

## RECENT RESULTS ON THE CONTROLLABILITY OF LINEAR COUPLED PARABOLIC PROBLEMS: A SURVEY

FARID AMMAR-KHODJA

Laboratoire de Mathématiques, Université de Franche-Comté  
16 route de Gray  
25030 Besancon cedex, France

ASSIA BENABDALLAH

Centre de Mathématiques et Informatique, Université Aix-Marseille 1  
Technopôle Château-Gombert, 39, rue F. Joliot Curie,  
13453 Marseille Cedex 13, France.

MANUEL GONZÁLEZ-BURGOS

Dpto, E.D.A.N., Universidad de Sevilla  
Aptdo. 1160  
41080 Sevilla, Spain.

LUZ DE TERESA

Instituto de Matemáticas, Universidad Nacional Autónoma de México  
Circuito Exterior, C.U.  
04510 D.F., México.

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**ABSTRACT.** This paper tries to summarize recent results on the controllability of systems of (several) parabolic equations. The emphasis is placed on the extension of the Kalman rank condition (for finite dimensional systems of differential equations) to parabolic systems. This question is itself tied with the proof of global Carleman estimates for systems and leads to a wide field of open problems.

**1. Introduction.** The control of coupled parabolic systems is a challenging issue, which has attracted the interest of the control community in the last decade. These parabolic systems arise, for example, in the study of chemical reactions (see e.g. [26], [16]), and in a wide variety of mathematical biology and physical situations (see e.g. [44], [60], [49]). In this paper we present the state of the art regarding this subject.

We deal with coupled scalar equations and we do not present results related to the controllability properties of fluid equations such as Stokes, Navier-Stokes, etc. There are, of course, other kinds of models which involve coupled PDE's. Some of them arise in thermoelasticity and are, in general, equations of different

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nature (e.g. parabolic-hyperbolic). The control problems related to them are also challenging with an extensive variety of open problems. For results related to these subjects we refer to e.g. [19], [38], [34], [35], [21], [29] and [18] for fluids and [51], [55] and [12] in the case of thermoelastic systems (see also the bibliography therein).

To focus on parabolic structures, for systems of (several) parabolic equations, the main issue is often to reduce the number of control functions acting on the system. In this sense, algebraic necessary conditions for the controllability are expected exactly as for linear finite dimensional systems (see Section 3 for the Kalman rank condition). We will see that in the constant coefficient case, depending on the nature of the control functions (acting on an open subset of the domain or on part of its boundary), this rank condition will be sufficient or not. This suggests a difference between the action of boundary and distributed controls respectively. For scalar parabolic problems, this difference of nature disappears: boundary controllability and distributed controllability are equivalent (see Section 2 for a precise statement). Moreover, when the principal part of the elliptic matrix-operator involved is not diagonal of order two (in space), the control problem is widely open. Even if the number of independent control functions is equal to the number of equations, apart from particular cases ( $2 \times 2$  parabolic systems as in [33], one dimensional setting, constant coefficients case, etc), it does not yet exist a general answer. So we want to present to the PDE mathematical control community this challenging subject that needs new ideas to be well-understood.

We start recalling in this section basic concepts of controllability. To this end, let us fix  $T > 0$  and let  $H$  and  $U$  be two separable Hilbert spaces with scalar product and associated norm respectively denoted by  $(\cdot, \cdot)_H$ ,  $(\cdot, \cdot)_U$ ,  $\|\cdot\|_H$  and  $\|\cdot\|_U$ . Let us consider  $T_0 \in (0, T)$  and the system:

$$\begin{cases} y' = A(t)y + B(t)u & \text{in } (T_0, T), \\ y(T_0) = y^0 \in H. \end{cases} \quad (1)$$

In this system  $y^0 \in H$  is the initial datum at  $t = T_0$  and  $u \in L^2(T_0, T; U)$  is the control which is exerted on the system by means of the operator  $B(t)$ . Assume that this problem is well-posed for any  $T_0 \in (0, T)$ , i.e., for any  $(y^0, u) \in H \times L^2(T_0, T; U)$  there exists a unique weak solution  $y \in C^0([T_0, T]; H)$  to (1) which depends continuously on the data. Let us denote by  $y(t; T_0, y^0, u) \in H$  the solution to the system at time  $t \in [T_0, T]$  and, for simplicity, let us denote by  $y(t; y^0, u) = y(t; 0, y^0, u)$ . With this notation, we have the following definitions:

- **Exact Controllability:** System (1) is exactly controllable at time  $T$  if for all  $(y^0, y^1) \in H \times H$ , there exists  $u \in L^2(0, T; U)$  such that the solution  $y$  of (1) satisfies

$$y(T; y^0, u) = y^1.$$

- **Controllability to trajectories:** System (1) is controllable to trajectories at time  $T$  if for every  $(y^0, \hat{y}^0) \in H \times H$  and  $\hat{u} \in L^2(0, T; U)$ , there exists  $u \in L^2(0, T; U)$  such that the corresponding weak solution to (1) satisfies

$$y(T; y^0, u) = y(T; \hat{y}^0, \hat{u}).$$

- **Null Controllability:** System (1) is null controllable at time  $T$  if for every  $y^0 \in H$  there exists  $u \in L^2(0, T; U)$  such that the corresponding weak solution to (1) satisfies

$$y(T; y^0, u) = 0.$$

Observe that in the linear case controllability to trajectories and null controllability are equivalent.

- **Approximate Controllability:** System (1) is approximately controllable at time  $T$  if for every  $(y^0, y^1) \in H \times H$ , and every  $\varepsilon > 0$ , there exists  $u \in L^2(0, T; U)$  such that the corresponding weak solution to (1) satisfies

$$\|y(T; y^0, u) - y^1\|_H < \varepsilon.$$

**Remark 1.** It is clear that the exact controllability of System (1) at time  $T$  implies the controllability to trajectories, the null and the approximate controllability of the system at time  $T$ . As said before, for linear systems controllability to trajectories and null controllability at time  $T$  are also equivalent. Nevertheless the two last definitions, in general, are independent (it is well-known that the wave equation could be approximately controllable at a time  $T$  and not null controllable for any positive time; on the other hand, the transport equation could be null controllable at a time  $T$  and not approximately controllable at this time).

**Remark 2.** Observe that in the previous definitions we have always fixed the initial time at  $t = 0$  and the final time at  $t = T$ . For the non-autonomous equation

$$y' = A(t)y + B(t)u \quad \text{in } (0, T), \quad (2)$$

it is possible to give stronger definitions of controllability. In this sense, it will be said that equation (2) is totally exactly controllable on  $(0, T)$  if for any  $T_0, T_1 \in (0, T)$ , with  $T_0 < T_1$ , and for any  $(y^0, y^1) \in H \times H$  there exists  $u \in L^2(T_0, T_1; H)$  such that the solution to (1) in  $(T_0, T_1)$  satisfies

$$y(T_1; T_0, y^0, u) = y^1.$$

Following the previous definition we can also define the concepts for equation (2): totally controllable to trajectories on  $(0, T)$ , totally null controllable on  $(0, T)$  and totally approximately controllable on  $(0, T)$ . In the autonomous case the different concepts of controllability at time  $T$  and total controllability on  $(0, T)$  coincide.

Let  $N \geq 1$  be fixed. Throughout the paper,  $\Omega \subset \mathbb{R}^N$  will be an open and bounded set with boundary  $\partial\Omega$  of class  $C^2$ ,  $\omega \subset \Omega$  a nonempty open subset and  $\Gamma_0 \subset \partial\Omega$  a nonempty relative open subset of the boundary  $\partial\Omega$ . For  $T > 0$  we denote  $Q_T = \Omega \times (0, T)$  and  $\Sigma_T = \partial\Omega \times (0, T)$ . For any set  $\mathcal{O} \subset \Omega$  or  $\mathcal{O} \subset \partial\Omega$ ,  $\mathcal{O}_T$  and  $1_{\mathcal{O}}$  will denote resp. the set  $\mathcal{O} \times (0, T)$  and the characteristic function of the set  $\mathcal{O}$ .

Let  $L$  be the time-dependent second order elliptic operators given by:

$$\begin{cases} L(t)y(x, t) = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( \alpha_{ij}(x, t) \frac{\partial y}{\partial x_j}(x, t) \right) + \sum_{i=1}^N b_i(x, t) \frac{\partial y}{\partial x_i}(x, t) \\ \quad + c(x, t)y(x, t). \end{cases} \quad (3)$$

The coefficients of  $L$  satisfy

$$\begin{cases} \alpha_{ij} \in W^{1,\infty}(Q_T), \quad b_i, c \in L^\infty(Q_T), \quad 1 \leq i, j \leq N, \\ \alpha_{ij}(x, t) = \alpha_{ji}(x, t) \forall (x, t) \in Q_T, \end{cases} \quad (4)$$

and the principal part of  $L$  satisfies a uniform elliptic condition: there exists  $a_0 > 0$  such that

$$\sum_{i,j=1}^N \alpha_{ij}(x, t) \xi_i \xi_j \geq a_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \forall (x, t) \in Q_T. \quad (5)$$

Of course, when  $\alpha_{ij} = \delta_{ij}$  and  $b_i = c = 0$  ( $1 \leq i, j \leq N$ ) we obtain  $L(t) = -\Delta$ .

For simplicity, in this paper we will consider the parabolic operator  $L(t)$  with homogeneous Dirichlet conditions. Most of the controllability results stated in this paper are valid for Neumann or Fourier boundary conditions, as soon as all the components of  $y$  satisfy the same boundary condition.

The plan of the paper is the following: In Section 2, we will address the null controllability problem for scalar parabolic equations with distributed or boundary controls. In Section 3 we recall some well-known results on controllability of linear systems in the finite dimensional case. In order to show the important differences between controllability results for scalar and non scalar parabolic problems, in Section 4 we will study the controllability problem for two simple examples. In Sections 5 and 6 we will exhibit necessary and sufficient condition for the null controllability problem of some classes of non scalar parabolic equations with distributed controls, localized on a subdomain of the domain, and with boundary controls, supported on a part of the boundary of the domain. Section 7 is devoted to give some sufficient conditions for the null controllability of other class of non scalar parabolic equations. Finally, in Section 8 we give some comments and open problems.

**2. Controllability results for scalar parabolic problems.** Controllability for scalar parabolic equations has been investigated for a long time. In 1971 and 1974 H.O. Fattorini and D.L. Russell proved the first results related to the null boundary controllability for the one dimensional heat equation [27, 28]. They used the so-called method of moments (see Section 2.2). In [58] the author proved a null controllability result for the heat equation in the  $N$ -dimensional case with a boundary control supported on the whole boundary of the domain. In fact, he proved that the null controllability of the wave equation at a positive time implies the null controllability of the heat equation at any positive time. In 1995-1996, the  $N$ -dimensional null controllability problem for parabolic equations (with boundary or distributed controls) was solved independently by G. Lebeau and L. Robbiano, [54] (for the heat equation), and A. Fursikov and O. Imanuvilov, [39] (for a general parabolic equation). The result in [54] was obtained through a spectral inequality and this inequality was proved by the authors using (and proving) local Carleman estimates. The null controllability was obtained from the dissipation effect of the operator. The result in [39] was obtained by proving Carleman estimates that imply an observability inequality equivalent to the null controllability or controllability to trajectories of the parabolic equation (see Section 2.1). Carleman inequalities have been introduced by [17] for proving uniqueness results for some PDE's and have been widely extended by Hörmander (see [45, 46]). See also [52] where different Carleman inequalities are presented and compared and where some applications to the controllability of the heat equation is also done.

Let us start recalling some results on the controllability problem for scalar parabolic problems. To this end, let us consider the operator  $L(t)$  given by (3) with coefficients which satisfy (4) and (5).

Let us consider the following scalar parabolic problems:

$$\begin{cases} \partial_t y + L(t)y = v1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \quad (6)$$

and

$$\begin{cases} \partial_t y + L(t)y = 0 & \text{in } Q_T, \\ y = h1_{\Gamma_0} & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega. \end{cases} \quad (7)$$

In (6) the initial datum  $y^0 \in L^2(\Omega)$  is given and  $v \in L^2(Q_T)$  is the control (distributed control). In (7),  $y^0$  and  $h$  are also given in appropriate spaces (see below). In this case  $h$  is a boundary control. We are interested in recalling null controllability properties of both problems.

It is well-known that System (6) is well-posed. To be precise, for every  $y^0 \in L^2(\Omega)$  and  $v \in L^2(Q_T)$ , System (6) admits a unique solution  $y \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$  which depends continuously on the data, i.e., there is a positive constant  $C_1$  such that

$$\|y\|_{L^2(0, T; H_0^1(\Omega))} + \|y\|_{C^0([0, T]; L^2(\Omega))} \leq C_1 (\|y^0\|_{L^2(\Omega)} + \|v\|_{L^2(Q_T)}).$$

On the other hand, when in (3) we take  $b_i = 0$  in  $Q$  for any  $i : 1 \leq i \leq N$ , given  $y^0 \in H^{-1}(\Omega)$  and  $h \in L^2(\Sigma_T)$ , System (7) also has a unique solution  $y \in L^2(Q_T) \cap C^0([0, T]; H^{-1}(\Omega))$  (defined by transposition; see [32] for a proof in the case  $N = 1$ ) which depends continuously on the data, i.e., there exists a positive constant  $C_2$  such that

$$\|y\|_{L^2(Q_T)} + \|y\|_{C^0([0, T]; H^{-1}(\Omega))} \leq C_2 (\|y^0\|_{H^{-1}(\Omega)} + \|h\|_{L^2(\Sigma_T)}).$$

In the general case, the system is well-posed if we take  $h \in X(\Gamma_0)$  with

$$X(\Gamma_0) = \{h : h = H|_{\Sigma_T} \text{ with } H \in L^2(0, T; H_0^1(\tilde{\Omega})), H_t \in L^2(0, T; H^{-1}(\tilde{\Omega}))\}, \quad (8)$$

where  $\tilde{\Omega}$  is an open set with a boundary of class  $C^2$  and such that  $\Omega \subset \tilde{\Omega}$ ,  $\partial\Omega \cap \tilde{\Omega} \subset\subset \Gamma_0$  and  $\tilde{\Omega} \setminus \bar{\Omega} \neq \emptyset$ . Indeed, if  $h \in X(\Gamma_0)$ , then  $h = H|_{\Sigma_T}$  with  $H \in L^2(0, T; H_0^1(\tilde{\Omega}))$  and  $H_t \in L^2(0, T; H^{-1}(\tilde{\Omega}))$ . If we perform the change  $w = y - H$  in  $Q_T$ , then System (7) is equivalent to a similar system for  $w$  with homogeneous Dirichlet boundary condition and a right hand side in  $L^2(0, T; H^{-1}(\Omega))$ .

In fact, we are going to prove the boundary null controllability result for System (7) with controls which belong to  $X(\Gamma_0)$ .

Let us start presenting the distributed null controllability property of System (6). It is well-known (e.g., see [24]) that this property for System (6) is equivalent to an appropriate property of the associated adjoint problem

$$\begin{cases} -\partial_t \varphi + L^*(t)\varphi = 0 & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = \varphi^T & \text{in } \Omega, \end{cases} \quad (9)$$

where  $\varphi^T \in L^2(\Omega)$  is given and  $L^*(t)$  is the operator given by

$$\begin{cases} L^*(t)\varphi(x, t) = - \sum_{i, j=1}^N \frac{\partial}{\partial x_i} \left( \alpha_{ij}(x, t) \frac{\partial \varphi}{\partial x_j}(x, t) \right) - \sum_{i=1}^N \frac{\partial}{\partial x_i} (b_i(x, t)\varphi) \\ \quad + c(x, t)\varphi(x, t), \quad \text{a.e. in } Q_T. \end{cases}$$

This problem is also well-posed and the solution depends continuously on  $\varphi^T$ : there exists a positive constant  $\tilde{C}$  such that for every  $\varphi^T \in L^2(\Omega)$  System (9) has only one solution  $\varphi \in L^2(0, T; H_0^1(\Omega)) \cap C^0([0, T]; L^2(\Omega))$  and it satisfies

$$\|\varphi\|_{L^2(0, T; H_0^1(\Omega))} + \|\varphi\|_{C^0([0, T]; L^2(\Omega))} \leq \tilde{C} \|\varphi^T\|_{L^2(\Omega)}.$$

One has:

**Theorem 2.1.** *Under the previous assumptions, System (6) is null controllable at time  $T > 0$  if and only if there exists a positive constant  $C$  such that*

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega_T} |\varphi|^2 dxdt, \quad \forall \varphi^T \in L^2(\Omega), \quad (10)$$

where  $\varphi$  is the solution of (9) associated to  $\varphi^T$ .

*Proof.* An easy way to prove this result is the following functional analysis argument: Let us set  $H = L^2(\Omega)$ , and let  $G$  and  $P$  be the following linear and continuous operators:

$$P : v \in L^2(Q_T) \mapsto Pv = y(\cdot, T) \in H,$$

where  $y$  is the solution to (6) corresponding to  $v$  and  $y^0 \equiv 0$ , and

$$G : y^0 \in H \mapsto G(y^0) = y(\cdot, T) \in H,$$

where  $y$  is the solution to (6) with data  $y^0$  and  $v \equiv 0$ . Then, the null controllability property for System (6) is equivalent to

$$R(G) \subset R(P).$$

Both  $G$  and  $P$  are linear and bounded operator with values in  $H$ . Then, the previous property holds if and only if (see e.g. [67], Theorem 2.2, p. 208) there exists  $C > 0$  such that

$$\|G^*(\varphi^T)\|_H \leq C \|P^*(\varphi^T)\|_{L^2(Q_T)}, \quad \forall \varphi^T \in H. \quad (11)$$

It is not difficult to see that  $G^* \in \mathcal{L}(H)$  and  $P^* \in \mathcal{L}(H; L^2(Q_T))$  are given by

$$G^*(\varphi^T) = \varphi(\cdot, 0) \quad \text{and} \quad P^*(\varphi^T) = \varphi 1_\omega$$

where  $\varphi$  is the solution to the adjoint System (9) and (11) is precisely (10).  $\square$

**Remark 3.** Inequality (10) is called the observability inequality for the adjoint System (9) and characterizes the null controllability properties of problem (6) at time  $T$ . In fact, it is possible to prove that, if the observability inequality (10) holds then, for every  $y^0 \in L^2(\Omega)$  there exists a distributed control  $v \in L^2(Q_T)$  such that

$$\|v\|_{L^2(Q_T)}^2 \leq C \|y^0\|_{L^2(\Omega)}^2 \quad \text{and} \quad y(\cdot, T) = 0,$$

where  $y$  is the solution to (6) corresponding to  $y^0$  and  $C > 0$  is the constant appearing in (10). For a proof, see for instance [37] and [20] (Theorem 2.44, p. 56).

