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On recognizable languages in divisibility monoids

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Abstract. Kleene's theorem on recognizable languages in free monoids is considered to be of eminent importance in theoretical computer science. It has been generalized into various directions, including trace and rational monoids. Here, we investigate divisibility monoids which are defined by and capture algebraic properties sufficient to obtain a characterization of the recognizable languages by certain rational expressions as known from trace theory. The proofs rely on Ramsey's theorem, distributive lattice theory and on Hashigushi's rank function generalized to our divisibility monoids. We obtain Ochmański's theorem on recognizable languages in free partially commutative monoids as a consequence.

1 Introduction

In the literature, Kleene's theorem on recognizable languages of finite words has been generalized in several directions, e.g. to formal power series by Schützenberger [17], to infinite words by Büchi [5], and to rational monoids by Sakarovitch [16]. In all these cases, the notions of recognizability and of rationality where shown to coincide. In concurrency theory, several authors investigated recognizable languages in trace monoids (free partially commutative monoids) which generalize free monoids. It is known that here the recognizable languages only form a proper subclass of the rational languages, but a precise description of them using c-rational expressions could be given by Ochmański [13]. A further generalization of Kleene's and Ochmański's results to concurrency monoids was given in [8]. It is the goal of this paper to derive such a result for even more general monoids. At the same time, we obtain that well known combinatorial methods crucial in trace theory (like Levi's Lemma) are intimately related with algebraic properties (like distributivity) from classical lattice theory [3] or the theory of event structures [19].

Trace theory provides an important mathematical model for the sequential behavior of a parallel system in which the order of two independent actions is regarded as irrelevant. One considers pairs (T, I) where T is the set of actions, and I is a symmetric and irreflexive binary relation on T describing the independence of two actions. The trace monoid or free partially commutative monoid $\mathbb{M}(T, I)$ is then defined as the quotient $T^\star/\!\sim$ where $\sim$ is the congruence on the free monoid $T^\star$ generated by all pairs (ab, ba) with $(a, b) \in I$. For surveys on the many results obtained for trace monoids, we refer the reader to the collection [7].

An algebraic characterization of trace monoids was given by Duboc [10]. Here we use a lattice theoretically easy generalization of these algebraic conditions for the definition of divisibility monoids.

As for trace monoids, a divisibility monoid has a finite system of irreducible generators. They could be viewed as atomic transitions in a concurrent system. However, in comparison with trace monoids we allow much more general commutation possibilities for these generators. In our monoids it is possible, e.g., that $ab = cd$ or $ab = cc$ where a, b, c, d are four pairwise different irreducible generators. This would mean that the different sequential transformations ab and cd (cc, resp.) give rise to the same effect. It is clear that this is a much more general situation than in trace theory where $ab = cd$ implies $\{a, b\} = \{c, d\}$ (a, b, c, d generators as above) and even than in the situation of the automata with dynamic (situation dependent) independence of actions investigated in [8]. However, as for traces, we assume that any two sequential representations (i.e., products) by irreducible generators of a given monoid element have the same length. This is ensured by requiring that the divisibility monoid is cancellative and that the prefix (= left divisibility) relation satisfies natural distributivity laws well known from lattice theory (Birkhoff [3]). These classical distributivity laws suffice to deduce our results. Also, they enable us to develop and use a calculus of residuals similar to the one used e.g. in lambda calculus [2], term rewriting [4] and the models for concurrency considered in [14, 18].

In these divisibility monoids, we investigate closure properties of the class of recognizable languages under rational operations, analogously as in trace theory. To achieve this, we develop an extension of the notion of the *rank* of a language, which was already shown to be very useful in trace theory by Hashigushi [11], cf. [7, 6]. Under the assumption of a finiteness condition on the commutation behavior of the monoid elements, we can prove that the product of recognizable languages is again recognizable.

To deal with the iteration, analogously as in trace theory, we define when a monoid element is connected (intuitively, it cannot be split into disjoint components) using classical lattice-theoretic concepts. In trace theory, the iteration of a recognizable language consisting only of connected elements is again recognizable. We show (cf. Example 1) that, somewhat surprisingly, this fails in general in divisibility monoids. However, using the residuum operation mentioned above, we can define when a language is residually closed. Then we can show, using also Ramsey's Theorem, that the iteration of a recognizable residually closed language consisting only of connected elements is again recognizable. We call a language c-rational if it can be constructed from finite languages using the operations union, product and this restricted version of iteration. Thus, the closure properties indicated so far ensure that any c-rational language is recognizable.

