^T$$

[30] Below # is a matrix conjugation.

(see [20, 42]), whereas (4.9) implies

$$(C^T f)(t) = \rho^{\frac{1}{2}} f(t) + \int_0^\infty \left[\frac{1}{2}\int_{|t-s|}^{2T-t-s} r(\eta)\,\mathrm{d}\eta\right] f(s)\,\mathrm{d}s, \qquad 0 \leqslant t \leqslant T. \tag{4.10}$$

All operators of the system α^T, as well as the reply function $r\,|_{[0,2T]}$, are determined by the behavior of the coefficients A, B in $\Omega^{c_1 T}$. Hence, the natural setup of the dynamical IP taking into account such a locality is to recover A, B in $\Omega^{c_1 T}$ via given R^{2T}. However, we shall see that R^{2T} does not determine the coefficients in $\Omega^{c_1 T}$ uniquely and this setup has to be corrected: R^{2T} must be supplemented with certain additional data.

4.1.2. Slow waves. Here we describe a nice physical effect, which plays a key role in a procedure solving the IP.

If $A = 0$ and B is diagonal, then problem (4.1)–(4.3) is decoupled to a pair of separate scalar problems. The modes do not interact and propagate independently with velocities c_1 and c_2, the second (slow) mode u_2 satisfying $\operatorname{supp} u_2^f(\cdot, t) \subset \overline{\Omega}_2^t = [0, c_2 t]$. An interesting and rather unexpected fact is that *slow waves* ever exist: in spite of the interaction, a certain mixture of the modes can propagate along the beam with the slow velocity c_2.

Lemma 6. *There exists a unique (scalar) function* $l \in C^\infty[0,\infty)$ *such that the solution* u^f *of (3.1)–(3.3) satisfies* $\operatorname{supp} u^f(\cdot, t) \subset \overline{\Omega}_2^t$, $t \geqslant 0$ *if and only if the components of the control* f *are linked as*

$$f_1(t) = \int_0^t l(t-s) f_2(s)\,\mathrm{d}s, \qquad t \geqslant 0. \tag{4.11}$$

The function l *depends on the coefficients locally: for every* $T > 0$, *the restriction* $l|_{0 \leqslant t \leqslant (1-\frac{c_2}{c_1})T}$ *is determined by* $A, B|_{0 \leqslant x \leqslant c_2 T}$.

Proof. See in [33]. □

We call l a *delaying function*.

In the system α^T, the delaying function determines the subspaces of controls

$$\mathcal{F}_l^{T,s} := \left\{ f \in \mathcal{F}^{T,s} \,\middle|\, f_1(t) = \int_{T-s}^t l(t-s) f_2(s)\,\mathrm{d}s,\, t \in \left[T-s, T-\frac{c_1}{c_2}s\right]\right\}$$

$(0 \leqslant s \leqslant T)$ such that $f \in \mathcal{F}_l^{T,s}$ implies

$$\operatorname{supp} u^f \subset \left\{(x,t) \in \overline{\Omega}_1^T \times [0,T] \,\middle|\, t \geqslant T - s + \frac{x}{c_2}\right\} \tag{4.12}$$

(in figure 8, $\operatorname{supp} u^f$ is shadowed; $\left[T-s, T-\frac{c_1}{c_2}s\right]$ is the interval, where the components of f are linked by (4.11)).

4.1.3. Controllability. The 'standard' reachable sets of the system α^T are $\mathcal{U}^s := W^T \mathcal{F}^{T,s}$, $0 \leqslant s \leqslant T$. Since W^T is an isomorphism on its range, each $\mathcal{U}^s$ is closed, whereas (4.5) implies $\mathcal{U}^s \subset \mathcal{H}^{c_1 s}$. A principal distinguishing feature of two-velocity systems is that this embedding is not dense: $\dim \mathcal{H}^{c_1 s} \ominus \mathcal{U}^s = \infty$ [33, 42]. So, in contrast to the acoustic and Maxwell systems (which are one-velocity systems!), in the case of the beam we encounter a lack of local boundary controllability: the set of waves filling the subdomain Ω_1^s is not complete in $L_2(\Omega_1^s; \mathbb{R}^2)$. The structure of $\mathcal{U}^s$ is studied in [42]: as is shown, the elements of $\mathcal{H}^{c_1 s} \ominus \mathcal{U}^s$ are supported in $\overline{\Omega}_1^s \backslash \Omega_2^s$, whereas the relation

$$X^{c_2 s} \mathcal{U}^s = \mathcal{H}^{c_2 s}, \qquad 0 \leqslant s \leqslant T \tag{4.13}$$

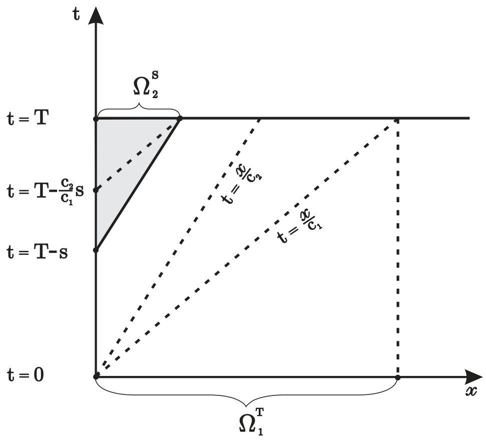

Figure 8. Slow waves.

holds, where X^ξ is the projection in $\mathcal{H}^{c_1T}$ onto $\mathcal{H}^\xi$ (X^{c_2s} cuts off functions on Ω_2^s). In a certain sense, this relation means that the forward parts of the waves u^f do not possess completeness but the tails do.

One more family of reachable sets specific for two-velocity systems is associated with the controls producing slow waves:

$$\mathcal{U}_l^s := W^T\mathcal{F}_l^{T,s}, \qquad 0 \leqslant s \leqslant T. \tag{4.14}$$

By (4.12), we have $\mathcal{U}_l^s \subset \mathcal{H}^{c_2s}$ and a remarkable fact is that the equality

$$\mathcal{U}_l^s = \mathcal{H}^{c_2s}, \qquad 0 \leqslant s \leqslant T \tag{4.15}$$

holds [42]. So, loosely speaking, the slow waves restore the controllability in filled domains but in the *slow domains* Ω_2^s only.

In accordance with (4.15), the projection $P_{\mathcal{U}_l^s}$ onto $\mathcal{U}_l^s$ cuts off functions on Ω_2^s:

$$P_{\mathcal{U}_l^s} = X^{c_2s}, \qquad 0 \leqslant s \leqslant T \tag{4.16}$$

whereas the projection onto $\mathcal{U}^s$ is of more complicated character.

4.1.4. Characterization of data and IP. Return to the IP. The object we are going to recover is a pair of matrix-functions A, B satisfying (4.4). Such an object is determined by four independent scalar functions (parameters)—the entries $a_{12}, b_{11}, b_{12}, b_{22}$. In the meantime, the reply function r playing the role of inverse data, is a symmetric matrix-function determined by three parameters r_{11}, r_{12}, r_{22}. Therefore, to hope for uniqueness of determination is not reasonable and we need to supplement r with 1-parameter data. A well-motivated choice is to add the delaying function l, which leads to the following results.

Theorem 5

(i) A $2\times$ 2-matrix function $r \in C^\infty\left([0, 2T]; \mathbb{M}^2\right)$, $r^\# = r$ is a reply function of a system α^T iff the operator C^T determined by the rhs of (4.10) is strictly positive definite in $\mathcal{F}^T$.[31]

(ii) Let r satisfy (i). Then, for any function $l \in C^\infty\left[0, T - \frac{c_2}{c_1}T\right]$, there exists a (not unique) system α^T such that r and l coincide with its reply and delaying functions.

[31] That is, $(C^T f, f)_{\mathcal{F}^T} \geqslant \kappa\|f\|^2_{\mathcal{F}^T}$ for all $f \in \mathcal{F}^T$ with a constant $\kappa > 0$.

(iii) Let A', B' and A'', B'' be the coefficients of two systems α'^T and α''^T, r', l' and r'', l'' their reply and delaying functions. The equalities $r'(t) = r''(t)$, $0 \leqslant t \leqslant 2T$ and $l'(t) = l''(t)$, $0 \leqslant t \leqslant T - \frac{c_2}{c_1}T$ imply $A'(x) = A''(x)$, $B'(x) = B''(x)$, $x \in \overline{\Omega_2^T} = [0, c_2 T]$.

Proof. See in [42]. □

Thus, the determination in the slow subdomain Ω_2^T, in which the local boundary controllability occurs, is unique. In the meantime, the data $r|_{[0,2T]}$, $l|_{[0,T-\frac{c_2}{c_1}T]}$ do not determine A and B in $\Omega_1^T \backslash \Omega_2^T$ uniquely. In [42], the character of this nonuniqueness is clarified and the set of all pairs A, B corresponding to the given data is described. Note that such a description invokes an extension of the reply function to a bigger interval $[0, 2T']$, $T > T'$, the extension preserving the positiveness of the operator (4.10) (see sections 2.3.7 and 2.3.11).

4.1.5. Visualization and solving IP. The proof of theorem 5 is constructive: an efficient procedure recovering A, B is proposed. The procedure exploits a relevant modification of the amplitude formula (2.36) which looks (by components) as follows:

$$\begin{aligned} u_1^f(c_2 s, T) &= -\big((W^T)^*[\mathbb{I} - P_{W^T\mathcal{F}_l^{T,s}}]W^T f\big)\big|_{t=T-\frac{c_2}{c_1}s-0} \\ u_2^f(c_2 s, T) &= -\big((W^T)^*[\mathbb{I} - P_{W^T\mathcal{F}_l^{T,s}}]W^T f\big)\big|_{t=T-\xi-0}^{t=T-\xi+0} \end{aligned} \tag{4.17}$$

$(0 \leqslant s \leqslant T)$. The derivation uses the geometric optics formulae, the relation (4.16) playing the key role.

Given R^{2T} and $l|_{[0,T-\frac{c_2}{c_1}T]}$, we can recover $A, B|_{\Omega_2^T}$ by the following scheme.

Step 1. Determine the (constant) matrix ρ from (4.9) and the operator C^T by (4.10). Construct the dynamical model (2.35).

Step 2. Replacing W^T by $\tilde{W}^T$ in (4.17), recover the waves $u^f(\cdot, T)$ on $\Omega_{c_2}^T = [0, c_2 T]$.[32]

Step 3. Determine the Laplacian on waves $L_0^T = -\frac{d^2}{dx^2} + A\frac{d}{dx} + B$ restricted on Ω_2^T, through its graph $\{\{u^f(\cdot, T), -u^{f_{tt}}(\cdot, T)\}|_{\Omega_2^T} \mid f \in C_0^\infty((0, T); \mathbb{R}^2)\}$ and find its coefficients $A, B|_{\Omega_2^T}$. The IP is solved.

In conclusion, let us discuss the following question of independent interest. To construct the dynamical model (2.35) one does not need the delaying function but the response operator R^{2T} only. Assume that the external observer ignores knowledge of the function l and applies not the modified AF (4.17) but the straightforward analog of (2.36), i.e., defines

$$w^f(s, T) := \lim_{t \to T-s-0} \big((\tilde{W}^T)^*[\tilde{\mathbb{I}} - P_{\tilde{W}^T\mathcal{F}^{T,s}}]\tilde{W}^T f\big)(t), \qquad 0 \leqslant s \leqslant T.$$

What will the observer get as result of such a visualization? Does w^f describe a wave process? The answer is affirmative: the $\mathbb{R}^2$-valued function $w^f(x, t)$ turns out to be a solution of the problem

$$\begin{aligned} &w_{tt} - w_{xx} + Qw = 0, && 0 < x < T, 0 < t < T \\ &w = 0, && \{(x, t) \mid x \in [0, T], 0 \leqslant t \leqslant x\} \\ &\rho^{-\frac{1}{2}} w|_{x=0} = f, && 0 \leqslant t \leqslant T \end{aligned}$$

which describes an *one-velocity* dynamical system α_Q^T with a matrix potential $Q = Q^\#$, the system possessing the response operator $R_Q^{2T} = R^{2T}$ (see [41]). Thus, we encounter the same phenomenon as in section 2.3.10: the canonical BCm-realization α_Q^T of the system α^T is

[32] More precisely, we are visualizing not the waves but their pictures, which are in this case identical to the waves themselves.

not identical to the system itself. Here we succeeded in recovering the original two-velocity system α^T owing to additional *a priori* information about its structure: namely, we invoked the delaying function l.

- Two-velocity systems is the intensively studied object: see [43, 44, 84, 85]; the interest is stimulated by the engineering applications. The paper [85] deals with the case $c_1 = \infty, c_2 < \infty$ (a composite beam); the equation governing the evolution of the corresponding system contains fourth-order derivatives w.r.t. the space variable. The authors apply the gradient methods for the recovery of parameters. We hope for future extension of the BCm to such systems.

4.2. *Lamé system*

4.2.1. Lamé parameters. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $\Gamma := \partial\Omega \in C^\infty$; c_{ijkl} an *elasticity tensor* in $\overline{\Omega}$. In the isotropic Lamé model, the tensor is

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

the *Lamé parameters* ρ, λ, μ are smooth functions satisfying the conditions $\rho > 0, \mu > 0, 3\lambda + 2\mu > 0$ in $\overline{\Omega}$. These conditions provide the ellipticity of the Lamé operator acting on $\mathbb{R}^3$-valued functions by the rule

$$(Lu)_i = -\rho^{-1} \sum_{i,j,k,l=1} \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right), \qquad i = 1, 2, 3.$$

The velocities $c_p := \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}}, c_s := \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$ of the pressure and shear waves ($c_p > c_s$) endow Ω with two conformal metrics $\mathrm{d}\tau_\alpha^2 := \frac{|\mathrm{d}x|^2}{c_\alpha^2}, \alpha = p, s$. The metrics determine the corresponding distances dist_α, the eikonals $\tau_\alpha := \mathrm{dist}_\alpha(\cdot, \Gamma)$, and the subdomains

$$\Omega_\alpha^\xi[\sigma] := \{x \in \Omega \mid \mathrm{dist}_\alpha(x, \sigma) < \xi\}, \qquad \xi > 0, \qquad \sigma \subset \Gamma,$$

$\Omega_\alpha^\xi := \Omega_\alpha^\xi[\Gamma]$. The inequality $c_s < c_p$ implies $\Omega_s^\xi[\sigma] \subset \Omega_p^\xi[\sigma]$. Also, we set $T_\alpha^\sigma := \inf\left\{\xi > 0 \mid \Omega_\alpha^\xi[\sigma] = \Omega\right\}$.

