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RECOGNIZABILITY IN THE LATTICE OF CONVEX  $\ell$ -SUBGROUPS  
OF A LATTICE-ORDERED GROUP

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The intent of this note is to show that several types of lattice-ordered groups and convex  $\ell$ -subgroups are impossible to recognize from the lattice structure of the lattice of convex  $\ell$ -subgroups of a lattice-ordered group. In particular we show that one cannot recognize if a lattice-ordered group belongs to any nontrivial proper variety, if it is archimedean, or if it is completely distributive, and also that one cannot tell if a given convex  $\ell$ -subgroup is closed. It is already known that no nontrivial proper variety other than  $\mathcal{N}$  (the variety of normal-valued  $\ell$ -groups) is recognizable, but our proof that  $\mathcal{N}$  is not recognizable yields the fact for all nontrivial proper varieties at once. Our method is to consider two specific  $\ell$ -groups which we label  $G$  and  $G'$ . We show that the lattices  $\mathcal{C}(G)$  and  $\mathcal{C}(G')$  of convex  $\ell$ -subgroups are isomorphic as lattices, and then consider the recognizability questions.

Let  $A(\mathbb{R})$  represent the lattice-ordered group of order-preserving permutations of the real numbers with pointwise meet and join as lattice operations, and composition for the group operation. Let  $C(\mathbb{R})$  be the  $\ell$ -group of continuous functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ , with pointwise meet, join and addition. We define a continuous function  $f: \mathbb{R} \rightarrow \mathbb{R}$  to be *finitely piecewise linear* if there is a finite set of real numbers  $a_1 < a_2 < \dots < a_n$  such that for each interval  $I$  of  $\mathbb{R}$  which has no  $a_k$  in its interior, there are real numbers  $m$  and  $b$  such that  $f(x) = mx + b$  for each  $x \in I$ . We will say that such a function  $f$  has *finitely many pieces* over any interval  $J \subseteq \mathbb{R}$ , where a *piece of  $f$  over  $J$*  is defined to be  $f$  restricted to a subinterval  $J'$  of  $J$  that is maximal with respect to the property that there are real numbers  $m$  and  $b$  with  $f(x) = mx + b$  for all  $x$  in  $J'$ . We also define the 0-support of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  to be  $\{x \in \mathbb{R} \mid f(x) \neq 0\}$  and the 1-support of  $f = \{x \in \mathbb{R} \mid f(x) \neq x\}$ . We will use  $\text{supp}(f)$  to represent both the 0-support of a function in  $C(\mathbb{R})$  and the 1-support of a function in  $A(\mathbb{R})$ , since it will be clear from the context just which is intended.

We let  $G = \{f \in C(\mathbb{R}) \mid f \text{ is finitely piecewise linear and } \text{supp}(f) \text{ is bounded}\}$  and let  $G' = \{f \in A(\mathbb{R}) \mid f \text{ is finitely piecewise linear and } \text{supp}(f) \text{ is bounded}\}$ . Since  $G$  is an  $\ell$ -subgroup of  $C(\mathbb{R})$  and  $G'$  is an  $\ell$ -subgroup of  $A(\mathbb{R})$ , we know that  $G$  and  $G'$  are  $\ell$ -groups. Ball invented the group  $G'$  to show that an  $\ell$ -group can have the DCC on regular subgroups and still fail to be normal-valued. In fact,  $G'$  has no nontrivial

proper normal subgroups at all. A variation of this example due to Conrad can be used to show that a locally flat  $\ell$ -group need not be archimedean (take  $G'' = G' \cap A(\emptyset)$ ; neither  $G'$  nor  $G''$  is archimedean).

Now we show that the lattices  $\mathcal{C}(G)$  and  $\mathcal{C}(G')$  are isomorphic. We begin by identifying all of the prime subgroups of  $G$ . For each  $r \in \mathbb{R}$ , let

$$\begin{cases} G_r &= \{f \in G \mid f(r) = 0\} \\ G_{r,+} &= \{f \in G \mid \text{for some } \varepsilon > 0 \text{ and for all } x \text{ in } [r, r + \varepsilon], f(x) = 0\} \\ G_{r,-} &= \{f \in G \mid \text{for some } \varepsilon > 0 \text{ and for all } x \text{ in } [r - \varepsilon, r], f(x) = 0\}. \end{cases}$$

It is easily seen that these are prime subgroups, using the finiteness in the cases of  $G_{r,+}$  and  $G_{r,-}$ . The next three lemmas together show that these are the only prime subgroups of  $G$ .