On the other hand, using Hahn-Banach Theorem one can see that the approximate controllability at time  $T$  for System (6) can be characterized by means of the following property of the adjoint problem (9) (the unique continuation property):

“If  $\varphi$  is solution to (9) associated to  $\varphi^T \in L^2(\Omega)$  and  $\varphi \equiv 0$  in  $\omega \times (0, T)$ , then  $\varphi \equiv 0$  in  $Q_T$  (and, evidently,  $\varphi^T \equiv 0$ )”.

Let us now study the boundary control problem (7). It is well-known that the boundary null controllability result for System (7) can be obtained from the corresponding result for System (6). In our following result we will state that, in fact, the distributed control problem for (6) ( $\omega \subset \Omega$  being an arbitrary nonempty open set) and the boundary control problem for (7) ( $\Gamma_0 \subset \partial\Omega$  being an arbitrary nonempty open subset of  $\partial\Omega$ ) are equivalent.

**Theorem 2.2.** *Let us fix  $T > 0$ . The following conditions are equivalent*

1. For any  $\Omega \subset \mathbb{R}^N$ , bounded open set with  $\Omega$  having a  $C^2$  boundary, any  $\omega \subset \Omega$ , nonempty open subset, and any coefficients  $\alpha_{ij}$ ,  $b_i$  and  $c$  ( $1 \leq i, j \leq N$ ), satisfying (4) and (5), System (6) is null controllable in  $L^2(\Omega)$  at time  $T > 0$  with distributed controls  $v \in L^2(Q_T)$ .
2. For any  $\Omega \subset \mathbb{R}^N$ , bounded open set with  $\Omega$  having a  $C^2$  boundary, any  $\Gamma_0 \subset \partial\Omega$ , nonempty relative open subset, and any coefficients  $\alpha_{ij}$ ,  $b_i$  and  $c$  ( $1 \leq i, j \leq N$ ), satisfying (4) and (5), System (7) is null controllable at time  $T > 0$  with boundary controls  $h \in L^2(0, T; H^{1/2}(\partial\Omega))$ .

*Proof.* We will use some ideas from [14] and [40]. Without loss of generality and to simplify notation we will consider the case in which  $\alpha_{ij} = \delta_{ij}$ . In the first part of the proof it would be needed to extend these coefficients  $\alpha_{ij}$  to an appropriate regular open set  $\Omega_0$  in order to get properties (4) and (5) in  $\Omega_0$ .

Let us see that point 1 implies 2. Let us fix  $\Omega \subset \mathbb{R}^N$  and  $\Gamma_0 \subset \partial\Omega$  as in the statement and take  $\tilde{\Omega}$ , a nonempty open set of class  $C^2$ , with  $\Omega \subset \tilde{\Omega}$ ,  $\partial\Omega \cap \tilde{\Omega} \subset \subset \Gamma_0$  and  $\tilde{\Omega} \setminus \bar{\Omega} \neq \emptyset$ . Let us consider a nonempty open subset  $\tilde{\omega} \subset \subset \tilde{\Omega} \setminus \bar{\Omega}$  and define  $\tilde{b}_i$ ,  $\tilde{c}$  and  $\tilde{y}^0$  as the extension by zero of  $b_i$ ,  $c$  and  $y^0$  to  $\tilde{\Omega}$ , and

$$\tilde{L}(t) = -\Delta + \sum_{i=1}^N \tilde{b}_i(x, t) \frac{\partial}{\partial x_i} + \tilde{c}(x, t).$$

Now, we solve the distributed control problem

$$\begin{cases} \partial_t \tilde{y} + \tilde{L}(t) \tilde{y} = \tilde{v} 1_{\tilde{\omega}} & \text{in } \tilde{Q}_T = \tilde{\Omega} \times (0, T), \\ \tilde{y} = 0 & \text{on } \tilde{\Sigma}_T = \partial\tilde{\Omega} \times (0, T), \\ \tilde{y}(\cdot, 0) = \tilde{y}^0, \quad \tilde{y}(\cdot, T) = 0 & \text{in } \tilde{\Omega}. \end{cases}$$

Then,  $y = \tilde{y}|_{\Omega}$  satisfies  $y(T) = 0$  and solves (7) with  $h = \tilde{y}|_{\partial\Omega} \in X(\Gamma_0)$  (see (8)).

Let us now show that point 2 implies 1. Again, fix  $\Omega \subset \mathbb{R}^N$  and  $\omega \subset \Omega$  as in the statement. We take  $\hat{\omega} \subset \subset \omega$  such that  $\Gamma_0 = \partial\hat{\omega}$  is of class  $C^2$  and we denote  $\hat{\Omega} = \Omega \setminus \hat{\omega}$ . Clearly,  $\hat{\Omega}$  is a bounded open set with a boundary of class  $C^2$ . We also take  $\theta \in C^\infty(\mathbb{R}^N)$  and  $\eta \in C^\infty(\mathbb{R})$  such that

$$\theta \equiv 1 \text{ in } \bar{\Omega} \setminus \omega, \quad \theta \equiv 0 \text{ in } \hat{\omega}, \quad \eta \equiv 1 \text{ in } [0, T/4] \quad \text{and} \quad \eta \equiv 0 \text{ in } [3T/4, T].$$

Let  $Y$  be the solution to (6) associated to  $y^0 \in L^2(\Omega)$  and  $v \equiv 0$ . We solve now the boundary control problem

$$\begin{cases} \partial_t \hat{y} + L(t) \hat{y} = 0 & \text{in } \hat{Q}_T = \hat{\Omega} \times (0, T), \\ \hat{y} = h 1_{\Gamma_0} & \text{on } \hat{\Sigma}_T = \partial\hat{\Omega} \times (0, T), \\ \hat{y}(\cdot, 0) = y^0 1_{\hat{\Omega}}, \quad \hat{y}(\cdot, T) = 0 & \text{in } \hat{\Omega}, \end{cases}$$

with  $h \in L^2(0, T; H^{1/2}(\partial\hat{\Omega}))$ . It is not difficult to check that  $y(x, t) = \theta(x) \hat{y} + (1 - \theta(x)) \eta(t) Y(x, t)$ , with  $(x, t) \in Q_T$ , satisfies  $y(\cdot, T) = 0$  in  $\Omega$  and is the solution to (6) corresponding to

$$v \equiv (1 - \theta) \eta'(t) Y + 2 \nabla \theta \cdot \nabla [\hat{y} - \eta(t) Y] + (\Delta \theta) [\hat{y} - \eta(t) Y] + \left( \sum_{i=1}^N b_i \frac{\partial \theta}{\partial x_i} \right) [\hat{y} - \eta(t) Y].$$

Using the properties of functions  $\theta$  and  $\eta$ , it is clear that  $\text{supp } v \subset \subset \omega \times (0, T]$ . Moreover, using the local regularizing effect of the parabolic operator  $\partial_t + L(t)$  we can conclude that  $v \in L^\infty(Q_T)$  and this regularity property is independent of the

regularity of  $h$  and  $y^0$ . Indeed, for any  $\delta \in (0, T)$ , any open subset  $\mathcal{O}_0 \subset\subset \Omega$  and any  $p \in [1, \infty)$  one has

$$Y \in X^p(\delta, T; \mathcal{O}_0) = \{y : y \in L^p(\delta, T; W^{2,p}(\mathcal{O}_0)), y_t \in L^p(\mathcal{O}_0 \times (\delta, T))\}.$$

Again, if we set  $z = [\hat{y} - \eta(t)Y]$ , then  $z$  is the solution to

$$\begin{cases} \partial_t z + L(t)z = -\eta'(t)Y|_{\hat{\Omega}} & \text{in } \hat{Q}_T, \\ z = h1_{\Gamma_0} - \eta(t)Y & \text{on } \hat{\Sigma}_T, \\ z(\cdot, 0) = 0, & \text{in } \hat{\Omega}, \end{cases}$$

the regularizing effect of  $\partial_t + L(t)$  also provides

$$z \in X^p(0, T; \mathcal{O}_1) = \{y : y \in L^p(0, T; W^{2,p}(\mathcal{O}_1)), y_t \in L^p(\mathcal{O}_1 \times (0, T))\},$$

for any  $p \in [1, \infty)$  and any  $\mathcal{O}_1 \subset\subset \hat{\Omega}$ . Finally, using the continuous embedding (see for instance Lemma 3.3 in [50])

$$X^p(0, T; \mathcal{O}) \hookrightarrow C^{1+\alpha, (1+\alpha)/2}(\bar{\mathcal{O}} \times [0, T]), \text{ with } p > N + 2 \text{ and } \alpha = 1 - \frac{N + 2}{p},$$

valid for any bounded domain  $\mathcal{O} \subset \mathbb{R}^N$  with boundary of class  $C^2$ , and the expression of  $v$ , we get  $v \in L^\infty(Q)$ . This proves the result.  $\square$

**Remark 4.** In the next sections we will prove that problem (6) (resp., problem (7)) is null controllable at time  $T$  for any  $T > 0$  and any nonempty open set  $\omega \subset \Omega$  (resp., nonempty relative open set  $\Gamma_0 \subset \partial\Omega$ ). Following the ideas of the previous proof it is possible to prove that the distributed and boundary null controllability problem for the scalar parabolic operator  $\partial_t + L(t)$  can be solved with controls  $v \in L^\infty(Q_T)$  (for problem (6)) and  $h \in L^\infty(\Sigma_T)$  (for problem (7)) and even better if  $b_i \equiv 0$  for every  $i : 1 \leq i \leq N$ . In [54] the authors also obtained the null controllability result with a control  $v \in C_0^\infty(Q_T)$  (in the distributed case) or  $v \in C_0^\infty(\Sigma_T)$  (for the boundary problem).

**Remark 5.** In the proof of Theorem 2.2 we have strongly used that the parabolic operator  $\partial_t + L(t)$  is scalar. In this paper we will see that, when we deal with non scalar parabolic operators, the equivalence between distributed and boundary controllability is no longer valid. In fact in Section 4 we give precise examples which show this notorious difference between coupled parabolic systems and scalar parabolic operators.

In the present article, the main tools for proving the null controllability property for coupled parabolic systems are the moment method and Carleman inequalities. In the two following subsections, we will present them briefly in the scalar case.

**2.1. Carleman inequalities.** In this section we recall the proof given in [39] of the distributed null controllability result for the linear parabolic operator  $L(t)$  given by (3) with coefficients which satisfy (4) and (5) using the technique of Carleman inequalities.

We will consider the following parabolic equation:

$$\begin{cases} -\partial_t z + L_0(t)z = F_0 + \sum_{i=1}^N \frac{\partial F_i}{\partial x_i} & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(x, T) = z^T(x) & \text{in } \Omega, \end{cases} \quad (12)$$



with  $z^T \in L^2(\Omega)$ ,  $F_i \in L^2(Q_T)$ ,  $i = 0, 1, \dots, N$ , and  $L_0(t)$  the self-adjoint parabolic operator given by

$$L_0(t)y(x, t) = - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( \alpha_{ij}(x, t) \frac{\partial y}{\partial x_j}(x, t) \right)$$

with coefficients  $\alpha_{ij}$  satisfying (4) and (5).

In several parts of the paper, we will use the following two functions:

$$\gamma(t) = t^{-1}(T-t)^{-1} \quad \beta(x, t) = \beta_0(x)/t(T-t) \quad (13)$$

where  $\beta_0(x)$  will be a particular function of class  $C^2$  (see [39] for a construction of this function).

The following *Carleman inequality* will be the main tool in order to prove (10):

**Lemma 2.3** ([39],[47]). *Let  $\mathcal{B} \subset \Omega$  be a nonempty open subset and  $d \in \mathbb{R}$ . Then, there exist a function  $\beta_0 \in C^2(\bar{\Omega})$  (only depending on  $\Omega$  and  $\mathcal{B}$ ) and two positive constants  $\tilde{C}_0$  and  $\tilde{\sigma}_0$  (which only depend on  $\Omega$ ,  $\mathcal{B}$  and  $d$ ) such that for  $\beta(x, t)$  and  $\gamma(t)$  given in (13) and for every  $z^T \in L^2(\Omega)$ , the solution  $z$  to (12) satisfies*

$$\left\{ \begin{array}{l} \mathcal{I}(d, z) \leq \tilde{C}_0 \left( s^d \iint_{\mathcal{B} \times (0, T)} e^{-2s\beta} \gamma(t)^d |z|^2 \right. \\ \left. + s^{d-3} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-3} |F_0|^2 + s^{d-1} \sum_{i=1}^N \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-1} |F_i|^2 \right), \end{array} \right. \quad (14)$$

for all  $s \geq \tilde{\sigma}_0 = \tilde{\sigma}_0(T+T^2)$ , with

$$\mathcal{I}(d, z) \equiv s^{d-2} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-2} |\nabla z|^2 + s^d \iint_{Q_T} e^{-2s\beta} \gamma(t)^d |z|^2.$$

When  $F_i \equiv 0$  for  $1 \leq i \leq N$ , there exist positive constants  $\tilde{C}_1$  and  $\tilde{\sigma}_1$  (which only depend on  $\Omega$ ,  $\mathcal{B}$  and  $d$ ) such that, for every  $z^T \in L^2(\Omega)$ , the solution  $z$  to (12) satisfies

$$\mathcal{I}_1(d, z) \leq \tilde{C}_1 \left( s^d \iint_{\mathcal{B} \times (0, T)} e^{-2s\beta} \gamma(t)^d |z|^2 + s^{d-3} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-3} |F_0|^2 \right), \quad (15)$$

for all  $s \geq \tilde{\sigma}_1 = \tilde{\sigma}_1(T+T^2)$  where

$$\mathcal{I}_1(d, z) \equiv s^{d-4} \iint_{Q_T} e^{-2s\beta} \gamma(t)^{d-4} (|\partial_t z|^2 + |\Delta z|^2) + \mathcal{I}(d, z).$$

The proof of this result can be found in [47] and is not included in this survey paper because it is very technical. Also an accessible proof can be found in [31].

Let us see how inequality (14) implies the observability inequality (10) for the adjoint problem (9) and, in view of Theorem 2.1, the distributed null controllability result for problem (6).

**Corollary 1.** *Suppose  $T > 0$  and let  $\Omega \subset \mathbb{R}^N$  and  $\omega \subset \Omega$  be two nonempty bounded open sets with  $\Omega \in C^2$ . Then, there exists a positive constant  $C$  (only depending on  $\Omega$ ,  $\omega$  and  $T$ ) such that for every  $\varphi^T \in L^2(\Omega)$  and  $\varphi$  the corresponding solution to (9), the following inequality holds:*

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega_T} |\varphi|^2 dx dt.$$

*Proof.* We follow [36] and [25]. Observe that (14) applied to problem (9) implies

$$\begin{aligned} & s^{d-2} \iint_{Q_T} e^{-2s\beta\gamma(t)^{d-2}} |\nabla\varphi|^2 + s^d \iint_{Q_T} e^{-2s\beta\gamma(t)^d} |\varphi|^2 \\ & \leq \tilde{C}_0 \left( s^d \iint_{\mathcal{B}_T} e^{-2s\beta\gamma(t)^d} |\varphi|^2 \right. \\ & \left. + s^{d-3} \|c\|_\infty^2 \iint_{Q_T} e^{-2s\beta\gamma(t)^{d-3}} |\varphi|^2 + s^{d-1} \|B\|_\infty^2 \iint_{Q_T} e^{-2s\beta\gamma(t)^{d-1}} |\varphi|^2 \right), \end{aligned}$$

where  $B = (b_i)_{1 \leq i \leq N}$ . It is easy to check the inequality

$$\gamma(t) \geq CT^{-2}, \quad \forall s \geq C(T + T^2).$$

As a consequence we can prove that for  $s \geq C(T + T^2 + T^2(\|c\|_\infty^{2/3} + \|B\|_\infty^2))$  one has

$$[s\gamma(t)]^3 - \tilde{C}_0 \|c\|_\infty^2 - \tilde{C}_0 [s\gamma(t)] \|B\|_\infty^2 \geq \frac{1}{2} [s\gamma(t)]^3,$$

(see [25] for details). Consequently, we have for  $d = 3$  and  $s = C(T + T^2 + T^2(\|c\|_\infty^{2/3} + \|B\|_\infty^2))$  (with  $C$  an appropriate positive constant only depending on  $\Omega$ ,  $\omega$  and  $T$ ) that

$$\iint_{Q_T} e^{-2s\beta t^{-3}} (T-t)^{-3} |\varphi|^2 \leq C_0 \iint_{\mathcal{B}_T} e^{-2s\beta t^{-3}} (T-t)^{-3} |\varphi|^2$$

and therefore

$$\iint_{\Omega \times (T/4, 3T/4)} |\varphi|^2 \leq e^{C(1+1/T+\|c\|_\infty^{2/3}+\|B\|_\infty^2)} \iint_{\mathcal{B}_T} |\varphi|^2.$$

This last inequality combined with energy estimates implies (10) and the proof is complete.  $\square$

**Corollary 2.** *Suppose  $T > 0$  and let  $\Omega \subset \mathbb{R}^N$  and  $\omega \subset \Omega$  be two nonempty bounded open sets with  $\Omega$  having a  $C^2$  boundary. Then, there exists a positive constant  $C$  (only depending on  $\Omega$ ,  $\omega$  and  $T$ ) such that for every  $y^0 \in L^2(\Omega)$  there is a control  $v \in L^2(\Omega)$  which satisfies*

$$\|v\|_{L^2(Q_T)}^2 \leq e^{C(1+1/T+\|c\|_\infty^{2/3}+\|B\|_\infty^2)} \|y^0\|^2,$$

and  $y(\cdot, T) = 0$  in  $\Omega$ , with  $y$  the solution to (6) associated to  $y^0$  and  $v$ .