Recall that an equation $ab = cd$ with irreducible generators a, b, c, d of M states that the different sequential executions ab and cd give rise to the same effect. If now $a \neq c$, the effect of a in the execution cd has to be resumed by that of d. Therefore, we consider the least equivalence on the irreducible generators of M identifying a and d that occur in an equation $ab = cd$ with $a \neq c$. Requiring that a and c are not equivalent whenever $ab = cd$ and $a \neq c$, we can prove the converse of the above result, i.e., we can show that any recognizable language is c-rational. With this requirement, our divisibility monoids are more similar to, but still more general than trace monoids. Our results can be summarized as follows (see the subsequent sections for the precise definitions)

Theorem 1. *Let $(M, \cdot, 1)$ be a labeled divisibility monoid with finite commutation behavior and $L \subseteq M$. Then L is recognizable iff L is c-rational.*

From these results, we obtain Ochmański's theorem for recognizable trace languages as an immediate consequence. Furthermore, a strengthening of the results from [8] for recognizable languages in concurrency monoids follows from our results (see the full paper [9]).

As the above examples and many others show, the class of divisibility monoids is much larger than the class of all concurrency monoids investigated in [8] which in turn is larger than the class of trace monoids.

The present divisibility monoids can hence be viewed as a general model for concurrent behaviors where it is still possible to describe recognizable sets of behaviors by certain rational expressions.

The complete proofs are contained in the full paper [9].

2 Preliminaries

Let $(M, \cdot, 1)$ be a monoid and $L \subseteq M$. A monoid morphism $\eta : M \to S$ into a finite monoid $(S, \cdot, 1)$ *recognizes* L if $\eta^{-1}\eta(L) = L$. The language L is *recognizable* if there exists a monoid morphism that recognizes L. For $x \in M$ let $x^{-1}L := \{y \in M \mid x \cdot y \in L\}$, the left quotient of L with respect to x. Then a classical result states that L is recognizable iff the set $\{x^{-1}L \mid x \in M\}$ is finite iff there is a finite M-automaton recognizing L.

Let $L, K \subseteq M$. Then $L \cdot K := \{l \cdot k \mid l \in L, k \in K\}$ is the product of L and K. By $\langle L \rangle$ we denote the submonoid of M generated by L, i.e., $\langle L \rangle = \{l_1 \cdot l_2 \cdot \ldots l_n \mid n \in \mathbb{N}, l_i \in L\}$. For a set T, $T^\star$ denotes the free monoid generated by T. Now let M be a free monoid and $L \subseteq M$. Then $\langle L \rangle$ is a subset of M while $L^\star$ is a set of words whose letters are elements of M. Classical formal language theory usually identifies the set $L^\star$ of words over L and the submonoid $\langle L \rangle$ of M generated by L. In this paper, we have to distinguish between them.

A language $L \subseteq M$ is *rational* if it can be constructed from the finite subsets of M by union, multiplication and iteration.

Now let T be a finite set and $L \subseteq M := T^\star$. By Kleene's Theorem, L is recognizable iff it is rational. In any monoid, the set of recognizable languages is closed under the usual set-theoretic operations, like complementation, intersection and difference.

For $x \in T^\star$, let $\alpha(x)$ denote the alphabet of x comprising all letters of T occurring in x. Then $L_B := \langle B \rangle \cap L \setminus (\bigcup_{A \subset B} \langle A \rangle)$ with $B \subseteq T$ is the set of elements x of L with $\alpha(x) = B$. If L is rational, the language L_B is rational, too. The language L is *monoalphabetic* if $L = L_B$ for some $B \subseteq T$. A language

$L \subseteq M$ is *monoalphabetic-rational* if it can be constructed from the finite subsets of M by union, multiplication and iteration where the iteration is applied to monoalphabetic languages, only. One can easily show that in a finitely generated free monoid any rational language is monoalphabetic-rational.

Let $(P, \leq)$ be a partially ordered set and $x \in P$. Then $\downarrow x$ comprises all elements dominated by x, i.e., $\downarrow x := \{y \in P \mid y \leq x\}$. If $A \subseteq P$, we write $A \leq x$ to denote that $A \subseteq \downarrow x$, i.e., that $a \leq x$ for all $a \in A$. The partially ordered set $(P, \leq)$ is a *lattice* if for any $x, y \in P$ the least upper bound $\sup(x, y) = x \vee y$ and the largest lower bound $\inf(x, y) = x \wedge y$ exist. The lattice $(P, \leq)$ is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in P$. This is equivalent to $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in P$. For properties of finite distributive lattices, we refer the reader to [3].

3 Divisibility monoids

In this section, we introduce divisibility monoids and investigate their basic properties.

Let $M = (M, \cdot, 1)$ be a monoid where $1 \in M$ is the unit element. We call M *cancellative* if $x \cdot y \cdot z = x \cdot y' \cdot z$ implies $y = y'$ for any $x, y, y', z \in M$. This in particular ensures that M does not contain a zero element and will be a very natural assumption (trivially satisfied in free monoids). For $x, y \in M$, *x is a left divisor of y* (denoted $x \leq y$) if there is $z \in M$ such that $x \cdot z = y$. In general, the relation $\leq$ is not antisymmetric, but we require this for a divisibility monoid.

Let $T := (M \setminus \{1\}) \setminus (M \setminus \{1\})^2$. The set T consists of those nonidentity elements of M that do not have a proper divisor, its elements are called *irreducible*. Note that T has to be contained in any set generating M.