**1. Lemma.** *Every proper convex  $\ell$ -subgroup of  $G$  is contained in some  $G_r$ . Hence each prime subgroup lies in a unique  $G_r$ .*

*Proof.* Let  $C \in \mathcal{C}(G)$  and suppose that for all  $r \in \mathbb{R}$ ,  $C \not\subseteq G_r$ . Let  $0 < g \in G$ , and let  $K = \overline{\text{supp}(g)}$ .  $K$  is compact, and so  $g(K)$  is bounded. Let  $M = \sup \{g(x) \mid x \in \mathbb{R}\}$ . For each  $r \in K$ ,  $C \not\subseteq G_r$ , so we can find  $0 < f_r \in C \setminus G_r$ . Since  $f_r(r) > 0$ , we have  $\frac{1}{2}f_r(r) > 0$ . Let  $U_r = \{x \in \mathbb{R} \mid f_r(x) > \frac{1}{2}f_r(r)\}$ . Since each  $f_r$  is continuous, each  $U_r$  is an open neighbourhood of  $r$ , and so  $\{U_r \mid r \in K\}$  is an open cover of  $K$ . Let  $\{U_{r_i} \mid 1 \leq i \leq n\}$  be a finite subcover, and let  $m = \min \{\frac{1}{2}f_{r_i}(r_i) \mid 1 \leq i \leq n\}$ . We have  $m > 0$ , so there must be a positive integer  $N$  with  $mN > M$ . Let  $f = \bigvee_{i=1}^n Nf_{r_i}$ .

Since  $0 < g \leq f \in C$ , we have  $g \in C$ , so  $C = G$ , and every proper convex  $\ell$ -subgroup lies in some  $G_r$ . Now, since  $G_r$  and  $G_s$  are incomparable if  $r \neq s$ , and since for any prime subgroup the set of convex  $\ell$ -subgroups containing it is totally ordered, each prime subgroup lies in a unique  $G_r$ .  $\square$

**2. Lemma.** *If  $P$  is a prime subgroup of  $G$  with  $P \not\subseteq G_r$ , then either  $P \subseteq G_{r,+}$  or  $P \subseteq G_{r,-}$ .*

*Proof.* Suppose  $P$  is a prime subgroup of  $G$  with  $P \not\subseteq G_{r,+}$  and  $P \not\subseteq G_{r,-}$ , but  $P \subseteq G_r$ . Let  $0 < h_1 \in P \setminus G_{r,+}$  and  $0 < h_2 \in P \setminus G_{r,-}$ . Let  $0 < g \in G_r$ . As in the proof of Lemma 1, let  $M = \sup \{g(x) \mid x \in \mathbb{R}\}$ . Since  $0 < h_1 \in P \subseteq G_r$  and  $h_1 \notin G_{r,+}$ , there must exist  $\varepsilon_1 > 0$  and  $m_1 > 0$  with  $h_1(x) = m_1(x - r)$  for all  $x$  in  $[r, r + \varepsilon_1]$ . Similarly, there exist  $\varepsilon_2 > 0$  and  $m_2 < 0$  with  $h_2(x) = m_2(x - r)$  for all  $x$  in  $[r - \varepsilon_2, r]$ . Since  $g$  consists of finitely many pieces over  $[r, r + \varepsilon_1]$ , we let  $m_+ = \max \{\text{slopes of pieces of } g \text{ over } [r, r + \varepsilon_1]\}$ . Similarly, we set  $m_- = \min \{\text{slopes of pieces of } g \text{ over } [r - \varepsilon_2, r]\}$ . We let  $N$  be a positive integer such that  $m_1N \geq m_+$  and  $m_2N \leq m_-$ . Then for all  $x$  in  $[r - \varepsilon_2, r + \varepsilon_1]$  we have  $0 \leq g(x) \leq [N(h_1 \vee h_2)](x)$ , with  $N(h_1 \vee h_2) \in P$ . Now let  $K_0 = \overline{\text{supp}(g)} \setminus (r - \varepsilon_2, r + \varepsilon_1)$ .  $K_0$  is compact, and  $r \notin K_0$ . Since  $P$  is prime, we know from Lemma 1 that if  $s \in K_0$ , then  $P \not\subseteq G_s$ . As in

the proof of Lemma 1, we find  $f \in P$  with  $f > 0$  and  $f(x) \geq g(x)$  for all  $x \in K_0$ . Then  $0 < g \leq f \vee N(h_1 \vee h_2)$ , with  $f \vee N(h_1 \vee h_2) \in P$ , so  $g \in P$ , and  $P = G_r$ .  $\square$

