RECENT RESULTS ON REFINEMENTS OF THE CENTRAL LIMIT THEOREM

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1. Introduction and summary

Let $\{X_n\}$ be a sequence of independent and identically distributed (i.i.d.) random vectors in R^k with zero mean vector and identity covariance matrix. The distribution Q_n of the normalized sum $n^{-1/2}(X_1 + \cdots + X_n)$ converges weakly to the k dimensional standard normal distribution Φ. Although many important results on rates of convergence have been obtained in the past, most of them refer to approximations of the distribution function F_n of Q_n by the normal distribution function. An exception to this is the case where Q_n is assumed to have a density with respect to Lebesgue measure or to have a lattice distribution. In this situation, one obtained local limit theorems as well (see [14], Chapter 16, and [19]). The first notable exception was a result of Esseen [13] which states that *if fourth moments are finite, then, uniformly over all spheres S (open or closed) with center at the origin, one has*

$$(1.1) \qquad Q_n(S) - \Phi(S) = O(n^{-k/(k+1)}), \qquad n \to \infty.$$

Esseen showed the remarkable depth of this result by relating a special case of this to the *lattice point problem* of analytic number theory. In 1960, Ranga Rao [29] investigated the rate of convergence over the class $\mathscr{C}$ of all measurable convex sets and proved that *if fourth moments are finite, then*

$$(1.2) \qquad \sup_{C \in \mathscr{C}} |Q_n(C) - \Phi(C)| = O\big(n^{-1/2}(\log n)^{(k-1)/2(k+1)}\big), \qquad n \to \infty.$$

He also obtained a number of asymptotic expansions extending some results of Cramér [9] (Chapter 7) and Esseen [13] for distribution functions. The present author [1] and von Bahr [34] independently obtained rates of convergence for general classes of sets; a typical application gives the following precise bound for $\mathscr{C}$:

$$(1.3) \qquad \sup_{C \in \mathscr{C}} |Q_n(C) - \Phi(C)| = O(n^{-1/2}), \qquad n \to \infty.$$

In [1], this was proved under the assumption of finiteness of moments of order $3 + \delta$ for some positive δ, while in [34] $E|X_1|^{k+1}$ was assumed finite for $k \geq 2$.

Sazonov [32] has now shown that (1.3) holds if $E|X_1|^3$ is finite. However, Sazonov's method does not seem to extend to general classes of sets and functions. In [2], [3], the author has given an account of his results on the estimation of $\int g\, dQ_n - \int g\, d\Phi$ for arbitrary real valued, bounded, almost surely (Φ) continuous functions g.

In addition to the results mentioned above, various useful refinements of other aspects of the central limit theorem have been obtained over the years. We mention only a few. Petrov [27], Richter [30], [31], and Linnik [21], [22], [23] improved Cramér's results on large deviations. Petrov [28], and Bikjalis [6], [7] sharpened some mean central limit theorems and asymptotic expansions. Ibragimov [17], [18] and Heyde [16] investigated exact rates of convergence for F_n. Nagaev [24], [25] obtained rates of convergence and asymptotic expansions for Markov chains. Recently, Stein [33] has devised a general method for dealing with dependent sequences. In another direction, Cramér [10] has obtained the first significant results on the speeds of convergence to other stable laws.

In Section 2 of this article, two lemmas are proved for an arbitrary separable metric group. They estimate the effect of a "small" perturbation by convolution on a finite signed measure. Although the only uses of these lemmas made here are in proving the main results in Section 4, they may be used to compute convergence rates for limit theorems in locally compact abelian groups when estimates of rates of convergence of characteristic functions are available. In [4], the author has used them to obtain some convergence rates in the Levy–Itô–Kawada theorem. Section 3 collects a number of lemmas of a different nature. Some of them provide Cramér type expansions of the characteristic function of Q_n. Others deal with truncation and choice of proper kernels. Section 4 contains the main results of this article. Theorems 4.1 and 4.2 improve previous results of [2], [3] by relaxing the assumption of finiteness of moments of order $3 + \delta$ for some positive δ to finiteness of third moments. Theorem 4.3 gives an asymptotic expansion under Cramér's condition (3.31), for an arbitrary bounded, almost surely (Φ) continuous function. It shows that even for functions which are not very smooth the error of approximation is of the order $o(n^{-(s-2)/2})$ if sth moments are finite, s being an integer not smaller than 3. Theorems 4.4 and 4.5 on asymptotic expansions when X_1 has a nonzero absolutely continuous component and when it has a lattice distribution, respectively, are stated for the sake of completeness. In their present form they are due to Bikjalis [6], [7]. However, Ranga Rao [29] was the first to show that an expansion like (4.113) holds uniformly over all Borel sets in the lattice case. Ranga Rao's estimate of the remainder involved a logarithmic factor, which has been removed in [7]. Theorem 4.6 gives a "distribution free" asymptotic expansion for a special class of functions. Theorem 4.7 deals with the case of summands with independent coordinates. The multidimensional extension of one dimensional results is particularly simple here, and one gets a good hold on the constants involved in the error bounds. Theorems 4.8 and 4.9 due, respectively, to Heyde [16] and Ibragimov [17], are concerned with exact rates of convergence.

2. Two lemmas for separable metric groups

In this section, G denotes a separable metric group, unless otherwise specified. We write the group operation as $+$ (addition) and the identity (or zero) element as "0". Let ρ be a left invariant metric on G, that is,

$$(2.1) \qquad\qquad \rho(u + x, u + y) = \rho(x, y), \qquad\qquad u, x, y \in G.$$

That there always exists such a metric defining the topology of G is well known (see [15], Theorem (8.3)). As is usual, μ^+, μ^-, and $|\mu|$ will denote, respectively, the positive, negative, and total variations of a finite signed measure μ. Also, $\|\mu\|$ denotes the variation norm of μ: $\|\mu\| = |\mu|(G)$. All measures or signed measures mentioned here are defined over the Borel σ-field $\mathscr{B}$ generated by the open sets of G. The convolution $\mu * \nu$ of finite signed measures μ, ν, is defined by

$$(2.2) \qquad\qquad \mu * \nu(B) = \int_G \mu(B - x)\, d\nu(x), \qquad\qquad B \in \mathscr{B}.$$

The n-fold convolution of a finite signed measure μ with itself is written as μ^{*n}. Let f be a real valued function on G. Define

$$S(x, \varepsilon) = \{y; \rho(x, y) < \varepsilon\}, \qquad\qquad x \in G,$$

$$w_f(A) = \sup \{|f(x) - f(y)|; x, y \in A\}, \qquad\qquad A \subset G,$$

$$(2.3)$$

$$f^{s, \varepsilon}(x) = \sup \{f(y); y \in S(x, \varepsilon)\}; f^{i, \varepsilon}(x) = \inf \{f(y); y \in S(x, \varepsilon)\}$$

$$w_f(x; \varepsilon) = f^{s, \varepsilon}(x) - f^{i, \varepsilon}(x) = w_f(S(x, \varepsilon)).$$

One can show that $f^{s, \varepsilon}$ is lower semicontinuous and $f^{i, \varepsilon}$ is upper semicontinuous if f is bounded (see [3], relation 2.18).

LEMMA 2.1. *Let μ be a finite measure, ν a finite signed measure, and K_ε a probability measure with all its mass in the sphere $S(0, \varepsilon)$. Let f be a real valued, bounded, Borel measurable function on G. Define*

$$\gamma(\varepsilon) = \max \left\{ \int_G f^{s, \varepsilon}\, d(\mu - \nu) * K_\varepsilon, \, -\int_G f^{i, \varepsilon}\, d(\mu - \nu) * K_\varepsilon \right\},$$

$$(2.4)$$

$$\tau(\varepsilon) = \max \left\{ \int_G (f^{s, 2\varepsilon} - f)\, d\nu^+, \int_G (f - f^{i, 2\varepsilon})\, d\nu^+ \right\}.$$

Then for all positive ε,

$$(2.5) \qquad\qquad \left| \int_G f\, d(\mu - \nu) \right| \leqq \gamma(\varepsilon) + \tau(\varepsilon).$$

PROOF.   We have

$$(2.6) \quad \gamma(\varepsilon) \geqq \int_G f^{s,\varepsilon} \, d(\mu - \nu) * K_\varepsilon$$

$$= \int_{S(0,\varepsilon)} \left[\int_G f^{s,\varepsilon}(y + x) \, d(\mu - \nu)(y) \right] dK_\varepsilon(x)$$

$$= \int_{S(0,\varepsilon)} \left[\int_G f^{s,\varepsilon}(y + x) \, d\mu(y) - \int_G f(y) \, d\nu(y) \right.$$

$$\left. - \int_G \left(f^{s,\varepsilon}(y + x) - f(y) \right) d\nu(y) \right] dK_\varepsilon(x)$$

$$\geqq \int_{S(0,\varepsilon)} \left[\int_G f(y) \, d\mu(y) - \int_G f(y) \, d\nu(y) \right.$$

$$\left. - \int_G \left(f^{s,\varepsilon}(y + x) - f(y) \right) d\nu^+(y) \right] dK_\varepsilon(x)$$

$$\geqq \int_G f d(\mu - \nu) - \int_G (f^{s,2\varepsilon} - f) \, d\nu^+ \geqq \int_G f d(\mu - \nu) - \tau(\varepsilon).$$

Similarly,

$$(2.7) \quad - \gamma(\varepsilon) \leqq \int_G f^{i,\varepsilon} \, d(\mu - \nu) * K_\varepsilon$$

$$= \int_{S(0,\varepsilon)} \left[\int_G f^{i,\varepsilon}(y + x) \, d\mu(y) \right.$$

$$\left. - \int_G f(y) \, d\nu(y) + \int_G \left(f(y) - f^{i,\varepsilon}(y + x) \right) d\nu(y) \right] dK_\varepsilon(x)$$

$$\leqq \int_G f d(\mu - \nu) + \int_G (f - f^{i,2\varepsilon}) \, d\nu^+ \leqq \int_G f d(\mu - \nu) + \tau(\varepsilon).$$

If $\int_G f d(\mu - \nu)$ is positive, then (2.5) follows from (2.6). Otherwise, it follows from (2.7). *Q.E.D.*

COROLLARY 2.1.   *Under the hypothesis of Lemma 2.1, one has*

$$(2.8) \quad \left| \int_G f d(\mu - \nu) \right| \leqq \left| \int_G f d(\mu - \nu) * K_\varepsilon \right| + \int_G w_f(\cdot\,; \varepsilon) \, d|(\mu - \nu) * K_\varepsilon|$$

$$+ \int_G w_f(\cdot\,; 2\varepsilon) \, d|\nu|.$$

If, in addition,

$$(2.9) \qquad\qquad (\mu - \nu) * K_\varepsilon(G) = 0,$$

which will be true in all our applications, then

$$(2.10) \quad \left| \int_G f d(\mu - \nu) \right| \leqq w_f(G) \| (\mu - \nu) * K_\varepsilon \| + \int_G w_f(\cdot\,; 2\varepsilon) \, d|\nu|.$$

The corollary follows easily from Lemma 2.1 and definitions of $\gamma(\varepsilon)$ and $\tau(\varepsilon)$. We choose the kernel probability measure K'_ε for the next lemma to satisfy

$$(2.11) \qquad\qquad \alpha \equiv \int_{S(0,\,\varepsilon)} dK'_\varepsilon > \frac{1}{2}.$$

For a real valued function f on G, we define the *translate* f_u of f by

$$(2.12) \qquad\qquad f_u(x) = f(x + u), \qquad\qquad u \in G.$$

LEMMA 2.2. *Let μ be a finite measure, v be a finite signed measure, and K'_ε be a probability measure satisfying* (2.11). *Let f be a real valued, bounded, Borel measurable function on G. Define*

$$(2.13)
\begin{aligned}
\gamma_1(\varepsilon) &= \sup_{u \in G} \max \left\{ \int_G (f_u)^{s,\varepsilon}\, d(\mu - v) * K'_\varepsilon, \; -\int_G (f_u)^{i,\varepsilon}\, d(\mu - v) * K'_\varepsilon \right\}, \\
\tau_1(\varepsilon) &= \sup_{u,\,u \in G} \max \left\{ \int_G \left(((f_u)^{s,2\varepsilon} - f_u)_{u'}\right) dv^+, \int_G \left((f_u - (f_u)^{i,2\varepsilon})_{u'}\right) dv^+ \right\}.
\end{aligned}$$

Then one has

$$(2.14) \qquad \delta \equiv \sup_{u \in G} \left| \int_G f_u\, d(\mu - v) \right| \leq (2\alpha - 1)^{-1} [\gamma_1(\varepsilon) + \tau_1(\varepsilon)],$$

where α is defined by (2.11).