Definition 1. *A monoid $(M, \cdot, 1)$ is called a* (left) divisibility monoid *provided the following hold*

1. *M is cancellative and its irreducible elements form a finite set of generators of M,*
2. *$(M, \leq)$ is a partial order such that any two elements $x, y \in M$ with an upper bound have a supremum, and*
3. *$(\downarrow m, \leq)$ is a distributive lattice for any $m \in M$.*

Since by condition 1 above a divisibility monoid $(M, \cdot, 1)$ is generated by the set T of its irreducible elements, there is a natural epimorphism from the free monoid $T^\star$ onto M. This epimorphism will be denoted by $[.]$.

Condition 2 is well known from domain theory and often regarded as "consistent completeness". It means that whenever two computations x and y from M allow a joint extension, there is a least such extension of them. In fact, the partial order $(M, \leq)$ can be seen as the compact elements of a Scott-domain. But $(M, \leq)$ is not necessarily a lattice since it may contain unbounded pairs of elements.

Using basic properties of distributive lattices, from conditions 1 and 3 one can infer that $\downarrow x$ is finite for any $x \in M$. It follows that any finite subset A

of M has an infimum in $(M, \leq)$, and if A has an upper bound, it also has a supremum. This supremum of A can be viewed as the least common multiple of A, whereas the infimum of A is the greatest common (left-)divisor of A. Observe that the distributivity required is a direct generalization of the triviality that in the multiplicative monoid $(\mathbb{N}, \cdot, 1)$ least common multiple and greatest common divisor distribute (i.e., $\gcd(x, \operatorname{lcm}(y, z)) = \operatorname{lcm}(\gcd(x, y), \gcd(x, z))$ for any x, y, z). In our general setting, the finiteness of $\downarrow x$ ensures that $(M, \leq)$ is even the set of compacts of a dI-domain. For the theory of dI-domains and their connection with lambda calculus we refer the reader to [1]. In particular we have $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ whenever the left hand side is defined.

Note that in left divisibility monoids the partial order is the prefix relation. Ordered monoids where the order relation is the intersection of the prefix and the suffix relation were investigated e.g. in [3] under the name "divisibility monoid". Since such monoids will not appear in this paper any more, we will simply speak of "divisibility monoids" as an abbreviation for "left divisibility monoid".

Next we show that for elements of a divisibility monoid a length can be defined in a natural way making the correspondence to computations even clearer: Let $x = x_1 x_2 \ldots x_n \in M$ with $x_i \in T$. Then $\{1, x_1, x_1 x_2, \ldots, x\}$ is a maximal chain in the finite distributive lattice $\downarrow x$. Since maximal chains in finite distributive lattices have the same size, any word u over T with $[u] = x$ has length n. Hence we can define the *length* of x to be $|x| = n$.

Divisibility monoids are defined algebraically, using classical notions from lattice theory. They can also be described combinatorially (and more similar to the original definition of trace monoids) using commutation conditions for their irreducible generators. A first step towards such a representation is provided by the following proposition.

Proposition 1. *Let M be a divisibility monoid and T the set of its irreducible elements. Let $\sim$ denote the least congruence on the free monoid $T^\star$ containing $\{(ab, cd) \mid a, b, c, d \in T$ and $a \cdot b = c \cdot d\}$. Then $\sim$ is the kernel of the natural epimorphism $[.] : T^\star \to M$. In particular, $M \cong T^\star/{\sim}$.*

On the other hand, there are sets of equations of the form $ab = cd$ such that $T^\star/{\sim}$ is not a divisibility monoid. In [12], those sets of equations are described that give rise to divisibility monoids.

Let M be a divisibility monoid. Two elements x and y are *independent* (denoted by $x \parallel y$) if $x \wedge y = 1$ and $\{x, y\}$ is bounded above. Intuitively, this means that the computations x and y have no nontrivial joint past and are consistent. In this case the supremum $x \vee y$ exists in M. Since M is cancellative, there is a unique element z such that $y \cdot z = x \vee y$. This element z is called *the residuum of x after y* and denoted by $x {\uparrow} y$. Intuitively, $x {\uparrow} y$ denotes the computation that has to be performed after y in order to obtain the least common extension of x and y. Note that the residuum is defined for independent elements x and y only. Clearly, $x {\uparrow} y$ is defined iff $y {\uparrow} x$ is defined and in this case $x(y {\uparrow} x) = y(x {\uparrow} y) = x \vee y$.

Now assume M to be a trace monoid. Then two traces $x = [u]$ and $y = [v]$ in M are independent iff each letter occurring in u is independent from each letter occurring in v. This coincides with the usual definition of independence in trace theory. If x and y are independent, then it is known that $y \cdot x = x \cdot y = x \vee y$ and hence $x{\uparrow}y = x$ and similarly $y{\uparrow}x = y$.