**3. Lemma.** *If  $P$  is a prime subgroup of  $G$  with  $P \subseteq G_{r+}$ , then  $P = G_{r+}$ . If  $P$  is a prime subgroup of  $G$  with  $P \subseteq G_{r-}$  then  $P = G_{r-}$ . Thus  $\{C \in \mathcal{C}(G) \mid \text{for some } r \in \mathbb{R}, C = G_r \text{ or } G_{r+} \text{ or } G_{r-}\}$  is a complete list of the prime subgroups of  $G$ .*

*Proof.* Let  $P \in \mathcal{C}(G)$ , with  $P \not\subseteq G_{r+}$ . Let  $0 < g \in G_{r+} \setminus P$ . Then there exists  $\varepsilon > 0$  with  $g(x) = 0$  for all  $x$  in  $[r, r + \varepsilon]$ . Define  $f$  by:

$$f(x) = \begin{cases} x - r, & \text{if } x \in [r, r + \frac{1}{2}\varepsilon] \\ r + \varepsilon - x, & \text{if } x \in [r + \frac{1}{2}\varepsilon, r + \varepsilon] \\ 0, & \text{if } x \notin [r, r + \varepsilon]. \end{cases}$$

Now,  $f \notin G_{r+}$ , so  $f \notin P$ , and  $g \notin P$ , but  $f \wedge g = 0$ , so  $P$  is not prime. A similar example shows that if  $P \in \mathcal{C}(G)$  and  $P \not\subseteq G_{r-}$ , then  $P$  is not prime.  $\square$

In  $G'$ , we let  $G'_r = \{f \in G' \mid f(r) = r\}$ ,  $G'_{r+} = \{f \in G' \mid \text{for some } \varepsilon > 0 \text{ and for all } x \in [r, r + \varepsilon], f(x) = x\}$ , and  $G'_{r-} = \{f \in G' \mid \text{for some } \varepsilon > 0 \text{ and for all } x \in [r - \varepsilon, r], f(x) = x\}$ . Then, as part of Ball's example, it is known that  $\{C' \in \mathcal{C}(G') \mid \text{for some } r \in \mathbb{R}, C' = G'_r \text{ or } G'_{r+} \text{ or } G'_{r-}\}$  is a complete list of the prime subgroups of  $G'$ . (The three lemmas we have just proved can be modified slightly to obtain this result.)

Now, recall (Bigard, et al, [1], 2.5.5) that a convex  $\ell$ -subgroup is the intersection of the prime subgroups containing it. We use this to set up a lattice isomorphism between  $\mathcal{C}(G)$  and  $\mathcal{C}(G')$  via a lattice that describes the collections of prime subgroups above elements of  $\mathcal{C}(G)$  and  $\mathcal{C}(G')$ . This isomorphism will send  $G_r$  to  $G'_r$ ,  $G_{r+}$  to  $G'_{r+}$ , and  $G_{r-}$  to  $G'_{r-}$ , as one would expect.

Let  $\mathcal{S} = \{(S, S_+, S_-) \mid S_+ \cup S_- \subseteq S \subseteq \mathbb{R}, S \text{ is closed in } \mathbb{R}, \text{ and for each sequence } \{s_n\}_{n=1}^\infty \text{ in } S \text{ with } \lim_{n \rightarrow \infty} s_n = s, \text{ we have that if } \{s_n\}_{n=1}^\infty \text{ is strictly increasing, then } s \in S_-, \text{ while if } \{s_n\}_{n=1}^\infty \text{ is strictly decreasing, then } s \in S_+\}$ . Order  $\mathcal{S}$  by setting  $(S, S_+, S_-) \leq (T, T_+, T_-)$  if and only if  $S \supseteq T$  and  $S_+ \supseteq T_+$  and  $S_- \supseteq T_-$ . This is a lattice order on  $\mathcal{S}$ , with  $(S, S_+, S_-) \vee (T, T_+, T_-) = (S \cap T, S_+ \cap T_+, S_- \cap T_-)$  and  $(S, S_+, S_-) \wedge (T, T_+, T_-) = (S \cup T, S_+ \cup T_+, S_- \cup T_-)$ .