4.2.2. System α^T. The dynamical *Lamé system* α^T is

$$u_{tt} + Lu = 0 \qquad \text{in} \quad \Omega_p^T \times (0, T) \tag{4.18}$$

$$u = 0 \qquad \text{in} \quad \left\{(x, t) \,\middle|\, x \in \overline{\Omega}_p^T, 0 \leqslant t \leqslant \tau_p(x)\right\} \tag{4.19}$$

$$u = f \qquad \text{on} \quad \Gamma \times [0, T] \tag{4.20}$$

where $f = \mathrm{col}\{f_i(\gamma, t)\}_{i=1}^3$ is a boundary control, $u = u^f(x, t) = \mathrm{col}\left\{u_i^f(x, t)\right\}_{i=1}^3$ is a solution (wave).

The outer space of the system is $\mathcal{F}^T := L_2(\Gamma \times [0, T]; \mathbb{R}^3)$; it contains the subspaces $\mathcal{F}^{T,\xi}[\sigma] := \mathrm{clos}\{f \in \mathcal{F}^T \mid \mathrm{supp}\, f \subset \sigma \times [T - \xi, T]\}$ of controls acting from an open $\sigma \in \Gamma$. Let $\mathcal{H} := L_{2,\rho}(\Omega; \mathbb{R}^3)$ (with the measure $\rho\,\mathrm{d}x$); for $A \subset \overline{\Omega}$, we set $\mathcal{H}A := \mathrm{clos}\,\{y \in \mathcal{H} \mid \mathrm{supp}\, y \subset A\}$. The subspace $\mathcal{H}^T := \mathcal{H}\Omega_p^T$ plays the role of the inner space of α^T.

The control operator is $W^T : \mathcal{F}^T \to \mathcal{H}^T, W^T f := u^f(\cdot, T)$; by the results of [51], W^T is continuous. The response operator (the map 'displacement $\mapsto$ tension') $R^T : \mathcal{F}^T \to \mathcal{F}^T$ is defined on smooth controls vanishing near $\Gamma \times \{t = 0\}$ by

$$(R^T f)_i = \sum_{i,j,k,l=1}^{3} \nu_j c_{ijkl} \frac{\partial u_k^f}{\partial x_l} \qquad \text{on} \qquad \Gamma \times [0, T],$$

where $\nu = \mathrm{col}\{\nu_j(\gamma)\}_{j=1}^3$ is the outward Euclidean normal on Γ. The continued response operator $R^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$ is associated with the extended system

$$\begin{aligned} &u_{tt} + Lu = 0 && \text{in} \quad \{(x,t) | x \in \Omega_p^T, 0 \leqslant t \leqslant 2T - \tau_p(x)\} \\ &u = 0 && \text{in} \quad \{(x,t) | x \in \overline{\Omega}_p^T, 0 \leqslant t \leqslant \tau_p(x)\} \\ &u = f && \text{on} \quad \Gamma \times [0, 2T] \end{aligned}$$

and acts on $\Gamma \times [0, 2T]$ by the same rule as R^T. The connecting operator of the system is $C^T := (W^T)^* W^T : \mathcal{F}^T \to \mathcal{F}^T$; the general relation (2.15) remains valid.

4.2.3. Boundary controllability and slow waves. By hyperbolicity of the problem (4.18)–(4.20), the embedding

$$\mathcal{U}^\xi[\sigma] := W^T \mathcal{F}^{T,\xi}[\sigma] \subset \mathcal{H}\Omega_p^\xi[\sigma]$$

holds. Let $X_s^\xi[\sigma]$ be the projection in $\mathcal{H}$ onto $\mathcal{H}\Omega_s^\xi[\sigma]$ cutting off the vector-valued functions on $\Omega_s^\xi[\sigma]$.

Theorem 6. *For any $T > 0$, $\xi \in [0, T]$ and $\sigma \subset \Gamma$, the equality*

$$\mathrm{clos}\, X_s^\xi[\sigma]\mathcal{U}^\xi[\sigma] = \mathcal{H}\Omega_s^\xi[\sigma] \tag{4.21}$$

is valid.

Proof. See in [51]. □

This result has the same meaning as (4.13): we have a lack of the boundary controllability in the subdomain $\Omega_p^\xi[\sigma]$ filled with waves, whereas the tails of such waves constitute a complete system in the *slow subdomain* $\Omega_s^\xi[\sigma] \subset \Omega_p^\xi[\sigma]$. In the meantime, one can show the examples of $\mathcal{H}\Omega_p^\xi[\sigma] \ominus \mathrm{clos}\,\mathcal{U}^\xi[\sigma] \neq \{0\}$.[33]

Let $\sigma = \Gamma$ and $T < T_s^\Gamma$, so that Ω_s^T does not cover Ω. By (4.21), the elements of the unreachable subspace $\mathcal{D}^T := \mathcal{H}\Omega_p^T \ominus \mathrm{clos}\,\mathcal{U}^T[\Gamma]$ are supported in $\Omega \backslash \Omega_s^T$. The presence of $\mathcal{D}^T \neq \{0\}$ leads to the following curious effect. Consider the system

$$\begin{aligned} &v_{tt} + Lv = 0 && \text{in} \quad \Omega \times (0, T) \\ &v|_{t=T} = 0, v_t|_{t=T} = y && \text{in} \quad \overline{\Omega} \\ &v = 0 && \text{on} \quad \Gamma \times [0, T]. \end{aligned}$$

Applying the Holmgren–John–Tataru uniqueness theorem for the Lamé system [60], one can show that taking $y \in \mathcal{D}^T$, we get the solution $v = v^y(x,t)$, whose forward front propagates (in the reversed time) in Ω from $\Omega_p^T \backslash \Omega_s^T$ toward Γ with the speed $\leqslant c_s$. Such solutions can be reasonably regarded as 3-dim *slow waves*. Some geophysicists claim that in the case of variable density ρ such waves cannot exist in principle. At the moment, we have no proof but the conjecture 'for $T < T_s^\Gamma$, the relation $\mathcal{D}^T \neq \{0\}$ always holds' looks well motivated.

4.2.4. Data continuation. One more result of [51] concerns the possibility of extending the response operator R^T on times $T' > T$ (see section 2.3.7). For an open $\sigma \subset \Gamma$, denote $\mathcal{F}^T[\sigma] := \mathcal{F}^{T,T}[\sigma]$ and introduce a 'partial' response operator $R_\sigma^T : \mathcal{F}^T[\sigma] \to \mathcal{F}^T[\sigma]$, $\mathrm{Dom}\, R_\sigma^T = \mathrm{Dom}\, R^T \cap \mathcal{F}^T[\sigma]$, $R_\sigma^T = [R^T \cdot]|_{\sigma \times [0,T]}$.

Lemma 7. *The operator R_σ^T given for a fixed $T > 2T_s^\sigma$ determines the operators $R_\sigma^{T'}$ for all $T' \in [0, \infty)$.*

[33] For instance, the system α^T with constant ρ, λ, μ.

Proof. See in [51], where a constructive procedure extending R_σ^T is proposed. The procedure uses an appropriate variant of the dynamical model (2.35); in fact, it is just a version of the trick [22]. The condition $T > 2T_s^\sigma$ provides the controllability $\operatorname{clos}\mathcal{U}^{\frac{T}{2}} = \mathcal{H}$, which the trick is based upon. □

4.2.5. Lamé-type system. For a fixed $T > 0$, the continued response operator R^{2T} of system (4.18)–(4.20) is determined by the Lamé parameters in the subdomain Ω_p^T. Therefore, a natural (time-optimal) setup of the dynamical IP is: given R^{2T} to recover λ, μ, ρ (or c_p, c_s) in Ω_p^T. It is one of the most required problems in applications (acoustics, geophysics, nondestructive testing, etc). However, this problem is far from being solved[34]. Here we present the results of the paper [31] dealing with the IP for a simplified version of problem (4.18)–(4.20). We call this version a *Lamé-type system*.

The system is

$$u_{tt} - \nabla\varkappa\,\operatorname{div} u + \operatorname{curl}\mu\,\operatorname{curl} u = 0 \qquad \text{in} \quad \Omega_p^T \times (0, T) \tag{4.22}$$

$$u = 0 \qquad \text{in} \quad \{(x,t) | x \in \overline{\Omega}_p^T, 0 \leqslant t \leqslant \tau_p(x)\} \tag{4.23}$$

$$u = f \qquad \text{on} \quad \Gamma \times [0, T] \tag{4.24}$$

where $\varkappa := \lambda + 2\mu$. Deriving (4.22) from (4.18), we put $\rho = 1$ and remove the low-order terms. This system possesses the same two velocities c_p and c_s. The character of controllability is also the same: theorem 6 remains in force.

Our choice of the response operator of system (4.22)–(4.24) is motivated by the Green formula for the operator $L = -\nabla\varkappa\,\operatorname{div} + \operatorname{curl}\operatorname{curl}$:

$$\begin{aligned}(Lu, v)_{\mathcal{H}} - (u, Lv)_{\mathcal{H}} &= (Nu, Dv)_{\mathcal{G}} - (Du, Nv)_{\mathcal{G}} \\ &= \int_\Gamma \mathrm{d}\Gamma \left[\begin{pmatrix} -\varkappa\,\operatorname{div} u \\ \mu\nu \times \operatorname{curl} u\end{pmatrix}\begin{pmatrix} v^\nu \\ v_\theta\end{pmatrix} - \begin{pmatrix} u^\nu \\ u_\theta\end{pmatrix}\begin{pmatrix} -\varkappa\,\operatorname{div} v \\ \mu\nu \times \operatorname{curl} v\end{pmatrix}\right]\end{aligned}$$

where $\mathcal{G} := L_2(\Gamma; \mathbb{R}^3)$, the matrix representation $u|_\Gamma = \binom{u^\nu}{u_\theta}$ corresponds to decomposition (3.2) at the boundary, $Du := u|_\Gamma = \binom{u^\nu}{u_\theta}$, $Nu := \binom{-\varkappa\,\operatorname{div} u}{\mu\nu\times\operatorname{curl} u}$. The response operator maps Du^f to Nu^f; it is defined on smooth controls vanishing near $\Gamma \times \{t = 0\}$ by

$$R^T f := \begin{pmatrix} -\varkappa\,\operatorname{div} u^f \\ \mu\nu \times \operatorname{curl} u^f\end{pmatrix} \qquad \text{on} \qquad \Gamma \times [0, T].$$

The continued response operator R^{2T} associated with the corresponding extended system acts by the same rule.

4.2.6. Dynamical IP. The main result of [31] is the following lemma.

Theorem 7. *For any $T > 0$, the operator R^{2T} determines $c_p|_{\Omega_p^T}$ and $c_s|_{\Omega_s^T}$ uniquely.*

[34] By speaking so, we mean a *time-optimal* reconstruction of parameters: the literature devoted to another setups and approaches is hardly observable. In the case of layered media reducible to 1-dim problems, time-optimal results are obtained by A S Blagoveschenskii [58].

The proof is constructive. We associate with system (4.22)–(4.24) the *acoustic (scalar) subsystem* α_p^T of the form

$$\begin{aligned} &\varphi_{tt} - c_p^2 \Delta\varphi = 0 && \text{in} \quad \Omega_p^T \times (0,T) \\ &\varphi = 0 && \text{in} \quad \left\{(x,t) \middle| x \in \overline{\Omega}_p^T, 0 \leqslant t \leqslant \tau_p(x)\right\} \\ &\varphi = g && \text{on} \quad \Gamma \times [0,T] \end{aligned}$$

with the response operator $R_p^T : g \mapsto \frac{\partial \varphi^g}{\partial \nu}|_{\Gamma\times[0,T]}$ and the *Maxwell (vector) subsystem* α_s^T

$$\begin{aligned} &e_{tt} - c_s^2 \operatorname{curl}\operatorname{curl} e = 0 && \text{in} \quad \Omega_s^T \times (0,T) \\ &e = 0 && \text{in} \quad \left\{(x,t) \middle| x \in \overline{\Omega}_s^T, 0 \leqslant t \leqslant \tau_s(x)\right\} \\ &e = h && \text{on} \quad \Gamma \times [0,T] \end{aligned}$$

with the response operator $R_s^T : h \mapsto \mu\nu \times \operatorname{curl} e^h|_{\Gamma\times[0,T]}$.

As was noted above, (4.22)–(4.24) is a simplified version of the genuine Lamé system (4.18)–(4.22). The principal simplification is that the Lamé-type system admits the *wave splitting*: the representation

$$u^f = \nabla\varphi^g + \operatorname{curl} e^h \tag{4.25}$$

holds with the proper g and h. As is shown in [31], this leads to a matrix splitting of the response operator: the representation

$$(R^T)^{-1} = (J^T)^2 \begin{pmatrix} R_p^T & -\operatorname{div}_\Gamma \\ \nabla_\Gamma & R_s^T \end{pmatrix} \tag{4.26}$$

is valid, where $J^T := \int_0^t [\cdot]\,\mathrm{d}s$. Its straightforward consequence is the relations

$$R_p^T g = \left\{ \frac{\partial^2}{\partial t^2} [R^T]^{-1} \begin{pmatrix} g \\ 0 \end{pmatrix} \right\}^\nu, \qquad R_s^T h = \left\{ \frac{\partial^2}{\partial t^2} [R^T]^{-1} \begin{pmatrix} 0 \\ h \end{pmatrix} \right\}_\theta. \tag{4.27}$$

Hence, given R^{2T} we can determine R_p^{2T} and R_s^{2T}. These operators, in turn, determine $c_p|_{\Omega_p^T}$ (see [9, 15]) and $c_s|_{\Omega_s^T}$ (see section 3.3.3, replacing the 'magnetic' μ by 1 and ϵ by c_s^{-2}). So, we arrive at the assertion of theorem 7. This result gives promise that time-optimal determination of the velocities $c_p|_{\Omega_p^T}$ and $c_s|_{\Omega_s^T}$ in the genuine Lamé system (4.18)–(4.22) is possible.