PROOF. Suppose η is a positive number such that $\delta - \eta - \tau_1(\varepsilon)$ is non-negative (if this is not possible, then $\delta \leq \tau_1(\varepsilon)$ and (2.14) surely holds). Now either (a) $\sup_{u \in G} \{ \int_G f_u\, d(\mu - v) \} = \delta$, or (b) $\sup_{u \in G} \{ -\int_G f_u\, d(\mu - v) \} = \delta$. If (a) holds, choose u_0 such that $\int_G f_{u_0}\, d(\mu - v) > \delta - \eta$. Then

$$(2.15) \quad \int_G (f_{u_0})^{s,\varepsilon}\, d(\mu - v) * K'_\varepsilon$$

$$
\begin{aligned}
&= \int_{S(0,\,\varepsilon)} \left[\int_G (f_{u_0})^{s,\varepsilon}(y + x)\, d(\mu - v)(y)\, dK'_\varepsilon(x) \right] \\
&\quad + \int_{G-S(0,\,\varepsilon)} \left[\int_G (f_{u_0})^{s,\varepsilon}(y + x)\, d(\mu - v)(y) \right] dK'_\varepsilon(x) \\
&\geq \int_{S(0,\,\varepsilon)} \left[\int_G f_{u_0}(y)\, d\mu(y) - \int_G f_{u_0}(y)\, dv(y) \right. \\
&\qquad\qquad \left. - \int_G \left((f_{u_0})^{s,\varepsilon}(y + x) - f_{u_0}(y)\right) dv(y) \right] dK'_\varepsilon(x) \\
&\quad + \int_{G-S(0,\,\varepsilon)} \left[\int_G f_{u_0}(y + x)\, d\mu(y) - \int_G f_{u_0}(y + x)\, dv(y) \right. \\
&\qquad\qquad \left. - \int_G \left((f_{u_0})^{s,\varepsilon}(y + x) - f_{u_0}(y + x)\right) dv(y) \right] dK'_\varepsilon(x)
\end{aligned}$$

$$\geqq \int_{S(0,\,\varepsilon)} \left[\delta - \eta - \int_G \left((f_{u_0})^{s,\,\varepsilon}(y + x) - f_{u_0}(y) \right) dv^+(y) \right] dK'_\varepsilon(x)$$

$$+ \int_{G-S(0,\,\varepsilon)} \left[-\delta - \int_G \left((f_{u_0})^{s,\,\varepsilon}(y + x) - f_{u_0}(y + x) \right) dv^+(y) \right] dK'_\varepsilon(x)$$

$$\geqq \int_{S(0,\,\varepsilon)} \left[\delta - \eta - \int_G \left((f_{u_0})^{s,\,2\varepsilon}(y) - f_{u_0}(y) \right) dv^+(y) \right] dK'_\varepsilon(x)$$

$$+ \int_{G-S(0,\,\varepsilon)} [-\delta - \tau_1(\varepsilon)] dK'_\varepsilon(x)$$

$$\geqq [\delta - \eta - \tau_1(\varepsilon)]\alpha + [-\delta - \tau_1(\varepsilon)](1 - \alpha)$$

$$= (2\alpha - 1)\delta - \tau_1(\varepsilon).$$

Since η may be taken arbitrarily small, one has

$$(2.16) \qquad \gamma_1(\varepsilon) \geqq \int_G (f_{u_0})^{s,\,\varepsilon} \, d(u - v) * K'_\varepsilon \geqq (2\alpha - 1)\delta - \tau_1(\varepsilon),$$

from which (2.14) follows. If (b) holds, choose u_0 such that $-\int_G f_{u_0} \, d(\mu - v)$ is larger than $\delta - \eta$, then by looking at $-f_{u_0}$ (instead of f_{u_0}) and noting that

$$(2.17) \qquad (-f_u)^{s,\,\varepsilon} = -(f_u)^{i,\,\varepsilon},$$

$$\int_G \left(f_u - (f_u)^{i,\,2\varepsilon} \right) dv^+ = \int_G \left((-f_u)^{s,\,2\varepsilon} - (-f_u) \right) dv^+,$$

one obtains, exactly as in (2.15),

$$(2.18) \qquad \gamma_1(\varepsilon) \geqq -\int_G (f_{u_0})^{i,\,\varepsilon} \, d(\mu - v) * K'_\varepsilon \geqq (2\alpha - 1)\delta - \tau_1(\varepsilon).$$

Again, (2.14) follows. *Q.E.D.*

REMARK. If the group G in Lemma 2.2 is abelian, one may replace $\tau_1(\varepsilon)$ by

$$(2.19) \qquad \tau'_1(\varepsilon) = \sup_{u \in G} \max \left\{ \int_G \left((f_u)^{s,\,2\varepsilon} - f_u \right) dv^+, \int_G \left(f_u - (f_u)^{i,\,2\varepsilon} \right) dv^+ \right\}.$$

COROLLARY 2.2.   *If G is abelian, then under the hypothesis of Lemma 2.2*

$$(2.20) \qquad \left| \int_G f \, d(\mu - v) \right| \leqq (2\alpha - 1)^{-1} \sup_{u \in G} \left[\left[\int_G f_u \, d(\mu - v) * K'_\varepsilon \right. \right.$$

$$\left. \left. + \int_G w_{f_u}(\cdot\,;\varepsilon) \, d|(\mu - v) * K'_\varepsilon| \right. \right] + \sup_{u \in G} \int_G w_{f_u}(\cdot\,;2\varepsilon) \, dv^+ \right].$$

If, in addition, $(\mu - v) * K'_\varepsilon(G) = 0$, then

$$(2.21) \qquad \left| \int_G f \, d(\mu - v) \right|$$

$$\leqq (2\alpha - 1)^{-1} w_f(G) \| (\mu - v) * K'_\varepsilon \| + \sup_{u \in G} \int_G w_{f_u}(\cdot\,;2\varepsilon) \, dv^+$$

For $G = R^k$, Lemmas 2.1 and 2.2 were proved by the author in [2], [3]. In Section 3, we shall make use of the inequalities (2.10) and (2.21). Although adequate for our purposes, these inequalities appear somewhat wasteful because of the presence of variation norms. For example, with $G = R^k$ and the indicator function of a measurable convex set as f, one may easily obtain the following inequalities from Lemmas 2.1 and 2.2:

$$
\begin{aligned}
(2.22) \quad & \sup_{C \in \mathscr{C}} |(\mu - \nu)(C)| \leqq \sup_{C \in \mathscr{C}} |(\mu - \nu) * K_\varepsilon(C)| + \sup_{C \in \mathscr{C}} \nu^+((\partial C)^\varepsilon), \\
& \sup_{C \in \mathscr{C}} |(\mu - \nu)(C)| \leqq (2\alpha - 1)^{-1}[\sup_{C \in \mathscr{C}} |(\mu - \nu) * K'_\varepsilon(C)| + \sup_{C \in \mathscr{C}} \nu^+((\partial C)^\varepsilon)],
\end{aligned}
$$

where $\mathscr{C}$ is the class of all measurable convex sets, ∂C is the boundary of the set C, and

$$
(2.23) \qquad A^\varepsilon = \{x; \rho(x, A) < \varepsilon\}, \qquad\qquad A \subset R^k,
$$

where ρ denotes Euclidean distance.

3. Some lemmas on characteristic functions

The random vectors introduced below are all defined on a probability space $(\Omega, \mathscr{B}, P)$, P being a probability measure on the σ-field $\mathscr{B}$ of subsets of the set Ω. For a random variable X defined on this space, $\int X$ will denote EX the expectation of X. Let $\{X_n = (X_{n,1}, \cdots, X_{n,k})\}$ be a sequence of independent and identically distributed random vectors with values in R^k satisfying

$$
(3.1) \qquad EX_1 \equiv (EX_{1,1}, \cdots, EX_{1,k}) = (0, \cdots, 0), \qquad \mathrm{Cov}\, X_1 = I,
$$

where $\mathrm{Cov}\, X_1$ is the covariance matrix of X_1, and I is the $k \times k$ identity matrix. For x, y in R^k, (x, y) denotes the usual inner product between x and y, $|x| = (x, x)^{1/2}$. For positive integers s, define

$$
(3.2) \qquad \beta_{s,i} = E|X_{1,i}|^s, \qquad \beta_s = \sum_{i=1}^k \beta_{s,i}, \qquad \rho_s = E|X_1|^s.
$$

If ρ_s is finite for a positive integer s, then define

$$
(3.3) \qquad \lambda_s(u) = i^{-s} \frac{\partial^s}{\partial t^s} [\log E(\exp\{it(X_1, u)\})]\big|_{t=0}, \qquad u \in R^k,
$$

where $\log$ is the principal branch of the logarithm. Thus, $\lambda_s(u)$ is the sth cumulant of (X_1, u). The definition goes over when u is a k-tuple of complex numbers. Suppose now that ρ_s is finite for some positive integer s not smaller than three. Let $P_j(u)$, $j = 0, 1, \cdots, s - 2$, be polynomials in u (k-tuple of complex numbers) obtained by equating coefficients of $n^{-j/2}$ on both sides of the formal identity

$$
(3.4) \qquad \exp\left\{\sum_{j=3}^\infty n^{-(j-2)/2} \lambda_j(u)(j!)^{-1}\right\} = \sum_{j=0}^\infty n^{-j/2} P_j(u).
$$

In particular,

$$(3.5) \qquad P_0 \equiv 1, \qquad P_1(u) = \frac{\lambda_3(u)}{6}, \qquad P_2(u) = \frac{\lambda_4(u)}{24} + \frac{\lambda_3^2(u)}{72}.$$

Note that $\sum_{j=0}^{s-2} n^{-j/2} P_j(it) \exp\{-\frac{1}{2}|t|^2\}$ for $t \in R^k$ is an approximation of the characteristic function f_n of the normalized sum Y_n,

$$(3.6) \qquad \begin{aligned} Y_n &= n^{-1/2} \sum_{j=1}^{n} X_j, \\ f_n(t) &= E[\exp\{(it, Y_n)\}] = f_1^n(tn^{-1/2}), \end{aligned} \qquad t \in R^k.$$

This important idea as well as some of the estimates below are essentially due to Cramér ([9], Chapter 7).

CONVENTION. *In this section and the next the positive constants a, b, c, with or without subscripts or superscripts, depend only on the indicated arguments.*

LEMMA 3.1 (see [5]). *Suppose ρ_s is finite for some positive integer s, $s \geq 3$. Then for $|t| \leq n^{1/2}/[8\rho_s^{1/(s-2)}]$, one has*

$$(3.7) \qquad \left| f_n(t) - \exp\{-\tfrac{1}{2}|t|^2\} \left[1 + \sum_{j=1}^{s-3} n^{-j/2} P_j(it) \right] \right|$$
$$\leq c_1(k, s)\rho_s |t|^s \exp\{-\tfrac{1}{4}|t|^2\} n^{-(s-2)/2}.$$

LEMMA 3.2 (see [5]). *Suppose ρ_s is finite for some integer s, $s \geq 3$. Then one has, for $|t| \leq c_2(k)n^{1/2}/\beta_s^{1/(s-2)}$,*

$$(3.8) \qquad \left| \frac{\partial^r}{\partial t_m^r} \left\{ f_n(t) - \exp\left\{ -\frac{1}{2}|t|^2 \right\} \left[1 + \sum_{j=1}^{s-3} n^{-j/2} P_j(it) \right] \right\} \right|$$
$$\leq c_3(k, s)\beta_s (|t|^{s-r} + |t|^{2(s-3)}) \exp\{-\tfrac{1}{4}|t|^2\} n^{-(s-2)/2},$$

if $0 \leq r \leq s$, $1 \leq m \leq k$.

LEMMA 3.3 (see [5]). *If ρ_3 is finite, then for $|t| \leq n^{1/2}/(4\rho_3)$, one has*

$$(3.9) \qquad |f_n(t)| \leq \exp\{-\tfrac{1}{3}|t|^2\}.$$

LEMMA 3.4. *If ρ_r is finite for some positive integer r, then one has*

$$(3.10) \qquad \left| \frac{\partial^r}{\partial t_m^r} f_n(t) \right| \leq c_4(r)\beta_{r,m} n^{r/2} |f_1(tn^{-1/2})|^{n-r}, \qquad 1 \leq m \leq k.$$

PROOF. Using Leibniz' formula, $(\partial^r/\partial t_m^r)f_n(t)$ may be expressed as the sum of n^r terms each of magnitude not exceeding $n^{-r/2}\beta_{r,m} |f_1(tn^{-1/2})|^{n-r}$. Q.E.D.