Again, let M be an arbitrary divisibility monoid. Fixing $x \in M$, we define a unary partial function c_x from M to M with domain $\mathrm{dom}(c_x) := \{y \in M \mid x \parallel y\}$ by letting $c_x(y) := y{\uparrow}x$. The function c_x will be called the *commutation behavior of* x. In this paper, as usual, an equation $c_x(y) = c_z(y')$ means "$c_x(y)$ is defined iff $c_z(y')$ is defined and in this case they are equal". In other words, y is independent from x iff y' is independent from z and in this case $y{\uparrow}x = y'{\uparrow}z$.

Let $\mathbb{C}_M$ denote the set of all commutation behaviors of elements of M, i.e., $\mathbb{C}_M = \{c_x \mid x \in M\}$. Note that $\mathbb{C}_M$ is a set of partial functions from M to M that might be infinite. If $\mathbb{C}_M$ is actually finite, we say that M is a *divisibility monoid with finite commutation behavior*.

Let M again be a trace monoid. Recall that $y{\uparrow}x = y$ whenever $y{\uparrow}x$ is defined. Hence the commutation behavior c_x is the identity on its domain. This in particular implies that two traces have the same commutation behavior iff they have the same alphabet. Thus, if M is finitely generated, as a divisibility monoid it has finite commutation behavior.

The following lemma lists some properties of the commutation behaviors our proofs rely on.

Lemma 1. *Let $(M, \cdot, 1)$ be a divisibility monoid and $x, x', y, z \in M$.*

1. *The commutation behavior c_x is injective and length-preserving on its domain.*
2. *$x \parallel yz$ iff $x \parallel y$ and $c_y(x) \parallel z$.*
3. *$c_{yz}(x) = c_z(c_y(x))$; in other words $x{\uparrow}(yz) = (x{\uparrow}y){\uparrow}z$.*
4. *$c_x(yz) = c_x(y) \cdot c_{c_y(x)}(z)$; equivalently $yz{\uparrow}x = (y{\uparrow}x) \cdot (z{\uparrow}(x{\uparrow}y))$.*
5. *If $c_x = c_{x'}$ and $y \parallel x$ then $c_{c_y(x)} = c_{c_y(x')}$.*

Note that the third statement of the lemma above in particular implies $c_z \circ c_y = c_{yz}$ where $\circ$ is the usual concatenation of partial functions. Hence $(\mathbb{C}_M, \circ, c_1)$ is a monoid, *the monoid of commutation behaviors of* M. The function $c : M \to \mathbb{C}_M : x \mapsto c_x$ is a monoid antihomomorphism. Thus, if M has finite commutation behavior, for any commutation behavior $c \in \mathbb{C}_M$, the set $\{x \in M \mid c_x = c\}$ of all elements of M with commutation behavior c is recognizable. This will be crucial for some proofs of our results. Unfortunately, we do not know whether actually each divisibility monoid has finite commutation behavior. This seems to be a difficult problem combining monoid theoretic, lattice theoretic and combinatorial concepts.

We will also need a lifting of the commutation behavior from a divisibility monoid M to the free monoid $T^\star$ which can be defined as follows. We define functions $d_u : T^\star \to T^\star$ for $u \in T^\star$ in such a way that equations like those from Lemma 1 hold: Recall that for $t \in T$ and $u \in T^\star$ with $[u] \parallel t$ we have $|t| = |c_{[u]}(t)|$ by Lemma 1 and therefore $c_{[u]}(t) \in T$. Hence $d_u(t) := c_{[u]}(t)$ (if $t \parallel [u]$) is a partial function mapping T to T. We extend it to a partial function from $T^\star$ to $T^\star$ by $d_u(tv) := d_u(t)d_{d_t(u)}(v)$. Then one gets properties similar to those listed in Lemma 1. In particular $d_u = d_v$ iff $c_{[u]} = c_{[v]}$ for any $u, v \in T^\star$.

Let $\mathbb{D}_M = \{d_u \mid u \in T^\star\}$ be the set of all commutation behaviors of words over T. Then $(\mathbb{D}_M, \circ, d_\varepsilon)$ is a monoid and $d : T^\star \to \mathbb{D}_M : u \mapsto d_u$ is a monoid antihomomorphism. Also, $d_u \mapsto c_{[u]}$ is a monoid isomorphism from $(\mathbb{D}_M, \circ, d_\varepsilon)$ to $(\mathbb{C}_M, \circ, c_1)$.

It is immediate that if $[u] \parallel [v]$ then $[v] = [w]$ implies $[d_u(v)] = c_{[u]}([v]) = c_{[u]}([w]) = [d_u(w)]$. The following lemma shows that not only the other implication holds as well but that even $\{d_u(w) \mid [v] = [w]\} = \{w' \mid [w'] = [d_u(v)]\}$. The proof relies on the fact that $(\downarrow x, \leq)$ is a distributive lattice and that projective intervals in distributive lattices are isomorphic.