4.2.7. Comments.

- The following fact is a simple consequence of representations (4.25)–(4.27): a control f produces the wave $u^f = \operatorname{curl} e^h$ (i.e., a slow wave propagating with the speed c_s) if and only if $f = \binom{-\operatorname{div}_\Gamma h}{R_s^T h}$, i.e., if the components of the control are linked by

$$f^\nu = -\operatorname{div}_\Gamma \left[R_s^T\right]^{-1} f_\theta =: \hat{l}^T f_\theta.$$

This relation[35] is a 3-dim analog of (4.11), which raises the following questions. In the case of the beam, the delaying function l is independent of the response operator R^{2T}, whereas in the Lamé type system the delaying operator $\hat{l}^T$ is determined by R^{2T}. In the meantime, since the slow waves exist in the genuine Lamé system (4.18)–(4.20)[36], the operator $\hat{l}^T$ should also exist. Is $\hat{l}^T$ determined by R^{2T}? If yes, is it possible to extract it from R^{2T}? Omitting the arguments, we could claim: the affirmative answer follows to the main objective that is time-optimal determination of c_p, c_s. We expect that, as well as in the case of the beam, the slow waves will play a key role in solving the 3-dim IP.

[35] We call $\hat{l}^T$ a *delaying operator*.

[36] We believe they exist!

- One more intriguing question is the following. Dealing with the Lamé system and repeating step by step the procedure (i)–(iii) of section 2.2.5, what object will be obtained as a result? In other words, what does the wave copy of the elastic body look like? The same question concerns the amplitude formula (2.36): what kind of pictures does it visualize? Can one extract information about λ, μ, ρ from the wave copy and these pictures? Does there exist a multidimensional analog of the system α^T_Q (see section 4.1.5) such that $R^{2T}_Q = R^{2T}_{\text{Lame}}$? All these questions are as yet open.
- In the case of variable parameters, our knowledge of reachable sets $\mathcal{U}^\xi[\sigma]$ of the Lamé system is very poor, which obstructs progress in the IP. For instance, the following question is not answered. Let σ' and σ'' be two open separated subsets of Γ and let $T > 0$ be such that $\Omega^T_p[\sigma'] \cap \Omega^T_p[\sigma''] \neq \{\emptyset\}$ but $\Omega^T_s[\sigma'] \cap \Omega^T_s[\sigma''] = \{\emptyset\}$. Do the subspaces $\operatorname{clos} \mathcal{U}^T[\sigma']$ and $\operatorname{clos} \mathcal{U}^T[\sigma'']$ intersect and, if yes, what does the intersection consist of? The answer may be very helpful for the IP. One more very important direction is the extension of the uniqueness and stability theorems providing the boundary controllability to more complicated models of elasticity theory: see [83].

5. Impedance tomography of manifolds

5.1. 2-dim EIT problem

5.1.1. General setup. The case $n = 2$. Let Ω be a smooth compact RM, $\dim \Omega = n \geqslant 2$, $\Gamma := \partial\Omega \in C^\infty$; g a metric tensor on Ω. Consider an elliptic boundary value problem

$$\Delta_g u = 0 \qquad \text{in} \quad \operatorname{int}\Omega \tag{5.1}$$

$$u = f \qquad \text{on} \quad \Gamma \tag{5.2}$$

let $u = u^f(x)$ be its solution. With problem (5.1)–(5.2) one associates the Dirichlet-to-Neumann (DN) map $\Lambda : L_2(\Gamma) \to L_2(\Gamma)$, $\operatorname{Dom}\Lambda = C^\infty(\Gamma)$, $\Lambda f := \frac{\partial u^f}{\partial \nu}\,|_\Gamma$ playing the role of the input/output correspondence.

An *electric impedance tomography* (EIT) *problem* is to determine (Ω, g) from the given Λ. Recall that 'to determine' means to construct a manifold $(\tilde\Omega, \tilde g)$ such that $\partial\tilde\Omega =: \tilde\Gamma = \Gamma$ and $\tilde\Lambda = \Lambda$. One more side of the problem is to describe the class of manifolds possessing the given Λ.

The well-known peculiarity of the 2-dim case is the following. If g' and g'' are two metrics on Ω such that $g'' = \rho g'$ with a smooth positive function ρ, then $\Delta_{g''} = \rho^{-1}\Delta_{g'}$ and, hence, the reserve of g'-harmonic and g''-harmonic functions on Ω is one and the same. By this, provided $\rho|_\Gamma = 1$ the DN maps of the manifolds (Ω, g') and (Ω, g'') coincide : $\Lambda' = \Lambda''$. This fact motivates the definition: two manifolds (Ω', g') and (Ω'', g'') with the common boundary Γ are said to be *conformal equivalent* if there exists a diffeomorphism $\beta : \Omega' \to \Omega''$, $\beta|_\Gamma = \text{id}$ and a positive function $\rho \in C^\infty(\Omega')$, $\rho|_\Gamma = 1$ such that β is an isometry of $(\Omega', \rho g')$ onto (Ω'', g'').

So, conformal equivalence of manifolds implies the coincidence of their DN maps. A remarkable fact is that the converse is also true.

Theorem 8. *Two 2-dim compact smooth orientable manifolds with a common boundary are conformal equivalent if and only if their DN maps coincide.*

This fact was established for the first time by Lassas and Uhlmann [80] by the use of analytic continuation. In the capacity of the representative of the equivalence class, the set of germs (a Riemannian surface) obtained as result of continuation from the boundary is proposed.

Here we present an approach [24] to the 2-dim EIT problem based on connections of the problem with *commutative Banach algebras (CBA)*. The approach is an (elliptic) version of the BCm: it exploits a relevant coordinatization of Ω. In more detail, to each $x \in \Omega$ we attach a Dirac measure δ_x and then identify these measures via Λ. The set of identified measures constitutes the required representative of the equivalence class of manifolds. The role of an operator visualizing the solutions u^f on the representative,[37] is played by the Gelfand transform.

5.1.2. Gelfand transform: algebra $\mathcal{A}(\Omega)$. Recall the basic definitions (e.g., see [86]). Let $\mathcal{A}$ be a (complex) CBA , $\mathcal{A}'$ the dual space (of continuous functionals). A functional $m \in \mathcal{A}'$ is called *multiplicative* if $m(ab) = m(a)m(b)$, $a, b \in \mathcal{A}$; the set $\mathcal{M}_{\mathcal{A}}$ of multiplicative functionals endowed with the $*$-weak topology is a compact set called a *spectrum* of the algebra $\mathcal{A}$.[38] The *Gelfand transform* (GT) of an element $a \in \mathcal{A}$ is a function $\tilde{a} : \mathcal{M}_{\mathcal{A}} \to \mathbb{C}$, $\tilde{a}(m) := m(a)$. The GT is a homeomorphism from $\mathcal{A}$ to a subalgebra $\tilde{\mathcal{A}} := \{\tilde{a} \mid a \in \mathcal{A}\}$ of the algebra $C(\mathcal{M}_{\mathcal{A}})$ of continuous functions on $\mathcal{M}_{\mathcal{A}}$.

Return to a 2-dim RM (Ω, g) and assume Ω oriented. Such a manifold carries a complex differentiable structure compatible with the metric g and supports the algebra of continuous analytic functions $\mathcal{A}(\Omega) := \{u + v\,\mathrm{i} \mid u, v \in C(\Omega), \Delta_g u = \Delta_g v = 0 \text{ in int}\,\Omega, \mathrm{d}u = \star\,\mathrm{d}v\}$. For each $x_0 \in \Omega$, the Dirac measure $\delta_{x_0} : a \mapsto a(x_0)$ is a continuous multiplicative functional; hence, $\mathcal{D} := \{\delta_{x_0} \mid x_0 \in \Omega\} \subset \mathcal{M}_{\mathcal{A}(\Omega)}$.

A peculiarity of the algebra $\mathcal{A}(\Omega)$ is that the set $\mathcal{D}$ exhausts $\mathcal{M}_{\mathcal{A}(\Omega)}$: for each $m \in \mathcal{M}_{\mathcal{A}(\Omega)}$ there is a unique $x_0 \in \Omega$ such that $m = \delta_{x_0}$ and, thus, the equality $\mathcal{D} = \mathcal{M}_{\mathcal{A}(\Omega)}$ holds (see [24, 86]). Any algebra possessing this property[39] turns out to be isometric to its GT; in our case, the isometry $\mathcal{A}(\Omega) \to \tilde{\mathcal{A}}(\Omega)$ is realized by a (spatial) isomorphism i^* induced by the map $i : \Omega \to \mathcal{M}_{\mathcal{A}(\Omega)}$, $i(x_0) := \delta_{x_0}$, the isomorphism acting by the rule $i^* : a \mapsto a \circ i^{-1} = \tilde{a}$. So, each $x_0 \in \Omega$ can be identified (through $x_0 \equiv \delta_{x_0}$) with a point of the spectrum $\mathcal{M}_{\mathcal{A}(\Omega)}$ of the algebra $\mathcal{A}(\Omega)$, whereas Ω is homeomorphic to $\mathcal{M}_{\mathcal{A}(\Omega)}$ (we write $\Omega \asymp \mathcal{M}_{\mathcal{A}(\Omega)}$).

5.1.3. Algebra $\mathcal{A}(\Gamma)$. A *trace map* $\mathrm{tr} : a \mapsto a|_\Gamma$ induces the map $\mathcal{A}(\Omega) \to \mathrm{tr}\,\mathcal{A}(\Omega) := \mathcal{A}(\Gamma)$. The algebra $\mathcal{A}(\Gamma)$ is a subalgebra of $C(\Gamma)$, whereas, by the maximal principle for analytic functions, tr maps $\mathcal{A}(\Omega)$ onto $\mathcal{A}(\Gamma)$ isometrically. By isometry of the algebras, we have $\mathcal{M}_{\mathcal{A}(\Gamma)} \asymp \mathcal{M}_{\mathcal{A}(\Omega)}$. Therefore, possessing $\mathcal{A}(\Gamma)$ and finding its spectrum, we get a compact set homeomorphic to Ω.[40] Moreover, constructing $\tilde{\mathcal{A}}(\Gamma)$ by GT, we get a function algebra on $\mathcal{M}_{\mathcal{A}(\Gamma)}$ isometric to the original $\mathcal{A}(\Omega)$.

The DN map determines the algebra $\mathcal{A}(\Gamma)$ as follows. Let $d := \frac{\mathrm{d}}{\mathrm{d}\gamma}$ be the differentiation along Γ compatible with the orientation, J an integration: $J\,\mathrm{d}f = f$ (mod const); denote $\dot{L}_2(\Gamma) := \{f \in L_2(\Gamma) \mid \int_\Gamma f = 0\}$. Since $\mathrm{Ker}\,\Lambda = \{\mathrm{const}\}$, the operators $\mathrm{d}\Lambda^{-1}d : L_2(\Gamma) \to L_2(\Gamma)$, $\mathrm{Dom}\,\mathrm{d}\Lambda^{-1}d = C^\infty(\Gamma)$ and $H : \dot{L}_2(\Gamma) \to \dot{L}_2(\Gamma)$, $\mathrm{Dom}\,H = C^\infty(\Gamma)$, $H := \Lambda J$ are well defined; moreover, by elliptic theory, H is an isomorphism. In the case of $\Omega = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$, H coincides with the classical Hilbert transform.

Lemma 8

(i) A real-valued function $f \in C^\infty(\Gamma)$ is a trace of the real part of a function $w = u + u^*\mathrm{i} \in \mathcal{A}(\Gamma)$ (i.e., $f = \mathrm{Re}\,\mathrm{tr}\,w$) if and only if it satisfies

$$[\mathbb{I} + H^2]\,\mathrm{d}f = 0. \tag{5.3}$$

[37] We are speaking about an analog of (2.29) and (2.30).
[38] Note that $\mathcal{M}_{\mathcal{A}}$ is canonically bijective to the set $\mathcal{I}_{\mathcal{A}}$ of maximal ideals of $\mathcal{A}$ [86].
[39] Such algebras are called *generic*.
[40] This is a key point for the future determination of Ω from Λ.

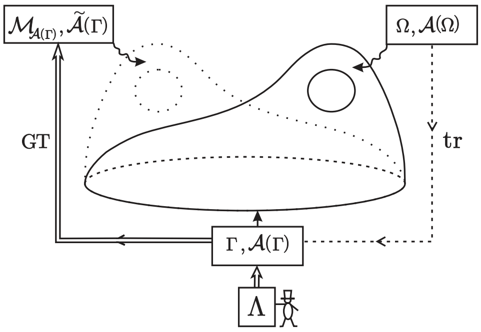

Figure 9. Reconstruction via GT.

(ii) If (5.3) holds (so that f possesses the conjugate function $f^* = \operatorname{Im} \operatorname{tr} w = u^*|_\Gamma$) then

$$\mathrm{d}f^* = H\,\mathrm{d}f. \tag{5.4}$$

(iii) The equality

$$\dim \operatorname{Ran}[\Lambda + \mathrm{d}\Lambda^{-1}d] = \beta_1(\Omega) \tag{5.5}$$

holds, where $\beta_1(\Omega)$ is the first Betti number of Ω.

Proof. See in [24]. □

Since $\mathcal{A}(\Gamma) \cap C^\infty(\Gamma)$ is dense in $\mathcal{A}(\Gamma)$, the lemma shows that the operator H (and, hence, the DN map Λ) determines the trace algebra $\mathcal{A}(\Gamma)$.

5.1.4. Reconstruction. A representative of the equivalence class of manifolds possessing the given DN map Λ, can be constructed by the following procedure.

Step 1. Find $\operatorname{Ker}[\mathbb{I} + H^2]$ and determine the (sub)algebra $\mathcal{A}(\Gamma) \cap C^\infty(\Gamma) = \{f + f^*\mathrm{i} \mid \mathrm{d}f \in \operatorname{Ker}[\mathbb{I} + H^2], \mathrm{d}f^* = H\,\mathrm{d}f\}$ by (5.3) and (5.4). Completing in $C(\Gamma)$-norm, recover the trace algebra $\mathcal{A}(\Gamma)$.