LEMMA 3.5 (see [5]). *If ρ_{j+2} is finite for some positive integer j, then for $1 \leq m \leq k$, $0 \leq r \leq j$, one has*

$$(3.11) \qquad \left| \frac{\partial^r}{\partial t_m^r} P_j(it) \right| \leq c_5(k, j)\rho_{j+2}(1 + |t|^{3j}).$$

We shall truncate the random vectors X_j following Bikjalis [6]. Define

$$(3.12) \quad X_{j,i;n} = \begin{cases} X_{j,i} & \text{if } |X_j| \leq n^{1/2}, \\ 0 & \text{if } |X_j| > n^{1/2}, \end{cases} \quad 1 \leq i \leq k,$$

$$X_{j,n} = (X_{j,1;n}, \cdots, X_{j,k;n}).$$

Let $D = ((\sigma_{i,j}))$ be the covariance matrix of $X_{1;n}$. The same symbol will be used to denote a linear operator and its matrix relative to the standard Euclidean basis. Thus, D will also denote the linear operator whose matrix is D.

LEMMA 3.6. *Suppose ρ_s is finite for some integer s, $s \geq 3$. Let $v_1, v_2, \cdots, v_k$ be nonnegative integers and let $v = v_1 + v_2 + \cdots + v_k$. Then for $1 \leq i, i' \leq k$, one has*

$$(3.13) \quad \begin{aligned} \left| E(X_{1,1;n}^{v_1} \cdots X_{1,k;n}^{v_k}) - E(X_{1,1}^{v_1} \cdots X_{1,k}^{v_k}) \right| &\leq n^{-(s-v)/2} \rho_s k^{v/2} \quad \text{if } v \leq s, \\ E(X_{1,1;n}^{v_1} \cdots X_{1,k;n}^{v_k}) &= E(X_{1,1}^{v_1} \cdots X_{1,k}^{v_k}) \\ &\quad + o(n^{-(s-v)/2}) \quad \text{if } v \leq s; \end{aligned}$$

$$(3.14) \quad \begin{aligned} E\left| X_{1,1;n}^{v_1} \cdots X_{1,k;n}^{v_k} \right| &\leq n^{(v-s)/2} \rho_s k^{v/2} \quad \text{if } v > s, \\ E\left| X_{1,1;n}^{v_1} \cdots X_{1,k;n}^{v_k} \right| &= o(n^{(v-s)/2}) \quad \text{if } v > s; \end{aligned}$$

$$(3.15) \quad \begin{aligned} |E X_{1;n}| &\leq \rho_s n^{-(s-1)/2}, \\ |E X_{1;n}| &= o(n^{-(s-1)/2}); \end{aligned}$$

$$(3.16) \quad \begin{aligned} |\text{Var}(X_{1,i;n}) - 1| &\leq \rho_s n^{-(s-2)/2} + \rho_s^2 n^{-(s-1)}, \\ \text{Var}(X_{1,i;n}) - 1 &= o(n^{-(s-2)/2}); \end{aligned}$$

$$(3.17) \quad \begin{aligned} |\text{Cov}(X_{1,i;n}, X_{1,i';n})| &\leq 2\rho_s n^{-(s-2)/2} + \rho_s n^{-(s-1)}, \\ \text{Cov}(X_{1,i;n}, X_{1,i';n}) &= o(n^{-(s-2)/2}). \end{aligned}$$

Lemma 3.6 may be proved along the lines of Bikjalis ([6], Section 4).

Let λ be the smallest eigenvalue of D. Since D is self adjoint and nonnegative definite,

$$(3.18) \quad \begin{aligned} \lambda &= \left(\inf_{|x|=1} |Dx|^2 \right)^{1/2} = \left[\inf_{|x|=1} \sum_{i=1}^k \left(\sum_{j=1}^k \sigma_{i,j} x_j \right)^2 \right]^{1/2} \\ &= \left(\inf_{|x|=1} \sum_{i=1}^k \left[\sigma_{i,i}^2 x_i^2 + \left(\sum_{j \neq i} \sigma_{i,j} x_j \right)^2 + 2\sigma_{i,i} x_i \sum_{j \neq i} \sigma_{i,j} x_j \right] \right)^{1/2} \\ &\geq 1 - c_6(k)\beta_3 n^{-1/2}, \end{aligned}$$

if n is sufficiently large, that is, if

$$(3.19) \quad \beta_3 < c_7(k) n^{1/2}.$$

We shall henceforth always assume that (3.19) holds, even though this will not usually be mentioned. In obtaining the inequality (3.18), we have made use of

(3.16) and (3.17) from Lemma 3.6. The largest eigenvalue Λ of D likewise satisfies

$$(3.20) \qquad \Lambda \leqq 1 + c_8(k)\beta_3 n^{-1/2}.$$

Similarly, if ρ_s is finite, $s \geqq 3$, then, denoting by $|D|$ the determinant of D, one obtains

$$(3.21) \quad \begin{aligned} \lambda &= 1 - o(n^{-(s-2)/2}), & \Lambda &= 1 + o(n^{-(s-2)/2}), \\ |D| &\geqq \lambda^k \geqq 1 - c_9(k)\beta_3 n^{-1/2}, & |D| &= 1 - o(n^{-(s-2)/2}). \end{aligned}$$

Let $D = \Sigma_{i=1}^p \lambda_i E_i$ be the spectral decomposition of the operator D. The distinct eigenvalues of D are $\lambda_1, \lambda_2, \cdots, \lambda_p$; and $E_1, E_2, \cdots, E_p$ are the corresponding orthogonal projections. Let $T = \Sigma_{i=1}^p \lambda_i^{-1/2} E_i$. Then T is self adjoint and positive definite, and

$$(322) \quad \begin{aligned} T'T = T^2 &= D^{-1}, & \|T\| &\leqq 1 + c_{10}(k)\beta_3 n^{-1/2}, \\ \|T - I\| &\leqq c_{11}(k)\beta_3 n^{-1/2}, & \|T\| &= 1 + o(n^{-(s-2)/2}), \\ \|T - I\| &= o(n^{-(s-2)/2}). \end{aligned}$$

We now define for each n an i.i.d. sequence $\{Z_{r;n}\}$ by

$$(3.23) \qquad Z_{r;n} = T(X_{r;n} - EX_{r;n}), \qquad r = 1, 2, \cdots.$$

Note that

$$(3.24) \qquad EZ_{r;n} = (0, 0, \cdots, 0), \qquad \text{Cov } Z_{r;n} = I.$$

Define

$$(3.25) \qquad \beta_{v,i;n} = E|Z_{1,i;n}|^v, \qquad \beta_{v;n} = \sum_{i=1}^k \beta_{v,i;n}, \qquad \rho_{v;n} = E|Z_{1;n}|^v.$$

Also, by Lemma 3.6 and (3.22), if ρ_s is finite, $s \geqq 3$, then

$$(3.26) \quad \begin{aligned} \rho_{v;n} &\leqq c_{12}(k)\beta_s, & \rho_{v;n} &= \rho_v + o(n^{-(s-2)/2}) & \text{if} \quad v \leqq s, \\ \rho_{v;n} &\leqq c_{13}(k)n^{(v-s)/2}\beta_s, & \rho_{v;n} &= o(n^{(v-s)/2}) & \text{if} \quad v > s; \end{aligned}$$

$$(3.27) \quad \begin{aligned} E\big(Z_{1,1;n}^{v_1} \cdots Z_{1,k;n}^{v_k}\big) \\ = \begin{cases} E\big(X_{1,1}^{v_1} \cdots X_{1,k}^{v_k}\big) + o(n^{-(s-v)/2}) & \text{if} \quad v = v_1 + \cdots + v_k \leqq s, \\ o(n^{(v-s)/2}) & \text{if} \quad v = v_1 + \cdots + v_k > s. \end{cases} \end{aligned}$$

Let the polynomials $\{P_{j;n}; j = 1, 2, \cdots, s - 2\}$ correspond to $\{Z_{r;n}\}$ as the $\{P_j; j = 1, 2, \cdots, s - 2\}$ correspond to $\{X_r\}$. Noting that

$$(3.28) \qquad P_j(u) = \frac{1}{(j+2)!} \lambda_{j+2}(u) + \frac{1}{j} \sum_{i=1}^{j-1} \frac{(j-i)\lambda_{j+2-i}(u)P_i(u)}{(j+2-i)!}$$

one may prove the following lemma using (3.26) and induction (see [5], relation (10)).

LEMMA 3.7. *If ρ_s is finite for some integer s, $s \geqq 3$, then*

$$(3.29) \qquad |P_j(it) - P_{j;n}(it)| = o(n^{-(s-j-2)/2}), \qquad 1 \leqq j \leqq s - 2.$$

It should be noted that Lemmas 3.1 to 3.5 hold if f_n, β_s, ρ_s, P_j are replaced by f_n', $\beta_{s;n}$, $\rho_{s;n}$, and $P_{j;n}$, respectively. Here f_n' is the characteristic function of the normalized sum Z_n,

$$(3.30) \qquad Z_n = n^{-1/2} \sum_{r=1}^{n} Z_{r;n}, \qquad f_n'(t) = E(\exp\{it, Z_n\}) = (f_1'(tn^{-1/2}))^n, \; t \in R^k.$$

LEMMA 3.8. *If f_1 satisfies Cramér's condition*

$$(3.31) \qquad \limsup_{|t| \to \infty} |f_1(t)| < 1,$$

then there exists an integer n_0 such that

$$(3.32) \qquad \sup_{n > n_0} \limsup_{|t| \to \infty} |f_1'(t)| < 1.$$

PROOF. Note that

$$(3.33) \quad \begin{aligned} |f_1'(t)| &= |E(\exp\{(it, Z_{1;n})\})| \\ &= |E(\exp\{(it, TX_{1;n})\})| = |E(\exp\{(iTt, X_{1;n})\})| \\ &\leq |E(\exp\{(iTt, X_1)\})| + 2P(X_{1;n} \neq X_1) \leqq |f_1(Tt)| + 2\rho_3 n^{-1/2}. \end{aligned}$$

By (3.22), for a sufficiently large integer n_0, $n > n_0$ implies

$$(3.34) \qquad |Tt| > \tfrac{1}{2}|t|, \qquad 2\rho_3 n^{-1/2} < \tfrac{1}{2}\big(1 - \limsup_{|t| \to \infty} |f_1(t)|\big).$$

Q.E.D.

The next two lemmas are concerned with the choice of proper kernels.

LEMMA 3.9. *There exists a probability measure M on the Borel σ-field $\mathcal{B}^k$ of R^k which concentrates all its mass in $S(0, 1)$ and which has a characteristic function ζ satisfying*

$$(3.35) \qquad \left| \frac{\partial^r}{\partial t_m^r} \zeta(t) \right| \leqq c_{14}(k, s) \exp\{-|t|u(|t|)\}, \qquad |t| \geqq 1, \; t \in R^k,$$

for $0 \leqq r \leqq k + s$, $1 \leqq m \leqq k$, where s is a given nonnegative integer, and u is a nonnegative decreasing function on $[1, \infty)$ satisfying

$$(3.36) \qquad \int_1^{\infty} x^{-1} u(x)\, dx < \infty.$$

PROOF. Let $\{U_n\}$ be a sequence of independent random variables, U_n having uniform distribution in the interval $(-r_n, r_n)$, where the r_n are positive numbers such that

$$(3.37) \qquad \sum_{n=1}^{\infty} r_n \leqq (k + s + 1)^{-1} k^{-1/2},$$

and, further, such that the characteristic function ζ_1 of the distribution M_1 of $\sum_{n=1}^{\infty} U_n$ (the sum converges almost surely) satisfies

$$(3.38) \qquad |\zeta_1(t)| \leqq c_{15}(k, s) \exp \{- |t| u(|t|)\}, \qquad |t| \geqq 1, t \in R^1.$$

This is possible by a result of Ingham [20]. Let $M_2 = M_1^{*(k+s+1)}$. Then, clearly, M_2 and its characteristic function $\zeta_2 = \zeta_1^{(k+s+1)}$ satisfy

$$(3.39) \qquad M_2[(-k^{-1/2}, k^{-1/2})] = 1,$$
$$\left| \frac{\partial^r}{\partial t^r} \zeta_1^{(k+s+1)}(t) \right| \leqq c_{16}(k, s) \exp \{- |t| u(|t|)\}, \qquad |t| \geqq 1, t \in R^1,$$

for $0 \leqq r \leqq k + s$. Now take $M = M_2 \times M_2 \times \cdots \times M_2$, a k dimensional product probability measure on $(R^k, \mathscr{B}^k)$. Q.E.D.

REMARK. Ingham [20] actually proved that in order that there exists a probability measure M_1 with compact support (in R^1) whose characteristic function ζ_1 obeys (3.38), it is necessary as well as sufficient that (3.36) holds.