Lemma 2. *Let $x \in M$, $u \in T^\star$ and $t_i \in T$ for $i = 1, 2, \ldots, n$ such that $c_{[u]}(x) = [t_1 t_2 \ldots t_n]$. Then there exist $s_i \in T$ for $i = 1, 2, \ldots, n$ such that $d_u(s_1 s_2 \ldots s_n) = t_1 t_2 \ldots t_n$. These elements s_i of T are unique.*

4 Commutation grids and the rank

In trace theory, the generalized Levi Lemma (cf. [6]) plays an important role. Here, we introduce a generalization to divisibility monoids using commutation grids. This enables us to obtain a concept of "rank" of a language in these monoids, similar to the one given by Hashiguchi [11] for trace monoids. Let M be a divisibility monoid and $x, y \in M$. Recall that $c_x(y) = y{\uparrow}x$. Similarly, we define $v{\uparrow}u := d_u(v)$ whenever the latter is defined for $u, v \in T^\star$.

Definition 2. *For $0 \leq i \leq j \leq n$ let $x^i_j, y^j_i \in T^\star$. The tuple $(x^i_j, y^j_i)_{0 \leq i \leq j \leq n}$ is a* commutation grid *provided $x^i_j \parallel y^{j-1}_i$, $x^i_j{\uparrow}y^{j-1}_i = x^{i+1}_j$, and $y^{j-1}_i{\uparrow}x^i_j = y^j_i$ for any $0 \leq i < j \leq n$ (see Fig. 1).*

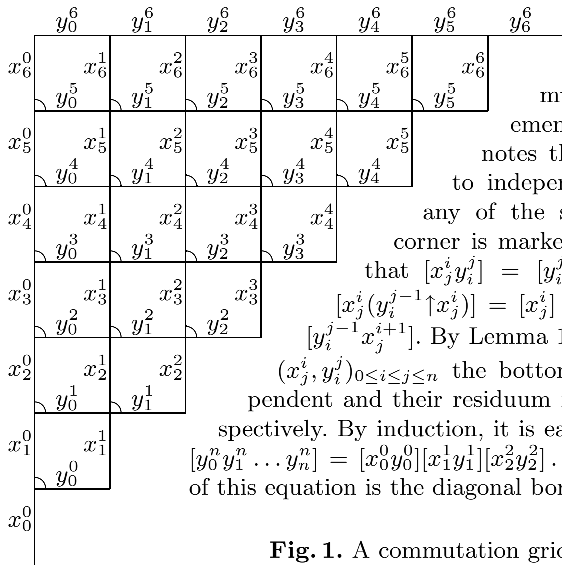
Lemma 3. *Let $z_0, z_1, \ldots, z_n, x, y \in T^\star$ with $[xy] = [z_0 z_1 \ldots z_n]$. Then there exists a commutation grid $(x^i_j, y^j_i)_{0 \leq i \leq j \leq n}$ such that $[x] = [x^0_0 x^0_1 \ldots x^0_n]$, $[y] = [y^n_0 y^n_1 \ldots y^n_n]$, and $[z_i] = [x^i_i y^i_i]$ for $i = 0, 1, \ldots, n$.*

Now we can introduce the notion of rank in the present context. Intuitively, it measures the amount of commutations of irreducible generators necessary to transform a product of two words into an equivalent word belonging to a given word language over T.

Definition 3. *Let $u, v \in T^\star$ and $X \subseteq T^\star$ such that $[uv] \in [X] := \{[w] \mid w \in X\}$. Let $\mathrm{rk}(u, v, X)$ denote the minimal integer n such that there exists a commutation grid $(u^i_j, v^j_i)_{0 \leq i \leq j \leq n}$ in $T^\star$ with $[u] = [u^0_0 u^0_1 \ldots u^0_n]$, $[v] = [v^n_0 v^n_1 \ldots v^n_n]$, and $u^0_0 v^0_0 u^1_1 v^1_1 \ldots u^n_n v^n_n \in X$.*

For $u, v \in T^\star$ and $X \subseteq T^\star$ with $[uv] \in [X] := \{[w] \mid w \in X\}$, one gets $\mathrm{rk}(u, v, X) \leq |uv|$. We define the *rank* $\mathrm{rk}(X)$ *of* X by

$$\mathrm{rk}(X) := \sup\{\mathrm{rk}(u, v, X) \mid u, v \in T^\star, [uv] \in [X]\} \in \mathbb{N} \cup \{\infty\}.$$



This picture depicts a commutation grid. Edges denote elements from $T^\star$ and an angle denotes that the two edges correspond to independent elements. Note that in any of the small squares, the lower left corner is marked by an angle. This indicates that $[x_j^i y_i^j] = [y_i^{j-1} x_j^{i+1}]$ because of $[x_j^i y_i^j] = [x_j^i (y_i^{j-1} \uparrow x_j^i)] = [x_j^i] \vee [y_i^{j-1}] = [y_i^{j-1}(x_j^i \uparrow y_i^{j-1})] = [y_i^{j-1} x_j^{i+1}]$. By Lemma 1, for any rectangle in the grid $(x_j^i, y_i^j)_{0 \le i \le j \le n}$ the bottom and the left side are independent and their residuum is the top (the right) side, respectively. By induction, it is easy to show that $[x_1^0 x_2^0 \dots x_n^0] \cdot [y_0^n y_1^n \dots y_n^n] = [x_0^0 y_0^0][x_1^1 y_1^1][x_2^2 y_2^2] \dots [x_n^n y_n^n]$. The right hand side of this equation is the diagonal border of the grid.