**Remark 6.** Taking into account Theorem 2.2, Corollary 2 also implies a boundary null controllability result for problem (7). It is important to point out that this boundary null controllability result for problem (7) can be obtained from an appropriate boundary Carleman inequality when the coefficients  $b_i$  of  $L(t)$  (see (3)) are regular enough. This Carleman inequality is like (15) for an appropriate weight function  $\tilde{\beta}_0 \in C^2(\bar{\Omega})$  (which depends only on  $\Omega$  and  $\Gamma_0$ ) instead of  $\beta_0$  and with the local term

$$s^{d-2} \iint_{\Gamma_0 \times (0, T)} e^{-2s \frac{\tilde{\beta}_0}{t(T-t)}} \gamma(t)^{d-2} \left| \frac{\partial z}{\partial n} \right|^2$$

instead of the integral over  $\mathcal{B} \times (0, T)$  in the right hand side of (14) ( $z$  is the solution to (12) associated to  $z^T \in L^2(\Omega)$ ). For more details, see [39].

**2.2. The moment problem.** This method has been successfully used to prove the null boundary controllability result for scalar one-dimensional parabolic equations with coefficients independent of  $t$  (see [27]). Let us briefly recall this method in the case of the scalar one-dimensional heat equation.

It is well-known that the operator  $-\partial_{xx}$  on  $(0, \pi)$  with homogenous Dirichlet boundary conditions admits a sequence of eigenvalues and normalized eigenfunctions given by

$$\lambda_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad k \geq 1, \quad x \in (0, \pi) \quad (16)$$

which is a Hilbert basis of  $L^2(0, \pi)$ . Thus, if  $y \in L^2(0, \pi)$  there exists a unique sequence  $\{y_k\}_{k \geq 1} \subset \mathbb{R}$  such that

$$y = \sum_{k \geq 1} y_k \phi_k.$$

Let us consider the problem

$$\begin{cases} y_t - y_{xx} = 0 & \text{in } Q_T = (0, \pi) \times (0, T), \\ y(0, \cdot) = v, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y^0 & \text{in } (0, \pi), \end{cases} \quad (17)$$

with  $y^0 \in H^{-1}(0, \pi)$  and  $v \in L^2(0, T)$ . Again, problem (17) is well-posed and the solution  $y$  (defined by transposition) depends continuously on the data  $y^0$  and  $v$ . Let us study the null controllability properties of this problem.

Given  $y^0 \in H^{-1}(0, \pi)$ , there exists a control  $v \in L^2(0, T)$  such that the solution  $y$  to (17) satisfies  $y(\cdot, T) = 0$  in  $(0, \pi)$  if and only if there exists  $v \in L^2(0, T)$  satisfying

$$-\langle y^0, e^{-\lambda_k T} \phi_k \rangle_{H^{-1}(0, \pi), H_0^1(0, \pi)} = \int_0^T v(t) e^{-\lambda_k (T-t)} \partial_x \phi_k(0) dt, \quad \forall k \geq 1.$$

In the previous equality  $\lambda_k$  and  $\phi_k$  are given in (16).

Using the Fourier decomposition of  $y^0$ ,  $y^0 = \sum_{k \geq 1} y_k^0 \phi_k$ , this is equivalent to the existence of  $v \in L^2(0, T)$  such that

$$k \sqrt{\frac{2}{\pi}} \int_0^T e^{-\lambda_k (T-t)} v(t) dt = -e^{-\lambda_k T} y_k^0 \quad \forall k \geq 1. \quad (18)$$

This problem is called a *moment problem*.

We have the following result:

**Theorem 2.4.** *For any  $y^0 \in L^2(0, \pi)$  and  $T > 0$ , there exists  $v \in L^2(0, T)$  solution to the moment problem (18). That is,  $v$  is a null control for equation (17).*

*Idea of the proof to Theorem 2.4.* Let us recall that a family  $\{p_k\}_{k \geq 1} \subset L^2(0, T)$  is biorthogonal to  $\{e^{-\lambda_k t}\}_{k \geq 1}$  if it satisfies

$$\int_0^T e^{-\lambda_k t} p_l(t) dt = \delta_{kl}, \quad \forall (k, l) : k, l \geq 1.$$

In [27] and [28], the authors solve the previous moment problem by proving the existence of a biorthogonal family  $\{p_k\}_{k \geq 1}$  to  $\{e^{-\lambda_k t}\}_{k \geq 1}$  which satisfies the additional property: for every  $\epsilon > 0$  there exists a constant  $C(\epsilon, T) > 0$  such that

$$\|p_k\|_{L^2(0, T)} \leq C(\epsilon, T) e^{\epsilon \lambda_k}.$$

In fact, the control is obtained as a linear combination of  $\{p_k\}_{k \geq 1}$ , that is,

$$v(T-s) = \sqrt{\frac{\pi}{2}} \sum_{k \geq 1} \frac{1}{k} e^{-\lambda_k T} y_k^0 p_k(s)$$

and the previous bounds are used to prove that this combination converges in  $L^2(0, T)$ .  $\square$

**Remark 7.** As said before, the proof of Theorem 2.4 is a consequence of the existence of a biorthogonal family to the sequence  $\{e^{-\lambda_k t}\}_{k \geq 1}$ , with  $\lambda_k$  given by (16), which satisfies appropriate bounds. In fact, in [27] and [28] the authors prove a general result on existence of a biorthogonal family to  $\{e^{-\Lambda_k t}\}_{k \geq 1}$  which satisfies appropriate bounds for sequences  $\Lambda = \{\Lambda_k\}_{k \geq 1} \subset \mathbb{R}_+$  such that

$$\sum_{k \geq 1} \frac{1}{\Lambda_k} < \infty \quad \text{and} \quad |\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1.$$

for a constant  $\rho > 0$ .

In Section 6 we will deal with the boundary controllability of non scalar parabolic systems using the moment technique. The main difficulty in that framework is that, in general, the associated sequences  $\Lambda = \{\Lambda_k\}_{k \geq 1}$  are complex, with nonzero real and imaginary parts, and does not satisfy the ‘‘gap condition’’  $|\Lambda_k - \Lambda_l| \geq \rho |k - l|$  ( $k, l \geq 1$ ), for a positive  $\rho$ . Moreover, the sequence  $\Lambda$  may contain multiple eigenvalues  $\Lambda_k$ , having different geometric and algebraic multiplicities.

**3. Controllability of linear finite dimensional systems.** Let us consider System (1) in the finite dimensional case:  $H = \mathbb{C}^n$  and  $U = \mathbb{C}^m$  with  $n, m \geq 1$ ,

$$y' = Ay + Bu \quad \text{on } [0, T], \quad y(0) = y^0, \quad (19)$$

where  $A \in \mathcal{L}(\mathbb{C}^n)$  and  $B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$  are constant matrices,  $y^0 \in \mathbb{C}^n$  and  $u \in L^2(0, T; \mathbb{C}^m)$  is the control.

Controllability of such autonomous linear ordinary differential system is completely solved. To describe the controllability result for System (19), let us define the controllability matrix, called also the *Kalman matrix*:

$$[A | B] = [B | AB | A^2 B | \dots | A^{n-1} B] \in \mathcal{L}(\mathbb{C}^{nm}; \mathbb{C}^n). \quad (20)$$

On the other hand, let  $\{\theta_l\}_{1 \leq l \leq \hat{p}} \subset \mathbb{C}$  be the set of distinct eigenvalues of  $A^*$ . For  $l : 1 \leq l \leq \hat{p}$ , we denote by  $m_l$  the geometric multiplicity of  $\theta_l$ . The sequence  $\{w_{l,j}\}_{1 \leq j \leq m_l}$  will denote a basis of the eigenspace associated to  $\theta_l$ . Finally, all along the paper  $I_n \in \mathcal{L}(\mathbb{R}^n)$  will denote the identity matrix. With this notation, one has:

**Theorem 3.1.** *The following conditions are equivalent*

1. System (19) is exactly controllable at time  $T$ , for every  $T > 0$ .
2. There exists  $T > 0$  such that System (19) is exactly controllable at time  $T$ .
3.  $\text{rank } [A | B] = n$  (Kalman rank condition).
4.  $\ker [A | B]^* = \{0\}$ .
5. Hautus test:

$$\text{rank} \begin{pmatrix} A^* - \theta_l I_n \\ B^* \end{pmatrix} = n, \quad \forall l : 1 \leq l \leq \hat{p}.$$

6.  $\text{rank } [B^* w_{l,1} | B^* w_{l,2} | \dots | B^* w_{l,m_l}] = m_l$ , for every  $l$ , with  $1 \leq l \leq \hat{p}$ .

Item 1 to 5 are well-known and the proofs can be found for instance in [66] and [67]. Item 6 can be easily deduced from the Hautus test and reciprocally.

**Remark 8.** The previous result provides a complete answer to the controllability problem for System (19). In this case (finite dimensional case) the four controllability concepts are equivalent and are independent of the final observation time  $T > 0$ .

Consider now the case of time dependent matrices:

$$x' = A(t)x + B(t)u \quad \text{on } [0, T], \quad (21)$$

where  $A \in C^{n-2}([0, T]; \mathcal{L}(\mathbb{R}^n))$  and  $B \in C^{n-1}([0, T]; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$  are given and  $u \in L^2(0, T; \mathbb{R}^m)$  is a control.

In this case it is also possible to study the controllability properties of System (21). To this end, let us define

$$\begin{cases} B_0(t) = B(t), \\ B_i(t) = A(t)B_{i-1}(t) - \frac{d}{dt}B_{i-1}(t), \end{cases} \quad (22)$$

( $1 \leq i \leq n-1$ ) and, as in the autonomous case, we introduce the Kalman matrix denoted by  $[A | B] \in C^0([0, T]; \mathcal{L}(\mathbb{R}^{nm}; \mathbb{R}^n))$  and given by:

$$[A | B](t) = (B_0(t) | B_1(t) | \cdots | B_{n-1}(t)). \quad (23)$$

Let us remark that when  $A$  and  $B$  are constant matrices,  $[A | B] \in \mathcal{L}(\mathbb{R}^{nm}, \mathbb{R}^n)$  is the matrix defined in (20). With this notation, one has:

**Theorem 3.2 (Silverman-Meadows [61]).** *Under the previous assumptions, one has:*

1. *If there exists  $t_0 \in [0, T]$  such that  $\text{rank } [A | B](t_0) = n$ , then System (21) is exactly controllable at time  $T$ .*
2. *System (21) is totally exactly controllable on  $(0, T)$  if and only if there exists  $E$ , a dense subset of  $(0, T)$ , such that  $\text{rank } [A | B](t) = n$  for every  $t \in E$ .*

In the particular case in which  $A$  and  $B$  are constant matrices, the exact controllability of System (21) is equivalent to the Kalman rank condition (20).

**Remark 9.** The first item in Theorem 3.2 gives a sufficient condition for the controllability of System (21) on  $(0, T)$ . In this time-dependent case this condition is not necessary (see [20]). Nevertheless, when  $A$  and  $B$  are analytic on  $(0, T)$  this condition is also necessary.

Again, the four controllability concepts for System (21) are equivalent but, in this case the positive controllability result depends on the final observation time  $T > 0$ .

A natural question is: what is the suitable extension of the Kalman rank condition to parabolic systems? Extensions of this rank condition to infinite dimensional systems have already been discussed by R. Triggiani [65] (see also [59, Proposition 7.1]). The work [65] is dedicated essentially to the characterization of the approximate controllability of abstract parabolic systems.

This extension is also the main interest of this paper. We will see that in the case of some non scalar parabolic systems with constant coefficients or time dependent coefficients, one can still define a Kalman condition which characterizes (in the sense of Theorems 3.1 and 3.2) both, the null and approximate controllability of these systems. Nevertheless for some systems this extended Kalman condition could be different if we deal with an approximate or a null controllability problem (see Remark 14, Subsection 5.3 and Remark 28). Besides, even if one can give

sufficient conditions for controllability of systems of 2 equations or *cascade* systems (see subsection 7.1), the general question of controllability of parabolic systems of non constant coefficients is widely open. Note that this question is open even if the number of controls  $m$  is equal to the number of equations  $n$  when  $n \geq 4$  (see [33]).

**4. Two examples of non scalar parabolic systems.** In this section we will study the controllability properties of two  $2 \times 2$  linear reaction-diffusion systems. Let us first consider the one-dimensional system

$$\begin{cases} y_t - Dy_{xx} = Ay + Bv1_\omega & \text{in } Q_T = (0, \pi) \times (0, T), \\ y = 0 & \text{on } \Sigma_T = \{0, \pi\} \times (0, T), \\ y(\cdot, 0) = y^0 & \text{in } (0, \pi). \end{cases} \quad (24)$$

Here  $\omega \subset (0, \pi)$  is a nonempty open interval,  $T > 0$ ,  $y^0 \in L^2(0, \pi; \mathbb{R}^2)$  is given,  $v \in L^2(Q_T)$  is the control function,  $y = (y_1, y_2)^*$  is the state and  $D, A \in \mathcal{L}(\mathbb{R}^2)$  and  $B \in \mathbb{R}^2$  are given by

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

On the other hand, let us also consider the boundary controllability problem:

$$\begin{cases} y_t - Dy_{xx} = Ay & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y^0 & \text{in } (0, \pi), \end{cases} \quad (25)$$

with  $y^0 \in H^{-1}(0, \pi; \mathbb{R}^2)$  given,  $v \in L^2(0, T)$  is the control and  $D, A$  and  $B$  as before.

Both problems are well-posed and the corresponding solution depends continuously on the data  $y^0$  and  $v$ . Observe that in the case of System (25) the solution  $y$  is defined by transposition.

In both cases, we want to control to zero the two variables  $y_1$  and  $y_2$  and to this end we have at our disposal only one control. This control  $v$  acts only in the first equation as a distributed control for System (24) or a boundary control for System (25).

The controllability properties of Systems (24) and (25) at time  $T$  are related to some properties of the adjoint system:

$$\begin{cases} -\varphi_t - D\varphi_{xx} = A^*\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = \varphi^T & \text{in } (0, \pi). \end{cases} \quad (26)$$

To be precise, the null controllability at time  $T > 0$  of System (24) is equivalent to the following *observability inequality* for (26): there exists a constant  $C > 0$  such that for any  $\varphi^T \in L^2(0, \pi; \mathbb{R}^2)$  the solution  $\varphi$  to (26) satisfies

$$\|\varphi_1(\cdot, 0)\|_{L^2(0, \pi; \mathbb{R}^2)}^2 + \|\varphi_2(\cdot, 0)\|_{L^2(0, \pi; \mathbb{R}^2)}^2 \leq C \iint_{\omega_T} |\varphi_1(x, t)|^2. \quad (27)$$

On the other hand, System (25) is null controllable at time  $T$  if and only if the following observability inequality for (26) holds: there exists a constant  $C > 0$  such that for any  $\varphi^T \in H_0^1(0, \pi; \mathbb{R}^2)$  the solution  $\varphi$  to (26) satisfies

$$\|\varphi_1(\cdot, 0)\|_{H_0^1(0, \pi; \mathbb{R}^2)}^2 + \|\varphi_2(\cdot, 0)\|_{H_0^1(0, \pi; \mathbb{R}^2)}^2 \leq C \int_0^T |\varphi_{1,x}(0, t)|^2 dt. \quad (28)$$

One has (see [6]):

**Theorem 4.1.** *System (24) is null controllable at time  $T$  if and only if*

$$\det[A | B] \neq 0.$$

*Proof.* It is clear that condition  $\det[A | B] \neq 0$  (i.e.  $a_{2,1} \neq 0$ ) is necessary (if not, Systems (24) and (26) are decoupled).

Let us see the sufficient condition. Following [63], take  $\mathcal{B} = \omega_1 \subset\subset \omega$  a new open interval and apply the Carleman estimate (15) with  $d = 3$  to each scalar equation of the adjoint System (26). If we add these two inequalities and we choose  $s$  large enough, it is possible to get rid of the coupling terms and obtain

$$\sum_{i=1}^2 \mathcal{I}_1(3, \varphi_i) \leq s^3 \iint_{\omega_{1,T}} e^{-2s\beta\gamma(t)^3} (|\varphi_1|^2 + |\varphi_2|^2), \quad (29)$$

for all  $s \geq \tilde{s}_2 = \tilde{\sigma}_2(T + T^2)$  ( $\omega_{1,T} = \omega_1 \times (0, T)$ ).