COROLLARY 3.1. *There exists a probability measure M in R^k which concentrates all its mass in $S(0, 1)$ and whose characteristic function ζ satisfies the inequality*

$$(3.40) \qquad \left| \frac{\partial^r}{\partial t_m^r} \zeta(t) \right| \leqq c_{17}(k, s) \exp \{- |t|^{1/2}\}, \qquad |t| \geqq 1,$$

for all r, $0 \leqq r \leqq k + s$.

LEMMA 3.10. *There exists a probability measure K' on the Borel σ-field $\mathscr{B}^k$ of R^k with a characteristic function ζ' such that*

$$(3.41) \qquad \int_{S(0,1)} dK' > \frac{3}{4}, \qquad \int_{R^k} |x|^{k+s} dK'(x) < \infty, \qquad \zeta'(t) = 0$$
$$if \quad |t| \geqq c_{18}(k, s),$$

where s is a given positive integer and $c_{18}(k, s)$ is a suitable positive constant.

PROOF. One can construct a probability measure obeying (3.41) in various ways. We give just one such construction. In R^1, let U_a be the uniform distribution in the interval $(-a, a)$, $a > 0$. The characteristic function p_a of U_a is given by

$$(3.42) \qquad p_a(t) = (at)^{-1} \sin at, \qquad t \in R^1.$$

The characteristic function $p_a^{2(k+s)}$ of $U_a^{*2(k+s)}$ is nonnegative real valued and integrable. Let the constant $c'_{18}(a, k, s)$ be chosen so that

$$(3.43) \qquad \int_{-\infty}^{+\infty} c'_{18}(a, k, s) p_a^{2(k+s)}(x) dx = 1.$$

The probability measure N_1, whose density is given by the integrand in (3.43), has finite moments of all orders up to $2(k + s) - 2$, and has a characteristic function which is equal to a constant multiple of the density of $U_a^{*2(k+s)}$, and therefore vanishes outside the interval $[-2a(k + s), 2a(k + s)]$. Let $N_2 = N_1 \times N_1 \times \cdots \times N_1$, a product probability measure in R^k. To get K' from N_2 one only has to choose a suitably. Q.E.D.

4. Main results

We continue to use the notation introduced earlier.

THEOREM 4.1. *If β_3 is finite then for every real valued, bounded, Borel measurable function g on R^k, one has*

$$(4.1) \qquad \left| \int_{R^k} g \, d(Q_n - \Phi) \right| \leqq a(k) w_g(R^k) \beta_3 n^{-1/2} + 2 \sup_{u \in R^k} \int_{R^k} w_{g_u}(\,\cdot\,; \varepsilon_n) \, d\Phi,$$

for all n such that $\beta_3 < a_1(k) n^{1/2} (\log n)^{-k}$, where

$$(4.2) \qquad \varepsilon_n = a_2(k) \beta_3 n^{-1/2},$$

and the positive constants $a(k)$, $a_1(k)$, $a_2(k)$ depend only on k.

PROOF. We shall apply Lemmas 3.1 to 3.5 throughout this proof to the sequence $\{Z_{r;n}\}$. Let Q'_n denote the distribution of $Z_n = \Sigma_{j=1}^n Z_{r;n} n^{-1/2}$. Let K'_ε be the distribution of εX, where X has distribution K' of Lemma 3.10 with $s = 2$, and K''_ε that of $\varepsilon T X$. The characteristic function ζ''_ε of K''_ε satisfies

$$(4.3) \qquad \zeta''_\varepsilon(t) = \zeta'(\varepsilon T t) = 0$$

if $|t| \geqq a_3(k)/\varepsilon$. We first show that, for a suitable choice of ε,

$$(4.4) \qquad \| (Q'_n - \Phi) * K''_\varepsilon \| \leqq a'(k) \beta_3 n^{-1/2}$$

if $\beta_3 < a_1(k) n^{1/2} (\log n)^{-k}$. Let h_n denote the density, and ξ_n the Fourier–Stieljes transform of $(Q'_n - \Phi) * K''_\varepsilon$. Then as in Bikjalis [6],

$$(4.5) \quad \| Q'_n - \Phi) * K''_\varepsilon \|$$

$$= \int_{R^k} |h_n(x)| \, dx$$

$$= \int_{R^k} \left(\prod_{m=1}^k (1 + x_m^{2(k+2)})^{1/2k} \right) |h_n(x)| \prod_{m=1}^k (1 + x_m^{2(k+2)})^{-1/2k} \, dx$$

$$\leqq \left[\int_{R^k} \prod_{m=1}^k (1 + x_m^{2(k+2)})^{-1/k} \, dx \right]^{1/2} \left[\int_{R^k} \left(\prod_{m=1}^k (1 + x_m^{2(k+2)})^{1/k} \right) h_n^2(x) \, dx \right]^{1/2}$$

$$\leqq a_4(k) \prod_{m=1}^k \left[\int_{R^k} (1 + x_m^{2(k+2)}) h_n^2(x) \, dx \right]^{1/2k}$$

$$\leqq a_4(k) \prod_{m=1}^k (I + I_m)^{1/2k},$$

where

$$(4.6) \qquad I = \int_{R^k} h_n^2(x) \, dx, \qquad I_m = \int_{R^k} x_m^{2(k+2)} h_n^2(x) \, dx.$$

Let us write

$$(4.7) \qquad \gamma_n(t) = \sum_{j=0}^{k-1} n^{-j/2} P_{j;n}(it) \exp \left\{ -\tfrac{1}{2} |t|^2 \right\}.$$

By the Plancherel theorem, and noting that $i^{-(k+2)}(\partial^{k+2}/\partial t_m^{k+2})\xi_n(t)$ is the Fourier transform of $x_m^{k+2}h_n(x)$, one has

$$(4.8) \quad I = (2\pi)^{-k}\int_{R^k}|\xi_n(t)|^2\,dt, \qquad I_m = (2\pi)^{-k}\int_{R^k}\left|\frac{\partial^{k+2}}{\partial t_m^{k+2}}\xi_n(t)\right|^2\,dt.$$

Now let

$$(4.9) \qquad \varepsilon = a_5(k)\beta_3 n^{-1/2}, \qquad a_5(k) \geqq \frac{a_3(k)8\rho_{3;n}}{\beta_3}.$$

Using Lemma 3.1 with $s = 3$, one then has

$$(4.10) \qquad I \leqq \frac{a_6(k)\rho_{3;n}^2}{n} \leqq \frac{a_7(k)\beta_3^2}{n}.$$

The last inequality of (4.10) follows from (3.26) and (3.27). Also, $I_m \leqq I_{m,1} + I_{m,2}$, where

$$(4.11) \begin{aligned} I_{m,1} &= 2(2\pi)^{-k}\int\left|\frac{\partial^{k+2}}{\partial t_m^{k+2}}\{(f_n'(t) - \gamma_n(t))\zeta'(\varepsilon Tt)\}\right|^2\,dt, \\ I_{m,2} &= 2(2\pi)^{-k}\int\left|\frac{\partial^{k+2}}{\partial t_m^{k+2}}\{(\gamma_n(t) - \exp\{-\tfrac{1}{2}|t|^2\})\zeta'(\varepsilon Tt)\}\right|^2\,dt, \end{aligned}$$

where both integrals are taken over the set $\{|t| \leqq n^{-1/2}/(4\rho_{3,n})\}$ so that

$$(4.12) \qquad I_{m,1} \leqq a_8(k)\left(\int_{B_1}\alpha_n(t)\,dt + \int_{B_2}\alpha_n(t)\,dt\right),$$

where

$$\alpha_n(t) = \left|\sum_{r=0}^{k+2}\binom{k+2}{r}\left[\frac{\partial^r}{\partial t_m^r}(f_n'(t) - \gamma_n(t))\right]\frac{\partial^{k+2-r}}{\partial t_m^{k+2-r}}\zeta'(\varepsilon Tt)\right|^2,$$

$$(4.13) \quad B_1 = \{|t| \leqq c_2(k)\beta_{k+2;n}^{-1/k}n^{1/2}\},$$

$$B_2 = \left\{c_2(k)\beta_{k+2;n}^{-1/k}n^{1/2} < |t| \leqq \frac{n^{1/2}}{4\rho_{3;n}}\right\}.$$

Noting that

$$(4.14) \qquad \left|\frac{\partial^{k+2-r}}{\partial t_m^{k+2-r}}\zeta'(\varepsilon Tt)\right| \leqq a_9(k), \qquad 0 \leqq r \leqq k+2,$$

it follows, by Lemma 3.2 with $s = k + 2$, that

$$(4.15) \qquad \int_{B_1}\alpha_n(t)\,dt \leqq a_{10}(k)\rho_{k+2;n}^2 n^{-k} \leqq \frac{a_{11}(k)\beta_3^2}{n}.$$

The last inequality is a consequence of (3.26) and (3.27). To evaluate $\int_{B_2}\alpha_n(t)\,dt$ note that, by (3.26) and (3.27),

$$(4.16) \qquad c_2(k)\beta_{k+2;n}^{-1/k}n^{1/2} \geqq a_{12}(k)\beta_3^{-1/k}n^{1/2k}.$$

Hence, by Lemmas 3.3 and 3.4, and letting $B_3 = \{|t| \geq a_{12}(k)\beta_3^{-1/k}n^{1/2k}\}$,

$$(4.17) \quad \int_{B_2} \left| \frac{\partial^r}{\partial t_m^r} f_n'(t) \right|^2 dt \leq c_4^2(r)\rho_{r;n}^2 n^r \int_{B_2} |f_1'(tn^{-1/2})|^{2(n-r)} dt$$

$$\leq a_{13}(k)\beta_3^2 n^{2r} \int_{B_3} \exp\left\{-\frac{2}{3}\frac{(n-r)|t|^2}{n}\right\} dt$$

$$\leq \frac{a_{14}(k)\beta_3^2}{n \log n},$$

if $\beta_3 < a_1(k)n^{1/2}/(\log n)^k$. Also, by Lemma 3.5, if $\beta_3 < a_1(k)n^{1/2} (\log n)^{-k}$, then

$$(4.18) \quad \sum_{r=0}^{k+2} \int_{B_2} \left| \frac{\partial^r}{\partial t_m^r} (\gamma_n(t) - \exp\{-\tfrac{1}{2}|t|^2\}) \right|^2 dt + \int_{B_2} \left| \frac{\partial^r}{\partial t_m^r} \exp\{-\tfrac{1}{2}|t|^2\} \right|^2 dt$$

$$\leq \sum_{j=1}^{k-1} a_{15}(k,j)n^{-j}\rho_{j+2;n}^2 + \frac{a_{15}'(k)\beta_3^2}{n}$$

$$\leq \frac{a_{16}(k)\beta_3^2}{n}.$$

From (4.14), (4.17), and (4.18), one gets

$$(4.19) \quad \int_{B_2} \alpha_n(t) \, dt \leq \frac{a_{17}(k)\beta_3^2}{n}.$$

The inequalities (4.15) and (4.19) yield

$$(4.20) \quad I_{m,1} \leq \frac{a_{18}(k)\beta_3^2}{n}.$$

The estimation of $I_{m,2}$ is simpler. In fact, using Lemma 3.5 as in obtaining (4.18), one gets

$$(4.21) \quad I_{m,2} \leq \frac{a_{19}(k)\beta_3^2}{n}.$$

Combining (4.20) and (4.21), we obtain

$$(4.22) \quad I_m \leq \frac{a_{20}(k)\beta_3^2}{n}.$$

Finally, (4.10) and (4.22) lead to (4.4). We shall now show that

$$(4.23) \quad \|(Q_n - \Phi) * K_\varepsilon'\| \leq a''(k)\beta_3 n^{-1/2},$$

if $\beta_3 < a_1(k)n^{1/2} (\log n)^{-k}$. Let $Q_{n,1}$ and $Q_{n,2}$ denote the distributions of $n^{-1/2} \sum_{r=1}^n X_{r,n}$, and $n^{-1/2} \sum_{r=1}^n (X_{r;n} - EX_{r;n})$, respectively. Also, Φ_1 will denote the normal distribution with mean vector $-n^{-1/2} \sum_{r=1}^n EX_{r;n}$ and covariance matrix I, while Φ_2 stands for a normal distribution with mean vector $-n^{-1/2} \sum_{r=1}^n T(EX_{r;n})$ and covariance matrix D^{-1}. Now

$$(4.24) \quad \|(Q_n - \Phi) * K'_\varepsilon - (Q_{n,1} - \Phi) * K'_\varepsilon\| \leq \|Q_n - Q_{n,1}\| \leq 2nP(|X_1| > n^{1/2})$$
$$\leq 2\rho_3 n^{-1/2} \leq 2k^{1/2}\beta_3 n^{-1/2}.$$