**4. Proposition.** *There is a lattice isomorphism between  $\mathcal{C}(G)$  and  $\mathcal{S}$ .*

*Proof.* Let  $C \in \mathcal{C}(G)$ , and set  $S_C = \{r \in \mathbb{R} \mid C \subseteq G_r\}$ ,  $S_{C+} = \{r \in \mathbb{R} \mid C \subseteq G_{r+}\}$ , and  $S_{C-} = \{r \in \mathbb{R} \mid C \subseteq G_{r-}\}$ . We want  $(S_C, S_{C+}, S_{C-}) \in \mathcal{S}$ . Obviously,  $S_C \subseteq \mathbb{R}$ , and since  $G_{r+} \subseteq G_r$  and  $G_{r-} \subseteq G_r$  for each  $r \in \mathbb{R}$ ,  $S_{C+} \cup S_{C-} \subseteq S_C$ . Let  $\{s_n\}_{n=1}^\infty$  be a sequence in  $S_C$ , with  $\lim_{n \rightarrow \infty} s_n = s \in \mathbb{R}$ . Let  $f \in C$ . Then  $f(s_n) = 0$  for all positive integers  $n$ , and by continuity,  $f(s) = 0$ , so  $f \in G_s$ , and  $C \subseteq G_s$ . Then  $s \in S_C$ , and  $S_C$  is closed. If, in addition,  $\{s_n\}_{n=1}^\infty$  is strictly increasing, then because  $f \in G$ , there must be  $\varepsilon > 0$  such that  $f(x) = mx + b$  on  $[s - \varepsilon, s]$ . Since  $\lim_{n \rightarrow \infty} s_n = s$ , there must be a positive integer  $N$  such that if  $n \geq N$ , then  $|s_n - s| < \varepsilon$ . Therefore, if  $n_1 > n_2 \geq N$

$s_{n_1} \in [s - \varepsilon, s]$  and  $s_{n_2} \in [s - \varepsilon, s]$  with  $s_{n_1} \neq s_{n_2}$ . Hence  $f(s_{n_1}) = f(s_{n_2}) = 0$ , so  $f(x) \equiv 0$  on  $[s - \varepsilon, s]$ , and  $s \in S_{C^-}$ . Similarly if  $\{s_n\}_{n=1}^\infty$  is strictly decreasing, then  $s \in S_{C^+}$ , and so  $(S_C, S_{C^+}, S_{C^-}) \in \mathcal{S}$  for each  $C \in \mathcal{C}(G)$ . We let  $\theta : \mathcal{C}(G) \rightarrow \mathcal{S}$  by setting  $\theta(C) = (S_C, S_{C^+}, S_{C^-})$ . If  $C, D \in \mathcal{C}(G)$  with  $C \neq D$ , then the sets of primes exceeding  $C$  is not equal to the set of primes containing  $D$ , and so  $\theta(C) \neq \theta(D)$ . Thus  $\theta$  is one-to-one. Let  $(S, S_+, S_-) \in \mathcal{S}$ , and set  $C = (\bigcap_{s \in S} G_s) \cap (\bigcap_{s \in S_+} G_{s^+}) \cap (\bigcap_{s \in S_-} G_{s^-})$ .

Obviously  $C \in \mathcal{C}(G)$ , and  $\theta(C) \leq (S, S_+, S_-)$ . If  $s \notin S$ , then since  $S$  is closed, there is  $\varepsilon > 0$  such that if  $x \in (s - 2\varepsilon, s + 2\varepsilon)$ , then  $x \notin S$ . Define  $f \in G$  by

$$f(x) = \begin{cases} x - s + \varepsilon, & \text{if } s - \varepsilon \leq x \leq s \\ -x + s + \varepsilon, & \text{if } s \leq x \leq s + \varepsilon \\ 0, & \text{if } x \notin [s - \varepsilon, s + \varepsilon]. \end{cases}$$

If  $x \in S$ , then  $x \notin [s - \varepsilon, s + \varepsilon]$ , so  $f(x) = 0$ . If  $x \in S_+ \cup S_-$ , then  $x \in S$  and  $f(x) = 0$ , and in addition, for each  $y$  with  $|y - x| < \varepsilon$ , we have  $y \notin [s - \varepsilon, s + \varepsilon]$ , since  $x \notin [s - 2\varepsilon, s + 2\varepsilon]$ . Thus  $f(y) = 0$ , so  $f \in G_r$  for all  $r \in S$ ,  $f \in G_{r^+}$  for all  $r \in S_+$ , and  $f \in G_{r^-}$  for all  $r \in S_-$ ; hence  $f \in C$ . But  $f(s) = \varepsilon > 0$ , so  $f \notin G_s$ , and  $C \not\subseteq G_s$ . Thus  $S_C = S$ . Suppose  $s \notin S_+$ . By definition there is  $\varepsilon > 0$  such that if  $x \in (s, s + 2\varepsilon)$ , then  $x \notin S$ . Define  $g \in G$  by

$$g(x) = \begin{cases} 2x - 2s, & \text{if } x \in [s, s + \frac{1}{2}\varepsilon] \\ -2x + 2s + 2\varepsilon, & \text{if } x \in [s + \frac{1}{2}\varepsilon, s + \varepsilon] \\ 0, & \text{if } x \notin [s, s + \varepsilon]. \end{cases}$$