Our next task will be to remove the local term for  $\varphi_2$  of the previous inequality. To this end, we will use the assumption  $a_{2,1} \neq 0$  and the first equation of (26):

$$a_{2,1}\varphi_2 = -\varphi_{1,t} - d_1\varphi_{1,xx} - a_{1,1}\varphi_1 \quad \text{in } Q_T.$$

Let  $\xi \in C_c^\infty(\omega)$  be a truncation function satisfying  $0 \leq \xi(x) \leq 1$  in  $\omega$  and  $\xi \equiv 1$  in  $\omega_1$ . Multiplying the first equation of (26) by  $s^3\xi e^{-2s\beta\gamma(t)^3}\varphi_2$ , one obtains (after some integrations by parts) that for all  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that

$$\begin{aligned} s^3 \iint_{\omega_{1,T}} e^{-2s\beta\gamma(t)^3} |\varphi_2|^2 &\leq s^3 \iint_{\omega_T} e^{-2s\beta\gamma(t)^3} \xi |\varphi_2|^2 \\ &= \frac{-s^3}{a_{2,1}} \iint_{\omega_T} e^{-2s\beta\gamma(t)^3} \xi \varphi_2 (\varphi_{1,t} + d_1\varphi_{1,xx} + a_{1,1}\varphi_1) \\ &\leq \varepsilon \mathcal{I}_1(3, \varphi_2) + C_\varepsilon s^7 \iint_{\omega_T} e^{-2s\beta\gamma(t)^7} |\varphi_1|^2, \end{aligned}$$

for all  $s \geq \tilde{s}_2 = \tilde{\sigma}_2(T + T^2)$ . This last estimate together with (29) gives the existence of a constant  $C > 0$  such that

$$\mathcal{I}_1(3, \varphi_1) + \mathcal{I}_1(3, \varphi_2) \leq C s^7 \iint_{\omega_T} e^{-2s\beta\gamma(t)^7} |\varphi_1|^2 \quad \forall s \geq \tilde{s}_2.$$

As usual, using that  $D\partial_{xx} + A^*$  generates a strongly continuous semigroup on  $L^2(0, \pi; \mathbb{R}^2)$ , this implies the desired observability estimate (27) for the adjoint problem (26).  $\square$

**Remark 10.** For simplicity we have presented Theorem 4.1 in the one-dimensional case. Nevertheless this result and the proof are also valid in any  $N$ -dimensional bounded domain.

In order to prove Theorem 4.1 we have adapted the proof of a similar result given in [63] (see also [15]) to the one-dimensional case. On the other hand, this approach has been generalized to some non scalar parabolic systems (cascade systems) in [41] (see also Section 7.1).

Concerning the case of boundary control, in [32] the following result has been proved:

**Proposition 1.** *Assume that  $d_1 \neq d_2$ ,  $\sqrt{d_1/d_2} \in \mathbb{Q}$ ,  $a_{1,1} = a_{1,2} = a_{2,2} = 0$  and  $a_{2,1} = 1$ . Then, for all  $T > 0$ , System (25) is not null controllable at time  $T$ .*

*Proof.* Suppose that  $d_1 \neq d_2$ . Then we have the following expression for the solution to (26):

$$\varphi(x, t) = \sum_{j \geq 1} \begin{pmatrix} \left( a_j - \frac{b_j}{(d_1 - d_2)\lambda_j} \right) e^{-d_1\lambda_j(T-t)} + \frac{b_j}{(d_1 - d_2)\lambda_j} e^{-d_2\lambda_j(T-t)} \\ b_j e^{-d_2\lambda_j(T-t)} \end{pmatrix} \sin jx,$$

with

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \int_0^\pi \varphi^T(x) \sin(jx) dx \in \mathbb{R}^2$$

and

$$\varphi_{1,x}(0, t) = \sum_{j \geq 1} j \left( \left( a_j - \frac{b_j}{(d_1 - d_2)\lambda_j} \right) e^{-d_1\lambda_j(T-t)} + \frac{b_j}{(d_1 - d_2)\lambda_j} e^{-d_2\lambda_j(T-t)} \right).$$

Assume that  $\sqrt{d_1/d_2} \in \mathbb{Q}$ , then  $d_1/d_2 = j_0^2/i_0^2$  for some  $i_0, j_0 \geq 1$  and  $i_0 \neq j_0$ . Take now  $\varphi^T$  such that  $a_j = b_j = 0$  for  $j \neq i_0$  and  $j \neq j_0$ . Now, take

$$a_{i_0} = 1, \quad b_{i_0} = 0, \quad a_{j_0} = i_0/j_0 \quad \text{and} \quad b_{j_0} = -i_0j_0(d_1 - d_2).$$

One can check (see [32]) that this solution is such that

$$\varphi_{1,x}(0, t) = 0 \quad \text{and} \quad \varphi(x, 0) \neq 0,$$

so the observability inequality (28) is not true for any  $C > 0$ .  $\square$

**Remark 11.** The previous result, in fact, establishes that System (25) is not approximately controllable in this case: “Assume that  $d_1 \neq d_2$ ,  $\sqrt{d_1/d_2} \in \mathbb{Q}$ ,  $a_{1,1} = a_{1,2} = a_{2,2} = 0$  and  $a_{2,1} = 1$ . Then, for all  $T > 0$ , System (25) is not approximately controllable at time  $T$ ”.

Proposition 1 can be extended to general matrices  $A \in \mathcal{L}(\mathbb{R}^2)$ . Indeed, let us consider the eigenvalues  $\mu_{1,k}$  and  $\mu_{2,k}$  of the matrix  $-k^2D + A^*$ ,  $k \geq 1$ . Assume that, for some  $k, j \geq 1$ , with  $k \neq j$ , we have  $\mu_{1,k} = \mu_{2,j}$ . Then, for all  $T > 0$ , System (25) is not approximately controllable at time  $T$  (see [8]).

**Remark 12.** Even if System (25) is very close to System (24), their controllability properties are strongly different. For System (25) (distributed control) we have obtained a complete characterization of the null controllability property. In fact, we have proved a distributed Carleman estimate for the adjoint problem (26). Summarizing, the same non scalar parabolic problem can be controlled to zero with distributed controls supported on an interval  $\omega_1$  and, however, the null controllability result fails when the control acts by means of the Dirichlet condition on a part of the boundary. This shows the different nature of scalar and non scalar problems regarding controllability properties.

**Remark 13.** When  $d_1 = d_2$  the controllability properties of System (25) are well-known and are not equivalent to those of System (24) (see [32], [7] and Section 6).

**Remark 14.** Let us go back to the boundary controlled system (25) in the simple case

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

At first sight the controllability properties of this system could seem very simple. Nevertheless, the situation is very intricate:



1. **Approximate controllability:** In [32] the authors give a necessary and sufficient condition for the boundary approximate controllability of System (25): “Under the previous assumptions, System (25) is approximately controllable at time  $T$  if and only if

$$d_1 = d_2 \text{ or } \sqrt{d_1/d_2} \notin \mathbb{Q}.”$$

2. **Null controllability:** When  $d_1 = d_2$  System (25) is null controllable at time  $T$ , then, a natural question is the following one: Is System (25) null controllable at time  $T$  when  $\sqrt{d_1/d_2} \notin \mathbb{Q}$ ? i.e., are approximate controllability and null controllability equivalent for System (25)? The answer is negative. In [57], the authors provide an example of matrix  $D$  satisfying  $\sqrt{d_1/d_2} \notin \mathbb{Q}$  (and therefore, the system is approximately controllable at any positive time  $T$ ) and such that System (25) is not controllable at any time  $T > 0$ . Then, approximate controllability and null controllability are not equivalent for the boundary controlled System (25) (see Subsection 5.3 for a similar result).

Observe that in this case the system has a sequence of simple positive real eigenvalues  $\Lambda = \{d_1 \lambda_l, d_2 \lambda_l\}_{l \geq 1}$  ( $\lambda_l$  given in (16)) which satisfies the condition

$$\sum_{k \geq 1} \frac{1}{\Lambda_k} < \infty.$$

This condition assures the existence of a biorthogonal family  $\{p_k\}_{k \geq 1}$  to the exponential family  $\{e^{-\Lambda_k t}\}_{k \geq 1}$  (see [32]). However the “gap condition”

$$|\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1$$

fails and this condition is strongly connected with the bounds of the  $L^2$ -norm of the biorthogonal family  $\{p_k\}_{k \geq 1}$ .

**5. Distributed control.** The main goal of this section is to give an extension of the algebraic Kalman rank condition to a class of parabolic systems where the controls act in the right hand side of the system and are supported on a subdomain  $\omega$  of the domain  $\Omega$  (distributed controls).

For  $n, m \in \mathbb{N}^*$  and  $T > 0$ , we consider the following  $n \times n$  parabolic system

$$\begin{cases} \partial_t y = (-DL(t) + A(t))y + B(t)v1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \quad (30)$$

where  $L$  is given by (3), and satisfies (4) and (5). The diffusion matrix  $D$  is assumed to be diagonalizable with positive real eigenvalues, i.e., for  $J = \text{diag}(d_i)_{n \times n}$ , with  $d_1, d_2, \dots, d_n > 0$ , one has

$$D = P^{-1}JP, \text{ with } P \in \mathcal{L}(\mathbb{R}^n), \det P \neq 0.$$

Moreover,

$$A(\cdot) \in C^{M-1}([0, T]; \mathcal{L}(\mathbb{R}^n)) \text{ and } B(\cdot) \in C^M([0, T]; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) \quad (31)$$

for an integer  $M \geq n$ . Moreover, in System (30),  $y^0 = (y_i^0)_{1 \leq i \leq n} \in L^2(\Omega; \mathbb{R}^n)$  and  $v \in L^2(Q_T; \mathbb{R}^m)$  (the control) are given and  $y = (y_i)_{1 \leq i \leq n}$  is the associated state. Observe that, again, we want to lead the solution to zero ( $n$  variables) and to this end we have at our disposal  $m$  distributed controls. Of course, the most interesting case is when  $m < n$ .

Thanks to the assumptions on  $L(t)$  (see (4) and (5)) and the structure of System (30), one can apply the Faedo-Galerkin method to obtain that the problem is well-posed, i.e., for any  $y^0 \in L^2(\Omega; \mathbb{R}^n)$  and  $v \in L^2(Q_T; \mathbb{R}^m)$  problem (30) admits a unique solution

$$y \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^n)) \cap C^0([0, T]; L^2(\Omega; \mathbb{R}^n)),$$

and this solution depends continuously on the data  $y^0$  and  $v$ .

When  $D$  is the identity matrix, the analysis of the controllability properties of System (30) is easier and the controllability results are very close to the results for finite dimensional systems. We will start analyzing this case.

**5.1. Identity diffusion matrix and non autonomous systems.** In this section we consider the non autonomous System (30) in the simpler case  $D \equiv I_n$ . The following result was proved in [5].

**Theorem 5.1.** *Assume that  $D = I_n$  and that the matrices  $A$  and  $B$  satisfy (31). Then, the following holds:*

1. *If there exist  $t_0 \in [0, T]$  and  $p \in \{1, \dots, M\}$  such that*

$$\text{rank}(B_0 | B_1 | \dots | B_{p-1})(t_0) = n, \quad (32)$$

*where  $B_i$  is given by (22), then System (30) is null controllable at time  $T$ .*

2. *System (30) is null controllable on every interval  $(T_0, T_1)$  with  $0 \leq T_0 < T_1 \leq T$  if and only if there exists  $E$  a dense subset of  $(0, T)$  such that  $\text{rank}[A | B](t) = n$  for every  $t \in E$ , (or, equivalently,*

$$\text{rank}(B_0 | B_1 | \dots | B_{p-1})(t) = n$$

*for all  $p \in \{n, \dots, M\}$  and  $t \in E$ ).*

*Sketch of the proof.* The proof uses in an essential way the assumption  $D = I_n$ . Thanks to the assumption (32) and using that  $M \geq n$ , it is possible to deduce the existence of an interval  $(T_0, T_1) \subseteq (0, T)$  such that

$$\text{rank} K_n(t) = n, \quad \forall t \in [T_0, T_1],$$

with  $K_n(t) = (B_0 | B_1 | \dots | B_{n-1})(t)$ . This last condition allows to perform a change of variables on the interval  $[T_0, T_1]$  and rewrite System (30) in a canonical form in the interval  $(T_0, T_1)$ . In particular this last system is a *cascade system* for which the result of Subsection 7.1 (see also [41]) can be applied. This implies the null controllability result on the interval  $(T_0, T_1)$  and then, at time  $T$ . For the details, see [5].  $\square$

**Remark 15.** It is interesting to point out that the assumptions in Theorems 3.2 and 5.1 are slightly different. Observe that the assumption in Theorem 3.2 is (32) for  $M = n$  and  $p = n$  and then, Theorem 5.1 has been proved under more general conditions. On the other hand if  $M = n$ , the regularity conditions on  $A$  and  $B$  (see (31)) in Theorem 5.1 are stronger than in Theorem 3.2.

Recall that  $\Omega \subset \mathbb{R}^N$  is an open set ( $N \geq 1$ ). In [5] the authors prove a better result: a Carleman type inequality for the system:

$$\begin{cases} -\partial_t \varphi + L^*(t)\varphi = A^*(t)\varphi + F_0 + \nabla \cdot F & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = \varphi^T & \text{in } \Omega, \end{cases} \quad (33)$$

where  $F_0 = (F_0^1, \dots, F_0^n)^* \in L^2(Q_T; \mathbb{R}^n)$ ,

$$F := (F^1 | F^2 | \dots | F^n) \in L^2(Q_T; \mathcal{L}(\mathbb{R}^n; \mathbb{R}^N)), \quad (\text{with } F^j : Q_T \rightarrow \mathbb{R}^N),$$

$\varphi^T \in L^2(\Omega; \mathbb{R}^n)$  and where, by means of  $\nabla \cdot F$ , we are denoting the column vector  $\nabla \cdot F := (\nabla \cdot F^1, \nabla \cdot F^2, \dots, \nabla \cdot F^n)^*$ . Thus, one has,

**Lemma 5.2.** *Let us assume that  $A$  and  $B$  satisfy hypothesis (31). Then, there exists a positive function  $\beta_0 \in C^2(\bar{\Omega})$  (only depending on  $\Omega$  and  $\omega$ ) such that, if (32) is fulfilled, there exist a time interval  $(T_0, T_1) \subseteq (0, T)$ , two positive constants*

$$\begin{cases} C_0 = C_0(\Omega, \omega, (\alpha_{ij})_{1 \leq i, j \leq N}, n, m, A(\cdot), B(\cdot)) \text{ and} \\ \sigma_0 = \sigma_0(\Omega, \omega, (\alpha_{ij})_{1 \leq i, j \leq N}, n, m, A(\cdot), B(\cdot)) \end{cases}$$

and integers  $\ell \geq 3$ ,  $\ell^1 \geq 0$  and  $\ell^2 \geq 2$  (only depending on  $n$ ,  $m$ ,  $A(\cdot)$  and  $B(\cdot)$ ) such that, for every  $\varphi^T \in L^2(\Omega; \mathbb{R}^n)$ , the solution  $\varphi$  to (33) satisfies

$$\begin{cases} \tilde{\mathcal{I}}(3, \varphi) \leq C_0 \left( s^\ell \iint_{\omega \times (T_0, T_1)} e^{-2s\tilde{\beta}} \tilde{\gamma}(t)^\ell |B^* \varphi|^2 \right. \\ \left. + s^{\ell^1} \iint_{\Omega \times (T_0, T_1)} e^{-2s\tilde{\beta}} \tilde{\gamma}(t)^{\ell^1} |F_0|^2 + s^{\ell^2} \iint_{\Omega \times (T_0, T_1)} e^{-2s\tilde{\beta}} \tilde{\gamma}(t)^{\ell^2} |F|^2 \right), \end{cases}$$

for every  $s \geq s_0 = \sigma_0 \left( \tilde{T} + \tilde{T}^2 + \tilde{T}^2 \|c\|_\infty^{2/3} + \tilde{T}^2 \|b\|_\infty^2 \right)$  with  $\tilde{T} = T_1 - T_0$ . In the previous inequality  $\tilde{\beta}(x, t)$ ,  $\tilde{\gamma}(t)$  and  $\tilde{\mathcal{I}}(d, z)$  are respectively given by  $\tilde{\beta}(x, t) \equiv \beta_0(x)/(t - T_0)(T_1 - t)$ ,  $\tilde{\gamma}(t) \equiv ((t - T_0)(T_1 - t))^{-1}$  and

$$\tilde{\mathcal{I}}(d, z) \equiv s^{d-2} \iint_{\Omega \times (T_0, T_1)} e^{-2s\tilde{\beta}} \tilde{\gamma}(t)^{d-2} |\nabla z|^2 + s^d \iint_{\Omega \times (T_0, T_1)} e^{-2s\tilde{\beta}} \tilde{\gamma}(t)^d |z|^2.$$

**Remark 16.** As in the case of finite-dimensional linear systems, it is interesting to point out that the existence of  $t_0 \in [0, T]$  satisfying (32) is not a necessary condition to have the null controllability at time  $T$  of System (30). For an explicit example, see [5].

As for finite-dimensional systems, when  $A$  and  $B$  are analytic functions on  $[0, T]$ , the condition of Theorem 5.1 1 is then a necessary and sufficient condition for the null controllability of System (30).

**Remark 17.** The results stated in Theorem 5.1 and Lemma 5.2 and their proofs are still valid if, instead of Dirichlet boundary conditions, Neumann or Fourier boundary conditions are considered in System (30).

**Remark 18.** The proofs of Theorem 5.1 and Lemma 5.2 strongly depend on the assumption on the diffusion matrix  $D$  ( $D$  is the identity matrix). In fact, the null controllability result for System (30) is similar to Theorem 3.2 and only depends on  $A$  and  $B$ . In the next subsection see will see that, even if  $D$  is a diagonalizable matrix, the null controllability of System (30) depends also on the matrix  $D$ .

