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ABSTRACT

The symmetries in a neighbourhood of a gray value image are modelled by conjugate harmonic function pairs. A harmonic function pair is utilized to represent a coordinate transformation defining a symmetry type. In this coordinate representation the image parts, which are symmetric with respect to the chosen function pair, have iso-gray value curves which are simple lines or parallel line patterns. The detection is modelled in the special Fourier domain corresponding to the new variables by minimizing an error function. It is shown that the minimization process or detection of these patterns can be carried out for the whole image *entirely in the spatial domain by convolutions.* The convolution kernel is complex valued, as is the the result. The magnitudes of the result are shown to correspond to a well defined certainty measure, while the orientation is the least square estimate of an orientation in the Fourier transform corresponding to the harmonic coordinates. Applications to four symmetries are given. These are circular, linear, hyperbolic and parabolic symmetries. Experimental results are presented.

1 INTRODUCTION

Describing events in neighbourhoods of a gray value image is an increasing need in Computer Vision. The generalized Hough transform, [4], is general and accurate enough to find arbitrary curves with the drawback of being computationally demanding. In the following we will give a method for detection of a large class of symmetries in a gray value neighbourhood which is a generalization of the work done in [3], for circular symmetry.

The detection is based on minimization of an error function in the Fourier domain, but computed entirely in *the spatial domain.* This minimization process is shown to be a convolution of the complex valued partial derivative image with a complex valued filter. The result delivers an angle corresponding to a subclass of neighbourhoods within the family of the neighbourhoods the a priori chosen function pair can handle.

2 MODELING THE LOCAL NEIGHBOURHOODS BY HARMONIC FUNCTIONS

Let v be the *conjugate harmonic function* of u. Moreover assume that both u and v are single valued. By definition, a curve pair defined by u, v:

$$\xi = u(x,y) \quad (1)$$

$$\eta = v(x,y) \quad (2)$$

are orthogonal to each other at their intersection points for any constants ξ and η. For non trivial $u(x,y)$ and $v(x,y)$, (1-2) define a coordinate transformation which is reversible almost everywhere.

Definition 1 *The local neighbourhood $f(x,y)$ represented in its local Cartesian coordinates, is said to be symmetric with respect to the coordinates $(\xi,\eta)^t$ if there exists a one dimensional function g so that $f(x,y) = g(a\xi + b\eta)$ for some real constants, a and b. Here $(\xi,\eta)^t = (u(x,y), v(x,y))^t$ and v is the harmonic conjugate function of u.*

This definition implies that the iso-gray value curves of a neighbourhood, which is symmetric with respect to a coordinate pair $(\xi,\eta)^t$, are *parallel lines* in this coordinate system.

Theorem 1 *A symmetric neighbourhood with respect to the coordinates $(\xi,\eta)^t$, that is $f(a\xi + b\eta)$, has a Fourier transform, in these coordinates, which is concentrated to a line through the origin.*

Any neighbourhood in the image will then have a Fourier transform which is not necessarily concentrated to a line through the origin. We will fit the best line to the corresponding Fourier domain in the least square sense. If there exists a symmetry according to an a priori model then the error will be low, [1].

Theorem 2 *Let $E(\theta)$ be the error defined by:*

$$E(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2(\bar{k}, \bar{k}_\theta)\, |F(\bar{k})|^2\, d\omega d\varsigma \quad (3)$$

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with $d(\bar{k}, \bar{k}_\theta)$ *being the Euiclidean distance between the coordinate vector* $\bar{k} = (\omega, \varsigma)^t$ *and the axis* $\bar{k}_\theta = (\cos\theta, \sin\theta)^t$.

The axis minimizing $E(\theta)$*, e.g.* $2\theta_{min}$ *is given by the formula:*

$$2\theta_{min} = \tan^{-1}(\omega_d^2 - \varsigma_d^2, 2p_d) \quad (4)$$

where

$$\begin{aligned}\omega_d^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega^2 |F(\omega,\varsigma)|^2 d\omega d\varsigma \\ \varsigma_d^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varsigma^2 |F(\omega,\varsigma)|^2 d\omega d\varsigma \\ p_d &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varsigma\omega |F(\omega,\varsigma)|^2 d\omega d\varsigma.\end{aligned}$$

The error difference corresponding to the lines given by θ_{max} *and* θ_{min} *is obtained through:*

$$C_{f1} = E(\theta_{max}) - E(\theta_{min}) = \sqrt{(\omega_d^2 - \varsigma_d^2)^2 + 4p_d^2} \quad (5)$$

and corresponds to certainty for the found axis. Moreover, the line corresponding to the angle maximizing the error $E(\theta)$*, is orthogonal to the line minimizing the error.*

This theorem allows us to consolidate the obtained orientation and the corresponding symmetry to a complex number z_1, [5]:

$$z_1 = \omega_d^2 - \varsigma_d^2 + i2p_d = C_{f1}\exp(i2\theta_{min}). \quad (6)$$

3 DETECTION OF LOCAL SYMMETRIES

In the previous section we have presented a model by which we could test the symmetry with respect to a fixed coordinate system in a local image. The Parseval relation along with the Cauchy-Riemann equations establishes:

$$z_1 = \sum_j (f_{xj} + i f_{yj})^2 w^j \quad (7)$$

with

$$w^j = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu_j(\bar{r}) \exp\left[-i2\arg(\frac{\partial\xi}{\partial x} + i\frac{\partial\xi}{\partial y})\right] dx dy. \quad (8)$$

z_1 is evaluated through a *spatial convolution for every neighbourhood* from the samples f_{xj} and f_{yj}. The latter are the values of the functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the discrete image position $\bar{r}_j$. The image given by $(f_{xj} + i f_{yj})^2$ will be referred to as the *partial derivative image*. The analytic function μ_j is the interpolation or interpixel function. The filter coefficients decrease rapidly as $\|\bar{r}_j\|$ become large when μ_j is chosen as a Gaussian, [1].