Next,

$$(4.25) \quad \|(Q_{n,1} - \Phi) * K'_\varepsilon\|$$

$$= 2 \sup_{B \in \mathscr{B}^k} \int_{R^k} (Q_{n,1}(B - x) - \Phi(B - x)) dK'_\varepsilon(x)$$

$$= 2 \sup_{B \in \mathscr{B}^k} \int_{R^k} \left(Q_{n,2}\left(B - n^{-1/2} \sum_{r=1}^n EX_{r;n} - x\right) \right.$$

$$\left. - \Phi_1\left(B - n^{-1/2} \sum_{r=1}^n EX_{r;n} - x\right) \right) dK'_\varepsilon(x)$$

$$= 2 \sup_{B \in \mathscr{B}^k} \int_{R^k} (Q_{n,2}(B - x) - \Phi_1(B - x)) dK'_\varepsilon(x)$$

$$= \|(Q_{n,2} - \Phi_1) * K'_\varepsilon\|.$$

Also, letting W_1, W_2, εX be independent random vectors with respective distributions $Q_{n,2}$, Φ_1, K'_ε,

$$(4.26) \quad \|(Q_{n,2} - \Phi_1) * K'_\varepsilon\|$$

$$= 2 \sup_{B \in \mathscr{B}^k} [P(W_1 + \varepsilon X \in B) - P(W_2 + \varepsilon X \in B)]$$

$$= 2 \sup_{B \in \mathscr{B}^k} [P(TW_1 + \varepsilon TX \in TB) - P(TW_2 + \varepsilon TX \in TB)]$$

$$= \|(Q'_n - \Phi_2) * K''_\varepsilon\|.$$

Finally,

$$(4.27) \quad \|(Q'_n - \Phi_2) * K''_\varepsilon - (Q'_n - \Phi) * K''_\varepsilon\| \leq \|\Phi_2 - \Phi\| \leq a_{21}(k)\beta_3 n^{-1/2}.$$

The last inequality is obtained in a straightforward manner using (3.22) and the fact, which follows from (3.15) in Lemma 3.6, that

$$(4.28) \quad n^{-1/2}\left|\sum_{r=1}^n EX_{r;n}\right| \leq n^{1/2}|EX_{r;n}| \leq \rho_3 n^{-1/2}.$$

The desired inequality (4.23) now follows from (4.4), and (4.24) through (4.27).

An immediate application of Corollary 2.2 with $\mu = Q_n$ and $\nu = \Phi$, and the kernel K'_ε as used here, completes the proof of the theorem. Q.E.D.

APPLICATION 4.1. In the space $\mathscr{P}$ of all probability measures on the Borel σ-field of R^k, define the distance d_0 by

$$(4.29) \quad d_0(P, Q) = \sup_{C \in \mathscr{C}} |P(C) - Q(C)|,$$

$\mathscr{C}$ being the class of all Borel measurable convex sets of R^k. If β_3 is finite, then

$$(4.30) \quad d_0(Q_n, \Phi) \leq a_{22}(k)\beta_3 n^{-1/2},$$

if $\beta_3 < a_1(k)n^{1/2}/(\log n)^k$. This follows from Theorem 4.1 by taking the indicator function I_C of an arbitrary Borel measurable convex set C for g, and by noting that the class $\mathscr{C}$ is translation invariant, that is, $(C - x)$ belongs to $\mathscr{C}$ for all x in R^k and for all C in $\mathscr{C}$, and that

$$(4.31) \qquad \sup_{C \in \mathscr{C}} \int_{R^k} w_{I_C}(\cdot\,; \varepsilon)\, d\Phi = \sup_{C \in \mathscr{C}} \Phi\big((\partial C)^\varepsilon\big) \leqq d(k)\varepsilon, \qquad \varepsilon > 0,$$

where ∂C is the boundary of C and $(\partial C)^\varepsilon$ is the ε neighborhood of ∂C as defined by (2.23). The constant $d(k)$ depends only on k. The inequality (4.31) was first obtained by Ranga Rao [29]. Later it was independently obtained by von Bahr [34] and Sazonov [32]. The last named author has shown

$$(4.32) \qquad d_0(Q_n, \Phi) \leqq ck^4\beta_3 n^{-1/2},$$

where c is an absolute constant. The usefulness of the metric d_0 derives from the richness of the class $\mathscr{C}$ which is large enough for many applications. It is also quite convenient to have a metric like d_0 which is *free of scale*; that is, if P and Q are two probability measures and L is an affine non singular linear transformation on R^k then d_0 satisfies

$$(4.33) \qquad d_0(P, Q) = d_0(P \circ L^{-1}, Q \circ L^{-1}).$$

Here $P \circ L^{-1}(B) = P(L^{-1}B)$, B being an arbitrary Borel set.

APPLICATION 4.2. Let $\mathscr{F}$ be the class of all real valued functions g on R^k satisfying

$$(4.34) \qquad w_g(R^k) \leqq 1, \qquad |g(x) - g(y)| \leqq |x - y|,$$

for all x, y in R^k. The distance d_1 on $\mathscr{P}$ defined by

$$(4.35) \qquad d_1(P, Q) = \sup_{g \in \mathscr{F}} \left| \int_{R^k} g\, d(P - Q) \right|,$$

is known to metrize the topology of weak convergence (see [11], Theorem 12). It is immediate from Theorem 4.1 that if β_3 is finite, then

$$(4.36) \qquad d_1(Q_n, \Phi) \leqq a_{23}(k)\beta_3 n^{-1/2}$$

if $\beta_3 < a_1(k)n^{1/2}/(\log n)^k$.

Several other applications are given in [2]. There are Borel measurable functions g for which

$$(4.37) \qquad \left| \int_{R^k} w_g(\cdot\,; \varepsilon)\, d\Phi \right| = O(\varepsilon), \quad \sup_{x \in R^k} \int_{R^k} w_{g_x}(\cdot\,; \varepsilon)\, d\Phi \nrightarrow 0 \qquad \text{as} \quad \varepsilon \downarrow 0.$$

For example, see [3]. Clearly, for such functions Theorem 4.1 is useless. The following theorem provides effective bounds in this situation.

THEOREM 4.2. *If β_3 is finite then for every real valued, bounded, Borel measurable function g on R^k, one has*

$$(4.38) \qquad \left| \int_{R^k} g \, d(Q_n - \Phi) \right| \leqq b(k) w_g(R^k) \beta_3 n^{-1/2} + \int_{R^k} w_g(\cdot \, ; \varepsilon_n) \, d\Phi$$

if $\beta_3 < b_1(k) n^{1/2} (\log n)^{-k}$, where

$$(4.39) \qquad \varepsilon_n = b_2(k) \beta_3 n^{-1/2} \log n,$$

and the positive constants $b(k)$, $b_1(k)$, $b_2(k)$ depend only on k.

PROOF. Let the probability measure M be as in the Corollary 3.1. Let M_ε be the distribution of $\varepsilon T Y$, where the random vector Y has distribution M. Let $K_{p,\varepsilon} = M_\varepsilon^{*p}$, p being a positive integer (depending on n) specified by (4.41) below. We continue to use the notation introduced in the course of proving Theorem 4.1. We first show that if β_3 is less than $b_1(k) n^{1/2} (\log n)^{-k}$, then

$$(4.40) \qquad \left\| (Q'_n - \Phi) * K_{p,\varepsilon} \right\| \leqq b'(k) \beta_3 n^{-1/2},$$

where, denoting by $[x]$ the smallest integer larger than x,

$$(4.41) \qquad p = [\log n], \qquad \varepsilon = b'_2(k) \rho_{3;n} n^{-1/2}.$$

The constant $b'_2(k)$ will be appropriately chosen in the sequel. Now define

$$(4.42) \qquad \xi'_n(t) = \left[f'_n(t) - \exp\left\{ -\tfrac{1}{2} |t|^2 \right\} \right] \zeta^p(\varepsilon T t),$$

where ζ is the characteristic function of M. As in (4.5),

$$(4.43) \qquad \left\| (Q'_n - \Phi) * K_{p,\varepsilon} \right\| \leqq b_3(k) \prod_{m=1}^{k} (J + J_m)^{1/2k},$$

where

$$(4.44) \qquad \begin{aligned} J &= (2\pi)^{-k} \int_{R^k} |\xi'_n(t)|^2 \, dt, \\ J_m &= (2\pi)^{-k} \int_{R^k} \left| \frac{\partial^{k+2}}{\partial t_m^{k+2}} \xi'_n(t) \right|^2 \, dt. \end{aligned}$$

Now

$$(4.45) \qquad J \leqq 3(2\pi)^{-k} (J'_1 + J'_2 + J'_3),$$

where

$$(4.46) \qquad \begin{aligned} J'_1 &= \int \{|t| \leqq n^{1/2}/(8\rho_{3;n})\} |f'_n(t) - \exp\left\{ -\tfrac{1}{2}|t|^2 \right\}|^2 \, dt, \\ J'_2 &= \int \{|t| \geqq n^{1/2}/(8\rho_{3;n})\} |f'_n(t) \zeta^p(\varepsilon T t)|^2 \, dt, \\ J'_3 &= \int \{|t| > n^{1/2}/(8\rho_{3;n})\} \exp\left\{ -|t|^2 \right\} \, dt. \end{aligned}$$

By Lemma 3.1, with $s = 3$,

$$(4.47) \qquad J'_1 \leqq \frac{b_4(k) \beta_3^2}{n}.$$

It is elementary to check that

$$(4.48) \qquad J_3' \leqq \frac{b_5(k)\beta_3^2}{n}.$$

By (3.22) and (3.40),

$$(4.49) \qquad J_2' \leqq \int \{|t| > n^{1/2}/(8\rho_{3;n})\}|\zeta(\varepsilon Tt)|^{2p} \, dt$$

$$\leqq b_6^{2p}(k) \int \{|t| > n^{1/2}/(8\rho_{3;n})\} \exp\{-2p|\varepsilon t|^{1/2}\} \, dt$$

$$\leqq b_6^{2p}(k)b_7(k) \int_{n^{1/2}/(8\rho_{3;n})}^{\infty} u^{k-1} \exp\{-2p\varepsilon^{1/2}u^{1/2}\} \, du$$

$$\leqq b_6^{2p}(k)b_8(k)c_n^{-2k+1} \exp\{-a_n\},$$

where

$$(4.50) \qquad c_n = 2p\varepsilon^{1/2}, \qquad a_n = 2p\varepsilon^{1/2}n^{1/4}(8\rho_{3;n})^{-1/2}.$$

It is clearly possible to choose $b_2'(k)$ such that

$$(4.51) \qquad J_2' \leqq \frac{b_9(k)\beta_3^2}{n}.$$

Combining (4.47), (4.48), and (4.51), we get

$$(4.52) \qquad J \leqq \frac{b_{10}(k)\beta_3^2}{n}.$$

To estimate J_m write

$$(4.53) \qquad J_m \leqq 2(2\pi)^{-k} \int_{R^k} \left| \frac{\partial^{k+2}}{\partial t_m^{k+2}} \left((f_n'(t) - \gamma_n(t))\zeta^p(\varepsilon Tt)\right) \right|^2 dt$$

$$+ \int_{R^k} \left| \frac{\partial^{k+2}}{\partial t_m^{k+2}} \left((\gamma_n(t) - \exp\{-\tfrac{1}{2}|t|^2\})\zeta^p(\varepsilon Tt)\right) \right|^2 dt.$$

Estimation of the integrals over the set $\{|t| \leqq n^{1/2}/(8\rho_{3;n})\}$ is exactly like that of I_m. The integrals need to be estimated, therefore, only over the set $B_3 = \{|t| > n^{1/2}/(8\rho_{3;n})\}$. Clearly,

$$(4.54) \qquad \int_{B_3} \left| \frac{\partial^{k+2}}{\partial t_m^{k+2}} \left((\gamma_n(t) - \exp\{-\tfrac{1}{2}|t|^2\})\zeta^p(\varepsilon Tt)\right) \right|^2 dt \leqq \frac{b_{11}(k)\beta_3^2}{n}.$$

Also,

$$(4.55) \qquad \int_{B_3} \left| \frac{\partial^{k+2}}{\partial t_m^{k+2}} \left((f_n'(t) - \gamma_n(t))\zeta^p(\varepsilon Tt)\right) \right|^2 dt \leqq 2(J_{m,1} + J_{m,2}),$$

where

$$(4.56) \qquad \begin{aligned} J_{m,1} &= \int_{B_3} \left| \frac{\partial^{k+2}}{\partial t_m^{k+2}} \left(f_n'(t) \zeta^p(\varepsilon Tt) \right) \right|^2 dt, \\ J_{m,2} &= \int_{B_3} \left| \frac{\partial^{k+2}}{\partial t_m^{k+2}} \left(\gamma_n(t) \zeta^p(\varepsilon Tt) \right) \right|^2 dt. \end{aligned}$$