Step 2. Find the (topologized) spectrum $\mathcal{M}_{\mathcal{A}(\Gamma)} =: \tilde{\Omega}$[41] and construct $\tilde{\mathcal{A}}(\Gamma)$ by the GT. Identifying $\Gamma \ni \gamma \equiv \delta_\gamma \in \mathcal{M}_{\mathcal{A}(\Gamma)}$, attach $\tilde{\Omega}$ to Ω along Γ.

Step 3. Using the functions $\tilde{a} \in \tilde{\mathcal{A}}(\Gamma)$ as local complex coordinates on $\tilde{\Omega}$, recover the differentiable structure and, thereafter, the metric $\tilde{g}$.[42]

Figure 9 illustrates the procedure. The set $\tilde{\Omega}$ is an analog of the wave copy in dynamical IPs. We stress once again that the determination $\Lambda \Rightarrow \tilde{\Omega}$ is realized through a relevant coordinatization of the original manifold Ω.

5.1.5. Reconstruction by Henkin. There is one more approach, which recovers Ω as a Riemann surface:

Step 1. Take a triple of real-valued functions $\{f_k\}_{k=1}^3 \subset C^\infty(\Gamma)$ provided $\mathrm{d}f_k \in \operatorname{Ker}[\mathbb{I} + H^2]$ and determine the conjugate functions $\{f_k^*\}_{k=1}^3$ by (5.4). In the generic case, the functions

[41] At this moment, we get a homeomorphic copy of Ω!
[42] Up to a conformal deformation: see [24] for detail.

$\{f_k + f_k^*\mathrm{i}\}_{k=1}^3$ generate the algebra $\mathcal{A}(\Gamma)$, whereas the (unknown) functions $\{w_k\}_{k=1}^3$, $\operatorname{tr} w_k = f_k + f_k^*\mathrm{i}$ determine the (unknown) embedding $j : \Omega \to \mathbb{C}^3$, $j(x) := \operatorname{col}\{w_k(x)\}_{k=1}^3$.

Step 2. Since the image $j(\Gamma) = \{f_k(\gamma) + f_k^*(\gamma)\,\mathrm{i} \mid \gamma \in \Gamma\}_{k=1}^3 \subset \mathbb{C}^3$ is known, one can recover $j(\Omega) =: \tilde{\Omega}$ as a (unique) *minimal surface* in $\mathbb{C}^3$ spanned on $j(\Gamma)$[43] and endow $\tilde{\Omega}$ with the metric $\tilde{g}$ induced by $\mathbb{C}^3$- metric.

Step 3. Identifying the boundaries by $\gamma \equiv j(\gamma)$, we get a representative $(\tilde{\Omega}, \tilde{g})$ of the class of conformally equivalent manifolds possessing the given DN map Λ.

This scheme (surely, in a refined form) is realized in [65], the reconstruction being implemented by the use of Cauchy type integrals: roughly speaking, $\tilde{\Omega}$ is recovered as a graph of a meromorphic function. This makes the scheme available for numerical realization. Moreover, the authors obtain a characterization of Λ, which settles the problem in the 2-dim case.

5.2. *3*-dim *case: formulae for Betti numbers*

5.2.1. Friedrichs decomposition. In section 5.2, we deal with a compact smooth oriented RM (Ω, g), $\dim \Omega = 3$ and use the notation accepted in sections 3.1.1 and 3.1.2. For the sake of simplicity, we assume $\Gamma = \partial\Omega$ connected.

The space of square integrable vector fields on Ω can be represented in the form

$$\vec{L}_2(\Omega) = \mathcal{C}_0 \oplus \mathcal{H} \oplus \mathcal{G}_0 \tag{5.6}$$

known as the *Hodge–Morrey decomposition* [82, 93]; here

$$\begin{aligned}
&\mathcal{C}_0 := \{\operatorname{curl} y \mid y \in \vec{H}^1(\Omega),\, \nu \times y|_\Gamma = 0\},\\
&\mathcal{H} = \{h \in \vec{L}_2(\Omega) \mid \operatorname{div} h = 0, \operatorname{curl} h = 0\},\\
&\mathcal{G}_0 := \{\nabla p \mid p \in H_0^1(\Omega)\};
\end{aligned}$$

the elements of $\mathcal{H}$ are called *harmonic fields*. The *Friedrichs decomposition* specifies the structure of $\mathcal{H}$:

$$\mathcal{H} = \mathcal{D} \oplus \mathcal{C}_{\text{harm}} = \mathcal{N} \oplus \mathcal{G}_{\text{harm}},$$

where $\mathcal{G}_{\text{harm}} := \{y \in \mathcal{H} \mid y = \nabla p\}$ and $\mathcal{C}_{\text{harm}} := \{y \in \mathcal{H} \mid y = \operatorname{curl} h\}$,

$$\mathcal{N} := \{h \in \mathcal{H} \mid \nu \cdot h|_\Gamma = 0\}, \qquad \mathcal{D} := \{m \in \mathcal{H} \mid \nu \times m|_\Gamma = 0\}$$

are the subspaces of the *Neumann and Dirichlet harmonic fields*. These subspaces are of finite dimension: $\dim \mathcal{N} = \beta_1(\Omega)$, $\dim \mathcal{D} = \beta_2(\Omega)$, where $\beta_{1,2}$ are the Betti numbers of Ω [93].

We use also the Hodge–Morrey decomposition at the boundary

$$\vec{L}_2(\Gamma) = \mathcal{C}_\Gamma \oplus \mathcal{H}_\Gamma \oplus \mathcal{G}_\Gamma,$$

where

$$\begin{aligned}
&\mathcal{G}_\Gamma := \{\nabla_\Gamma f \mid f \in H^1(\Gamma)\}, \qquad \mathcal{C}_\Gamma := \{\nu \times h \mid h \in \mathcal{G}_\Gamma\},\\
&\mathcal{H}_\Gamma := \{\eta \in \vec{L}_2(\Gamma) \mid \operatorname{div}_\Gamma \eta = \operatorname{div}_\Gamma \nu \times \eta = 0\}.
\end{aligned}$$

By $\operatorname{div}_\Gamma$ and ∇_Γ we denote the intrinsic divergence and gradient on Γ.

[43] So, $\tilde{\Omega}$ solves a complex Plateu problem.

5.2.2. DN maps and Hilbert transforms. An *electrostatics problem* is to find a function $u = u^f(x)$ satisfying

$$\Delta u = 0 \qquad \text{in} \quad \operatorname{int}\Omega \tag{5.7}$$

$$u = f \qquad \text{on} \quad \Gamma. \tag{5.8}$$

An operator $\Lambda : L_2(\Gamma) \to L_2(\Gamma)$, $\operatorname{Dom}\Lambda = C^\infty(\Gamma)$, $\Lambda f := \nu \cdot \nabla u^f\big|_\Gamma = \frac{\partial u^f}{\partial \nu}\big|_\Gamma$ is called an *electric DN map*. Integration by parts (3.1) easily leads to

$$\int_\Omega \nabla u^{f'} \cdot \nabla u^{f''} = \int_\Gamma \Lambda f' \cdot f''; \tag{5.9}$$

hence, Λ is a positive operator. By the relations $\operatorname{Ker}\Lambda = \{\text{const}\}$, $\operatorname{Ran}\Lambda = \operatorname{div}_\Gamma \vec{C}^\infty(\Gamma) = \dot{C}^\infty(\Gamma) := \{f \in C^\infty(\Gamma) \mid \int_\Gamma f = 0\}$, the composition $\nabla_\Gamma \Lambda^{-1} \operatorname{div}_\Gamma$ is well defined on smooth fields.

Denote $\dot{L}_2(\Gamma) := \{f \in L_2(\Gamma) \mid \int_\Gamma f = 0\}$ and define an operator $J : \mathcal{G}_\Gamma \to \dot{L}_2(\Gamma)$ by $\nabla_\Gamma J = \text{id}$. The operator $H := \Lambda J : \mathcal{G}_\Gamma \to \dot{L}_2(\Gamma)$ is well defined on $\mathcal{G}_\Gamma^\infty := \mathcal{G}_\Gamma \cap \vec{C}^\infty(\Gamma)$; by elliptic theory, H can be extended to $\mathcal{G}_\Gamma$ up to isomorphism onto $\dot{L}_2(\Gamma)$. We name H an *electric Hilbert transform*.

A *magnetostatics problem* is to find a field $h = h^j(x)$ satisfying

$$\Delta h = 0, \qquad \operatorname{div} h = 0 \qquad \text{in} \quad \operatorname{int}\Omega \tag{5.10}$$

$$\nu \times h = j \qquad \text{on} \quad \Gamma \tag{5.11}$$

with $j \in \vec{C}^\infty(\Gamma)$. This problem is solvable but not uniquely: a field h satisfies (5.10), (5.11) with $j = 0$ iff $h \in \mathcal{D}$ (see [93], lemma 3.3.6). In the following, h^j is the (unique) solution orthogonal to $\mathcal{D}$. With the problem we associate a *magnetic DN map* $\vec{\Lambda} : \vec{L}_2(\Gamma) \to \vec{L}_2(\Gamma)$, $\operatorname{Dom}\vec{\Lambda} = \vec{C}^\infty(\Gamma)$, $\vec{\Lambda} j := (\operatorname{curl} h^j)_\theta|_\Gamma$. Integration by parts implies

$$\int_\Omega \operatorname{curl} h^{j'} \cdot \operatorname{curl} h^{j''} = \int_\Gamma \vec{\Lambda} j' \cdot j''; \tag{5.12}$$

hence, $\vec{\Lambda}$ is a positive operator. As is shown in [28], the embedding $\operatorname{Ran}\vec{\Lambda} \supset \mathcal{G}_\Gamma \cap \vec{C}^\infty(\Gamma)$ and $\operatorname{Ker}\vec{\Lambda} \subset \mathcal{C}_\Gamma$ hold. By this, the composition $\operatorname{div}_\Gamma \vec{\Lambda}^{-1} \nabla_\Gamma$ is well defined on smooth functions.

Introduce an operator $\vec{J} : \dot{L}_2(\Gamma) \to \mathcal{G}_\Gamma$ defined on $\dot{C}^\infty(\Gamma)$ by the relation $\vec{J} \operatorname{div}_\Gamma = \text{id}$. The operator $\vec{H} := \vec{\Lambda}\vec{J} : \dot{L}_2(\Gamma) \to \vec{L}_2(\Gamma)$ is well defined on $\dot{C}^\infty(\Gamma)$ and can be extended to $\dot{L}_2(\Gamma)$ up to isomorphism onto its image. We name $\vec{H}$ a *magnetic Hilbert transform*.

The transforms H and $\vec{H}$ are the relevant analogs of the scalar Hilbert transform associated with 2-dim problem (5.1), (5.2) (see section 5.1.3).

5.2.3. Formulae. The main result of the paper [28] is the 3-dim analogs of the formula (5.5). For a harmonic field h, we denote by $\operatorname{tr} h := h|_\Gamma$ its trace on Γ. Recall that the tangent component $h_\theta(\gamma)$, $\gamma \in \Gamma$ is identified with the corresponding vector of $T_\gamma(\Gamma)$.

Theorem 9. *The representations*

$$\operatorname{tr}\mathcal{N} = \operatorname{Ran}[\vec{\Lambda} + \nabla_\Gamma \Lambda^{-1} \operatorname{div}_\Gamma], \qquad \operatorname{tr}\mathcal{D} = \{\operatorname{Ran}[\Lambda + \operatorname{div}_\Gamma \vec{\Lambda}^{-1} \nabla_\Gamma]\}\nu$$

hold and imply

$$\beta_1(\Omega) = \dim \operatorname{Ran}[\vec{\Lambda} + \nabla_\Gamma \Lambda^{-1} \operatorname{div}_\Gamma], \qquad \beta_2(\Omega) = \dim \operatorname{Ran}[\Lambda + \operatorname{div}_\Gamma \vec{\Lambda}^{-1} \nabla_\Gamma].$$

Proof. See in [28]. □

In the 2-dim case, the Betti number $\beta_1(\Omega)$ determines Ω up to homeomorphism. This is not true for $\dim\Omega \geqslant 3$; however, our formulae for $\beta_{1,2}(\Omega)$ provide a certain information about the topology of Ω, which can be extracted from the DN maps.

5.2.4. Existence of conjugate. We say a function u and a solenoidal field h are *conjugated* (and write $u^* = h, u = h^*$) if $\nabla u = \operatorname{curl} h$ in Ω. This definition evidently implies $\Delta u = 0$ and $\Delta h = 0$, so that $\nabla u, \operatorname{curl} h \in \mathcal{H}$.

If Ω is of nontrivial topology, neither u satisfying $\Delta u = 0$ nor h satisfying $\Delta h = 0$ has a conjugate. A remarkable fact is that the existence of the conjugate can be checked in terms of the traces on Γ by use of the Hilbert transforms.

Lemma 9

(i) For $f \in \dot{C}^\infty(\Gamma)$, the function u^f has the conjugate field $h^j = (u^f)^*$ iff $[\mathbb{I} + \vec{H} H]\nabla_\Gamma f = 0$. In this case, the equality $\operatorname{div}_\Gamma j = -H\nabla_\Gamma f$ holds and determines $j \in \mathcal{G}_\Gamma^\infty$.

(ii) For $j \in \mathcal{G}_\Gamma^\infty$, the field h^j has the conjugate function $u^f = (h^j)^*$ iff $[\mathbb{I} + H\vec{H}]\operatorname{div}_\Gamma j = 0$. In this case, the equality $\nabla_\Gamma f = \vec{H} \operatorname{div}_\Gamma j$ holds and determines $f \in \dot{C}^\infty(\Gamma)$.