Fig. 1. A commutation grid

A word language $X \subseteq T^\star$ is *closed* if $[u] \in [X]$ implies $u \in X$ for any $u \in T^\star$. Since $\mathrm{rk}(u, v, X) = 0$ whenever $uv \in X$, the rank of a closed language equals 0.

We just note here that if M is a trace monoid then these notions coincide with the corresponding ones known from trace theory. Hence the following result generalizes [6, Thm. 3.2].

Theorem 2. *Let $(M, \cdot, 1)$ be a divisibility monoid with finite commutation behavior. Let $X \subseteq T^\star$ be recognizable and $n := \mathrm{rk}(X)$ be finite. Then $[X]$ is recognizable in M.*

Proof. Let η be a homomorphism into a finite monoid S recognizing X with $d_u = d_v$ whenever $\eta(u) = \eta(v)$. For $x \in M$ let $R(x)$ denote the subset

$$\{(\eta d(x_0), \eta d(x_1) \dots \eta d(x_n))_{d \in \mathbb{D}_M} \mid x_0, x_1, \dots, x_n \in T^\star \text{ and } x = [x_0 x_1 \dots x_n]\}$$

of $(S^{n+1})^{|\mathbb{D}_M|}$. Hence there are only finitely many sets $R(x)$. We show $R(x) = R(z) \Rightarrow x^{-1}[X] = z^{-1}[X]$, which implies that $[X]$ is recognizable.

So let $R(x) = R(z)$ and let $y \in x^{-1}[X]$. Since $\mathrm{rk}(X) = n$, there exists a commutation grid $(u_j^i, v_i^j)_{0 \le i \le j \le n}$ such that $x = [u_0^0 u_1^0 \dots u_n^0]$, $y = [v_0^n v_1^n \dots v_n^n]$, and $u_0^0 v_0^0 u_1^1 v_1^1 \dots u_n^n v_n^n \in X$. Then $(\eta d(u_0^0), \eta d(u_1^0) \dots \eta d(u_n^0))_{d \in \mathbb{D}_M} \in R(x) = R(z)$. Hence there exist words $w_j^0 \in T^\star$ with $\eta d(w_j^0) = \eta d(u_j^0)$ for each $0 \le j \le n$ and $d \in \mathbb{D}_M$, and $z = [w_0^0 w_1^0 \dots w_n^0]$. Then $d_{w_j^0} = d_{u_j^0}$ implying the existence of a commutation grid $(w_j^i, v_i^j)_{0 \le i \le j \le n}$. Then one gets $zy = [w_0^0 v_0^0 w_1^1 v_1^1 \dots w_n^n v_n^n] \in [X]$. Hence $y \in z^{-1}[X]$ and therefore $x^{-1}[X] = z^{-1}[X]$ as claimed above. $\square$

5 On recognizable and c-rational languages

First, we prove closure properties of the set of recognizable languages in a divisibility monoid.

Lemma 4. *The set of recognizable languages in a divisibility monoid with finite commutation behavior is closed under multiplication.*

To prove this lemma, one shows that for any closed languages $X, Y \subseteq T^\star$, the product XY has rank at most 1, i.e., $\mathrm{rk}(X\,Y) \leq 1$. Then the lemma follows from Kleene's Theorem and from Thm. 2. We note that the distributivity assumption on divisibility monoids is crucial for this result to hold, cf. [8, Ex. 4.11].

As in a trace monoid, the set of recognizable languages in a divisibility monoid is not closed under iteration. Therefore, we need some more notions: An element $x \in M$ of a divisibility monoid $(M, \cdot, 1)$ is *connected* if there are *no* independent $y, z \in M \setminus \{1\}$ such that $x = y \vee z = yc_y(z)$. A set $X \subseteq M$ ($X \subseteq T^\star$, respectively) is *connected* if all of its elements are connected ($[X] \subseteq M$ is connected, respectively). For trace monoids, this lattice theoretic definition is equivalent to the usual one via alphabets, and the iteration of a recognizable connected language is again recognizable. The following example shows that the latter is not the case for divisibility monoids.