By (3.22), (3.26), and (3.27),

$$(4.57) \qquad \begin{aligned} J_{m,1} &\leqq \sum_{r=0}^{k+2} b_{12}(r, k) \beta_{r,m;n}^2 \int_{B_3} \left| \frac{\partial^{k+2-r}}{\partial t_m^{k+2-r}} \zeta^p(\varepsilon Tt) \right|^2 dt \\ &\leqq \sum_{r=0}^{k+2} b_{13}(r, k) \beta_3^2 n^r \|T\|^{2(k+2-r)} \varepsilon^{2(k+2-r)} p^{2(k+2-r)}, \end{aligned}$$

$$(4.58) \qquad b_6^{2p}(k) \int_{B_3} \exp\left\{ -(p + r - k - 2)\varepsilon^{1/2}|t|^{1/2} \right\} dt \leqq b_{14}(k)\beta_3^2,$$

if the constant $b_2'(k)$ is suitably chosen. Note that there is no conflict between this choice and that made in getting (4.51). In both cases one has to take it sufficiently large. Estimation of $J_{m,2}$ is much simpler. In fact, $J_{m,2}$ is of the order of $\int_{B_3}(\partial^{k+2}/\partial t_m^{k+2}) \exp\left\{ -\frac{1}{2}|t|^2 \right\} dt$. Hence,

$$(4.59) \qquad J_{m,2} \leqq \frac{b_{15}(k)\beta_3^2}{n}.$$

It then follows that if β_3 is less than $b_1(k)n^{1/2}(\log n)^{-k}$, then

$$(4.60) \qquad J_m \leqq \frac{b_{16}(k)\beta_3^2}{n}.$$

By (4.52) and (4.60), one finally obtains the inequality (4.40). It now follows exactly as in the proof of Theorem 4.1 that if β_3 is less than $b_1(k)n^{1/2}(\log n)^{-k}$, then

$$(4.61) \qquad \|(Q_n - \Phi) * K_{p,\varepsilon}\| \leqq b(k)\beta_3 n^{-1/2}.$$

The proof of the theorem is now completed by applying Corollary 2.1 with $\mu = Q_n$, $\nu = \Phi$, replacing ε by $p\varepsilon$. Q.E.D.

APPLICATION 4.3.   The Prokhorov distance d_2 is defined in $\mathscr{P}$ by

$$(4.62) \quad d_2(P, Q) = \inf\{\varepsilon; \varepsilon > 0, Q(A) \leqq P(A^\varepsilon) + \varepsilon, P(A) \leqq Q(A^\varepsilon) + \varepsilon, A \in \mathscr{B}^k\}.$$

This distance metrizes the topology of weak convergence of probability measures on the Borel σ-field of a separable metric space (see, for example, [8], pp. 237–239). It may be shown (see [12], Proposition 1) that

$$(4.63) \qquad d_2(P, Q) = \inf\{\varepsilon; \varepsilon > 0, Q(A) \leqq P(A^\varepsilon) + \varepsilon, A \in \mathscr{B}^k\}.$$

Now letting $\mu = Q$, and $\nu = P$, in Lemma 2.1, one obtains the following inequality from the inequalities (2.6).

$$(4.64) \qquad \int_{R^k} f^{s,\,\varepsilon} d(Q - P) * K_\varepsilon \geq \int_{R^k} f d(Q - P) - \int_{R^k} (f^{s,\,2\varepsilon} - f) \, dP,$$

where P and Q are arbitrary probability measures on $\mathscr{B}^k$, and f is an arbitrary real valued, bounded, Borel measurable function on R^k. Specializing to indicator functions of Borel sets, one gets

$$(4.65) \qquad Q(A) - P(A) \leq (Q - P) * K_\varepsilon(A^\varepsilon) + P(A^{2\varepsilon} - A).$$

Now take $Q = Q_n$, $P = \Phi$, and replace ε by $p\varepsilon$, to get

$$(4.66) \qquad \begin{aligned} Q_n(A) &\leq \Phi(A) + \Phi(A^{2p\varepsilon} - A) + \| (Q_n - \Phi) * K_{p,\,\varepsilon} \| \\ &= \Phi(A^{2p,\,\varepsilon}) + \| (Q_n - \Phi) * K_{p,\,\varepsilon} \|, \end{aligned}$$

from which, using (4.61), and (4.63), we obtain

$$(4.67) \qquad d_2(Q_n, \Phi) \leq \max \{2p\varepsilon, \, b(k)\beta_3 n^{-1/2}\} \leq b_{17}(k)\beta_3 n^{-1/2} \log n,$$

for all n such that β_3 is less than $b_1(k) n^{1/2} (\log n)^{-k}$.

REMARKS. The author does not know if the factor $\log n$ in (4.39) may be removed. The method of computation of d_1 and d_2 as given here is valid in the more general context mentioned in the introductory section (see [4] in this connection).

We next turn to asymptotic expansions. Suppose ρ_{j+2} is finite for a positive integer j. Let $P_j(-\varphi)$ be the real valued function on R^k whose Fourier transform (evaluated at t) is $P_j(it) \exp\{-\frac{1}{2}|t|^2\}$. Since $\exp\{-\frac{1}{2}|t|^2\}$ is the value at t of the Fourier transform of the standard normal density which we denote by φ, $P_j(-\varphi)$ is obtained by formally replacing $(it_1)^{v_1}(it_2)^{v_2} \cdots (it_k)^{v_k}$ $\exp\{-\frac{1}{2}|t|^2\}$ in $P_j(it) \exp\{-\frac{1}{2}|t|^2\}$ by $(-\partial/\partial x_1)^{v_1}(-\partial/\partial x_2)^{v_2} \cdots (-\partial/\partial x_k)^{v_k}\varphi$. For example, from the expression for $P_1(it)$ given in (3.5) one obtains

$$(4.68) \qquad P_1(-\varphi)(x) = -\frac{1}{6} \sum_{j,\,j',\,j''} E(X_{1,j}X_{1,j'}X_{1,j''}) \frac{\partial^3}{\partial x_j \partial x_{j'} \partial x_{j''}} \varphi(x).$$

Let $P_j(-\Phi)$ be the finite signed measure whose density (with respect to Lebesgue measure) is $P_j(-\varphi)$. We write $P_0(-\varphi)$ for φ, and $P_0(-\Phi)$ for Φ. For the random vectors $\{Z_{r;\,n}\}$, we similarly define $P_{j;\,n}(-\varphi)$ and $P_{j;\,n}(-\Phi)$ to correspond to $P_{j;\,n}(it)$.

THEOREM 4.3. *If ρ_s is finite for some integer s, $s \geq 3$, and if the characteristic function f_1 of X_1 satisfies Cramér's condition (3.31), then for every real valued, bounded, Borel measurable function g on R^k one has*

$$(4.69) \qquad \left\| \int_{R^k} g \, d\left[Q_n - \sum_{j=0}^{s-2} n^{-j/2} P_j(-\Phi) \right] \right\|$$

$$\leq w_g(R^k)\delta_1(n) + \int_{R^k} w_g(\cdot; e^{-\beta n}) \, d\left| \sum_{j=0}^{s-2} n^{-j/2} P_j(-\Phi) \right|,$$

and, also,

$$(4.70) \qquad \left\| \int_{R^k} g \, d \left[Q_n - \sum_{j=0}^{s-2} n^{-j/2} P_j(-\Phi) \right] \right\|$$

$$\leqq w_g(R^k) \delta_1'(n) + \big(1 + \delta(n)\big) \int_{R^k} w_g(\cdot\,; e^{-\beta^n}) \, d\Phi,$$

where $\delta_1(n) = o(n^{-(s-2)/2})$, $\delta_1'(n) = o(n^{-(s-2)/2})$, $\delta(n) = o(1)$, and β is a positive constant. The quantities $\delta_1(n)$, $\delta_1'(n)$, $\delta(n)$, and β depend only on the distribution of X_1 and not on the function g.

PROOF.   We continue to use the notation introduced earlier. In addition, let

$$\Psi_r = \sum_{j=0}^{r} n^{-j/2} P_j(-\Phi), \qquad \Psi_{r;n} = \sum_{j=0}^{r} n^{-j/2} P_{j,n}(-\Phi),$$

$$(4.71) \qquad \gamma_r'(t) = \sum_{j=0}^{r} n^{-j/2} P_j(it) \exp\left\{-\tfrac{1}{2}|t|^2\right\},$$

$$\gamma_{r;n}(t) = \sum_{j=0}^{r} n^{-j/2} P_{j;n}(it) \exp\left\{-\tfrac{1}{2}|t|^2\right\}.$$

Let us first show that

$$(4.72) \qquad \left\| (Q_n' - \Psi_{s-2;n}) * M_\varepsilon \right\| = o(n^{-(s-2)/2}),$$

where ε is defined by $\varepsilon = e^{-\alpha n}$, α being a suitable positive constant to be chosen later. The probability measure M_ε is the distribution of εTY, Y having distribution M of corollary to Lemma 3.9. Now remembering how the density of $P_{j;n}(-\Phi)$ is formally obtained from the polynomial $P_{j;n}(it)$, it is easy to see that

$$(4.73) \qquad \left\| n^{-j/2} P_{j;n}(-\Phi) \right\| \leqq n^{-j/2} b_{18}(k, j) \rho_{j+2;n}$$

$$= o(n^{-(s-2)/2}), \qquad \text{for } j > s - 2.$$

Therefore, (4.72) will be proved if we prove

$$(4.74) \qquad \left\| (Q_n' - \Psi_{k+s-1;n}) * M_\varepsilon \right\| = o(n^{-(s-2)/2}).$$

As in (4.5) (also see [6], p. 413),

$$(4.75) \quad \left\| (Q_n' - \Psi_{k+s-1;n}) * M_\varepsilon \right\|^2 \leqq b_{18}'(k) \prod_{m=1}^{k} \left[\int_{R^k} (1 + x_m^{2(k+s)}) \, p_n^2(x) \, dx \right]^{1/k}$$

$$\leqq b_{18}'(k) \prod_{m=1}^{k} (L_0 + L_m)^{1/k},$$

where p_n is the density of $(Q_n' - \Psi_{k+s-1;n}) * M_\varepsilon$, and L_0 and L_m are given by

$$L_0 = (2\pi)^{-k} \int_{R^k} |f_n'(t) - \Psi_{k+s-1;n}(t)|^2 |\zeta(\varepsilon Tt)|^2 \, dt,$$

$$(4.76)$$

$$L_m = (2\pi)^{-k} \int_{R^k} \left| \frac{\partial^{k+s}}{\partial t_m^{k+s}} \big((f_n'(t) - \gamma_{k+s-1;n}(t)) \zeta(\varepsilon Tt) \big) \right|^2 \, dt.$$

Again,

(4.77)
$$L_0 \leqq 4(2\pi)^{-k}(L_{0,1} + L_{0,2} + L_{0,3} + L_{0,4}),$$

where

(4.78)
$$L_{0,1} = \int_{B_1} |f_n'(t) - \gamma_{k+s-1;n}(t)|^2 \, dt,$$

$$L_{0,2} = \int_{B_2 - B_1} |f_n'(t)|^2 \, dt,$$

$$L_{0,3} = \int_{R^k - B_2} |f_n'(t)\zeta(\varepsilon T t)|^2 \, dt,$$

$$L_{0,4} = \int_{R^k - B_1} |\gamma_{k+s-1;n}(t)|^2 \, dt,$$

the sets B_1, B_2 being defined by

(4.79) $B_1 = \{|t| \leqq \tfrac{1}{8} n^{1/2} \rho_{k+s+2;n}^{-1/(k+s)}\}, \qquad B_2 = \{|t| \leqq \tfrac{1}{4} n^{1/2} \rho_{3;n}^{-1}\}.$

By Lemma 3.1, (3.26), and (3.27),

(4.80)
$$L_{0,1} \leqq b_{19}(k, s)\rho_{k+s+2;n}^2 n^{-(k+s)} = o(n^{-(s-2)}).$$

By (3.26) and (3.27),

(4.81)
$$\tfrac{1}{8} n^{1/2} \rho_{k+s+2;n}^{-1/(k+s)} \geqq b_{20}(k, s)\rho_3 n^{1/2 - (k+2)/[2(k+s)]}.$$