Proof. See in [28]. □

5.2.5. Comments

- Let u and h be a function and a field in Ω; a pair $q := \{u, h\} \in \mathcal{Q}(\Omega) := C(\Omega) \times \vec{C}(\Omega)$ can be considered as a *quaternion field* on Ω; we denote $u := \operatorname{Re} q, h := \operatorname{Im} q$. The space $\mathcal{Q}(\Omega)$ is an algebra w.r.t. the H-multiplication; it contains a subspace $\mathcal{Q}_{\text{harm}}(\Omega) := \{q \mid (\operatorname{Re} q)^* = \operatorname{Im} q\}$ of *harmonic quaternion fields*. However, $\mathcal{Q}_{\text{harm}}(\Omega)$ is not a subalgebra: $q, q' \in \mathcal{Q}_{\text{harm}}(\Omega)$ does not necessarily imply $(\operatorname{Re}(q'q''))^* = \operatorname{Im}(q'q'')$. Therefore, an attempt to repeat the 2-dim trick (that is the coordinatization by multiplicative functionals or, the same, by maximal ideals) for reconstruction fails. So, the important and challenging problem of recovering the 3-dim manifold from its DN map remains unsolved. The only known result [81][44] concerns a real analytic case but such analyticity is not natural in this kind of IP.
- Understanding δ_{x_0} as a focused wave or a visualizing functional (see (2.29)), we succeed in solving the dynamical IPs. Interpreting δ_{x_0} as a multiplicative functional (maximal ideal) of a proper algebra, we solve the 2-dim elliptic IP. Probably, what we need in the 3-dim case is to recognize the invariant meaning of Dirac measures in elliptic problems.
- One more reason to look for another (not algebraic) coordinatization in elliptic problems is the following. Dealing with determination $\Lambda \Rightarrow L = -\Delta + q$ (with nonanalytic potential q) we have no chance of retaining the algebraic structures. In the meantime, by ellipticity, the Dirac measures remain in the role of the *generating kernels* in the subspace of L-harmonic solutions. Perhaps, the problem will be solved if a relevant invariant description of these kernels is obtained.

5.3. DN operator on differential forms

5.3.1. Forms. Here Ω is assumed to be an n-dimensional smooth compact oriented $RM, \partial\Omega =: \Gamma \in C^\infty$. Let $\Phi^k(\Omega)$ be the space of smooth real differential forms of degree k on Ω, $\Phi(\Omega) = \oplus_{k=0}^n \Phi^k(\Omega)$ the graded algebra of such forms. We use standard operations on $\Phi(\Omega)$: the differential d, the Hodge operator $\star$, the codifferential δ and the Hodge Laplacian $\Delta := d\delta + \delta d$. The space $\Phi^k(\Omega)$ is endowed with L_2-metric, $(\alpha, \beta) := \int_\Omega \alpha \wedge \star\beta$.[45] The embedding $\Gamma \to \Omega$ is denoted by i; with a slight abuse of terms we call the form $i^*\alpha \in \Phi(\Gamma)$ the trace of α.

[44] It is shown that, for $\dim \Omega \geqslant 3$, the operator Λ determines a real analytic (Ω, g) up to isometry.
[45] We also set $\int_\Omega \varphi = 0$ for $\varphi \in \Phi^k(\Omega)$ with $k < n$.

The Hodge–Morrey decomposition is an L_2-orthogonal decomposition of the form

$$\Phi^k(\Omega) = \mathcal{C}^k(\Omega) \oplus \mathcal{H}^k(\Omega) \oplus \mathcal{E}^k(\Omega)$$

generalizing (5.6), where

$$\begin{aligned}
\mathcal{E}^k(\Omega) &:= \{\mathrm{d}\alpha \mid \alpha \in \Phi^{k-1}(\Omega), i^*\alpha = 0\},\\
\mathcal{C}^k(\Omega) &:= \{\delta\alpha \mid \alpha \in \Phi^{k+1}(\Omega), i^*(\star\alpha) = 0\},\\
\mathcal{H}^k(\Omega) &:= \{\lambda \in \Phi^k(\Omega) \mid \mathrm{d}\alpha = 0, \delta\alpha = 0\},
\end{aligned}$$

the elements of $\mathcal{H}^k(\Omega)$ are called harmonic fields. The Friedrichs decomposition details the structure of the harmonic subspaces:

$$\mathcal{H}^k(\Omega) = \mathcal{D}^k(\Omega) \oplus \mathcal{H}^k_{\mathrm{co}}(\Omega) = \mathcal{N}^k \oplus \mathcal{H}^k_{\mathrm{ex}}(\Omega), \tag{5.13}$$

where $\mathcal{H}^k_{\mathrm{co}}(\Omega) := \{\lambda \in \mathcal{H}^k(\Omega) \mid \lambda = \delta\alpha\}$ and $\mathcal{H}^k_{\mathrm{ex}}(\Omega) := \{\lambda \in \mathcal{H}^k(\Omega) \mid \lambda = \mathrm{d}\alpha\}$ are the subspaces of coexact and exact fields,

$$\mathcal{D}^k := \{\lambda \in \mathcal{H}^k(\Omega) \mid i^*\lambda = 0\}, \qquad \mathcal{N}^k := \{\lambda \in \mathcal{H}^k(\Omega) \mid i^*(\star\lambda) = 0\},$$

are the Dirichlet and Neumann subspaces, $\dim \mathcal{N}^k = \dim \mathcal{D}^{n-k} = \beta_k(\Omega)$. Note that $\star$ maps $\mathcal{N}^k$ onto $\mathcal{D}^{n-k}$ isometrically. A Neumann field $\lambda \in \mathcal{N}^k$ is uniquely determined by its trace $i^*\lambda$ (e.g., see [93]).

5.3.2. DN operator. Given $\varphi \in \Phi^k(\Gamma)$, the boundary value problem

$$\begin{aligned}
&\Delta\omega = 0, \qquad \delta\omega = 0 \qquad \text{in} \quad \operatorname{int}\Omega\\
&i^*\omega = \varphi \qquad \text{on} \quad \Gamma
\end{aligned}$$

is solvable (see [93], lemma 3.4.7), its solution $\omega = \omega^\varphi(x)$ is unique up to an element of $\mathcal{D}^k$. Therefore, the form

$$\Lambda f := i^*(\star\mathrm{d}\omega^\varphi) \qquad \text{on} \quad \Gamma$$

is independent of the choice of solution and, hence, Λ is a well-defined operator acting from $\Phi^k(\Gamma)$ to $\Phi^{n-1-k}(\Gamma)$. We call it a *DN operator*.

As is easy to check, redefining $\tilde{\Lambda} := \star_\Gamma \Lambda$ on $\Phi(\Gamma)$, we get an operator reducible by the graduation: $\tilde{\Lambda}$ maps $\Phi^k(\Gamma)$ to $\Phi^k(\Gamma)$. The relation $(\mathrm{d}\omega^{\varphi'}, \mathrm{d}\omega^{\varphi''})_\Omega = (\tilde{\Lambda}\varphi', \varphi'')_\Gamma$ generalizing (5.9) and (5.12) holds and shows that $\tilde{\Lambda}$ is a positive operator. In the case of $\dim\Omega = 3$, the operators Λ and $\vec{\Lambda}$ associated with problems (5.7), (5.8) and (5.10), (5.11) can be interpreted as the parts of $\tilde{\Lambda}$. So, there is a freedom in defining the DN operator[46] and the following result shows that our choice is also well motivated. We denote $\mathcal{H}(\Omega) := \oplus_{k=0}^n \mathcal{H}^k(\Omega)$.

Theorem 10

(i) The relation $\operatorname{Ker}\Lambda = \operatorname{Ran}\Lambda = i^*\mathcal{H}(\Omega)$ holds.
(ii) The operator equalities $\Lambda d = \mathrm{d}\Lambda = 0$, $\Lambda^2 = 0$ are valid.
(iii) For every $k = 0, 1, \ldots, n-1$, the operator $\Lambda + (-1)^{kn+k+n}\,\mathrm{d}\Lambda^{-1}d : \Phi^k(\Gamma) \to \Phi^{n-1-k}(\Gamma)$ is well defined; its range coincides with the subspace $i^*\mathcal{N}^{n-1-k}$ and $\dim[\Lambda + (-1)^{kn+k+n}\,\mathrm{d}\Lambda^{-1}d]\Phi^k(\Gamma) = \beta_{n-1-k}(\Omega)$.

Proof. See in [53]. This result generalizes the formulae for Betti numbers obtained above. □

[46] For another definition, see in [68].

5.3.3. Hilbert transform. An analog of the transforms H^{-1} and $\vec{H}^{-1}$ (see section 5.2.2) on forms is introduced as follows:

$$T := \mathrm{d}\Lambda^{-1} : i^*\mathcal{H}(\Omega) \to i^*\mathcal{H}(\Omega).$$

This is a well-defined operator since the equation $\Lambda\varphi = \psi$ has a solution for any $\psi \in i^*\mathcal{H}(\Omega)$ (see theorem 10(i)), and $\mathrm{d}\varphi$ is determined by ψ. Operator T maps k-forms to n–k-forms. In particular, T is defined on exact boundary forms and preserves the exactness: $T : \mathcal{E}^k(\Gamma) \to \mathcal{E}^{n-k}(\Gamma)$. One more property of this version of the Hilbert transform is that it maps the traces of Neumann fields to the traces of Neumann fields: $T : i^*\mathcal{N}^k \to i^*\mathcal{N}^{n-k}$ [53].

Let $\omega \in \Phi^k(\Omega)$ and $\varepsilon \in \Phi^{n-2-k}(\Omega)$ $(0 \leqslant k \leqslant n-2)$ be two coclosed forms: $\delta\omega = 0, \delta\varepsilon = 0$. The form ε is named a conjugate form of ω (we write $\omega^* = \varepsilon$) if $\mathrm{d}\omega = \star\,\mathrm{d}\varepsilon$. This implies $\Delta\omega = 0$, $\Delta\varepsilon = 0$ and $\varepsilon^* = (-1)^{kn+k+n+1}\omega = \star\star\omega$. As well as in the case of $n = 2, 3$, the existence of ω^* can be checked in terms of the trace $i^*\omega$ and the operator Λ.

Lemma 10. *A form $\omega \in \Phi^k(\Omega)$, $\delta\omega = 0$ satisfying $\Delta\omega = 0$ has a conjugate form iff its trace $\varphi = i^*\omega$ satisfies $[\Lambda + (-1)^{nk+n+k}\,\mathrm{d}\Lambda^{-1}d]\varphi = 0$. In this case, if $\varepsilon = \omega^*$ and $\psi = i^*\varepsilon$ then $\mathrm{d}\psi = T d\varphi$.*

Proof. See in [53]. □

One more result (V A Sharafutdinov [53]) is that the DN operator Λ determines the *real additive cohomology structure of* Ω. This is proven by constructing an isomorphic copy of the exact cohomology sequence of the pair (Ω, Γ). The elements of the copy are the cohomology spaces on Γ constituting a complex; the operators Λ and T play the role of coboundary and imbedding operators of this complex. Perhaps such a role inscribes these operators in the list of objects of algebraic topology.

6. Problems on graphs

6.1. Forward and inverse problems

6.1.1. Graph. Let $\Omega = E \cup V \cup \Gamma \subset \mathbb{R}^2$ be a finite connected planar graph, $E = \{e_j\}_{j=1}^p$ the *edges* (nonintersecting open intervals of straight lines on the plane), $V = \{v_k\}_{k=1}^q$ the *interior vertices* (of multiplicity $\geqslant 3$), $\Gamma = \{\gamma_l\}_{l=1}^n$ the *boundary vertices* (of multiplicity $= 1$). A function ρ on $\Omega\backslash V$ is said to be a *density* if $\rho(\cdot) \geqslant \rho_0 > 0$ and for each $e_j \in E$ the function $\rho|_{e_j}$ can be extended to $\bar{e}_j$ so that $\rho|_{\bar{e}_j} \in C^2(\bar{e}_j)$. A pair (Ω, ρ) is said to be an *equipped graph*.

The density determines an *optic metric* $\mathrm{d}\tau^2 = \rho|\mathrm{d}x|^2$ and the corresponding (*optic*) distance

$$\tau(a, b) = \begin{cases} \min \displaystyle\int_a^b \rho^{\frac{1}{2}}|\mathrm{d}x|, & a \neq b \\ 0, & a = b \end{cases}$$

where min is taken over the paths connecting a with b. If Ω is a *tree* (does not contain cycles), such a path is unique.

A set of functions $\{\tau(\cdot, \gamma_l)\}_{l=1}^n$ determines a map $i{:}\Omega \to \mathbb{R}^n$, $i(x) := \mathrm{col}\{\tau(x, \gamma_l)\}_{l=1}^n$; $\Omega_{\mathrm{opt}} := i(\Omega)$ is a graph in $\mathbb{R}^n$ named an *optic image* of Ω. We equip

$\Omega_{\rm opt}$ with the density $\rho_{\rm opt} := \rho \circ i^{-1}$. If Ω is a tree, $\{\tau(\cdot, \gamma_l)\}_{l=1}^n$ constitutes a coordinate system and the map i is an isometry from (a metric space) (Ω, τ) to $\left(\Omega_{\rm opt}, \frac{1}{\sqrt{n}}\,{\rm dist}_{\mathbb{R}^n}\right)$.

Let (Ω', ρ'), (Ω'', ρ'') be equipped trees, $(\Omega'_{\rm opt}, \rho'_{\rm opt})$, $(\Omega''_{\rm opt}, \rho''_{\rm opt})$ their optic images. As is shown in [26], the equality $(\Omega'_{\rm opt}, \rho'_{\rm opt}) = (\Omega''_{\rm opt}, \rho''_{\rm opt})$ holds iff the trees are *spatially isometric*, i.e., there is an $\mathbb{R}^2$-isometry $I : \Omega' \to \Omega''$ such that $\rho' = \rho'' \circ I$. In other words, a tree is determined by its optical image up to a spatial isometry.