**5.2. Diagonal diffusion matrix and autonomous systems.** In this subsection we consider the controllability problem for System (30) in the autonomous case with general diagonalizable diffusion matrix  $D$ . So, let us consider the scalar parabolic operator  $L$  given in (3) and satisfying (4) and (5), with the assumptions:

$$\alpha_{ij}(x, t) = \alpha_{ij}(x), \quad b_i = 0, \quad c(x, t) = c(x) \quad (1 \leq i, j \leq n),$$

and  $A \in \mathcal{L}(\mathbb{R}^n)$  and  $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ .

In particular, the operator  $L$  with homogenous Dirichlet boundary conditions has a sequence of positive eigenvalues  $\{\lambda_k\}_{k \geq 1}$  and normalized eigenfunctions  $\{\phi_k\}_{k \geq 1}$ .

Unlike the previous case, it is not possible to reduce the controllability of System (30) to an algebraic Kalman condition involving only matrices  $A$  and  $B$ . Let us see this point. Again, the null controllability property for System (30) is equivalent to the *observability inequality*: there exists a positive constant  $C$  such that

$$\|\varphi(\cdot, T)\|_{L^2(\Omega; \mathbb{R}^n)} \leq C \iint_{\omega_T} |B^* \varphi(x, t)|^2, \quad \forall \varphi^T \in L^2(\Omega; \mathbb{R}^n), \quad (34)$$

with  $\varphi$  the solution of the adjoint system

$$\begin{cases} -\partial_t \varphi = (-D^* L + A^*) \varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(\cdot, T) = \varphi^T & \text{in } \Omega, \end{cases} \quad (35)$$

corresponding to  $\varphi^T \in L^2(\Omega; \mathbb{R}^n)$ . Observe that if we take  $\varphi^T = a \phi_k$  with  $k \geq 1$  and  $a \in \mathbb{R}^n$  arbitrary, in particular, inequality (34) provides the controllability of the finite-dimensional system

$$z' = (-\lambda_k D + A)z + Bu, \quad \text{on } [0, T], \quad z(0) = z^0 \in \mathbb{R}^n.$$

Applying Theorem 3.1 we deduce the necessary condition

$$\text{rank}[-\lambda_k D + A | B] = n, \quad \forall k \geq 1. \quad (36)$$

An interesting question is the following one: is condition (36) sufficient to assure the null controllability of System (30)? Let us see this question.

We denote by  $W$  the operator given by  $W := -DL + A$  with domain given by  $\mathcal{D}(W) = \mathcal{D}(L)^n = (H^2(\Omega) \cap H_0^1(\Omega))^n$ . We define the Kalman operator associated with  $(W, B)$  as the unbounded matrix operator

$$\begin{cases} \mathcal{K} := [W | B] = [-DL + A | B] : \mathcal{D}(\mathcal{K}) \subset L^2(\Omega; \mathbb{R}^{nm}) \longrightarrow L^2(\Omega; \mathbb{R}^n), \text{ with} \\ \mathcal{D}(\mathcal{K}) := \{u \in L^2(\Omega; \mathbb{R}^{nm}) : \mathcal{K}u \in L^2(\Omega; \mathbb{R}^n)\}, \end{cases}$$

where  $[W | B]$  is defined as in (20). One can check (see [6]) that (36) is equivalent to the condition

$$\text{Ker}(\mathcal{K}^*) = \{0\}. \quad (37)$$

In [6] is proved the following result:

**Theorem 5.3.** *Let  $L$  be the operator given by (3) and satisfying (4) and (5) with  $b_i = 0$  and time independent coefficients  $\alpha_{ij}$  and  $c$  ( $1 \leq i, j \leq N$ ). Then, the following conditions are equivalent:*

1. System (30) is null controllable at any time  $T > 0$ .
2. System (30) is approximately controllable at any time  $T > 0$ .
3. The Kalman operator  $\mathcal{K}$  satisfies (37).

*Sketch of the proof.* First, the previous reasoning shows that (37) (or equivalently, condition (36)) is a necessary condition for the null and approximate controllability at time  $T > 0$  of System (30).

The sufficient conditions are based on two main arguments:

**First argument:** In [6] the authors prove a Carleman type estimate for the solutions of adjoint System (35). To be precise:

**Lemma 5.4 (Carleman estimate).** *Given  $d \in \mathbb{R}$  and  $k \in \mathbb{N}$ , there exist two positive constants  $C$  and  $\sigma$  (only depending on  $\Omega$ ,  $\omega$ ,  $n$ ,  $L$ ,  $D$ ,  $A$ ,  $k$  and  $d$ ) such that for every  $\varphi^T \in L^2(\Omega; \mathbb{R}^n)$  the corresponding solution  $\varphi$  to (35) satisfies*

$$\int_0^T (s\gamma(t))^d e^{-2sM_0\gamma(t)} \|L^k \mathcal{K}^* \varphi\|_{L^2(\Omega)^{nm}}^2 \leq C \iint_{\omega_T} (s\gamma(t))^{d+K} e^{-2s\beta} |B^* \varphi|^2, \quad (38)$$

for every  $s \geq \sigma \left( T + T^2 + T^2 \|c\|_\infty^{2/3} \right)$ . In (38), the functions  $\beta_0$ ,  $\gamma$  and  $\beta$  are given in Lemma 2.3 and  $M_0$  and  $K$  are respectively given by:  $M_0 = \max_{\bar{\Omega}} \beta_0$  and  $K = 4k - 4 + r(n)$ , for some  $r$  only depending on  $n$ .

It is interesting to point out that estimate (38) is obtained by means of a scalar parabolic equations of order  $2n$  and gives a partial observability inequality. It is valid even if condition (37) fails. On the other hand, combining condition (37) and inequality (38) for  $k = 0$ , we deduce the continuation property for the solutions of (35): “If  $\varphi$  is the solution to (35) associated to  $\varphi^T \in L^2(\Omega; \mathbb{R}^n)$  and  $\varphi \equiv 0$  in  $\omega_T$ , then  $\varphi \equiv 0$  in  $Q_T$ ”. As in the scalar case, this property is equivalent to the approximate controllability of (30) at time  $T$ . Therefore, point 3 implies point 2.

**Second argument:** In order to prove that condition (37) implies the null controllability result at time  $T$  for System (30), we need a better property of the operator  $\mathcal{K}^*$ : The coercivity of  $\mathcal{K}^*$ . One has,

**Lemma 5.5 (Invertibility of  $\mathcal{K}^*$  and continuity of the inverse).** *Fix  $k \geq (n-1)(2n-1)$ . Then, condition (37) holds if and only if there exists  $C > 0$  such that*

$$\|\varphi\|_{L^2(\Omega; \mathbb{R}^n)}^2 \leq C \|L^k \mathcal{K}^* \varphi\|_{L^2(\Omega; \mathbb{R}^{nm})}^2 \quad \forall \varphi \in L^2(\Omega; \mathbb{R}^n) \text{ satisfying } \mathcal{K}^* \varphi \in \mathcal{D}(L^k)^{nm}. \quad (39)$$

Using assumption (37), Lemma 5.5 and inequality (38) for  $k = (n-1)(2n-1)$ , we directly deduce the observability inequality (34). This proves that point 3 implies point 1 and ends the proof.  $\square$

**Remark 19.** Observe that, in general, inequality (39) is not valid if  $k < (n-1)(2n-1)$ . For instance, let us take  $n = 2$ ,  $k = 0$  and  $B \equiv e_1$ . Then inequality (39) amounts to the property  $0 \in \rho(\mathcal{K}\mathcal{K}^*)$ , where  $\rho(\mathcal{K}\mathcal{K}^*)$  is the resolvent set of the unbounded operator  $\mathcal{K}\mathcal{K}^*$ . In this example, if  $\mathcal{K}_k = [-\lambda_k D + A \mid B]$  (recall that  $(\lambda_k)_{k \geq 1}$  is the sequence of positive eigenvalues of  $L$ ) then  $\sigma(\mathcal{K}\mathcal{K}^*) = \cup_{k \geq 1} \sigma(\mathcal{K}_k \mathcal{K}_k^*)$  as it can be checked by expanding any function  $\varphi \in L^2(\Omega; \mathbb{R}^2)$  as  $\varphi = \sum_{k \geq 1} V_k \phi_k$ ,  $V_k \in \mathbb{R}^2$  ( $k \geq 1$ ). But

$$\mathcal{K}_k \mathcal{K}_k^* = \begin{bmatrix} 1 + (-d_1 \lambda_k + a_{11})^2 & a_{12} (-d_1 \lambda_k + a_{11}) \\ a_{12} (-d_1 \lambda_k + a_{11}) & a_{12}^2 \end{bmatrix}, \quad k \geq 1.$$

Thus  $\lambda$  is an eigenvalue of  $\mathcal{K}\mathcal{K}^*$  if and only if there exists  $k \geq 1$  such that

$$\lambda^2 - \left( 1 + (-d_1 \lambda_k + a_{11})^2 + a_{12}^2 \right) \lambda + a_{12}^2 = 0.$$

This gives two families of eigenvalues

$$\mu_k^\pm = \frac{1}{2} \left( \alpha_k \pm \sqrt{\alpha_k^2 - 4a_{12}^2} \right), \quad \left( \alpha_k := 1 + (-d_1 \lambda_k + a_{11})^2 + a_{12}^2 \right), \quad k \geq 1.$$

Since  $\alpha_k \rightarrow +\infty$  as  $k \rightarrow \infty$ , it appears that  $\mu_k^- \rightarrow 0$  as  $k \rightarrow \infty$ . Then  $0 \notin \rho(\mathcal{K}\mathcal{K}^*)$ .

**Remark 20.** Condition (37) is independent of  $\omega$ . Therefore, if it fails, System (30) is not null controllable at time  $T$  even if  $\omega \equiv \Omega$ .

On the other hand, System (30) can be exactly controlled to zero with one control force ( $m = 1$ ) even if  $A \equiv 0$ . Indeed, let us assume that  $D = \text{diag}(d_i)_{n \times n}$ , with  $d_i > 0$  for every  $1 \leq i \leq n$ , and  $B = (b_1, \dots, b_n)^* \in \mathbb{R}^n$ . Then, (36) holds if and only if  $b_i \neq 0$  for every  $i$  and the diffusion coefficients  $d_i$  are distinct.

**Remark 21.** As said before, the Kalman condition (37) is equivalent to (36). Analyzing this last condition, one can prove that either there exists  $k_0 \in \mathbb{N}^*$  such that  $\text{rank } \mathcal{K}_k = n$  for every  $k > k_0$  or  $\text{rank } \mathcal{K}_k < n$  for every  $k \in \mathbb{N}^*$ , where  $\mathcal{K}_k = [-\lambda_k D + A | B]$ . For more details, see [6].

**Remark 22.** Theorem 5.3 provides a necessary and sufficient condition, condition (36), for the distributed controllability of System (30) at time  $T > 0$ . In particular, the previous result shows that, as in the scalar case, System (30) is null controllable at time  $T$  if and only if it is approximate controllable at time  $T$ . In Section 4, we gave a boundary controlled system (see Remark 14) where this equivalence fails. In the next subsection we will give another example, this time a distributed controlled system, where the previous property also fails.

**5.3. Approximate controllability without null controllability: the example of an abstract parabolic system.** Let  $L$  be as in the previous section and let us consider the corresponding sequence of positive eigenvalues  $\{\lambda_k\}_{k \geq 1}$  and normalized eigenfunctions  $\{\phi_k\}_{k \geq 1}$ . Let us also consider a real function  $f$  defined on the spectrum of  $L$ ,  $\sigma(L) = \{\lambda_k\}_{k \geq 1}$ , and define

$$f(L) := \sum_k f(\lambda_k) \langle \cdot, \phi_k \rangle \phi_k.$$

Assume that  $f : \sigma(L) \rightarrow \mathbb{R}$  satisfies

$$0 < |f(s)| < s, \quad \forall s > 0, \quad \text{and} \quad f(s) = o(s) \quad \text{as} \quad s \rightarrow \infty.$$

We now consider the parabolic system

$$\begin{cases} \partial_t y = (-I_2 L + A f(L))y + Bv & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \quad (40)$$

where  $n = 2$ ,  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  ( $m = 1$ ). As before  $v \in L^2(Q_T; \mathbb{R}^2)$  is a distributed control. Under these assumptions, in [4] the following result has been proved:

**Theorem 5.6.** *Under the previous assumptions, one has:*

- 1) System (40) is approximate controllable at time  $T$ , for any  $T > 0$ .
- 2) System (40) is null-controllable at time  $T > 0$  if and only if the function  $s^{-3} f^2(s) e^{2Ts}$  stays bounded away from zero as  $s \rightarrow \infty$ .

*Sketch on the proof.* The approximate controllability property can be easily checked proving a unique continuation property for the adjoint problem to (40).

On the other hand, as above, the null controllability result for System (40) amounts to an appropriate *observability inequality* for the corresponding adjoint

system. In terms of quadratic forms, this observability inequality reads as follows:

$$e^{-2WT} \leq C_T \int_0^T e^{-Wt} P e^{-Wt} dt \quad (41)$$

where  $W$  and  $P$  are the operators given by

$$W = \begin{pmatrix} L & -f(L) \\ -f(L) & L \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

One can see that the eigenvalues of the  $2 \times 2$  matrix  $W$  are  $\lambda_{\pm}(L)$ , where  $\lambda_{\pm}(s) = s \pm |f(s)|$ , and that the orthogonal matrix

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

diagonalizes  $W$  with

$$\Lambda = \begin{pmatrix} \lambda_+(L) & 0 \\ 0 & \lambda_-(L) \end{pmatrix} = VWV^*.$$

For details, see [4].

For any real  $s$  we set  $\eta(s) = (e^s - 1)/s$ . Computing explicitly the integral at the right-hand side of (41), one can rewrite the null controllability condition in the form

$$I \leq \frac{C_T}{2} B_T(L) \quad (42)$$

where

$$B_T(s) = \begin{pmatrix} \eta(2T\lambda_+(s)) & -\eta(-T\lambda_+(s) - T\lambda_-(s)) \\ -\eta(-T\lambda_+(s) - T\lambda_-(s)) & \eta(2T\lambda_-(s)) \end{pmatrix}.$$

Using that  $B_T(\cdot)$  is symmetric and after some computations, we can prove the result (see [4] for the details).  $\square$

**Remark 23.** Observe that System (40) is approximately controllable at any time  $T > 0$ . Nevertheless, if we take  $f(s) = e^{-s^{1+\varepsilon}}$  with  $\varepsilon > 0$ , from the previous result we deduce that the system is not null-controllable at any positive time  $T$ . Thus, we have another example of non scalar parabolic system which is approximately controllable at a given  $T > 0$  and is not null controllable at this time  $T$ .

Observe also that if  $f(s)$  has a polynomial decay when  $s \rightarrow \infty$ , then the system is null-controllable at any  $T > 0$ . On the other hand, taking  $f(s) = e^{-s}$ , as a consequence of Theorem 5.6 we deduce that System (40) is null controllable at time  $T$  if and only if  $T > 1$ . In this case the system behaves as a hyperbolic system where a minimal time is required for the null controllability.

**Remark 24.** It is interesting to note that, as in the previous subsection, the Kalman operator is still well-defined for System (40):  $\mathcal{K} = [-I_2L + Af(L) | B]$ . In fact, taking into account the expression

$$\mathcal{K}^* \varphi = (\varphi_2, f(L)\varphi_1 - L\varphi_2), \quad \varphi = (\varphi_1, \varphi_2),$$

we deduce that the system is approximately controllable at time  $T$  if only if  $\text{Ker } \mathcal{K}^* = \{0\}$ . This last condition is equivalent to  $f \neq 0$ .

On the other hand, the operator  $\mathcal{K}$  satisfies estimate (39) for some  $k$  if and only if the operator  $L^k f(L)$  is bounded from below in  $L^2(\Omega; \mathbb{R}^2)$ . But, this last property amounts to

$$\liminf_{s \rightarrow \infty} |s^k f(s)| > 0.$$

Using Theorem 5.6, we can conclude that  $\mathcal{K}$  satisfies estimate (39) if and only if System (40) is null controllable at time  $T$  for any positive  $T$ .

**6. Boundary Control.** In this Section we will deal with boundary controllability problems for non scalar parabolic systems. As in Section 5, we want to provide necessary and sufficient conditions which characterize the controllability properties of such parabolic systems.

There are very few results in the literature concerning the boundary null controllability of coupled parabolic systems. To our knowledge, there are two results stated in the one dimensional framework (see [32] and [7]) and a third result stated in several dimensions (see [1]). In [32] and [7], the authors provide a necessary and sufficient condition for the boundary controllability of parabolic system when  $D$ , the diffusion matrix, is the identity and  $A$  and  $B$ , the coupling and control matrices, are constant ( $2 \times 2$  system in [32] and  $n \times n$  system in [7]). In [1], the authors deal with the  $N$ -dimensional case and give some sufficient conditions imposing appropriate geometric conditions. These conditions are inherited from the method, that consists in proving a result for coupled hyperbolic equations and then, using the Kannai transform, they obtain the result for parabolic equations.

In the remainder of this section we will only focus on the boundary controllability results in the one-dimensional case established in [32] and [7]. For the multidimensional results of [1] see Subsection 7.2.4.