Hence, by Lemma 3.3,

(4.82)
$$L_{0,2} \leqq \int_{B_2 - B_1} \exp\{-\tfrac{2}{3}|t|^2\} \, dt = o(n^{-(s-2)}).$$

By Lemma 3.8, (3.26), and (3.27),

(4.83)
$$\eta \equiv \sup_{n > n_0} \sup_{|t| > (4\rho_{3;n})^{-1}} |f_1'(t)| < 1.$$

Hence,

(4.84) $L_{0,3} \leqq b_{21}(k, s)\eta^{2n} \int_{R^k + B_1} \exp\{-2|\varepsilon T t|^{1/2}\} \, dt \leqq b_{22}(k, s)\eta^{2n}(\varepsilon\|T\|)^{-k}.$

Now choose a number η' such that $\eta < \eta' < 1$, and let

(4.85)
$$\varepsilon = (\eta')^{2n/k} = \exp\{-\alpha n\}, \qquad \alpha = \left(\frac{2}{k}\right)\log\left(\frac{1}{\eta'}\right).$$

Then clearly $L_{0,3} = o(n^{-(s-2)})$. Lastly, because of (4.81) and the presence of the exponential term $\exp\{-\tfrac{1}{2}|t|^2\}$ in the integrand of $L_{0,4}$, $L_{0,4} = o(n^{-(s-2)})$. It follows that

(4.86)
$$L_0 = o(n^{-(s-2)}).$$

Next,

$$(4.87) \qquad L_m \leqq 4(2\pi)^{-k}(L_{m,1} + L_{m,2} + L_{m,3} + L_{m,4}),$$

where

$$(4.88) \quad
\begin{aligned}
L_{m,1} &= \int_{B_3} \left| \frac{\partial^{k+s}}{\partial t_m^{k+s}} \left((f_n'(t) - \gamma_{k+s-1;n}(t)) \zeta(\varepsilon T t) \right) \right|^2 dt, \\
L_{m,2} &= \int_{B_2-B_3} \left| \frac{\partial^{k+s}}{\partial t_m^{k+s}} \left(f_n'(t) \zeta(\varepsilon T t) \right) \right|^2 dt, \\
L_{m,3} &= \int_{R^k-B_2} \left| \frac{\partial^{k+s}}{\partial t_m^{k+s}} \left(f_n'(t) \zeta(\varepsilon T t) \right) \right|^2 dt, \\
L_{m,4} &= \int_{R^k-B_3} \left| \frac{\partial^{k+s}}{\partial t_m^{k+s}} \left(\gamma_{k+s-1;n}(t) \zeta(\varepsilon T t) \right) \right|^2 dt,
\end{aligned}$$

the set B_3 being defined by $B_3 = \{|t| \leqq c_2(k) n^{1/2} \beta_{k+s+2;n}^{-1/(k+s)}\}$. By Lemma 3.2, (3.26), and (3.27),

$$(4.89) \qquad L_{m,1} \leqq b_{23}(k,s) \beta_{k+s+2;n}^2 n^{-(k+s)} + b_{24}(k,s) \beta_{k+s+2;n}^2 (\varepsilon \|T\|)^{k+2s}$$

$$= o(n^{-(s-2)}).$$

By Lemmas 3.3 and 3.4,

$$(4.90) \quad L_{m,2} \leqq \int_{R^k-B_3} \left| \sum_{r=0}^{k+s} b_{25}(k,r,s) \beta_{r;n} n^{r/2} \varepsilon^{k+s-r} \exp\left\{ -\frac{1}{3} \frac{|t|^2(n-r)}{n} \right\} \right|^2 dt$$

$$= o(n^{-(s-2)}).$$

Again by Lemma 3.4 and (4.83),

$$(4.91) \quad L_{m,3} \leqq \int_{R^k-B_2} \left| \sum_{r=0}^{k+s} b_{26}(k,s,r) \beta_{r;n} n^{r/2} \eta^{n-r} \frac{\partial^{k+s-r}}{\partial t_m^{k+s-r}} \zeta(\varepsilon T t) \right|^2 dt$$

$$\leqq \sum_{r=0}^{k+s} b_{27}(k,s,r) \beta_{r;n}^2 n^r \eta^{2(n-r)} (\varepsilon \|T\|)^{k+2s-2r}$$

$$= o(n^{-(s-2)}).$$

Finally, the presence of an exponential term $\exp\{-\frac{1}{2}|t|^2\}$ in the integrand of $L_{m,4}$ immediately gives $L_{m,4} = o(n^{-(s-2)/2})$. We then have

$$(4.92) \qquad L_m = o(n^{-(s-2)/2}),$$

which combined with (4.86), when substituted in (4.75), gives (4.74) and, therefore, the desired inequality (4.72). We next show that

$$(4.93) \qquad \|(Q_n - \Psi_{s-2}) * M_\varepsilon'\| = o(n^{-(s-2)/2}),$$

where M_ε' is the distribution of εY, Y having distribution M of Corollary 3.1.

Now remembering the definitions of $Q_{n,1}$, $Q_{n,2}$, we see that

$$(4.94) \qquad \| (Q_n - \Psi_{s-2}) * M'_\varepsilon - (Q_{n,1} - \Psi_{s-2}) * M'_\varepsilon \|$$
$$\leq \| Q_n - Q_{n,1} \| \leq 2nP(|X_1| > n^{1/2}) = o(n^{-(s-2)/2}).$$

We next define two finite signed measures $\Psi_{s-2;1}$, $\Psi_{s-2;2}$, by

$$(4.95) \qquad \begin{aligned} \Psi_{s-2;1}(B) &= \Psi_{s-2}(B + n^{-1/2} \sum_{r=0}^{n} EX_{r;n}), \\ \Psi_{s-2;2}(B) &= \Psi_{s-2}(T^{-1}B), \end{aligned} \qquad B \in \mathscr{B}^k.$$

It is not difficult to show that

$$(4.96) \qquad \| \Psi_{s-2;1} - \Psi_{s-2} \| = o(n^{-(s-2)/2}), \quad \| \Psi_{s-2;2} - \Psi_{s-2} \| = o(n^{-(s-2)/2}).$$

The first assertion follows by estimating the density of $\Psi_{s-2;1} - \Psi_{s-2}$, using the readily verified fact that

$$(4.97) \qquad \left| n^{-1/2} \sum_{r=1}^{n} EX_{r;n} \right| = o(n^{-(s-2)/2}).$$

The second assertion in (4.96) follows in the same way, this time using the relation $\| T - I \| = o(n^{-(s-2)/2})$ given in (3.22). Since

$$(4.98) \qquad \| (Q_{n,1} - \Psi_{s-2}) * M'_\varepsilon \| = \| (Q_{n,2} - \Psi_{s-2;1}) * M'_\varepsilon \|,$$

the first assertion in (4.96) together with (4.94) imply

$$(4.99) \qquad \| (Q_n - \Psi_{s-2}) * M'_\varepsilon \| = \| (Q_{n,2} - \Psi_{s-2}) * M'_\varepsilon \| + o(n^{-(s-2)/2}).$$

But

$$(4.100) \qquad \| (Q_{n,2} - \Psi_{s-2}) * M'_\varepsilon \| = \| (Q'_n - \Psi_{s-2;2}) * M_\varepsilon \|.$$

Hence, the second assertion in (4.96) yields

$$(4.101) \qquad \| (Q_n - \Psi_{s-2}) * M'_\varepsilon \| = \| (Q'_n - \Psi_{s-2}) * M_\varepsilon \| + o(n^{-(s-2)/2}),$$

which, combined with

$$(4.102) \qquad \| \Psi_{s-2} - \Psi_{s-2;n} \| = o(n^{-(s-2)/2}),$$

leads to

$$(4.103) \qquad \| (Q_n - \Psi_{s-2}) * M'_\varepsilon \| = \| (Q'_n - \Psi_{s-2;n}) * M_\varepsilon \| + o(n^{-(s-2)/2}).$$

The assertion (4.102) is a simple consequence of Lemma 3.7 (see [6], pp. 420–421). The desired relation (4.93) now follows from (4.72) and (4.103). The first assertion (4.69) of Theorem 4.3 follows on applying Corollary 2.1 with $\mu = Q_n$, $v = \Psi_{s-2}$, $K_\varepsilon = M_\varepsilon$ (note that M concentrates all its mass in the sphere $S(0, \varepsilon \| T \|)$, and that $2\varepsilon \| T \| < \exp\{-\beta n\}$ for any positive β smaller than α, if n is sufficiently large). To obtain the second assertion (4.70) observe that

$$(4.104) \qquad \int_{R^k} w_g(\cdot\,; 2\varepsilon)\, d\big|\Psi_{s-2}\big| = \int_{S(0,r)} w_g(\cdot\,; 2\varepsilon)\, d\big|\Psi_{s-2}\big|$$

$$+ \int_{R^k - S(0,r)} w_g(\cdot\,; 2\varepsilon)\, d\big|\Psi_{s-2}\big|.$$

Take $r = (3s \log n)^{1/2}$. Then the first integral is bounded above by

$$(4.105) \qquad \big[1 + b_{28}(k)\rho_s n^{-1/2}(1 + r^{3s})\big] \int_{S(0,r)} w_g(\cdot\,; 2\varepsilon)\, d\Phi,$$

and the second is bounded above by

$$(4.106) \qquad w_g(R^k)\big|\Psi_{s-2}\big|\big(R^k - S(0, r)\big) = w_g(R^k)\delta_2(n),$$

where $\delta_2(n) = o(n^{-(s-2)/2})$. Hence,

$$(4.107) \qquad \int_{R^k} w_g(\cdot\,: 2\varepsilon)\, d\big|\Psi_{s-2}\big| \leq w_g(R^k)\delta_2(n) + \big(1 + o(1)\big) \int_{R^k} w_g(\cdot\,; 2\varepsilon)\, d\Phi.$$

Q.E.D.

Note that the error of approximation in Theorem 4.3 is of the order of $o(n^{-(s-2)/2})$ even for those functions g for which

$$(4.108) \qquad \int_{R^k} w_g(\cdot\,; \varepsilon)\, d\Phi = o\left[\left(\log \frac{1}{\varepsilon}\right)^{-(s-2)/2}\right], \qquad\qquad \varepsilon \downarrow 0.$$

It may also be noted that if X_1 has an integrable characteristic function, then no kernel measure is needed for smoothing and one may show more simply $\big\|(Q_n - \Psi_{s-2})\big\| = o(n^{-(s-2)/2})$, if sth moments are finite. However, Bikjalis [6] has the following better result.

THEOREM 4.4. *Suppose X_1 has a nonzero absolutely continuous component If ρ_s is finite for some integer s, $s \geq 3$, then*

$$(4.109) \qquad \left\|Q_n - \sum_{j=0}^{s-2} n^{-j/2} P_j(-\Phi)\right\| = o(n^{-(s-2)/2}).$$

Theorems 4.3 and 4.4 do not cover the important class of discrete distributions. For the one dimensional lattice distributions, Esseen [13] obtained asymptotic expansions of the distribution function of Q_n. Noting that Q_n has point masses of the order of $n^{-1/2}$ (for $k = 1$), it is clear that the so called Edgeworth type expansions above do not hold. But the fact that the distribution Q_n may be roughly viewed as a probability measure on a group isomorphic to the additive group of integers enables one almost trivially to express the "density" of Q_n with respect to counting measure on the group in terms of its characteristic function via the Fourier inversion theorem (the dual group of the group of integers being the circle group T, Fourier inversion always holds for summable functions on the integer group). Since the expansions of f_n, given by Lemmas 3.1 and 3.2, are still valid, one obtains an expansion of the Edgeworth type for the point masses of Q_n. This is the so called local limit theorem in the lattice case.