6.1.2. Spectral data. Introduce a Hilbert space $\mathcal{H} := L_{2,\rho}(\Omega)$ of functions on Ω,

$$(y, w)_{\mathcal{H}} := \int_\Omega yw\,\rho|{\rm d}x| = \sum_{e\in E}\int_e yw\,\rho|{\rm d}x|.$$

We assign a function y on Ω to a class $\mathcal{H}^1_0$ if $y \in C(\Omega)$, $y|_e \in H^1(e)$ for every $e \in E$, and $y|_\Gamma = 0$. We write $y \in \mathcal{H}^2$ if

(i) $y \in C(\Omega)$ and $y|_e \in H^2(e)$ for each $e \in E$

(ii) for every $v \in V$, the equality (*Kirchhoff law*)

$$\sum_{\bar e \ni v} \frac{{\rm d}y}{{\rm d}e}(v) = 0$$

(summation over the edges incident to the given v) holds, where

$$\frac{{\rm d}y}{{\rm d}e}(v) := \lim_{x\in e, x\to v} \frac{y(x) - y(v)}{|x - v|}, \qquad v \in V\backslash\Gamma.$$

An operator $L := \mathcal{H} \to \mathcal{H}$, ${\rm Dom}\, L = \mathcal{H}^2 \cap \mathcal{H}^1_0$,

$$(Ly)\,|_e := -\frac{1}{\rho}\frac{{\rm d}^2 y}{{\rm d}e^2}, \qquad e \in E \tag{6.1}$$

is positive definite; its spectrum $\{\lambda_k\}_{k=1}^\infty$, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ is discrete. Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis of eigenfunctions: $L\varphi_k = \lambda_k\varphi_k$, $(\varphi_k, \varphi_l)_{\mathcal{H}} = \delta_{kl}$. Recall also the well-known fact: the eigenfunction φ_1 does not change sign in ${\rm int}\,\Omega$.

Denote $\frac{{\rm d}\varphi_k}{{\rm d}e}\big|_\Gamma := {\rm col}\left\{\frac{{\rm d}\varphi_k}{{\rm d}e}(\gamma_1), \ldots \frac{{\rm d}\varphi_k}{{\rm d}e}(\gamma_n)\right\}$; a set of pairs $\Sigma := \left\{\lambda_k; \frac{{\rm d}\varphi_k}{{\rm d}e}\big|_\Gamma\right\}_{k=1}^\infty$ is said to be the *(Dirichlet) spectral data* of the equipped graph(Ω, ρ). We also assume $\frac{{\rm d}\varphi_1}{{\rm d}e}(\gamma_l) > 0, l = 1, \ldots, n$.

6.1.3. Spectral IP: main result. The *spectral IP* is: given Σ to recover (Ω, ρ). Our contribution to this problem is as follows.

Theorem 11. *If* (Ω, ρ) *is an equipped tree, then* Σ *determines the optic image* $(\Omega_{\rm opt}, \rho_{\rm opt})$ *and, hence, determines* (Ω, ρ) *up to a spatial isometry on the plane.*

In the rest of section 6, we prove theorem 11. The proof follows our basic pattern: to recover an unknown object (here, a graph) via a relevant coordinatization (here, by the Dirac measures). Namely,

(1) we identify Ω with the set of Dirac measures $\{\delta_{x_0} \mid x_0 \in \Omega\}$;

(2) using the spectral model (2.34), we select the set of spectral images $\tilde\Omega = \{\tilde\delta_{x_0} \big| x_0 \in \Omega\}$, the selection mechanism exploiting the waves produced by boundary controls;

(3) the set $\tilde\Omega$ is endowed with a metric which turns $\tilde\Omega$ into an isometric copy of (Ω, ρ); the optic image $(\Omega_{\rm opt}, \rho_{\rm opt})$ is also recovered.

6.2. *Waves on graphs*

6.2.1. Dynamical system. Let $L_* \supset L$ be an operator in $\mathcal{H}$ defined on $\mathcal{H}^2$ and acting by the rule (6.1). With the graph (Ω, ρ) one associates a dynamical system α^T of the form

$$\begin{aligned} &u_{tt} + L_* u = 0 && \text{in} \quad (\Omega\backslash\Gamma) \times (0, T) \\ &u|_{t=0} = u_t|_{t=0} = 0 && \text{in} \quad \Omega \\ &u = f && \text{on} \quad \Gamma \times [0, T] \end{aligned}$$

where $f = \operatorname{col}\{f_k(t)\}_{k=1}^n \in \mathcal{F}^T := L_2([0, T]; \mathbb{R}^n)$ is a boundary control, $u = u^f(x, t)$ is a solution (wave). The outer space $\mathcal{F}^T$ contains the subspaces

$$\mathcal{F}^T[\gamma] := \{f \in \mathcal{F}^T \mid \operatorname{supp} f(\cdot, t) \subset \{\gamma\}, \forall t\}$$

of controls acting from $\gamma \in \Gamma$, so that $\mathcal{F}^T = \oplus_{\gamma\in\Gamma}\mathcal{F}^T[\gamma]$. We use also the classes $\mathcal{F}_1^T[\gamma] := \mathcal{F}^T[\gamma] \cap H_0^1([0, T]; \mathbb{R}^n)$. The inner space is $\mathcal{H}$. The control operator $W^T : \mathcal{F}^T \to \mathcal{H}$, $W^T f := u^f(\cdot, T)$ acts continuously, the relation $W^T H_0^1([0, T]); \mathbb{R}^n) \subset \mathcal{H}_0^1$ being valid.

6.2.2. Propagators. Fix $\gamma \in \Gamma$ and choose $f = f_\gamma(t) := \operatorname{col}\{\delta_{\gamma\gamma'}\theta(t)\}_{\gamma'\in\Gamma} \in \mathcal{F}^T[\gamma]$, where $\theta(t) := \frac{1}{2}[1 + \operatorname{sign} t]$ is the Heaviside function. The solution $p_\gamma := u^{f_\gamma}$ is called a *propagator* corresponding to the boundary vertex γ. Simple analysis of propagation of singularities in Ω leads to the relation

$$\tau(\gamma', \gamma'') = 2 \inf\{t \geqslant 0 \mid (p_{\gamma'}(\cdot, t), p_{\gamma''}(\cdot, t))_{\mathcal{H}} \neq 0\} \tag{6.2}$$

expressing the optic hodograph $\tau|_{\Gamma\times\Gamma}$ of the graph Ω in wave terms.

Let $\mathcal{H}_0^1 \subset \mathcal{H} \subset \mathcal{H}^{-1} := \left(\mathcal{H}_0^1\right)'$ be a standard triple (a rigged Hilbert space). By the properties of W^T, for $h \in \mathcal{H}^{-1}$ and $f \in \mathcal{H}_0^1([0, T]); \mathbb{R}^n)$, the product $(h, u^f(\cdot, T))_{\mathcal{H}}$ is well defined. Along with it, we can define the time

$$t_\gamma[h] := \inf\left\{t \geqslant 0 \middle| (h, u^f(\cdot, t))_{\mathcal{H}} \neq 0,\ f \in \mathcal{F}_1^T[\gamma]\right\} \tag{6.3}$$

and specify it as the moment at which the waves produced at γ begin to interact with the functional h.

One more class of propagators is related to the system

$$\begin{aligned} &v_{tt} + Lv = 0 && \text{in} \quad \mathcal{H} \\ &v|_{t=0} = 0,\ v_t|_{t=0} = h && \text{in} \quad \mathcal{H} \end{aligned}$$

governed by the operator L (with zero Dirichlet boundary conditions on Γ). Its solution

$$v^h(\cdot, t) := \sum_{k=1}^{\infty} \frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}} (h, \varphi_k)_{\mathcal{H}}\, \varphi_k(\cdot) \tag{6.4}$$

is well defined for $h \in \mathcal{H}^{-1}$, $v^h \in C_{\text{loc}}([0, \infty); \mathcal{H})$. Fix $\xi \in \Omega\backslash\Gamma$; let $\delta_\xi \in \mathcal{H}^{-1}$ be a Dirac measure supported in ξ, $c_\xi := \|\delta_\xi\|_{\mathcal{H}^{-1}}^{-1}$. The solution $p_\xi := v^{c_\xi\delta_\xi}$ is said to be a propagator corresponding to the point ξ. Analyzing propagation of singularities, we arrive at the relation

$$\tau(\xi', \xi'') = 2 \inf\{t \geqslant 0 \mid (p_{\xi'}(\cdot, t), p_{\xi''}(\cdot, t))_{\mathcal{H}} \neq 0\} \tag{6.5}$$

for $\xi', \xi'' \in \Omega\backslash\Gamma$. This relation expresses the optic distance τ in wave terms.

6.2.3. Key lemma. A function $d : \mathcal{H}^{-1} \to \overline{\mathbb{R}}_+$,

$$d[h] := \operatorname{diam} \operatorname{supp} h$$

(diameter in the τ-metric) is called an *optic diameter*. A function $d_{\mathrm{w}} : \mathcal{H}^{-1} \to \overline{\mathbb{R}}$,

$$d_{\mathrm{w}}[h] := \max_{\gamma',\gamma''\in\Gamma} \left\{\tau\left(\gamma',\gamma''\right) - \left(t_{\gamma'}[h] + t_{\gamma''}[h]\right)\right\} \geqslant -\infty \tag{6.6}$$

is said to be a *wave diameter*; the term is motivated by the role of waves in (6.2) and (6.3). As we shall see, the wave diameter is determined by the spectral data and, hence, can be used for solving the IP.

Lemma 11. *For any graph Ω, the relation $d_{\mathrm{w}} \leqslant d$ holds. If Ω is a tree then $d_{\mathrm{w}} = d$.*

Proof. See in [26]. □

The proof is based on analysis of the boundary controllability of trees: see [27] and [54].

6.3. Solving the IP

6.3.1. Spectral model. By perfect analogy with (2.34), we define the space $\tilde{\mathcal{H}} := l_2$ and the unitary transform $U : \mathcal{H} \to \tilde{\mathcal{H}}$, $\tilde{y} = Uy := \{(y, \varphi_k)\}_{k=1}^{\infty}$. The following is the list of objects, whose U-images are determined by the spectral data Σ.[47]

(i) The classes $\mathcal{H}_0^1$ and $\mathcal{H}^{-1}$ are mapped to $\tilde{\mathcal{H}}_0^1 := U\mathcal{H}_0^1 = \left\{\{c_k\}_{k=1}^{\infty} \middle| \sum_{k=1}^{\infty} \lambda_k c_k^2 < \infty\right\}$ and $\tilde{\mathcal{H}}^{-1} := \left\{\{c_k\}_{k=1}^{\infty} \middle| \sum_{k=1}^{\infty} \lambda_k^{-1} c_k^2 < \infty\right\}$.

(ii) For $f \in \mathcal{F}^T$, the spectral image of the wave u^f is

$$\tilde{u}^f(t) := Uu^f(\cdot,t) = \left\{\sum_{\gamma\in\Gamma} \frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma) \int_0^t \frac{\sin\sqrt{\lambda_k}(t-s)}{\sqrt{\lambda_k}} f(\gamma,s)\,\mathrm{d}s\right\}_{k=1}^{\infty}, \tag{6.7}$$

so that the model control operator $\tilde{W}^T := UW^T : f \mapsto \tilde{u}^f(T)$ is also determined. By (6.7), we find the spectral images of the propagators p_γ:

$$\tilde{p}_\gamma(t) := Up_\gamma(\cdot,t) = \left\{\frac{\mathrm{d}\varphi_k}{\mathrm{d}e}(\gamma) \frac{1-\cos\sqrt{\lambda_k}t}{\lambda_k}\right\}_{k=1}^{\infty}.$$

As a consequence, Σ determines the hodograph of Ω: by isometry, we have

$$\tau(\gamma',\gamma'') = \langle\text{see (6.2)}\rangle = 2\inf\{t \geqslant 0 \mid (\tilde{p}_{\gamma'}(t), \tilde{p}_{\gamma''}(t))_{\tilde{\mathcal{H}}} \neq 0\}. \tag{6.8}$$

(iii) For any $\tilde{h} \in \tilde{\mathcal{H}}^{-1}$, we have

$$t_\gamma[U^*\tilde{h}] = \langle\text{see (6.3)}\rangle = \inf\left\{t \geqslant 0 \mid \left(\tilde{h}, \tilde{u}^f(t)\right)_{\tilde{\mathcal{H}}} \neq 0,\ f \in \mathcal{F}_1^T[\gamma]\right\}. \tag{6.9}$$

Hence, given a functional $\tilde{h}$, we can determine the wave diameter $d_{\mathrm{w}}[h]$ of its preimage $h = U^*\tilde{h}$ by substituting (6.8), (6.9) into the rhs of (6.6). In other words, a function $d_{\mathrm{w}}[U^*\cdot] : \tilde{\mathcal{H}}^{-1} \to \overline{\mathbb{R}}$ is determined by Σ. As a result, if Ω is a tree then, by lemma 11, its spectral data determine the optic diameter $d[U^*\cdot] : \tilde{\mathcal{H}}^{-1} \to \overline{\mathbb{R}}_+$.

(iv) A set of normalized Dirac measures $\left\{c_\xi \delta_\xi \middle| \xi \in \Omega\backslash\Gamma\right\}$ can be characterized as $\left\{h \in \mathcal{H}^{-1} \mid \|h\|_{\mathcal{H}^{-1}} = 1, d[h] = 0, (h, \varphi_1)_{\mathcal{H}} > 0\right\}$. Such a representation enables us to characterize the spectral images: $\{\tilde{h} \in \tilde{\mathcal{H}}^{-1} \mid \|\tilde{h}\|_{\tilde{\mathcal{H}}^{-1}} = 1, d_{\mathrm{w}}[U^*\tilde{h}] = 0, (\tilde{h}, \tilde{\varphi}_1)_{\tilde{\mathcal{H}}} > 0\} =: \tilde{\Omega}$. Thus, Σ determines the set $\tilde{\Omega}$: it is just the set of zeros of the function $d_{\mathrm{w}}[U^*\cdot]$.

[47] The derivation of these results uses nothing besides integration by parts: see [26].