Let us fix  $T > 0$  and let  $A \in \mathcal{L}(\mathbb{C}^n)$  and  $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$  be two given matrices. Thus, we consider the following one dimensional control problem: For all  $y^0 \in H^{-1}(0, \pi; \mathbb{C}^n)$ , find  $v \in L^2(0, T; \mathbb{C}^m)$  such that the corresponding solution to

$$\begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T = (0, \pi) \times (0, T), \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y^0 & \text{in } (0, \pi), \end{cases} \quad (43)$$

satisfies

$$y(\cdot, T) = 0 \quad \text{in } (0, \pi).$$

Let us remark that, for every  $v \in L^2(0, T; \mathbb{C}^m)$  and  $y^0 \in H^{-1}(0, \pi; \mathbb{C}^n)$ , System (43) possesses a unique solution (defined by transposition) which satisfies

$$y \in L^2(Q_T; \mathbb{C}^n) \cap C^0([0, T]; H^{-1}(0, \pi; \mathbb{C}^n))$$

and depends continuously on the data  $v$  and  $y^0$ , i.e., there exists a constant  $C = C(T) > 0$  such that

$$\|y\|_{L^2(Q_T; \mathbb{C}^n)} + \|y\|_{C^0([0, T]; H^{-1}(0, \pi; \mathbb{C}^n))} \leq C (\|y^0\|_{H^{-1}(0, \pi; \mathbb{C}^n)} + \|v\|_{L^2(0, T; \mathbb{C}^m)}).$$

In what follows, we set:

$$L_k = -\lambda_k I_n + A \in \mathcal{L}(\mathbb{C}^n) \text{ and } L_k^* = -\lambda_k I_n + A^* \in \mathcal{L}(\mathbb{C}^n), \quad \forall k \geq 1, \quad (44)$$

where  $\{\lambda_k\}_{k \geq 1}$  is the sequence of eigenvalues of the Dirichlet Laplacian on  $(0, \pi)$  and  $\phi_k$  is the normalized eigenfunction corresponding to  $\lambda_k$  (see (16)). For  $k \geq 1$ , let us introduce the matrices

$$B_k = \begin{pmatrix} B \\ \vdots \\ B \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^{nk}), \quad \mathcal{L}_k = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & L_k \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{nk}). \quad (45)$$



Let us define what we will call the *Kalman matrix* associated with the pair  $(\mathcal{L}_k, B_k)$ :

$$\mathcal{K}_k = [\mathcal{L}_k \mid B_k] = [B_k \mid \mathcal{L}_k B_k \mid \mathcal{L}_k^2 B_k \mid \cdots \mid \mathcal{L}_k^{nk-2} B_k \mid \mathcal{L}_k^{nk-1} B_k] \in \mathcal{L}(\mathbb{C}^{mnk}, \mathbb{C}^{nk}).$$

The following characterization of the null controllability at time  $T$  of sytem (43) explain why we have called Kalman matrix to the previous matrix. This result can be found in [7]:

**Theorem 6.1.** *Under the previous assumption, the following conditions are equivalent*

1. System (43) is null controllable at any time  $T$ .
2. System (43) is approximately controllable at any time  $T$ .
3. The pair  $(\mathcal{L}_k, B_k)$  is controllable for all  $k \geq 1$ , i.e.,

$$\text{rank } \mathcal{K}_k = nk, \quad \forall k \geq 1. \quad (46)$$

*Sketch of the proof.* The most important part of the proof is the implication 3  $\Rightarrow$  1. Its proof requires two main steps.

**First Step:** We reformulate the null controllability problem for System (43) as a vector moment problem:

$$\left\{ \begin{array}{l} \text{Find } v \in L^2(0, T; \mathbb{C}^m) \text{ such that} \\ -\langle y^0, \varphi(\cdot, 0) \rangle = \int_0^T (v(t), B^* \varphi_x(0, t))_{\mathbb{C}^m} dt, \quad \forall \varphi^T \in H_0^1(0, \pi; \mathbb{C}^n). \end{array} \right. \quad (47)$$

where for  $\varphi^T \in H_0^1(0, \pi; \mathbb{C}^n)$  and  $\varphi$  is the corresponding solution of the adjoint system. This solution is given by

$$\varphi(x, t) = \sum_{k \geq 1} e^{(-\lambda_k I_n + A^*)(T-t)} \phi_k(x) \varphi_k^T, \quad \text{with } \varphi_k^T = \int_0^\pi \varphi^T(x) \phi_k(x) dx \in \mathbb{C}^n.$$

Actually, condition (46) only has to be checked for a frequency: there exists a positive integer  $k_0$ , only depending on  $A$ , such that  $\text{rank } \mathcal{K}_{k_0} = nk_0$  if and only if  $\text{rank } \mathcal{K}_k = nk$  for every  $k \geq 1$ . Let us consider the finite-dimensional space

$$X_0 = \{w : w = \sum_{1 \leq k \leq k_0} w_k \phi_k \text{ with } w_k \in \mathbb{C}^n\}.$$

Given  $y \in H^{-1}(0, \pi; \mathbb{C}^n)$  (resp.  $y \in L^2(0, \pi; \mathbb{C}^n)$ ), we will use the notation  $y_k = \langle y, \phi_k \rangle \in \mathbb{C}^n$ , (resp.  $y_k = (y, \phi_k)_{L^2(0, \pi)}$ ), where  $\langle \cdot, \cdot \rangle$  stands for the usual duality pairing between  $H^{-1}(0, \pi)$  and  $H_0^1(0, \pi)$ . With this notation, consider

$$Y^0 = \left( \frac{\sqrt{\pi}}{k\sqrt{2}} y_k^0 \right)_{1 \leq k \leq k_0} \in \mathbb{C}^{nk_0}, \quad \text{with } y_k^0 = \langle y^0, \phi_k \rangle \in \mathbb{C}^n,$$

and  $\Phi^T = \left( k\sqrt{\frac{2}{\pi}} \varphi_k^T \right)_{1 \leq k \leq k_0} \in \mathbb{C}^{nk_0}$ . Then

$$\left\{ \begin{array}{l} B^* \varphi_x(0, t) = B_{k_0}^* e^{\mathcal{L}_{k_0}^*(T-t)} \Phi^T + \sum_{k > k_0} k \sqrt{\frac{2}{\pi}} B^* e^{(-\lambda_k I_n + A^*)(T-t)} \varphi_k^T, \quad t \in (0, T), \\ -\langle y^0, \varphi(\cdot, 0) \rangle = -(Y^0, e^{\mathcal{L}_{k_0}^* T} \Phi^T)_{\mathbb{C}^{nk_0}} - \sum_{k > k_0} (y_k^0, e^{(-\lambda_k I_n + A^*)T} \varphi_k^T)_{\mathbb{C}^n} \end{array} \right.$$

with  $(B_{k_0}, \mathcal{L}_{k_0})$  given by (45).

As it is proved in [7], (47) transforms into the problem

$$\left\{ \begin{array}{l} \text{Find } v \in L^2(0, T; \mathbb{C}^m) \text{ such that} \\ \int_0^T (v(T-t), B_{k_0}^* e^{\mathcal{L}_{k_0}^* t} \Phi^T)_{\mathbb{C}^m} dt = F(Y^0, \Phi^T), \quad \forall \Phi^T \in \mathbb{C}^{nk_0}, \\ \int_0^T (v(T-t), B^* e^{(-\lambda_k I_n + A^*)t} a)_{\mathbb{C}^m} dt = f_k(y^0, a), \quad \forall a \in \mathbb{C}^n, \forall k > k_0, \end{array} \right. \quad (48)$$

where we have introduced the bilinear forms  $F : \mathbb{C}^{nk_0} \times \mathbb{C}^{nk_0} \rightarrow \mathbb{C}$  and  $f_k : H^{-1}(0, \pi; \mathbb{C}^n) \times \mathbb{C}^n \rightarrow \mathbb{C}$  given by

$$\left\{ \begin{array}{l} F(Y^0, \Phi^T) = -(Y^0, e^{\mathcal{L}_{k_0}^* T} \Phi^T)_{\mathbb{C}^{nk_0}}, \quad \forall (Y^0, \Phi^T) \in \mathbb{C}^{nk_0} \times \mathbb{C}^{nk_0}, \\ f_k(y^0, a) = -\frac{1}{k} \sqrt{\frac{\pi}{2}} (y_k^0, e^{(-\lambda_k I_n + A^*)T} a)_{\mathbb{C}^n}, \quad \forall (y^0, a) \in H^{-1}(0, \pi; \mathbb{C}^n) \times \mathbb{C}^n. \end{array} \right.$$

Using the Kalman condition (46), one proves that (48) can be reduced to a scalar moment problem involving the family  $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$  where  $\{\Lambda_k\}_{k \geq 1}$  is the sequence of eigenvalues of the operator  $I_n \partial_{xx} - A$  with homogenous Dirichlet boundary conditions.

**Second step:** The aim of this step is to prove that the previous moment problem has a solution. Following [27], this will occur if one proves the existence of a biorthogonal family to  $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$  ( $\eta \geq 1$  is given) which satisfies appropriate bounds of their  $L^2$ -norms. Recall that the family  $\{\varphi_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta-1} \subset L^2(0, T; \mathbb{C})$  is biorthogonal to  $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$  if:

$$\int_0^T t^j e^{-\Lambda_k t} \varphi_{l,i}^*(t) dt = \delta_{kl} \delta_{ij}, \quad \forall (k, j), (l, i) : k, l \geq 1, 0 \leq i, j \leq \eta-1.$$

Thus, in [7] the authors prove the following result:

**Theorem 6.2 (Biorthogonal family).** *Let us fix  $\eta \geq 1$ , an integer,  $T \in (0, \infty]$  and  $\{\Lambda_k\}_{k \geq 1} \subset \mathbb{C}$  a sequence. Assume that for two positive constants  $\delta$  and  $\rho$  one has*

$$\left\{ \begin{array}{l} \Re \Lambda_k \geq \delta |\Lambda_k| \quad \forall k \geq 1, \quad \sum_{k \geq 1} \frac{1}{|\Lambda_k|} < \infty, \\ |\Lambda_k - \Lambda_l| \geq \rho |k - l|, \quad \forall k, l \geq 1. \end{array} \right.$$

*Then, there exists  $\{\varphi_{k,j}\}_{k \geq 1, 0 \leq j \leq \eta-1}$  biorthogonal to  $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$  such that, for every  $\epsilon > 0$ , there exists  $C(\epsilon, T) > 0$  satisfying*

$$\|\varphi_{k,j}\|_{L^2(0, T; \mathbb{C})} \leq C(\epsilon, T) e^{\epsilon \Re \Lambda_k}, \quad \forall (k, j) : k \geq 1, 0 \leq j \leq \eta-1.$$

With this last result the proof of the problem can be obtained exactly as in the scalar parabolic case (see [27]).  $\square$

**Remark 25.** 1. Note that the Kalman condition (46) contains the condition  $\text{rank}[A|B] = n$  (it corresponds to  $k = 1$ ). Then we see that  $\text{rank}[A|B] = n$  is a necessary condition, in general not sufficient, for the null controllability of System (43). In fact, Theorem 6.1 was proved in [32] for  $n = 2$  and, in this case, condition (46) is equivalent to  $\text{rank}[A|B] = 2$  and

$$\mu_1 - \mu_2 \neq k^2 - l^2, \quad \forall k, l \in \mathbb{N} \text{ with } k \neq l,$$

where  $\mu_1$  and  $\mu_2$  are the eigenvalues of  $A^*$ .

2. From Theorems 5.1 and 6.1 we can conclude once again that, unlike the scalar case  $n = 1$ , the distributed controllability property of parabolic systems is not equivalent to the boundary controllability property: the rank condition,  $\text{rank}[A|B] = n$ , is a necessary and sufficient condition for the controllability of system

$$\begin{cases} \partial_t y = \Delta y + Ay + Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases}$$

whereas the null controllability of System (43) needs some additional assumption on  $A$  and  $B$ : condition (46). This shows that there is an important difference between the controllability properties for scalar and non scalar parabolic problems.

**Remark 26.** As a consequence of Theorem 6.1, we can also state a controllability result for system:

$$\begin{cases} y_t = y_{xx} + Ay & \text{in } Q_T, \\ y(0, \cdot) = B_1 v_1, \quad y(\pi, \cdot) = B_2 v_2 & \text{on } (0, T), \\ y(\cdot, 0) = y^0 & \text{in } (0, \pi), \end{cases} \quad (49)$$

where  $A \in \mathcal{L}(\mathbb{C}^n)$ ,  $B_1 \in \mathcal{L}(\mathbb{C}^{m_1}; \mathbb{C}^n)$ ,  $B_2 \in \mathcal{L}(\mathbb{C}^{m_2}; \mathbb{C}^n)$  are given matrices and  $y^0 \in H^{-1}(0, \pi; \mathbb{C}^n)$  is the initial datum. Observe that in System (49),  $v_1 \in L^2(0, T; \mathbb{C}^{m_1})$  and  $v_2 \in L^2(0, T; \mathbb{C}^{m_2})$  are the control functions and they act on the system by means of the Dirichlet boundary condition at points  $x = 0$  and  $x = \pi$ .

If we set

$$L_k = -\lambda_k I_n + A \in \mathcal{L}(\mathbb{C}^n), \quad \tilde{B}_k = \begin{pmatrix} B_1 & B_2 \\ \vdots & \vdots \\ B_1 & (-1)^{k+1} B_2 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^{nk}), \quad k \geq 1.$$

with  $m = m_1 + m_2$ , and  $\tilde{\mathcal{K}}_k := [\mathcal{L}_k | \tilde{B}_k]$ , then, one has: “System (49) is exactly controllable to trajectories at any time  $T$  if and only if  $\text{rank} \tilde{\mathcal{K}}_k = nk$ , for any  $k \geq 1$ .”

**Remark 27.** As in [27], one can consider a distributed control that depends only on time

$$\begin{cases} y_t = y_{xx} + Ay + Bfv & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y^0 & \text{in } (0, \pi), \end{cases} \quad (50)$$

where  $A \in \mathcal{L}(\mathbb{C}^n)$  and  $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$  are two given matrices,  $y^0 \in L^2(0, \pi; \mathbb{C}^n)$  is the initial datum and  $f \in L^2(0, \pi; \mathbb{C})$  is a given function such that for every  $\varepsilon > 0$

$$\inf_{k \geq 1} |f_k| e^{\varepsilon \lambda_k} > 0 \quad (51)$$

where  $f_k = (f, \phi_k)_{L^2(0, \pi)} \in \mathbb{C}$ . This last condition can be checked if we take  $f(x) = 1_{(a, b)}$  with  $a = \pi/4$  and  $b = (1 + 2\sqrt{2})\pi/4$ . In System (50),  $v \in L^2(0, T; \mathbb{C}^m)$  is a control function that, of course, only depends on time. One has (see [7]):

**Theorem 6.3.** *Let us fix  $A \in \mathcal{L}(\mathbb{C}^n)$ ,  $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$  and  $f \in L^2(0, \pi; \mathbb{C})$  satisfying (51). Then, System (50) is exactly controllable to trajectories at any time  $T$  if and only if the pair  $(\mathcal{L}_k, B_k)$  is controllable for all  $k \geq 1$ , with  $(\mathcal{L}_k, B_k)$  defined in (45).*

**Remark 28.** Let us consider the system

$$\begin{cases} y_t = Dy_{xx} + Ay & \text{in } Q_T, \\ y(0, \cdot) = Bv, \quad y(\pi, \cdot) = 0 & \text{on } (0, T), \\ y(\cdot, 0) = y^0 & \text{in } (0, \pi), \end{cases} \quad (52)$$

where

$$D = \text{diag}(d_1, \dots, d_n), \quad A \in \mathcal{L}(\mathbb{C}^n), \quad B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n),$$

with  $d_i > 0$  for  $1 \leq i \leq n$ . In a forthcoming paper [8], we will show that we can define a Kalman condition for System (52) by replacing  $L_k$  (defined in (44)) by  $-\lambda_k D + A$ . The approximate controllability of this system is equivalent to the same Kalman rank condition (46). The null controllability problem is much more intricate as it is proved in [57] (see also Remark 14). In this case the sequence  $\{\Lambda_k\}_{k \geq 1}$  of distinct eigenvalues of the operator  $D\partial_{xx} + A^*$  satisfies the two first conditions in the statement of Theorem 6.2. These conditions imply the existence of a biorthogonal family to  $\{e^{-\Lambda_k t}\}_{k \geq 1}$  (and then to  $\{t^j e^{-\Lambda_k t}\}_{k \geq 1, 0 \leq j \leq \eta-1}$ , with  $\eta \geq 1$ ) and thus, the moment method can be applied to System (52) providing a control  $v$ . Nevertheless, the ‘‘gap condition’’ in general fails and it is crucial in order to prove the estimates of the biorthogonal family (and therefore, for showing that the control  $v$  satisfies  $v \in L^2(0, T; \mathbb{C}^m)$ ) (see [7] and [9]). Moreover in [9] we exhibit some examples where the system is null controllable at time  $T$  if and only if  $T$  is large enough (depending on  $D$ ,  $A$  and  $B$ ).

**Remark 29.** The main difficulty in Theorem 6.1 comes from having less controls than equations ( $m < n$ ). If  $\text{rank } B = n$ , it is possible to prove the null controllability of system

$$\begin{cases} y_t = DL(t)y + A(\cdot, t)y & \text{in } Q_T, \\ y = Bv1_{\Gamma_0}, & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases}$$

where  $T > 0$ ,  $\Omega \subset \mathbb{R}^N$  is a regular bounded open set,  $\Gamma_0 \subset \partial\Omega$  is a relative open subset of the boundary,  $L(t)$  is given by (3) and satisfies (4) and (5),  $A \in L^\infty(Q_T; \mathcal{L}(\mathbb{C}^n))$ ,  $B \in \mathcal{L}(\mathbb{C}^m; \mathbb{C}^n)$ , with  $\text{rank } B = n$ , and  $D \in \mathcal{L}(\mathbb{C}^n)$  is a matrix whose eigenvalues  $\{d_i\}_{1 \leq i \leq p}$  satisfy

$$\Re(d_i) > 0 \quad \text{and} \quad \text{Index}(d_i) \leq 4, \quad \forall i : 1 \leq i \leq p,$$

where  $\text{Index}(d_i)$  is the dimension of the largest Jordan block associated to  $d_i$ . In fact the null controllability result for the previous system can be deduced ( $\text{rank } B = n$ ) from a distributed null controllability result for a similar system. For a proof, see [33].