To get approximations for $Q_n(B)$, where B is an arbitrary Borel set, one then uses something like the Euler summation formula to express a sum as an integral. The multidimensional extensions of Esseen's result as well as expansions for arbitrary Borel sets are due to Ranga Rao [29]. Theorem 4.5 below is a refinement of these results due to Bikjalis [7]. To be able to state it, we need some additional notation. Let L be a lattice in R^k defined by

$$(4.110) \qquad L = \{x_0 + m; m \text{ is an integer vector}\}, \quad x_0 = (x_{0,1}, \cdots, x_{0,k}),$$

where by an integer vector we mean a point in R^k all of whose coordinates are integers. Without any essential loss of generality, we assume that the k dimensional lattice distribution of X_1 has all its point masses in L (and in no proper sublattice of L). We further assume, again without loss of generality, $EX_1 = (0, 0, \cdots, 0)$, $\text{Cov } X_1 = V$. Here V is a positive definite matrix. The following functions appear in the Euler summation formula. For t in R^1, define

$$(4.111) \qquad S_j(t) = \begin{cases} (-1)^{j/2-1} \sum\limits_{a=1}^{\infty} 2 \cos (2a\pi t)/(2a\pi)^j & \text{if } j \text{ is even} \\ (-1)^{(j-1)/2} \sum\limits_{a=1}^{\infty} 2 \sin (2a\pi t)/(2a\pi)^j & \text{if } j \text{ is odd,} \end{cases}$$

$j = 1, 2, \cdots$. The function $S_1(t)$ is periodic with period unity, and the expression for $S_1(t)$ in (4.111) is merely the Fourier series for $S_1(t) = t - [t] - \frac{1}{2}$ with $[t] = $ integer part of t. The function S_1 is right continuous, is linear in the open interval $(N, N+1)$ (for every integer N) with slope 1, and has a jump -1 at every integer point. Also, $(d/dt)S_j(t) = S_{j-1}(t), j \geq 2$. Thus, the functions S_j, $j \geq 2$, are all absolutely continuous. Now define the following operators $T_j^{(s)}$, $1 \leq j \leq k$,

$$(4.112) \qquad T_j^{(s)} = 1 + \sum_{r=1}^{s-2} (-1)^r n^{-r/2} S_r(x_j n^{1/2} - n x_{0,j}) \frac{\partial^r}{\partial x_j^r}.$$

Also, let us denote the formal product of $T_j^{(s)}$, $1 \leq j \leq k$, by $\prod_{j=1}^{k} T_j(s)$. Let Q_n here stand for the distribution of $n^{-1/2} \sum_{r=1}^{n} X_r$. Other notation will also remain unchanged.

THEOREM 4.5. *If X_1 has a lattice distribution as described above, and if sth moments are finite for some integer s, $s \geq 3$, then*

$$(4.113) \qquad Q_n(B) - \int_B d\left[\left(\prod_{j=1}^{k} T_j^{(s)}\right)\Psi_{s-2}\right] = o(n^{-(s-2)/2}),$$

uniformly over all Borel sets B. Here Ψ_{s-2} is as defined by (4.71).

REMARKS. Theorems 4.3, 4.4, and 4.5 do not cover nonlattice discrete distributions (as well as some singular distributions). For some special functions g the Edgeworth type expansion

$$(4.114) \qquad \int_{R^k} g \, dQ_n = \int_{R^k} g \, d\Psi_{s-2} + o(n^{-(s-2)/2}),$$

may be obtained, if ρ_s is finite, $s \geq 3$, no matter what the type of distribution X_1 may have. For example, if g is a trigonometric polynomial then Lemma 3.1 immediately provides such an expansion. Our next theorem provides a class of functions for which this expansion is always valid.

THEOREM 4.6.   *If ρ_s is finite for some integer s, $s \geq 3$, and if g (real or complex valued) is the Fourier–Stieljes transform of a finite signed measure μ satisfying*

$$(4.115) \qquad \int_{R^k} |x|^{s-2} \, d|\mu|(x) < \infty,$$

then one has

$$(4.116) \qquad \int_{R^k} g \, d\left(Q_n - \sum_{j=0}^{s-2} n^{-j/2} P_j(-\Phi) \right) = o(n^{-(s-2)/2}), \qquad n \to \infty.$$

PROOF.   We need the following sharpening of Lemma 3.1 (see [29], Theorem 5.4.1, or use truncation and Lemma 3.1), a one dimensional version of which appears in [13] (Lemma 2b, p. 44): *if ρ_s is finite for some integer s, $s \geq 3$, then for $|t| \leq n^{1/2}/(8\rho_s^{1/(s-2)})$, one has*

$$(4.117) \qquad \left| f_n(t) - \exp\left\{ -\tfrac{1}{2}|t|^2 \right\} \sum_{j=0}^{s-2} n^{-j/2} P_j(it) \right|$$
$$\leq c_1'(k, s)\rho_s |t|^s \exp\left\{ -\tfrac{1}{4}|t|^2 \right\} n^{-(s-2)/2} \delta(n),$$

where $\delta(n)$ goes to zero as n goes to infinity. Now by Parseval's relation (see [14], p. 480)

$$(4.118) \qquad \int_{R^k} g \, d\left(Q_n - \sum_{j=0}^{s-2} n^{-j/2} P_j(-\Phi) \right)$$
$$= \int_{R^k} \left(f_n(t) - \exp\left\{ -\tfrac{1}{2}|t|^2 \right\} \sum_{j=0}^{s-2} n^{-j/2} P_j(it) \right) d\mu(t).$$

The integral on the right is first estimated over the region $B = \{ |t| \leq n^{1/2}/(8\rho_s^{1/(s-2)}) \}$. This is of the order $o(n^{-(s-2)/2})$ by (4.117). Over the complement of B the integral is of the order of

$$(4.119) \qquad \int_{R^k - B} d|\mu| \leq \left(\frac{n^{1/2}}{8\rho_s^{1/(s-2)}} \right)^{-(s-2)} \int_{R^k - B} |t|^{s-2} \, d|\mu|(t) = o(n^{-(s-2)/2}),$$

by (4.115). Q.E.D.

Theorem 4.6 is a considerable improvement on a previous result of the author (see Theorem 3 in [3], where a different form of Parseval's relation was used). The next theorem is designed to show that with the additional assumption of independence of the coordinates of the random vectors, multidimensional results become simple consequences of one dimensional results. The proof is based on an easy trick involving Fubini's theorem. Let P, Q be two probability measures on $(R^k, \mathscr{B}^k)$. Let $U = (U_1, \cdots, U_k)$ and $V = (V_1, \cdots, V_k)$ be two random vectors with respective distributions P, Q. Denote by P_i, Q_i, the

(marginal) distributions of U_i, V_i, respectively. On the space of all probability measures $\mathscr{P}$ (on $(R^k, \mathscr{B}^K)$) define the pseudometrics $d_{0,i}$, $1 \leq i \leq k$, by

$$(4.120) \qquad d_{0,i}(P, Q) = d_0(P_i, Q_i) = \sup \{|P_i(I) - Q_i(I)|; I \text{ an interval of } R^1\}.$$

THEOREM 4.7. *If P and Q are product probability measures on $(R^k, \mathscr{B}^k)$, then*

$$(4.121) \qquad d_0(P, Q) \leq \sum_{i=1}^{k} d_{0,i}(P, Q).$$

PROOF. We have $P = P_1 \times P_2 \times \cdots \times P_k$, $Q = Q_1 \times Q_2 \times \cdots \times Q_k$. Let C be an arbitrary Borel measurable convex set in R^k. Denoting by $\theta_1, \theta_2, \cdots, \theta_k$, appropriate real numbers of magnitudes not exceeding unity, one has

$$(4.122) \qquad P(C) = \int_{R^k} P_1(C_{x_2 x_3 \cdots x_k}) d(P_2 \times P_3 \times \cdots \times P_k)$$

$$= \int_{R^k} Q_1(C_{x_2 x_3 \cdots x_k}) d(P_2 \times P_3 \times \cdots \times P_k) + \theta_1 \, d_{0,1}(P, Q)$$

$$= Q_1 \times P_2 \times \cdots \times P_k(C) + \theta_1 \, d_{0,1}(P, Q)$$

$$= \int_{R^k} P_2(C_{x_1 \cdot x_3 x_4 \cdots x_k}) d(Q_1 \times P_3 \times \cdots \times P_k) + \theta_1 \, d_{0,1}(P, Q)$$

$$= \int_{R^k} Q_2(C_{x_1 \cdot x_3 x_4 \cdots x_k}) d(Q_1 \times P_3 \times \cdots \times P_k) + \theta_1 \, d_{0,1}(P, Q) + \theta_2 \, d_{0,2}(P, Q)$$

$$= Q_1 \times Q_2 \times P_3 \times \cdots \times P_k(C) + \theta_1 \, d_{0,1}(P, Q) + \theta_2 \, d_{0,2}(P, Q) = \cdots$$

$$= Q_1 \times Q_2 \times \cdots \times Q_k(C) + \sum_{i=1}^{k} \theta_i \, d_{0,i}(P, Q),$$

where

$$(4.123) \qquad \begin{aligned} C_{x_2 x_3 \cdots x_k} &= \{x_1; x_1 \in R^1, (x_1, x_2, x_3, \cdots, x_k) \in C\}, \\ C_{x_1 \cdot x_3 x_4 \cdots x_k} &= \{x_2; x_2 \in R^1, (x_1, x_2, x_3, \cdots, x_k) \in C\}, \end{aligned}$$

and so on. Q.E.D.

As an immediate application it follows from the Berry–Esseen theorem (with Zolotarev's estimate of the absolute constant involved, as appears in [35]) that if third moments are finite and X_1 has independent coordinates, then $d_0(Q_n, \Phi) \leq 1.64 \beta_3 n^{-1/2}$. Similarly, one may obtain asymptotic expansions of $Q_n(C)$ for convex sets C using one dimensional Edgeworth expansions for distribution functions. It should be pointed out that the equalities in (4.122) hold for any set C (not necessarily convex) whose "sections" $C_{x_2 3 \cdots x_k}$, and so forth, are all line segments (empty, finite, or infinite). In fact, the method extends to all Borel sets B whose "sections" are disjoint unions of $m(B)$ line segments or less, $1 \leq m(B) < \infty$.

For the sake of completeness, we state now two theorems on exact rates of convergence. Theorem 4.8 is due to Heyde [16], while Theorem 4.9 is due to Ibragimov [17]. These are both one dimensional results.

THEOREM 4.8. *Let* $k = 1$. *Let* δ *be a positive number,* $0 < \delta < 1$. *Then*

$$(4.124) \qquad \sum_{n=1}^{\infty} d_0(Q_n, \Phi) n^{-1+\delta/2} < \infty,$$

if and only if $\beta_{2+\delta}$ *is finite. Also,*

$$(4.125) \qquad \sum_{n=1}^{\infty} d_0(Q_n, \Phi) n^{-1} < \infty$$

if and only if $E(X_1^2 \log(1 + |X_1|))$ *is finite.*

THEOREM 4.9. *Let* $k = 1$. *Let* δ *be a positive number,* $0 < \delta < 1$. *Then*

$$(4.126) \qquad d_0(Q_n, \Phi) = O(n^{-\delta/2})$$

if and only if

$$(4.127) \qquad \int_{|x|>z} x^2 \, dQ_1(x) = O(z^{-\delta}), \qquad\qquad z \to \infty.$$

Also,

$$(4.128) \qquad d_0(Q_n, \Phi) = O(n^{-1/2}),$$

if and only if (4.127) *holds with* $\delta = 1$, *and*

$$(4.129) \qquad \int_{-z}^{z} x^3 \, dQ_1(x) = O(1), \qquad\qquad z \to \infty.$$

Extensions of Theorems 4.8 and 4.9 to all k would be useful. In case the random vector X_1 has independent coordinates, these extensions are immediate in view of Theorem 4.7.

In conclusion, we make a few additional remarks. First, the assumption (3.1) is merely a convenient normalization and does not involve any essential loss of generality (see, for example, Section 4 of [2], and the concluding remarks in [3]). Second, most of the results presented in this article have extensions to the nonidentically distributed case. For example, Theorems 4.1 and 4.2 are proved for this case in [2], [3], respectively, under the assumption of finiteness of moments of order $3 + \delta$ for some positive δ. The average of the $(3 + \delta)$th moments of the first n random vectors appears in the bounds. For $k = 1, 2$, one may take $\delta = 0$, so that complete extensions of Theorems 4.1 and 4.2 are available in one and two dimensions (see [2], Theorems 2 and 3. Theorem 1 in [3] admits similar modifications for $k = 1, 2$). To do away with δ in the general case, one would require a suitable extension of Lemma 3.2, which, to the knowledge of the author, is not yet available. Third, Theorems 4.1 and 4.2 hold with β_3 replaced by $\beta_{2+\delta}$ and $n^{-1/2}$ replaced by $n^{-\delta/2}$ throughout, for δ satisfying $0 < \delta < 1$. This follows by truncation if one remembers, in the

notation of Section 3, that $\beta_{3;n} = o(n^{(1-\delta)/2})$, $\beta_{3;n} \leqq c(k)n^{(1-\delta)/2}\beta_{2+\delta}$. In fact, it is easily shown that the remainder in Theorem 4.1 is $o(n^{-\delta/2})$, while that in Theorem 4.2 is $o(n^{-\delta/2}\log n)$, if one merely assumes finiteness of $\beta_{2+\delta}$, $0 < \delta < 1$. It should be mentioned that Bikjalis [6] was the first to exploit the technique of truncation in the present context. Fourth, finally, a great deal remains to be done as far as efficient estimations of constants appearing in the bounds are concerned.

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