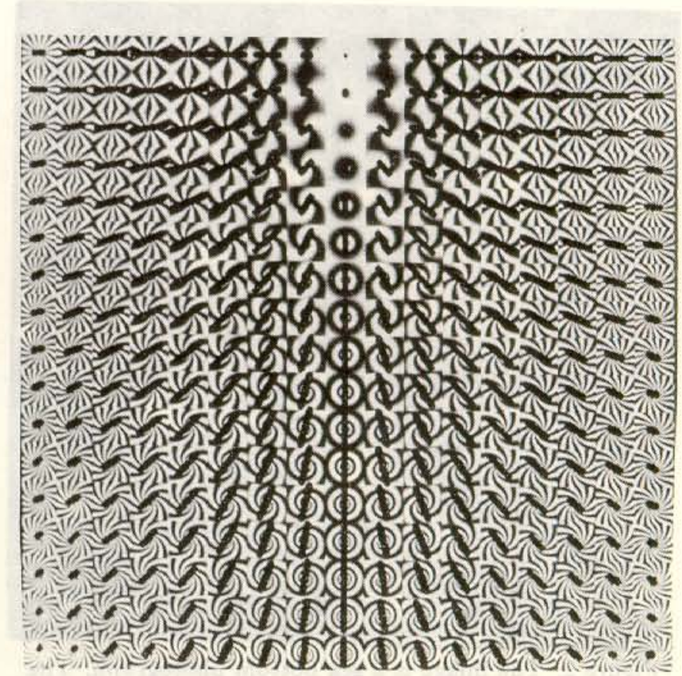

Figure 1: The figure illustrates orientations of the lines in ξ, η domain corresponding to the circular symmetry at selected parts of the original image. The lengths of the bars are proportional to the obtained certainties.

4 APPLICATIONS AND EXPERIMENTS

1. The function $\log z$, except for the origin, is analytic and single valued if one defines the *principal branch* as the value set. Since the imaginary part of an analytic function is the conjugate harmonic function of its real part, then

$$\log z = \ln r + i\varphi \quad (9)$$

where $r = |z|$, $\varphi = \arg(z)$ and z is the complex variable $x + iy$. Thus

$$\begin{aligned}\xi &= \ln r \\ \eta &= \varphi\end{aligned}$$

is obtained. According to the previous sections all neighbourhoods, with iso-gray values being $a\xi + b\eta$ with any real constants a and b, are included in the symmetry model associated with the coordinates $\xi = \ln r$ and $\eta = \varphi$. We will call this type of neighbourhood *circularly symmetric*. The result of the convolution proposed by (7) with the filter coeficients w^j which are obtained by (8), is overlayed the original image as bars, Figure 1. Experiments with natural images has been made. The circular symmetry detection is utilized in the classification tasks. Figure 2 illustrates a sea bottom image with sea anemones and their identification only by the circular symmetry image and the original image. Box classification is used.

2. Another pair of harmonic functions, the simplest one, is obtained by the analytic function z:

$$z = x + iy = \xi + i\eta \quad (10)$$

Figure 2: The image is a sea bottom photograph. The objective is to identify the sea anemones. The labels are obtained as a result of box classification.

This is simply a model of neighbourhoods having edge or line forms. It delivers the orientation of these lines together with a certainty, [2].

3. Choose the analytic function z^2 to generate the harmonic pair

$$z^2 = x^2 - y^2 + i2xy = \xi + i\eta \qquad (11)$$

generating a symmetry. The lines in this symmetry modell correspond to hyperbolas. The orientations of the lines uniquely correspond to the assymptotes of hyperbolas, [1].

4. Yet another symmetry will be generated by the real and the imaginary part of the analytic function $\sqrt{z}$, (the principal branch of the log is utilized):

$$\begin{aligned}\sqrt{z} &= \sqrt{r}\exp(i\frac{\varphi}{2}) \\ &= \sqrt{r}\cos(\frac{\varphi}{2}) + i\sqrt{r}\sin(\frac{\varphi}{2}) = \xi + i\eta.\end{aligned}$$

This operator is observed to be useful in finger print images to detect patterns having *parabolic symmetries*. The linear combinations of these coordinates, $a\xi + b\eta$, result in rotated versions of the parabolas, [1].

The list of symmetric patterns detectable by the formula (7) can be made long. It is sufficient to know one of the coordinate curves or its gradient, to be able to construct a symmetry model for the family of curves associated with this curve.

5 CONCLUSION

A method to model symmetries of the neighbourhoods in gray value images is derived. It is based on the form of the iso-gray value curves. For every neighbourhood a complex number is obtained through a convolution of a complex valued image with a complex valued filter. The magnitude of the complex number is the degree of symmetry with respect to the a priori chosen harmonic function pair. The degree of symmetry has a clear definition which is based on the error in the Fourier domain. The argument of the complex number is the angle representing the relative dominance of one of the harmonic pair functions compared to the other.

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