If  $x \in S$ , then  $x \notin (s, s + \varepsilon)$ , so  $g(x) = 0$ . If  $x \in S_-$ , then  $x \notin (s, s + 2\varepsilon)$ , so that for each  $y$  in  $[x - \varepsilon, x]$ ,  $y \notin (s, s + \varepsilon)$  and  $g(y) = 0$ . If  $x \in S_+$ , then  $x \neq s$  and  $x \notin (s, s + \varepsilon)$  so if  $x \geq s + \varepsilon$ , then for all  $y$  in  $[x, x + \varepsilon]$ ,  $y \geq s + \varepsilon$  and  $g(y) = 0$ , while if  $x < s$ , then set  $\delta = \frac{1}{2}(s - x)$ . We have  $\delta > 0$ , and for each  $y$  in  $[x, x + \delta]$ ,  $y < s$ , so  $g(y) = 0$ . Thus  $g \in C$ , but  $g \notin G_{s^+}$  since there is no  $\alpha > 0$  with  $g(x) = 0$  for all  $x$  in  $[s, s + \alpha]$ . Hence  $S_+ = S_{C^+}$ . A similar argument shows that  $S_- = S_{C^-}$ , so that  $\theta(C) = (S, S_+, S_-)$ , and  $\theta$  is onto. If  $C, D \in \mathcal{C}(G)$ , then since  $G_r$  is a prime subgroup of  $G$  for each  $r \in \mathbb{R}$ , we have that if  $C \wedge D \subseteq G_r$ , then  $C \subseteq G_r$  or  $D \subseteq G_r$ . Thus  $r \in S_{C \wedge D} \Leftrightarrow C \wedge D \subseteq G_r \Leftrightarrow C \subseteq G_r$  or  $D \subseteq G_r \Leftrightarrow r \in S_C \cup S_D$ , so  $S_{C \wedge D} = S_C \cup S_D$ . Similarly,  $S_{C \wedge D^+} = S_{C^+} \cup S_{D^+}$  and  $S_{C \wedge D^-} = S_C \cup S_{D^-}$ , so that  $\theta(C \wedge D) = \theta(C) \wedge \theta(D)$ . Also,  $r \in S_{C \vee D} \Leftrightarrow C \vee D \subseteq G_r \Leftrightarrow C \subseteq G_r$  and  $D \subseteq G_r \Leftrightarrow r \in S_C \cap S_D$ , so that  $S_{C \vee D} = S_C \cap S_D$ . Similarly,  $S_{C \vee D^+} = S_{C^+} \cap S_{D^+}$  and  $S_{C \vee D^-} = S_{C^-} \cap S_{D^-}$ , so that  $\theta(C \vee D) = \theta(C) \vee \theta(D)$ , and  $\theta$  is a lattice isomorphism of  $\mathcal{C}(G)$  onto  $\mathcal{S}$ .  $\square$

Essentially the same proof shows that if  $\theta' : \mathcal{C}(G') \rightarrow \mathcal{S}$ . By the rule  $\theta'(C') = (S_{C'}, S_{C'^+}, S_{C'^-})$ , where  $S_{C'} = \{r \in \mathbb{R} \mid C' \subseteq G'_r\}$ ,  $S_{C'^+} = \{r \in \mathbb{R} \mid C' \subseteq G'_{r^+}\}$ , and  $S_{C'^-} = \{r \in \mathbb{R} \mid C' \subseteq G'_{r^-}\}$ , then  $\theta'$  is a lattice isomorphism of  $\mathcal{C}(G')$  onto  $\mathcal{S}$ . Thus, the function  $(\theta')^{-1} \theta$  is a lattice isomorphism of  $\mathcal{C}(G)$  onto  $\mathcal{C}(G')$ , and it is easily

checked that this map sends  $G_r$  to  $G'_r$ ,  $G_{r+}$  to  $G'_{r+}$ , and  $G_{r-}$  to  $G'_{r-}$  for each  $r \in \mathbb{R}$ .