**Remark 30.** Theorem 6.2 generalizes the results on biorthogonal families to exponentials established in [27] and [28] in two directions: firstly, Theorem 6.2 is valid for general complex sequences  $\{\Lambda_k\}_{k \geq 1}$ ; secondly, Theorem 6.2 establishes the existence of a family biorthogonal to a sequence of complex matrix exponentials.

**7. Some sufficient conditions for space dependent coefficients.** In this section we will analyze some results on the literature that give sufficient conditions to the null controllability of some coupled parabolic equations. Historically, the first case that has been studied was that of two coupled equations,  $n = 2$ , (see [63] and [3]). The case  $n = 2$  is a particular case of a more general one: the *cascade systems*. Nowadays, the null controllability result for *cascade systems* is the only one

at our disposal for parabolic systems of  $n$  equations with space and time dependent coefficients. We will present it in the next subsection and give the main ideas of the proof.

**7.1. Cascade Systems.** We consider the linear parabolic system

$$\left\{ \begin{array}{l} \partial_t y_1 - L_1 y_1 + \sum_{j=1}^n C_{1j} \cdot \nabla y_j + \sum_{j=1}^n a_{1j} y_j = v 1_\omega \quad \text{in } Q_T, \\ \partial_t y_2 - L_2 y_2 + \sum_{j=1}^n C_{2j} \cdot \nabla y_j + \sum_{j=1}^n a_{2j} y_j = 0 \quad \text{in } Q_T, \\ \dots \\ \partial_t y_n - L_n y_n + \sum_{j=1}^n C_{nj} \cdot \nabla y_j + \sum_{j=1}^n a_{nj} y_j = 0 \quad \text{in } Q_T, \\ y_i = 0 \text{ on } \Sigma_T, \quad y_i(\cdot, 0) = y_i^0 \text{ in } \Omega, \quad 1 \leq i \leq n, \end{array} \right.$$

where  $a_{ij} = a_{ij}(x, t) \in L^\infty(Q_T)$ ,  $C_{ij} = C_{ij}(x, t) \in L^\infty(Q_T; \mathbb{R}^N)$  ( $1 \leq i, j \leq n$ ),  $y_i^0 \in L^2(\Omega)$  ( $1 \leq i \leq n$ ) and  $L_k$  is, for every  $1 \leq k \leq n$ , a second order operator as (3) satisfying (4) and (5).

Equivalently, the previous system can be written as

$$\left\{ \begin{array}{l} \partial_t y - Ly + C \cdot \nabla y + Ay = Bv 1_\omega \quad \text{in } Q_T, \\ y = 0 \quad \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 \quad \text{in } \Omega, \end{array} \right. \quad (53)$$

where  $L$  is the matrix operator given by  $L = \text{diag}(L_1, \dots, L_n)$ ,  $y = (y_i)_{1 \leq i \leq n}$  is the state and  $\nabla y = (\nabla y_i)_{1 \leq i \leq n}$ , and where

$$\left\{ \begin{array}{l} y^0 = (y_i^0)_{1 \leq i \leq n} \in L^2(\Omega; \mathbb{R}^n), \quad A(x, t) = (a_{ij}(x, t))_{1 \leq i, j \leq n} \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n)), \\ C(x, t) = (C_{ij}(x, t))_{1 \leq i, j \leq n} \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n; \mathbb{R}^{Nn})) \text{ and } B \equiv e_1 = (1, 0, \dots, 0)^* \end{array} \right.$$

are given. Let us observe that, for each  $y^0 \in L^2(\Omega; \mathbb{R}^n)$  and  $v \in L^2(Q_T)$ , System (53) admits a unique weak solution  $y \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^n)) \cap C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$ . By *cascade system* we mean that matrices  $A$  and  $C$  have the following structure:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n-1} & a_{nn} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ 0 & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{nn} \end{pmatrix} \quad (54)$$

with  $a_{ij} \in L^\infty(Q_T)$ ,  $C_{ij} \in L^\infty(Q_T; \mathbb{R}^N)$  ( $1 \leq i \leq j \leq n$ ) and  $a_{i,i-1} \in L^\infty(Q_T)$  ( $2 \leq i \leq n$ ).

In order to study the null controllability of System (53), we will consider the corresponding adjoint problem which, under assumption (54) (cascade system), has

the form

$$\begin{cases} -\partial_t \varphi_i - L_i \varphi_i - \sum_{j=1}^i [\nabla \cdot (C_{ji} \varphi_j) - a_{ji} \varphi_j] = -a_{i+1,i} \varphi_{i+1} & \text{in } Q_T, \\ \dots & (1 \leq i \leq n-1), \\ -\partial_t \varphi_n - L_n \varphi_n - \sum_{j=1}^n [\nabla \cdot (C_{jn} \varphi_j) - a_{jn} \varphi_j] = 0 & \text{in } Q_T, \\ \varphi_i = 0 \text{ on } \Sigma_T, \quad \varphi_i(\cdot, T) = \varphi_i^T \text{ in } \Omega, \quad 1 \leq i \leq n, \end{cases} \quad (55)$$

where  $\varphi_i^T \in L^2(\Omega)$  ( $1 \leq i \leq n$ ). It is well-known that the null controllability of System (53) (with  $L^2$ -controls) is equivalent to the existence of a constant  $C > 0$  such that the so-called *observability inequality*

$$\|\varphi(\cdot, 0)\|_{L^2(\Omega; \mathbb{R}^n)}^2 \leq C \iint_{\omega_T} |\varphi_1(x, t)|^2. \quad (56)$$

holds for every solution  $\varphi = (\varphi_1, \dots, \varphi_n)^*$  to (55). Inequality (56) can be deduced from an appropriate global Carleman inequality for the adjoint System (55). This Carleman inequality is established in the following result ([41]):

**Theorem 7.1.** *Let us suppose that the operator  $L_k$ ,  $A \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n))$  and  $C \in L^\infty(Q_T; \mathcal{L}(\mathbb{R}^n; \mathbb{R}^{N_n}))$  are given by (3) and (54) and satisfy (4), (5) and*

$$a_{i,i-1} \geq c_0 > 0 \text{ or } -a_{i,i-1} \geq c_0 > 0 \text{ in } \omega_0 \times (0, T), \quad \forall i : 2 \leq i \leq n, \quad (57)$$

for an open set  $\omega_0 \subset \omega$ . Let  $M_0 = \max_{2 \leq i \leq n} \|a_{i,i-1}\|_\infty$ . Then, there exist a positive function  $\beta_0 \in C^2(\bar{\Omega})$  (only depending on  $\Omega$  and  $\omega_0$ ), two positive constants  $N_0$  (only depending on  $\Omega$ ,  $\omega_0$ ,  $\tilde{c}_0$ ,  $\tilde{M}_0$ ,  $c_0$  and  $M_0$ ) and  $\sigma_0 = \sigma_0(\Omega, \omega_0, \tilde{c}_0, \tilde{M}_0, M_0)$  and  $l \geq 3$  (only depending on  $n$ ) such that, for every  $\varphi^T \in L^2(Q_T; \mathbb{R}^n)$ , the solution  $\varphi$  to (55) satisfies

$$\sum_{i=1}^n \mathcal{I}(3(n+1-i), \varphi_i) \leq N_0 s^l \iint_{\omega_0, T} e^{-2s\beta} \gamma(t)^l |\varphi_1|^2,$$

$\forall s \geq s_0 = \sigma_0 \left[ T + T^2 + T^2 \max_{i \leq j} \left( \|a_{ij}\|_\infty^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_\infty^{\frac{2}{3(j-i)+1}} \right) \right]$ . In the previous inequality,  $\beta(x, t)$  and  $\gamma(t)$  are given in (13) and  $\mathcal{I}(d, z)$  is given in the statement of Lemma 2.3.

Following the same reasoning as in Corollary 1, from Theorem 7.1 we can deduce the following result:

**Corollary 3.** *Under assumptions of Theorem 7.1, there exists a positive constant  $C$  (only depending on  $\Omega$ ,  $\omega$ ,  $n$  and  $c_0$ ) such that for every  $y^0 \in L^2(\Omega; \mathbb{R}^n)$  there is a control  $v \in L^2(\Omega)$  which satisfies*

$$\|v\|_{L^2(Q_T)}^2 \leq e^{C\mathcal{H}} \|y^0\|_{L^2(\Omega; \mathbb{R}^n)}^2,$$

and  $y(\cdot, T) = 0$  in  $\Omega$ , with  $y$  the solution to (53) associated to  $y^0$  and  $v$ . In the previous inequality,  $\mathcal{H}$  is given by

$$\mathcal{H} \equiv 1 + T + \frac{1}{T} + \max_{i \leq j} \left( \|a_{ij}\|_\infty^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_\infty^{\frac{2}{3(j-i)+1}} + T (\|a_{ij}\|_\infty + \|C_{ij}\|_\infty^2) \right).$$

*Sketch of the proof of Theorem 7.1.* Given  $\omega_0 \subset \omega$ , we choose  $\omega_1 \subset\subset \omega_0$ . Let  $\beta_0 \in C^2(\bar{\Omega})$  be the function provided by Lemma 2.3 and associated to  $\Omega$  and  $\mathcal{B} \equiv \omega_1$ , and let  $\beta(x, t)$  the function given by  $\beta(x, t) = \beta_0(x)/t(T - t)$ . We will do the proof in two steps:

**Step 1.** Let  $\varphi = (\varphi_1, \dots, \varphi_n)^*$  be the solution to (55) associated to  $\varphi^T \in L^2(\Omega; \mathbb{R}^n)$ . We begin applying inequality (14) with  $\mathcal{B} = \omega_1$  to each function  $\varphi_i$  ( $1 \leq i \leq n$ ) with  $L \equiv L_i$ ,  $d = 3(n + 1 - i)$  and

$$F \equiv \sum_{j=1}^i [\nabla \cdot (C_{ji}\varphi_j) - a_{ji}\varphi_j] - a_{i+1,i}\varphi_{i+1},$$

if  $1 \leq i \leq n - 1$ , and  $F \equiv \sum_{j=1}^n [\nabla \cdot (C_{jn}\varphi_j) - a_{jn}\varphi_j]$ , for  $i = n$ . Now if we take

$$s \geq s_0 = \sigma_0 \left( T + T^2 + T^2 \max_{i \leq j} \left( \|a_{ij}\|_{\infty}^{\frac{2}{3(j-i)+3}} + \|C_{ij}\|_{\infty}^{\frac{2}{3(j-i)+1}} \right) \right),$$

with  $\sigma_0 = \sigma_0(\Omega, \omega_0, \tilde{a}_0, \tilde{M}_0, M_0) > 0$ , we obtain the existence of a positive constants  $C_1 = C_1(\Omega, \omega_0, \tilde{a}_0, \tilde{M}_0, M_0)$  such that if  $s \geq s_0$ , then

$$\sum_{i=1}^n \mathcal{I}(3(n + 1 - i), \varphi_i) \leq C_1 \left( \sum_{i=1}^n s^{3(n+1-i)} \iint_{\omega_{1,T}} e^{-2s\beta} \gamma(t)^{3(n+1-i)} |\varphi_i|^2 \right). \quad (58)$$

**Step 2.** We will see that, thanks to assumptions (54) and (57), we can eliminate in (58) the local terms for  $2 \leq i \leq n$ . In order to carry this process out, we will need the following result:

**Lemma 7.2.** *Under assumptions of Theorem 7.1 and given  $l \in \mathbb{N}$ ,  $\varepsilon > 0$ ,  $k \in \{2, \dots, n\}$  and two open sets  $\mathcal{O}_0$  and  $\mathcal{O}_1$  such that  $\omega_1 \subset \mathcal{O}_1 \subset\subset \mathcal{O}_0 \subset \omega_0$ , there exist a positive constant  $C_k$  (only depending on  $\Omega$ ,  $\mathcal{O}_0$ ,  $\mathcal{O}_1$ ,  $\tilde{c}_0$ ,  $\tilde{M}_0$ ,  $c_0$  and  $M_0$ ) and  $l_{kj} \in \mathbb{N}$ ,  $1 \leq j \leq k - 1$  (only depending on  $l$ ,  $n$ ,  $k$  and  $j$ ), such that, if  $\varphi$  is the solution to (55) associated to  $\varphi^T \in L^2(Q_T; \mathbb{R}^n)$  and  $s \geq s_0$ , one has*

$$\left\{ \begin{array}{l} s^l \iint_{\mathcal{O}_{1,T}} e^{-2s\beta} \gamma(t)^l |\varphi_k|^2 \leq \varepsilon [\mathcal{I}(3(n + 1 - k), \varphi_k) + \mathcal{I}(3(n - k), \varphi_{k+1})] \\ \quad + C_k \left( 1 + \frac{1}{\varepsilon} \right) \sum_{j=1}^{k-1} s^{l_{kj}} \iint_{\mathcal{O}_{0,T}} e^{-2s\beta} \gamma(t)^{l_{kj}} |\varphi_j|^2. \end{array} \right.$$

(In this inequality we have taken  $\varphi_{k+1} \equiv 0$  when  $k = n$ ).

The proof of Theorem 7.1 is a consequence of inequality (58) and Lemma 7.2. For the details, see [41].  $\square$

**Remark 31.** As far as we know the first study of controllability of cascade systems of parabolic equations appears in the context of the so called ‘‘Insensitizing controls’’ introduced by J.L. Lions in [56]. In this context, the problem is to control two coupled (cascade) parabolic equations but the first equation is forward and the second equation is backward in time with zero initial data, and  $a_{21} = 1_{\mathcal{O}}$ . The results are much more intricate because this relation forward-backward poses an obstacle to obtain the observability inequality. The first results on existence of insensitizing controls were due to Bodart and Fabre in [13] in the approximate controllability context and a nonempty intersection of  $\mathcal{O}$  and  $\omega$  was required. This condition was used later in the paper of de Teresa [63] where the (null) insensitizing result was proved for zero initial data and a source term decaying to zero exponentially as

times approaches to  $T$ . Further results in this context are the results in [14], where a (slightly) superlinear result was proved, the results of [43], where the author proves some results when the cascade coupling is an operator of first or second order (see Section 7.2 for the case of two forward equations). In [64] a characterization of the initial data that can be insensitized is presented. The only paper that gives a complete result on the existence of approximate insensitizing controls for the heat equation when  $\omega \cap \mathcal{O} = \emptyset$  is the one by Kavian and de Teresa [48]. This result is also valid in the case of two forward-forward equations and is presented in Section 7.2.5. As far as we know only the papers of [62] and [23] treat this problem for the wave equation, [43] for the Stokes system and [30] for an ocean circulation model.

**Remark 32.** Observe that if the coefficients  $a_{i,i-1}$  are constant for any  $i : 2 \leq i \leq n$ , condition (57) is also a necessary condition for the null controllability of System (53). In general, this condition is not necessary for the approximate controllability (see [48]) neither for the null controllability of this system (see [1] and Subsection 7.2.4). Nevertheless, the null controllability result for System (53) is open if one only assumes that the function  $a_{i,i-1} \neq 0$  in  $Q_T$  ( $2 \leq i \leq n$ ) instead of condition (57).

**Remark 33.** In the proof of Theorem 7.1 we have strongly used that the first order terms  $C_{ij}$  in (53) satisfy the property

$$C_{ij} \equiv 0 \quad \text{in } Q_T, \quad \forall i, j : 1 \leq j < i \leq n.$$

In the next subsection we will describe null controllability results for some systems where the coupling terms are partial differential operators of first or second order.

## 7.2. Other results.

7.2.1. *Controllability of two parabolic equations coupled with first and second order partial differential operators.* All previous results presented before concern parabolic equations coupled by a matrix  $A$  (a zero order partial differential operator). In [42] the author considers two parabolic equations with coupling operators of second or first order. Even if in this paper “some general” first order and second order operators have been considered, let us describe the controllability results for the following system:

$$\begin{cases} \partial_t y - \Delta y + cy + E \cdot \nabla y = \partial_{x_1}(w\theta_1) + v1_\omega & \text{in } Q_T, \\ \partial_t w - \Delta w + hw + K \cdot \nabla w = \Delta(y\theta_2) & \text{in } Q_T, \\ y = w = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0, \quad w(\cdot, 0) = w^0 & \text{in } \Omega, \end{cases} \quad (59)$$

where  $c, h \in \mathbb{R}$  and  $E, K \in \mathbb{R}^N$  and  $\theta_1$  and  $\theta_2$  are two given functions.

One has:

**Theorem 7.3** ([42]). *Suppose that  $\theta_i \in C^2(\overline{\Omega})$  for  $i = 1, 2$ . Assume also that there exists a nonempty open subset  $\omega_2 \subset \omega$  and a positive constant  $C$  such that  $|\theta_2| \geq C > 0$  in  $\omega_2$ . Then System (59) is null controllable at any positive time  $T$ .*

**Remark 34.** As in Theorem 7.1, observe that in the previous result again the control open  $\omega$  have to meet the set where the function  $|\theta_2|$  is positive. The general case  $\theta_2 \in C^2(\overline{\Omega})$  with  $\theta_2 \neq 0$  in  $\Omega$  is still open.



7.2.2. *Parabolic systems of  $n$  equations controlled by  $n - 2$  controls.* In [11], the following  $3 \times 3$  control problem has been studied:

$$\begin{cases} \partial_t y = (\mathcal{L} + A)y + Bv1_\omega & \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \quad (60)$$

where  $\mathcal{L} = \text{diag}(L_1, L_2, L_3)$  with  $(L_i)_{1 \leq i \leq 3}$  operators as in (3) satisfying (4) and (5),  $A = (a_{ij})_{1 \leq i, j \leq 3} \in C^4(\overline{Q_T}; \mathcal{L}(\mathbb{R}^3))$ ,  $B = (1, 0, 0)^* \in \mathbb{R}^3$ ,  $v \in L^2(Q_T)$  is the control, and  $y^0 = (y_i^0)_{1 \leq i \leq 3} \in L^2(\Omega; \mathbb{R}^3)$  is the initial condition.