(v) For every $\tilde h \in \tilde\Omega$, one can find the image of the propagator p_ξ ($\xi = \operatorname{supp} U^*\tilde h$) by

$$\tilde p_\xi(t) = Up_\xi(\cdot, t) = \langle\text{see (6.4)}\rangle = \left\{\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}(\tilde h, \tilde\varphi_k)_{\tilde{\mathcal{H}}}\right\}_{k=1}^{\infty}.$$

Then, for $\tilde h', \tilde h'' \in \tilde\Omega$ and the corresponding points $\tilde\xi' = \operatorname{supp} U^*\tilde h'$, $\tilde\xi'' = \operatorname{supp} U^*\tilde h''$, we determine

$$\tau(\xi', \xi'') = \langle\text{see (6.5)}\rangle = 2\inf\{t \geqslant 0 \mid (\tilde p_{\xi'}(t), \tilde p_{\xi''}(t))_{\tilde{\mathcal{H}}} \neq 0\} =: \tilde\tau(\tilde h', \tilde h'').$$

Therefore, endowing the set $\tilde\Omega$ with a function $\tilde\tau : \tilde\Omega \times \tilde\Omega \to \overline{\mathbb{R}}_+$, we turn $\tilde\Omega$ into an isometric copy of the space (Ω, τ) (through the isometry $\tilde i : \tilde\Omega \to \Omega\backslash\Gamma, \tilde i(\tilde h) = \operatorname{supp} U^*\tilde h$). As is easy to recognize, $(\tilde\Omega, \tilde\tau)$ is just a version of the wave copy of the original (Ω, τ) reproduced via the spectral model.

6.3.2. Recovering (Ω, ρ). Let (Ω, ρ) be an equipped tree, Σ its spectral data. For determination $\Sigma \Rightarrow (\Omega, \rho)$ one needs just to summarize the results (i)–(v), i.e., to construct the copy $(\tilde\Omega, \tilde\tau)$. Thereafter, completing $\tilde\Omega$ and finding its (topological) boundary $\tilde\Gamma = \{\gamma_k\}_{k=1}^n$, we can provide $\tilde\Omega$ with the coordinates $\{\tilde\tau(\cdot, \tilde\gamma_k)\}_{k=1}^n$. The image $\tilde j(\tilde\Omega) \subset \mathbb{R}^n$ under the map $\tilde j : \tilde h \mapsto \operatorname{col}\{\tilde\tau(\cdot, \tilde\gamma_k)\}_{k=1}^n$ coincides with Ω_{opt} by construction. Some extra work is required to provide Ω_{opt} with the density ρ_{opt} but this can also be done: see [26].

Thus, the spectral data Σ determine the optic image $(\Omega_{\text{opt}}, \rho_{\text{opt}})$ and, hence, the tree (Ω, ρ) up to a spatial isometry. Theorem 11 is proven.

So, the IP is once again solved by the coordinatization $x_0 \leftrightarrow \delta_{x_0}$ and relevant interpretation of the Dirac measures: on trees these measures are characterized as the *functionals of zero wave diameter*.

6.3.3. Comments and hypotheses

- Let R^{2T} be the continued response operator of the system α^T associated with an equipped tree (Ω, ρ). Given for $T \geqslant \frac{1}{2}\operatorname{diam}\Omega$, the operator R^{2T} determines the tree up to a spatial isometry on the plane. To prove this, one needs just to repeat our considerations, using the dynamical model (2.35) instead of the spectral one. However, to justify the approach the results on the boundary controllability of trees [26, 27] are required.

  In [26] the list of problems on graphs reducible to the spectral or dynamical IPs is presented. In the case of a tree, each of these problems can be solved by a certain version of the BCm. For other approaches and results see [3, 73, 97].
- A challenging open problem is to recover graphs containing the cycles. Lemma 11 shows that $d_w \geqslant 0$ is a necessary condition for a graph to be a tree. In the meantime, it is easy to present a graph with cycles and a point ξ in the graph such that $d_w[\delta_\xi] < 0$ (see [26]). Does $d_w \geqslant 0$ characterize the trees? An affirmative answer would enable one to detect the presence of cycles via inverse data.

  A property which could help us to select the set $\tilde\Omega$ in $\tilde{\mathcal{H}}^{-1}$ is *positivity* of the Dirac measures: $\varphi \geqslant 0$ implies $(\delta_\xi, \varphi) \geqslant 0$, whereas these measures can be characterized as extreme points of the convex set of positive functionals normalized by $(h, 1) = 1$. In this connection, there was an attempt to characterize the cone $\mathcal{H}_+ := \{y \in \mathcal{H} \mid y \geqslant 0\} \subset \mathcal{H}$ in terms of Σ. The idea was to invoke the maximal principle for the heat equation. Consider the heat conductivity system

$$\begin{aligned} &u_t + L_* u = 0 && \text{in} \quad (\Omega\backslash\Gamma) \times (0, T)\\ &u|_{t=0} = 0 && \text{in} \quad \Omega\\ &u = f && \text{on} \quad \Gamma \times [0, T] \end{aligned}$$

let $u = u^f(x,t)$ be its solution (heat wave). As is known, $f \geqslant 0$ implies $u^f \geqslant 0$ and, hence, the reachable set $\mathcal{U}_+^T := \left\{u^f(\cdot,T) \middle| f \in C_0^\infty(\Gamma \times [0,T]),\, f \geqslant 0\right\}$ is embedded in $\mathcal{H}_+$. In the meantime, the image $\tilde{\mathcal{U}}_+^T := U\mathcal{U}_+^T$ is determined by the spectral data and, hence, the condition $(\tilde{h}, \tilde{\mathcal{U}}_+^T)_{\tilde{\mathcal{H}}} \geqslant 0$, necessary for the inclusion $\tilde{h} \in \tilde{\Omega}$, is checkable via Σ. Thereafter, selecting in $\tilde{\mathcal{H}}^{-1}$ the set of spectral images of the (properly normalized) positive functionals, we might determine the images of Dirac measures as the extreme points of this set and, thus, might get $\tilde{\Omega}$.

Unfortunately, the attempt fails: the set $\cup_{T>0}\mathcal{U}_+^T$ is not dense in $\mathcal{H}_+$ [29] and, hence, $(\tilde{h}, \tilde{\mathcal{U}}_+^T)_{\tilde{\mathcal{H}}} \geqslant 0$ does not ensure $\tilde{h} \geqslant 0$. However, perhaps, the idea is not exhausted yet and we can hope for determination of $\tilde{\mathcal{H}}_+$ by 'physically motivated' properties of another type of waves.

Presumably, the following class of graphs with cycles is available for the first attack by the BCm. We say a graph Ω is *transparent* if the shortest paths linking the boundary vertices pairwise exhaust Ω. The reason to single out this class is that transparency implies $d_{\mathrm{w}}[\delta_\xi] = 0$. However, if Ω contains cycles, not only the Dirac measures are of zero wave diameter and the open problem is to identify δ_ξ (surely, in terms of the inverse data!) among other zeros of d_{w}.

- The procedure (i)–(v) (section 6.3.1) is not available for elaborating the algorithms since the function $d_{\mathrm{w}}[U^*\cdot]$ is very irregular on $\tilde{\mathcal{H}}^{-1}$ (discontinuous at every point). In [54], we propose another approach based on a certain version of the Gelfand–Levitan–Krein equations. The results of numerical testing demonstrate the efficiency of the corresponding algorithm. In the same paper (see also [27]), the time-optimal and local procedure of controlling a tree from the boundary is described. Note that the possible lack of boundary controllability is one of the principle obstructions to extending the BCm to graphs with cycles.

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Appendix

A.1. Proof of lemma 1

The definition of $\omega_\gamma^{s,\varepsilon}$ easily implies

$$\omega_\gamma^s = \{x \in \Omega \mid \mathrm{d}(x,\gamma) \leqslant s, \tau(x) = s\} = \bar{\Omega}^s[\gamma] \cap \Gamma^s.$$

Let $x \in \omega_\gamma^s$; connect x with γ with a shortest geodesic. Since $x \in \bar{\Omega}^s[\gamma]$, the geodesic is of length $\leqslant s$; since $x \in \Gamma^T$, the length is $\geqslant s$. Therefore, the geodesic is of length s and connects Γ^s with Γ. Thus, x is an endpoint of a normal geodesic of length s, i.e., $x = x(\gamma, s)$. As a result, ω_γ^s can contain (if nonempty) only one point $x(\gamma, s)$, what occurs if $s \leqslant \tau_*(\gamma)$. $\square$

A.2. Proof of theorem 1

(i) The assertion of the theorem follows evidentially from the equality

$$\mathrm{cl}L_0^T = \mathrm{cl}\big(-\Delta_0^T\big) \tag{A.1}$$

for the closures of the operators. Prove (A.1).

(ii) The Friedrichs extension $-\Delta^T$ of $-\Delta_0^T$ is a positive definite and boundedly invertible operator and, by $-\Delta^T \supset -\Delta_0^T \supset L_0^T$, we have the equalities $(-\Delta^T)^{-1}\Delta_0^T = \mathbb{I}$ and $(-\Delta^T)^{-1}L_0^T = \mathbb{I}$. To close a boundedly invertible operator is to extend its inverse by continuity. Therefore, the equality (A.1) will be proven if we verify the relation

$$\mathrm{clos}\,\mathrm{Ran}L_0^T = \mathrm{clos}\,\mathrm{Ran}\big(-\Delta_0^T\big). \tag{A.2}$$

Prove (A.2).

(iii) The embedding $\mathrm{clos}\,\mathrm{Ran}L_0^T \subset \mathrm{clos}\,\mathrm{Ran}\big(-\Delta_0^T\big)$ is evident; to prove the opposite embedding is to prove that $\big\{y \perp \mathrm{Ran}L_0^T\big\}$ implies $\big\{y \perp \mathrm{Ran}\big(-\Delta_0^T\big)\big\}$. Prove this implication .

Take a nonzero $\big\{y \perp \mathrm{Ran}\, L_0^T\big\}$ and consider the system

$$v_{tt} - \Delta v = 0 \qquad \text{in} \quad (\mathrm{int}\,\Omega^T) \times \mathbb{R} \tag{A.3}$$

$$v|_{t=T} = 0, \qquad v_t|_{t=T} = y \qquad \text{in} \quad \overline{\Omega}^T \tag{A.4}$$

$$v = 0 \qquad \text{on} \quad (\Gamma \cup \Gamma^T) \times \mathbb{R} \tag{A.5}$$

with the solution $v = v^y(x, t)$,

$$v^f(\cdot, t) = \int_0^\infty \frac{\sin\sqrt{\lambda}(t-T)}{\sqrt{\lambda}}\,\mathrm{d}P_\lambda y,$$

where P_λ is the spectral measure of $-\Delta^T$.

Choose an even function $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \geqslant 0$, $\mathrm{supp}\,\varphi \subset [-1, 1]$ such that $\int_{-\infty}^\infty \varphi(t)\,\mathrm{d}t = 1$. For $\varepsilon > 0$, denote $\varphi_\varepsilon(t) := \varepsilon^{-1}\varphi(\varepsilon^{-1}t)$, $t \in \mathbb{R}$ and $v_\varepsilon^y := \varphi_\varepsilon * v^y$ (the convolution w.r.t. t). Straightforward calculation (see [22], pp 360–2) shows that v_ε^y is a solution of (A.3)–(A.5) with y replaced by y_ε, where

$$y_\varepsilon := \int_0^\infty \rho_\varepsilon(\lambda)\,\mathrm{d}P_\lambda y, \qquad \rho_\varepsilon(\lambda) := \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(\eta)\cos\sqrt{\lambda}\eta\,\mathrm{d}\eta,$$

so that $v_\varepsilon^y = v^{y_\varepsilon}$. The function ρ_ε possesses the properties:

(α) $|\,\rho_\varepsilon\,| \leqslant 1$, $\lim_{\varepsilon\to0}\rho_\varepsilon = 1$ uniformly on any compact set in $[0, \infty)$

(β) $\lim_{\lambda\to\infty}\lambda^k\,|\,\rho_\varepsilon(\lambda)\,| = 0$ for any $k \geqslant 0$.

They lead to the corresponding properties of y_ε:

(α') $\|y_\varepsilon\|_{\mathcal{H}} \leqslant \|y\|_{\mathcal{H}}$, $\|y - y_\varepsilon\|_{\mathcal{H}} \to 0$ as $\varepsilon \to 0$.

(β') $y_\varepsilon \in \bigcap_{k\geqslant0}\mathrm{Dom}(-\Delta^T)^k$ (and, hence, $y_\varepsilon|_\Gamma = 0$). By ellipticity of $-\Delta^T$, the last property implies $y_\varepsilon \in C^\infty(\Omega^T)$ and $v^{y_\varepsilon}(\cdot, t) \in C^\infty(\Omega^T)$, $t \in \mathbb{R}$.

Keeping $\varepsilon > 0$ fixed, choose $\delta > \varepsilon$ and take in (2.7)–(2.10) a control $f \in \mathcal{F}^{T,T-\delta}$; define

$$f_\varepsilon(\cdot, t) := \int_0^T [\varphi_\varepsilon(t-\eta) - \varphi_\varepsilon(2T - t - \eta)]\,f(\cdot, \eta)\,\mathrm{d}\eta.$$

As is easy to check, $f_\varepsilon \in C^\infty([0, T]; L_2(\Gamma)) \subset \mathcal{F}^T$ and $(f_\varepsilon)_{tt} = (f_{tt})_\varepsilon$ for $f \in \mathcal{F}^{T,T-\delta} \cap C^\infty([0, T]; L_2(\Gamma))$. A simple calculation leads to

$$(W^T f_\varepsilon, y)_{\mathcal{H}} = (W^T f, y_\varepsilon)_{\mathcal{H}} \tag{A.6}$$

(see [22], pp 361–2).