Now, we consider the questions of recognizability. First,  $G$  is a member of the variety of abelian lattice-ordered groups, which is a subvariety of all nontrivial varieties, while  $G'$  lies outside the variety of normal-valued  $\ell$ -groups, since each  $G'_r$  is a regular subgroup of  $G'$  but is not normal (however,  $G'_{r+} \triangleleft G'_r$  and  $G'_{r-} \triangleleft G'_r$ , for each  $r \in \mathbb{R}$ ). But  $\mathcal{N}$  contains all proper varieties, so  $G'$  cannot lie in any proper variety, while  $G$  is in all nontrivial varieties, and it is impossible to tell if a lattice-ordered group belongs to any nontrivial proper variety by considering only its lattice of convex  $\ell$ -subgroups.

Next,  $G$  is archimedean, while  $G'$  is not, and so it is impossible to recognize from the lattice of convex  $\ell$ -subgroups if a given  $\ell$ -group is archimedean.

Martinez [6] introduced the idea of a torsion class, which generalizes the notion of a variety. The class of archimedean  $\ell$ -groups is not a torsion class, but if we take only those archimedean  $\ell$ -groups with the additional property that every  $\ell$ -homomorphic image is archimedean, then we get the torsion class of hyperarchimedean  $\ell$ -groups. Several characterizations of hyperarchimedean  $\ell$ -groups are known (for a history, see Conrad [3]), including the fact that an  $\ell$ -group is hyperarchimedean if and only if each prime subgroup is a minimal prime subgroup, and so each prime subgroup is a maximal. Since a convex  $\ell$ -subgroup  $P$  is prime if and only if the set of convex  $\ell$ -subgroups containing  $P$  is totally ordered, we can recognize primes, and certainly maximals are recognizable, so it is possible to recognize if an  $\ell$ -group is hyperarchimedean. Hence some torsion classes are recognizable, while varieties are not. Conrad [4] has considered several other important torsion classes, and determined which are recognizable by looking at torsion radicals. He has shown that the torsion class of divisible abelian  $\ell$ -groups is not recognizable, while the following torsion classes are recognizable:

- $\mathcal{A}$  = all hyperarchimedean  $\ell$ -groups
- $\mathcal{F}$  = all  $\ell$ -groups such that each bounded disjoint subset is finite
- $\mathcal{F}_v$  = all finite-valued  $\ell$ -groups
- $\mathcal{D}$  = all  $\ell$ -groups with DCC on the set of regular subgroups
- $\mathcal{O}$  = all cardinal sums of o-groups
- $\mathcal{A}$  = all cardinal sums of archimedean o-groups
- $\mathcal{B}$  = all  $\ell$ -groups such that each prime exceeds a unique minimal prime.

Recall that in an  $\ell$ -group  $A$ , a convex  $\ell$ -subgroup  $C$  is closed if for each family  $\{x_i \mid i \in I\}$  with  $e \leq x_i \in C$  such that  $\bigvee_{i \in I} x_i = x$  exists in  $A$ , then  $x \in C$ . In Bigard, et al. [1] (11.1.10), it is shown that in an archimedean  $\ell$ -group, closed convex  $\ell$ -subgroups are polars, so that in  $G$ , no  $G_r$ ,  $G_{r+}$ , or  $G_{r-}$  can be closed, since they are not polars (for  $G_{r+}^\perp = G_{r-}^\perp = \{0\}$ ). On the other hand, for each  $r \in \mathbb{R}$ ,  $G'_r$  is closed in  $G'$ . This is a result of McCleary [7], which generalized the work of Lloyd [5]. Hence it is impossible to recognize closed convex  $\ell$ -subgroups in the lattice of convex  $\ell$ -subgroups of a lattice-ordered group.

Finally, Byrd and Lloyd [2] define the distributive radical of an  $\ell$ -group, and show that it is the intersection of all closed prime subgroups. They also show that an  $\ell$ -group is completely distributive if and only if it has a trivial distributive radical. In our examples, no prime subgroups of  $G$  are closed, so that  $G$  is its own radical, while all of the  $G'_i$  are closed in  $G'$ , so that the distributive radical of  $G'$  is  $\{1\}$ . Hence it is impossible to recognize the distributive radical, and one cannot tell if an  $\ell$ -group is completely distributive, from the lattice of convex  $\ell$ -subgroups.

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