Example 1. Let $T = \{a, b, c, d\}$ and let $\sim$ denote the least congruence on $T^\star$ with $ab \sim cd$ and $ba \sim dc$. Now we consider the monoid $M := T^\star/{\sim}$. Using the characterization from [12], one can show that M is a divisibility monoid. Moreover, it has finite commutation behavior. Since any irreducible element is trivially connected, $\{a, b\}$ is a recognizable connected language in M. Let L denote the iteration of this language in M, i.e., $L := \langle\{a, b\}\rangle \subseteq M$. To show that L is not recognizable, it suffices to prove that $X := \{w \in T^\star \mid [w] \in L\}$ is not recognizable in the free monoid $T^\star$. Note that X consists of those words that are equivalent to some word containing a's and b's, only. Clearly any such word has to contain the same number of c's and of d's. If X was recognizable, the language $Y = X \cap (ad)^\star(cb)^\star$ would be recognizable. We will derive a contradiction by showing $Y = \{(ad)^i(cb)^i \mid i \in \mathbb{N}\}$: By the observation above, $Y \subseteq \{(ad)^i(cb)^i \mid i \in \mathbb{N}\}$. Starting with $(ab)c \sim cdc \sim cba$, we obtain $(ab)^n c \sim c(ba)^n$ for any n. Thus $ad(ab)^n cb \sim adc(ba)^n b \sim aba(ba)^n b = (ab)^{n+2}$. Applying this equation to a word of the form $(ad)^i(cb)^i$ several times, one gets $(ad)^i(cb)^i \sim (ab)^{2i} \in X$ and therefore $Y = \{(ad)^i(cb)^i \mid i \in \mathbb{N}\}$. ☐

An analysis of this example leads to the following additional requirement on recognizable languages that we want to iterate: A language $X \subseteq T^\star$ is *residually closed* if it is closed under the application of d_u and d_u^{-1} for elements u of X (Note that in the example above $d_a^{-1}(b) = c \notin \{a, b\}$, i.e., this language is not residually closed.) A language $L \subseteq M$ is *residually closed* iff $\{w \in T^\star \mid [w] \in L\}$ is residually closed. Recall that in a trace monoid the commutation behaviors d_u are contained in the identity function on $T^\star$. Hence any trace language is residually closed.

Theorem 3. *Let $(M, \cdot, 1)$ be a divisibility monoid with finite commutation behavior. Let $X \subseteq T^\star$ be closed, connected, and residually closed. Then the rank* $\mathrm{rk}(\langle X \rangle)$ *of the iteration of X is finite.*

Proof. Let $u, v \in T^\star$ and $x_0, x_1, \dots, x_n \in X$ such that $[uv] = [x_0 x_1 \dots x_n]$. One can show that there exists a commutation grid $(u^i_j, v^j_i)_{0 \le i \le j \le n}$ in $T^\star$ such that $[u] = [u^0_0 u^0_1 \dots u^0_n]$, $[v] = [v^n_0 v^n_1 \dots v^n_n]$ and $[x_i] = [u^i_i v^i_i] \in [X]$.

Constructing a subgrid one shows that it is sufficient to consider the case $u^i_i \neq \varepsilon \neq v^i_i$ for all $0 \le i \le n$.

Now one can prove that there are no $1 \le \alpha \le \beta \le \gamma \le n$ with

$$d_{u^0_\alpha u^0_{\alpha+1} \dots u^0_{\beta-1}} = d_{u^0_\beta u^0_{\beta+1} \dots u^0_{\gamma-1}} = d_{u^0_\alpha u^0_{\alpha+1} \dots u^0_{\gamma-1}}$$

since otherwise $[u^\alpha_\alpha v^\alpha_\alpha]$ would not be connected. Since $\mathbb{D}_M$ is finite, Ramsey's Theorem [15] bounds n and therefore the rank of $\langle X \rangle$. ☐

Using Kleene's Theorem and Thm. 2 one gets that the iteration of a connected, recognizable and residually closed language is recognizable.

A language $L \subseteq M$ is *c-rational* if it can be constructed from the finite subsets of M by union, multiplication and iteration where the iteration is applied to connected and residually closed languages, only. Since any element $x \in M$ has only finitely many prefixes, finite languages are recognizable. By Lemma 4 and Thm. 3, we get

Theorem 4. *Let $(M, \cdot, 1)$ be a divisibility monoid with finite commutation behavior. Let $L \subseteq M$ be c-rational. Then L is recognizable.*

Next we want to show the inverse implication of the theorem above. Let $(M, \cdot, 1)$ be a divisibility monoid, E a finite set and $\ell : T \to E$ a function. Then ℓ is a *labeling function* and (M, ℓ) is a *labeled divisibility monoid* if $\ell(s) = \ell(s{\uparrow}t)$ and $\ell(s) \neq \ell(t)$ for any $s, t \in T$ with $s \parallel t$. We note that the monoid M from Example 1 becomes a labeled divisibility monoid by putting $\ell(a) = \ell(d) = 0$ and $\ell(b) = \ell(c) = 1$. Thus, our main Thm. 5 holds for this monoid which is not a trace monoid.