By the choice of $y \in \mathcal{H}^T \ominus \operatorname{Ran} L_0^T$, for $f \in \mathcal{F}^{T,T-\delta} \cap C^\infty([0,T]; L_2(\Gamma))$ we have

$$
\begin{aligned}
0 &= \left(L_0^T W^T f_\varepsilon, y\right)_{\mathcal{H}} = (W^T(f_\varepsilon)_{tt}, y)_{\mathcal{H}} = (W^T(f_{tt})_\varepsilon, y)_{\mathcal{H}} \\
&= \langle \text{see (A.6)} \rangle = (W^T f_{tt}, y_\varepsilon)_{\mathcal{H}} = (\Delta W^T f, y_\varepsilon)_{\mathcal{H}} \\
&= \langle \text{see } (\beta') \rangle = (W^T f, \Delta y_\varepsilon)_{\mathcal{H}}.
\end{aligned}
\tag{A.7}
$$

Since the set of f used is dense in $f \in \mathcal{F}^{T,T-\delta}$, the set of waves $W^T f$ is dense in $\operatorname{clos} \mathcal{U}^{T-\delta}$ and, hence, dense in $\mathcal{H}^{T-\delta}$ by controllability of (2.21). Therefore, (A.7) implies $\Delta y_\varepsilon = 0$ in $\Omega^{T-\delta}$ and, by arbitrariness of δ, in Ω^T. Tending $\varepsilon \to 0$, with regard to (α') we easily get $\Delta y = 0$ in Ω^T and $y|_\Gamma = 0$.

For $a \in \operatorname{Dom}(-\Delta^T)_0$, we have

$$
\left(-\Delta_0^T a, y\right)_{\mathcal{H}} = -\int_\Gamma \frac{\mathrm{d}a}{\mathrm{d}\nu} y \, \mathrm{d}\Gamma - (a, \Delta y)_{\mathcal{H}} = 0
$$

($\int_{\Gamma^T} = 0$ since a vanishes near Γ^T). So, $y \perp \operatorname{Ran} L_0^T$ implies $y \perp \operatorname{Ran}\left(-\Delta_0^T\right)$. □

A.3. Proof of theorem 3 and lemma 4

Since $T > 0$ is arbitrary, we can consider the case $s = T$ only.

(i) Prove the first relation in (3.13). Let $n \in \mathcal{J}^T[\sigma] \ominus \operatorname{clos} \mathcal{U}^T[\sigma]$; we shall show that n is harmonic on its support, i.e., $\operatorname{curl} n = 0$ in $\Omega^T[\sigma]$. By hyperbolicity, the problem

$$
e_{tt} - \Delta e = 0 \qquad \text{in} \quad \mathcal{A}^T \tag{A.8}
$$

$$
e|_{t=T} = 0, \qquad e_t|_{t=T} = y \in \mathcal{J}^T \tag{A.9}
$$

$$
e_\theta = 0 \qquad \text{on} \quad \Gamma \times [0,T] \tag{A.10}
$$

in the spacetime subdomain

$$
\begin{aligned}
\mathcal{A}^T &:= \{(x,t) \mid x \in \operatorname{int} \Omega^T, T - d(x, \Gamma^T) < t < T\} \\
&= [(\operatorname{int} \Omega^T) \times (0,T)] \backslash \{\text{backward domain of influence of } \Gamma^T \times \{t = T\}\}
\end{aligned}
$$

is well posed and said to be dual to (3.6)–(3.8). Let $e = e(x,t)$ be the solution; note that $\mathcal{A}^T \supset \operatorname{supp} u^f$, where u^f is a trajectory of the system α^T. Following [36], one can derive the duality relation

$$
(u^f(\cdot, T), y)_{\mathcal{J}} = (f, \nu \times \operatorname{curl} e^y)_{\mathcal{F}^T} \tag{A.11}
$$

for $f \in \mathcal{M}^T$, where the trace $\nu \times \operatorname{curl} e^y|_{\Gamma \times [0,T]}$ is defined as an element of $L_2([0,T]); \vec{\mathcal{H}}^{-1}(\Gamma))$ and the rhs is understood in the relevant sense. If $y = n$ in (A.9), the lhs of (A.11) equals zero for all $f \in \mathcal{F}^{T,T}[\sigma] \cap \mathcal{M}^T$. Hence, by the density of smooth f, we have

$$
\nu \times \operatorname{curl} e^n = 0 \qquad \text{on} \quad \sigma \times [0,T]. \tag{A.12}
$$

By the form of the Cauchy data (A.9), a vector-valued function

$$
E(\cdot, t) := \begin{cases} e^n(\cdot, t) & 0 \leqslant t \leqslant T \\ -e^n(\cdot, 2T - t) & T \leqslant t \leqslant 2T \end{cases}
$$

turns out to be a solution of the equation

$$
E_{tt} - \Delta E = 0 \qquad \text{in} \quad B^{2T}, \tag{A.13}
$$

where $B^{2T} := A^T \cup \{(x,t) \mid (x, 2T - t) \in A^T\} \cup \{(x,T) \mid x \in \operatorname{int} \Omega^T\}$. This solution satisfies

$$
E_\theta = \langle \text{see (A.10)} \rangle = 0, \quad \nu \times \operatorname{curl} E = \langle \text{see (A.12)} \rangle = 0 \qquad \text{on} \quad \sigma \times [0, 2T]. \tag{A.14}
$$

Assume $E \in C^4(\operatorname{supp} E)$. Multiplying (A.13) by ν on σ, we get

$$\nu \cdot E_{tt} = \langle \text{see (A.13)} \rangle = -\nu \cdot \operatorname{curl}\operatorname{curl} E = \langle \text{see section 2.1.1} \rangle$$
$$= -\operatorname{div}_\Gamma \nu \times \operatorname{curl} E = \langle \text{see (A.14)} \rangle = 0;$$

hence,

$$E_{tt} = (E_\theta + E \cdot \nu\nu)_{tt} = 0 \qquad \text{on} \quad \sigma \times [0, 2T]. \tag{A.15}$$

Since

$$\nu \cdot \operatorname{curl} E = \operatorname{div}_\Gamma \nu \times E = \operatorname{div}_\Gamma \nu \times E_\theta = \langle \text{see (A.14)} \rangle = 0,$$

it follows from (A.14) that

$$\operatorname{curl} E = 0 \qquad \text{on} \quad \sigma \times [0, 2T]. \tag{A.16}$$

The function $\Psi := \operatorname{curl} E$ satisfies

$$\Psi_{tt} - \Delta\Psi = 0 \qquad \text{in} \quad B^{2T} \tag{A.17}$$

and

$$\Psi = \langle \text{see (A.16)} \rangle = 0, \qquad \operatorname{curl} \Psi = \operatorname{curl}\operatorname{curl} E = -E_{tt}$$
$$= \langle \text{see (A.15)} \rangle = 0 \qquad \text{on} \quad \sigma \times [0, 2T].$$

The relations $\Psi(\cdot, t)|_\sigma = \operatorname{curl} \Psi(\cdot, t)|_\sigma = 0$ easily imply $\frac{D\Psi(\cdot,t)}{\partial\nu}|_\sigma = 0$. Hence,

$$\Psi = 0, \qquad \frac{D\Psi}{\partial\nu} = 0 \qquad \text{on} \quad \sigma \times [0, 2T],$$

so that Ψ is a solution of (A.17) possessing the zero Cauchy data at $\sigma \times [0, 2T]$. Applying the vectorial version of the Holmgren–John–Tataru theorem [60] and the standard procedure of data continuation from the time-like surface (see [67, 92]) we get

$$\Psi = 0 \qquad \text{in} \quad K^{2T}_\sigma := \{(x, t) \mid x \in \Omega^T, | T - t | \leqslant T - d^T(x, \sigma)\}. \tag{A.18}$$

Returning to (A.13), we conclude that $E_{tt} = -\operatorname{rot}\Psi = 0$ in K^{2T} and, hence, $E(\cdot, t) = (t - T)n$ by (A.9). The latter implies

$$\operatorname{curl} n = \operatorname{curl} E_t(\cdot, T) = \Psi_t(\cdot, T) = \langle \text{see (A.18)} \rangle = 0 \quad \text{in} \quad K^{2T}_\sigma \cap \{t = T\} = \Omega^T[\sigma]; \tag{A.19}$$

we have also

$$\nu \times n = \nu \times E_t(\cdot, T) = \nu \times e_{\theta t}(\cdot, T) = \langle \text{see (A.10)} \rangle = 0 \qquad \text{on} \quad \Gamma. \tag{A.20}$$

Thus, n is harmonic in $\Omega^T[\sigma]$.

Not assuming the C^4-smoothness of the solution E, we can reduce the problem to the smooth case applying the regularization $\varphi_\varepsilon *$ by perfect analogy to the proof of theorem 1 above (see also [36]). So, we arrive at $\mathcal{J}^T[\sigma] \approx \operatorname{clos} \mathcal{U}^T[\sigma]$ proving the first relation in (3.13)[48].

(ii) Prove the second relation in (3.13). Let e^y be the solution of the problem

$$e_{tt} - \Delta e = 0 \qquad \text{in} \quad \mathcal{A}^{T,r} \tag{A.21}$$

$$e|_{t=r} = 0, \quad e_t|_{t=r} = y \in \mathcal{J}^T \tag{A.22}$$

$$e_\theta = 0 \qquad \text{on} \quad \Gamma \times [0, r], \tag{A.23}$$

[48] Moreover, by (A.19) and (A.20), it is proven that $\mathcal{J}^T[\sigma] \ominus \operatorname{clos} \mathcal{U}^T[\sigma] = \mathcal{D}^T[\sigma] := \{y \in \mathcal{J}^T \mid \operatorname{curl} y = 0 \text{ in } \Omega^T[\sigma], y \times \nu|_\Gamma = 0\}$.

where $\mathcal{A}^{T,r} := \{(x,t) \mid x \in \operatorname{int}\Omega^T \times [0,r], r - d(x,\Gamma^T) < t < r\}$. Let w^h be the solution of problem (3.10)–(3.11) with $h \in \mathcal{G}^r_\omega$ provided $d^T(\omega,\Gamma^T) > 0$.[49] Integration by parts with regard to the positional relationship of $\operatorname{supp} e^h$ and $\operatorname{supp} w^y$ leads to the duality relation

$$\int_{\Omega^T} y \cdot w^h(\cdot,t) = -\int_{\omega\times[0,r]} e^y \cdot h.$$

Take $y = m \in \mathcal{J}^r[\omega] \ominus \operatorname{clos}\mathcal{U}^r[\omega]$; by this choice, the rhs of the last equality equals zero. Hence, by arbitrariness of h, we easily have $e^m(\cdot,t) \perp \mathcal{J}^0[\omega]$, $t \in [0,r]$, that implies $\operatorname{curl} e^m(\cdot,t) = 0$ in ω.

Extending by oddness

$$E(\cdot,t) := \begin{cases} e^m(\cdot,t) & 0 \leqslant t \leqslant r \\ -e^m(\cdot,2r-t) & r \leqslant t \leqslant 2r, \end{cases}$$

we get a solution of the problem

$$E_{tt} - \Delta E = 0 \qquad \text{in} \quad \Omega^T \times (0,2r).$$

If $E \in C^3(\operatorname{supp} E)$ then $\Psi = \operatorname{curl} E$ satisfies

$$\begin{aligned} \Psi_{tt} - \Delta\Psi &= 0 \qquad &&\text{in} \quad \Omega^T \times (0,2r) \\ \Psi &= 0 &&\text{in} \quad \omega \times [0,2r]. \end{aligned}$$

Applying the Holmgren–John–Tataru theorem [60] we conclude that

$$\Psi = 0 \qquad \text{in} \quad K^{2r}_\omega := \{(x,t) \mid x \in \Omega^T, \mid r - t \mid \leqslant r - d^T(x,\omega)\}.$$

This implies $\Psi(\cdot,r) = \operatorname{curl} m = 0$ in $\Omega^r[\omega]$ and, hence, $\operatorname{clos}\mathcal{U}^r[\omega] \approx \mathcal{J}^r[\omega]$. If E is not smooth enough to justify this derivation, one can invoke the regularization by $\varphi_\varepsilon *$. □

(iii) Here we prove lemma 4.

The embedding $\mathcal{U}^{s,\varepsilon}_\gamma \subset \mathcal{J}^0\big[\omega^{s,\varepsilon}_\gamma\big]$ If $h \in \mathcal{U}^{s,\varepsilon}_\gamma$ then $h \perp \operatorname{clos}\mathcal{U}^{s-\varepsilon}$ and, by (3.13) (for $\sigma = \Gamma$), h is harmonic in $\Omega^{s-\varepsilon}$. Since $\operatorname{supp} h \subset \overline{\Omega}^s[\sigma_\varepsilon(\gamma)]$, h vanishes in $\Omega^{s-\varepsilon}\backslash\Omega^s[\sigma_\varepsilon(\gamma)]$; hence, by the uniqueness theorem for harmonic fields (see e.g. [93]), $h = 0$ in $\Omega^{s-\varepsilon}$. This implies $\operatorname{supp} h \subset \omega^{s,\varepsilon}_\gamma$ and we get $h \in \mathcal{J}^0\big[\omega^{s,\varepsilon}_\gamma\big]$.

The embedding $\mathcal{U}^{s,\varepsilon}_\gamma \supset \mathcal{J}^0[\omega^{s,\varepsilon}_\gamma]$ By (3.13), any $\psi \in \mathcal{J}^s[\sigma_\varepsilon(\gamma)] \ominus \operatorname{clos}\mathcal{U}^s[\sigma_\varepsilon(\gamma)]$ is harmonic in $\Omega^s[\sigma_\varepsilon(\gamma)]$ and, in particular, in $\omega^{s,\varepsilon}_\gamma$. For small enough ε, the cap $\omega^{s,\varepsilon}_\gamma$ is homeomorphic to an $\mathbb{R}^3$-ball and, hence, $\psi|_{\omega^{s,\varepsilon}_\gamma} = \nabla p$. By this, for $h \in \mathcal{J}^0\big[\omega^{s,\varepsilon}_\gamma\big]$ we have $(h,\psi)_{\mathcal{J}} = 0$. Therefore, $h \perp \{\mathcal{J}^0[\sigma_\varepsilon(\gamma)] \ominus \operatorname{clos}\mathcal{U}^s[\sigma_\varepsilon(\gamma)]\}$, i.e., $h \in \operatorname{clos}\mathcal{U}^s[\sigma_\varepsilon(\gamma)]$. In the meantime, $\operatorname{supp} h \subset \omega^{s,\varepsilon}_\gamma$ implies $h \perp \operatorname{clos}\mathcal{U}^{s-\varepsilon}[\sigma_\varepsilon(\gamma)]$. As result, $h \in \mathcal{U}^{s,\varepsilon}_\gamma$. □

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