For each  $l \in \{1, 2, 3\}$ , let us denote by  $(\alpha_{ij}^{(l)})_{1 \leq i, j \leq N}$  the diffusion coefficient (defined in (3)) associated to  $L_l$ . The following theorem has been proved:

**Theorem 7.4** ([11]). *Suppose that  $a_{21}$  and  $a_{31}$  are time independent,  $\exists j \in \{2, 3\}$  such that  $|a_{j1}(x)| \geq C > 0$  for all  $x \in \omega$ , with  $C > 0$ , and that  $L_2 = L_3$ . For  $j \in \{2, 3\}$ , we set  $k_j = \frac{6}{j}$  and  $B_{k_j} \in C^3(\overline{Q_T}; \mathbb{R}^N)$  given by*

$$B_{k_j}^i := \sum_{\ell=1}^N \alpha_{i\ell}^{(2)} \left( \partial_\ell a_{k_j 1} - \frac{a_{k_j 1}}{a_{j1}} \partial_\ell a_{j1} \right), \quad 1 \leq i \leq N.$$

Assume that  $\partial\omega \cap \partial\Omega = \gamma$ , with  $|\gamma| \neq 0$ , and  $B_{k_j} \cdot \nu \neq 0$  on  $\gamma$ , where  $\nu$  is the outward unit normal vector. Then, System (60) is null controllable at time  $T$ .

**Remark 35.** The proof of this result is obtained through a Gauss elimination procedure. Using the first assumption of the theorem, the controllability of System (60) is equivalent to controllability of a  $2 \times 2$  system of this form where for simplicity we have considered the case where  $L_2 = L_3 = -\Delta$

$$\begin{cases} \partial_t y_1 = -L_2 y_1 + \tilde{a}_{11} y_1 + \tilde{a}_{12} y_2 + A_{11} \cdot \nabla y_1 + A_{12} \cdot \nabla y_2 + f\chi_\omega & \text{in } Q_T, \\ \partial_t y_2 = -L_2 y_2 + \tilde{a}_{22} y_2 + A_{22} \cdot \nabla y_2 + b_{k_j} y_1 + B_{k_j} \cdot \nabla y_1 & \text{in } Q_T, \\ y_1 = y_2 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot, 0) = y_1^0, \quad y_2(\cdot, 0) = y_2^0 & \text{in } \Omega. \end{cases} \quad (61)$$

where

$$b_{k_j} = \frac{2\nabla a_{j1} \cdot (\nabla a_{k_j 1} a_{j1} - \nabla a_{j1} a_{k_j 1})}{a_{j1}^2} + \frac{a_{k_j 1} \Delta a_{j1} - a_{j1} \Delta a_{k_j 1}}{a_{j1}} - (-1)^j \frac{\det[A|C]}{a_{j1}},$$

and  $\tilde{a}_{ij}$  and  $A_{ij}$  depend on the coefficients of the matrices  $\mathcal{L}$  and  $A$ . Following the ideas developed in Section 4, we can understand that a necessary condition for the controllability of System (61) is that

$$\mathcal{T}y_1 := b_{k_j} y_1 + B_{k_j} \cdot \nabla y_1 \neq 0 \quad \text{on } \omega_T.$$

As the controllability is obtained through Carleman estimates, the authors of [11] assume stronger assumptions: they assume the invertibility on  $\omega_T$  of the operator  $\mathcal{T}$ . This assumption is satisfied if  $\partial\omega \cap \partial\Omega = \gamma$ , with  $|\gamma| \neq 0$ , and  $B_{k_j} \cdot \nu \neq 0$  on  $\gamma$ .

**Remark 36.** 1. In [11] one can find an example where the algebraic Kalman rank condition  $\text{rank}[A|B] = n$ , with  $[A|B]$  given by (20), is not satisfied but the assumption in Theorem 7.4 is satisfied and then the corresponding system is null controllable at time  $T$ : Let  $\Omega$  any smooth domain in  $\mathbb{R}^2$  containing  $\omega = \{(x, y) \in \mathbb{R}^2 : y < -1, (x-2)^2 + (y+1)^2 < 1\}$  and  $\gamma = [1, 3] \times \{-1\}$ . Let  $a_{32} = a_{23} = a_{22} = a_{33} = 0$ ,  $a_{31}(x, y) = -y^2$ ,  $a_{21}(x, y) = x + y$  and  $L_i = -\Delta$  for  $i = 1, 2, 3$ . We have

$$a_{31} \neq 0 \quad \text{in } \omega \quad \text{and} \quad \left( \nabla a_{21} - \frac{a_{21}}{a_{31}} \nabla a_{31} \right) \cdot \nu(x) = 2x - 1 > 0 \quad \text{on } \gamma.$$

Then, Theorem 7.4 can be applied and the system is null controllable at any time  $T$  and  $\det[A, B] = 0$  in  $\Omega$ .

2. Theorem 7.4 has been generalized to systems of  $n$  equations controlled by  $(n - 2)$  controls (see [11]).
3. In [11], some sufficient conditions for the null controllability of two parabolic equations coupled by first order partial differential operators are also derived.

7.2.3. *Lebeau-Robbiano method.* The Lebeau-Robbiano method has also been applied for solving controllability problems for parabolic systems. In [4] the following system has been considered

$$\begin{cases} \partial_t y_1 = (-\Delta)^\alpha y_2 - (-\Delta)^\beta y_1 + v1_\omega & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + (-\Delta)^\alpha y_1 & \text{in } Q_T, \\ y_1 = y_2 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot, 0) = y_1^0, \quad y_2(\cdot, 0) = y_2^0 & \text{in } \Omega, \end{cases} \quad (62)$$

where  $\alpha, \beta > 0$  are given and  $\Omega \subset \mathbb{R}^N$  is a bounded open set having a  $C^\infty$  boundary.

The matrix operator

$$\begin{cases} \mathcal{L} = \begin{pmatrix} -(-\Delta)^\beta & (-\Delta)^\alpha \\ (-\Delta)^\alpha & \Delta \end{pmatrix}, \\ D(\mathcal{L}) = (D((-\Delta)^\beta) \cap D((-\Delta)^\alpha)) \times (D(-\Delta) \cap D((-\Delta)^\alpha)) \end{cases}$$

is self-adjoint and its eigenfunctions are obtained from those of  $(-\Delta)$  with homogeneous Dirichlet boundary conditions. The authors used the spectral inequality proved in [54] to show the following controllability result:

**Theorem 7.5** ([4]). *Assume that  $\beta > 0$  and  $2\alpha < \beta + 1$ . Then for any  $(y_1^0, y_2^0)^* \in L^2(\Omega; \mathbb{R}^2)$  and any  $T > 0$ , there exists a control  $v \in L^2(Q_T)$  such that the solution  $(y_1, y_2)^*$  of (62) satisfies  $y_1(\cdot, T) = y_2(\cdot, T) = 0$ .*

In [53] non-selfadjoint operators have been considered

$$\mathcal{L} = \begin{pmatrix} L_1 & a_{12} \\ a_{21} & L_2 \end{pmatrix},$$

with  $L_1$  and  $L_2$  two self-adjoint elliptic operators of second order. The coefficients  $(a_{12}, a_{21})$  are assumed to be in  $L^\infty(\Omega)$ . After proving a spectral inequality for the eigenfunctions of  $\mathcal{L}$ , the author obtained a controllability result at any  $T > 0$  for system

$$\begin{cases} \partial_t y_1 = L_1 y_1 + a_{12} y_2 + v1_\omega & \text{in } Q_T, \\ \partial_t y_2 = a_{21} y_1 + L_2 y_2 & \text{in } Q_T, \\ y_1 = y_2 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot, 0) = y_1^0, \quad y_2(\cdot, 0) = y_2^0 & \text{in } \Omega, \end{cases}$$

when  $N \leq 3$  (or for any dimension if  $a_{12} = a_{21}$ ) and  $|a_{21}(\cdot)| \geq a_0 > 0$  in some nonempty open subset of  $\omega$ . Even if the result was already known (see for instance [63]), the spectral inequality proved for  $\mathcal{L}$  leads to sharp estimates for the control of the low frequencies.

7.2.4. *Control domain and coupling terms.* In the case of non constant coefficients, almost all the parabolic systems for which positive controllability results have been obtained can be transformed to *cascade systems*. As it has been shown in Subsection 7.1, condition (57) is the main assumption for the controllability of these

systems. In [48] one finds that, at least for approximate controllability, this assumption is not necessary. Recently, in [1], it has been proved that even for null controllability this assumption is not sharp. More precisely, consider the following system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \delta p y_2 + v 1_\omega & \text{in } Q_T, \\ \partial_t y_2 = p y_1 + \Delta y_2 & \text{in } Q_T, \\ y_1 = y_2 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot, 0) = y_1^0, \quad y_2(\cdot, 0) = y_2^0 & \text{in } \Omega, \end{cases} \quad (63)$$

with  $p$  a smooth real-valued function,  $\delta > 0$  and  $\Omega$  a bounded domain of  $\mathbb{R}^N$  of class  $C^\infty$ . The following theorem is established

**Theorem 7.6** ([1]). *Let  $p \geq 0$  on  $\Omega$ . Assume that there exists  $p_0 > 0$  and  $\omega_p \subset \Omega$  satisfying the Geometric Control Condition (GCC) (see [10]) with  $p \geq p_0$  in  $\omega_p$ . Assume that  $\omega$  also satisfies GCC. Then there exists  $\delta_0 > 0$  such that for all  $0 < \sqrt{\delta} \|p\|_{L^\infty(\Omega)} \leq \delta_0$  System (63) is null controllable at any positive time  $T$ .*

With the same kind of arguments, in [1], a new boundary control result is proved. For simplicity, consider

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \delta p y_2 & \text{in } Q_T, \\ \partial_t y_2 = p y_1 + \Delta y_2 & \text{in } Q_T, \\ y_1 = b v, \quad y_2 = 0 & \text{on } \Sigma_T, \\ y_1(\cdot, 0) = y_1^0, \quad y_2(\cdot, 0) = y_2^0 & \text{in } \Omega, \end{cases} \quad (64)$$

where  $b$  is a smooth real-valued function on  $\partial\Omega$ ,  $p \in L^\infty(\Omega)$ ,  $\delta > 0$  and  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  of class  $C^\infty$ . One has

**Theorem 7.7** ([1]). *Let  $p$  satisfy assumptions of Theorem 7.6. Suppose that there exists a subdomain  $\Gamma_b$  of  $\partial\Omega$  satisfying GCC and  $b \geq b_0 > 0$  on  $\Gamma_b$ . Then there exists  $\delta_0 > 0$  such that for all  $0 < \sqrt{\delta} \|p\|_{L^\infty(\Omega)} \leq \delta_0$  System (64) is null controllable at any time  $T > 0$ .*

Even if the geometrical assumptions are obviously too strong, these two theorems give the first examples on controllability of cascade system with coupling terms vanishing on the control domain. Moreover it also gives the first result on boundary control of two coupled parabolic equations for  $N > 1$ .

**7.2.5. Approximate controllability of two coupled equations.** In this section we will consider only the case of two cascade coupled equations,

$$\begin{cases} y_t - D\Delta y = Ay + Bv 1_\omega & \text{in } Q_T, \\ y = 0 & \text{in } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \quad (65)$$

where

$$D = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ a(x) & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

with  $a(x) \geq 0$  is such that  $\text{supp } a \subset \mathcal{O}$ . That is, we consider a coupling that depends on  $x$ , with support in  $\mathcal{O}$ , and with a distributed control exerted in the open set  $\omega$ . We will present two different results related to the approximate controllability of System (65) when the control and coupling sets have empty intersection i.e.  $\mathcal{O} \cap \omega = \emptyset$ .

Using the Fourier decomposition of the solutions and the analyticity in time, in [48] the following result was proved:

**Theorem 7.8.** *Let  $\nu = 1$ . Then for any nonempty open sets  $\omega$ ,  $\mathcal{O} \subset \Omega$  and when  $a = 1_{\mathcal{O}}$  System (65) is approximately controllable at any time  $T > 0$ .*

On the other hand, the following unpublished counterexample is due to B. Dehman, M. Léautaud, J. Le Rousseau, L. Rosier and L. de Teresa:

**Theorem 7.9.** *Let  $\Omega = (0, 2\pi)$ . Then if  $\sqrt{\nu} \in \mathbb{Q}$ , for every  $\delta \in (0, 2\pi)$  and  $r \in \mathbb{N}$ , there exists  $a \in C^r(0, 2\pi)$  with  $a \geq 0$ ,  $\text{supp } a \subset (2\pi - \delta, 2\pi)$  such that System (65) is not approximately controllable for any data  $y^0 \in L^2(0, 2\pi)$  and any  $\omega \subset (0, 2\pi - \delta)$ .*

The proof of Theorem 7.9 is based on the following result:

**Proposition 2.** *For all  $\nu \neq 1$  such that  $\sqrt{\nu} \in \mathbb{Q}$ , for all  $\delta \in (0, 2\pi)$  and  $r \in \mathbb{N}$ , there exists  $a \in C^r(0, 2\pi)$  with  $a \geq 0$ ,  $a \neq 0$ ,  $\text{supp } a \subset (2\pi - \delta, 2\pi)$ , and an eigenvalue  $\lambda \in \mathbb{R}^+$  and associated eigenfunction  $(w_1, w_2) \neq (0, 0)$  satisfying  $w_2|_{(0, 2\pi - \delta)} = 0$  together with the system*

$$\begin{cases} -\nu w_{1,xx} = \lambda w_1 & \text{in } (0, 2\pi), \\ -w_{2,xx} + a(x)w_1 = \lambda w_2 & \text{in } (0, 2\pi), \\ w_i(0) = w_i(2\pi) = 0, & i = 1, 2. \end{cases}$$

This result seems to indicate that results concerning the null controllability of a system with a boundary control are very close to distributed null controllability results when the support of the coupling and the support of the control have empty intersection.

**8. Some comments and open problems.** We have showed the complexity of the problem of controlling coupled parabolic equations and also the very different behavior with respect to the scalar case (boundary controllability not equivalent to distributed controllability, approximate controllability not equivalent to null controllability, minimal time for controlling as in Remarks 28 and 23). The list of open problems is long and there is a lot of work to be done in order to fully understand this challenging subject. Of course, as in other equations, non linear problems are a big issue (for a presentation of some of them see [20]). As far as we know nonlinear problems have been studied only in some particular cases of two parabolic coupled equations, see e.g. [63], [3], [14], [2] and [22]. Unlike the four first references, in the last one, the linearized system around the trajectory 0 is not null controllable. The authors apply the return method in order to overcome this difficulty. There are also open problems for linear systems that are easy to state but which answer is far from being known.

In this paper we have presented some controllability results for some non scalar problems. Let us mention that in the two main results of this work (see Theorems 5.3 and 6.1) we have given a necessary and sufficient condition for the null controllability of these systems. To our knowledge these results together with the one in [53] are the only ones where a vectorial approach is used. For the moment, this approach requires using constant coefficients and also restricts the class of parabolic systems concerned. For us, extending this vectorial approach to general parabolic systems seems to be a challenging issue.

It is impossible to list all the open problems and we conclude the paper by presenting some of them. They are *simple* but we think that they can illustrate the complexity of controllability issues for linear parabolic systems:

1. Most of the difficulties for the null controllability of System (30) come from the fact of having less controls than equations ( $m < n$ ). But even in the case  $B = I_n$  ( $m = n$ ) and  $D$  a non diagonalizable matrix the null controllability of System (30) is open if  $n > 4$  (see [33] for some partial results).
2. Apart from the results in [1] the boundary controllability problem for parabolic systems in the  $N$ -dimensional case ( $N > 1$ ) is widely open (even in the case of identity diffusion matrices and constant coefficients).
3. In section 7.2.5 we have given an approximate controllability result of a cascade system when the support of the coupling and the control sets have empty intersection. This problem is far from being well-understood. In terms of unique continuation the following “simple” problem is open. Let us take  $a, b \in L^\infty(Q_T)$  and  $\omega \cap \mathcal{O} = \emptyset$  and consider the system:

$$\begin{cases} -\partial_t p - \Delta p + bp = 0 & \text{in } Q_T, \\ -\partial_t z - \Delta z + az = p1_{\mathcal{O}} & \text{in } Q_T, \\ p = z = 0 & \text{on } \Sigma_T, \\ p(\cdot, T) = p^T, \quad z(\cdot, T) = z^T & \text{in } \Omega, \end{cases}$$

where  $(p^T, z^T) \in L^2(\Omega; \mathbb{R}^2)$ . Assume that  $(p, z)$  is a solution of this system. The unique continuation problem to be answered is the following one: Does  $z \equiv 0$  in  $\omega \times (0, T)$  implies  $p = z = 0$  in  $Q_T$ ? Some partial answers can be found in [48] and [1].

4. In this paper, we have only considered the control of System (30) where the coupling is exerted by means of the matrix  $A$ . Of course this is not the general case and for example, one can consider systems with coupling in the first order terms. The controllability of such systems has been investigated in [42] and [11]. Some sufficient conditions for the null controllability have been proposed but the general question is again widely open.

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*E-mail address:* fammarkh@univ-fcomte.fr

*E-mail address:* assia@cmi.univ-mrs.fr

*E-mail address:* manoloburgos@us.es

*E-mail address:* deteresa@matem.unam.mx