Now let (M, ℓ) be a labeled divisibility monoid. The *label sequence* of a word $u_0 u_1 \dots u_n \in T^\star$ is the word $\ell(u_0)\ell(u_1)\dots\ell(u_n) \in E^\star$. We extend the mapping ℓ to words over T by $\ell(tw) = \{\ell(t)\} \cup \ell(w)$ and to elements of M by $\ell([u]) := \ell(u)$ for $u \in T^\star$. This latter is well defined by Prop. 1. Note that $\ell : M \to 2^E$ is a monoid homomorphism into the finite monoid $(2^E, \cup, \emptyset)$. One can show that $\ell(x) \cap \ell(y) = \emptyset$, $\ell(y) = \ell(y{\uparrow}x)$, and $\ell(x) \cup \ell(y) = \ell(x \vee y)$ for any $x, y \in M$ with $x \parallel y$.

A language $L \subseteq M$ is *monoalphabetic* if $\ell(x) = \ell(y)$ for any $x, y \in L$. It is an *mc-rational language* if it can be constructed from the finite subsets of M by union, multiplication and iteration where the iteration is applied to connected and monoalphabetic languages, only. Since, as we mentioned above, independent elements of M have disjoint label sets, any monoalphabetic language is residually closed. Hence mc-rational languages are c-rational.

Now let $\preceq$ be a linear order on the set E and let $x \in M$. The word $u \in T^\star$ with $x = [u]$ is *the lexicographic normal form of* x (denoted $u = \mathrm{lexNF}(x)$) if its label sequence is the least among all label sequences of words $v \in T^\star$ with $x = [v]$. This lexicographic normal form is unique since two word $u, v \in T^\star$ having the same label sequence with $[u] = [v]$ are equal. Let $\mathrm{LNF} = \{\mathrm{lexNF}(x) \mid x \in M\}$ denote the set of all words in $T^\star$ that are in lexicographic normal form. One can characterize the words from LNF similarly to trace theory. This characterization implies that LNF is recognizable in the free monoid $T^\star$.

The crucial point in Ochmański's proof of the c-rationality of recognizable languages in trace monoids is that whenever a square of a word is in lexicographic normal form, it is actually connected. This does not hold any more for labeled divisibility monoids. But whenever a product of $|E| + 2$ words having the same set of labels is in lexicographic normal form, this product is connected.

We need another notation: For a set $A \subseteq E$ and $u \in T^\star$ let $n_A(u)$ denote the number of occurrences of maximal factors w of u with $\ell(w) \subseteq A$ or $\ell(w) \cap A = \emptyset$. The number $n_A(u)$ is the number of blocks of elements of A and of $E \setminus A$ in the label sequence of u. Furthermore, we put $n_A(x) := n_A(\mathrm{lexNF}(x))$ for $x \in M$.

Lemma 5. *Let $(M, \cdot, 1, \ell)$ be a labeled divisibility monoid, $x, y \in M$ and $x \parallel y$. Then $n_{\ell(x)}(x \vee y) \leq |E| + 1$.*

Lemma 6. *Let $X \subseteq T^\star$ be a monoalphabetic language. Let $w \in X^{|E|+2} \cap \mathrm{LNF}$. Then $[w]$ is connected.*

Proof. Let $n = |E|+1$ and $x_i \in [X]$ with $[w] = x_0 x_1 \ldots x_n$. Furthermore assume $A = \ell(x_i)$ which is well defined since X is monoalphabetic. Now let $x, y \in M$ with $x \parallel y$ and $x \vee y = [w]$. Then $\ell(x) \cap \ell(y) = \emptyset$. If A contained an element from $\ell(x)$ and another one from $\ell(y)$, we would obtain $n_{\ell(x)}([w]) > n > |E| + 1$, contradicting Lemma 5. Hence $A \subseteq \ell(x)$ or $A \subseteq \ell(y)$. Now $\ell(x) \cup \ell(y) = \ell(x \vee y) = \ell(x_0 x_1 \ldots x_n) = A \subseteq \ell(x)$ implies $y = 1$. $\square$

Now one can show that in a labeled divisibility monoid (with possibly infinite commutation behavior) any recognizable set is mc-rational. This proof follows the lines of the corresponding proof by Ochmański for traces using Lemma 6. Summarizing, we get the following theorem which in particular implies Thm. 1.

Theorem 5. *Let $(M, \cdot, 1)$ be a labeled divisibility monoid with finite commutation behavior and $L \subseteq M$. Then L is recognizable iff L is c-rational iff L is mc-rational.*

6 Open problems

Sakarovitch's and Ochmański's results are important generalizations of Kleene's Theorem to rational and to trace monoids, respectively; thus into "orthogonal" directions since any rational trace monoid is free. Our further extension of Ochmański's result is not "orthogonal" to Sakarovitch's approach any more (for instance $\{a, b, c, d\}^\star / \langle ab = cd \rangle$ is both, a rational monoid and a divisibility monoid, but no free monoid). Hence our approach can be seen as a step towards a common generalization of Sakarovitch's and Ochmański's results.