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Zhi-Zhong Chen, Michelangelo Grigni, Christos H. Papadimitriou

Institutions: Tokyo Denki University, Emory University, University of California, Berkeley

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Recognizing Hole-Free 4-Map Graphs in Cubic Time

Zhi-Zhong Chen* Michelangelo Grigni† Christos H. Papadimitriou‡

Abstract

We present a cubic-time algorithm for the following problem: Given a simple graph, decide whether it is realized by adjacencies of countries in a map without holes, in which at most four countries meet at any point.

Key words. planar graphs, maps, map graphs, cliques, graph algorithms.

1 Introduction

The authors [2] introduced a modified notion of planarity, in which two countries of a map are considered adjacent when they share any *point* of their boundaries (not necessarily an *edge*, as planarity requires). Such adjacencies of countries in a map define a *map graph*.

In order to make the notions of map and map graph more clear, we need to recall several basic concepts in graph theory. Hereafter, a graph may have multiple edges but no loops, while a *simple* graph has neither multiple edges nor loops. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A *cycle* of a graph G is a connected subgraph H of G such that each $v \in V(H)$ is incident to exactly two edges of H. A graph is *planar* if it can be embedded in the sphere so that any pair of edges can only intersect at their endpoints; a *sphere* graph is a planar one together with such an embedding. Let $\mathcal{G}$ be a sphere graph. Consider the set of all points of the sphere that lie on no edge of $\mathcal{G}$. This set consists of a finite number of topologically connected regions; the closure of each such region is a *face* of $\mathcal{G}$. A face f of $\mathcal{G}$ is a *cycle-face* if its boundary is a cycle of $\mathcal{G}$.

A *map* $\mathcal{M}$ is a sphere graph such that some of its cycle-faces are labeled while the other faces are unlabeled. The labeled faces of $\mathcal{M}$ are the *countries* of $\mathcal{M}$, while the unlabeled faces are the *holes* of $\mathcal{M}$. Two countries are *adjacent* in $\mathcal{M}$ if their boundaries intersect (possibly, the intersection contains no edge of $\mathcal{M}$). The *map graph* of $\mathcal{M}$ is the simple graph G where $V(G)$ consists of the countries of $\mathcal{M}$ and $E(G)$ consists of all $\{f_1, f_2\}$ such that f_1 and f_2 are adjacent countries. We call G a *map graph*, call $\mathcal{M}$ a *map* of G, and say that $\mathcal{M}$ *realizes* G. If $\mathcal{M}$ has no hole, then it is a *hole-free* map and its map graph G is a *hole-free* map graph. To distinguish the elements of $V(\mathcal{M})$ from those of $V(G)$, we call the former *nodes* and call the latter *vertices* or *countries*. Moreover, we use lower-case Greek letters to denote nodes and use

*Department of Mathematical Sciences, Tokyo Denki University, Hatoyama, Saitama 350-0394, JAPAN. E-mail: chen@r.dendai.ac.jp.

†Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322. E-mail: mic@mathcs.emory.edu.

‡Computer Science Division, University of California at Berkeley, Berkeley, CA 94720. E-mail: christos@cs.berkeley.edu.

lower-case roman letters to denote vertices. For an integer k, a *k-node* is a node of $\mathcal{M}$ that appears on the boundaries of exactly k countries of $\mathcal{M}$; if $\mathcal{M}$ has no j-node with $j > k$, then it is a *k-map* and its map graph is a *k-map graph*. For example, Figure 2.2(2) is a hole-free 4-map and Figure 2.2(3) is a 4-map with one hole, realizing the same graph.

1.1 Motivations and Previous Results

In addition to having relevance to planarity, map graphs are related to the topological inference problem which arises from theoretical studies in geographic database systems. For the details and a comprehensive survey of known results on map graphs, we refer the reader to [3]. Here we only describe a brief history of research on map graphs. In [2] and [3], the authors gave a simple nondeterministic polynomial-time algorithm for recognizing map graphs and investigated the structure and the number of maximal cliques in a map graph. Subsequently, Thorup [7] presented a polynomial-time algorithm for recognizing map graphs. Unfortunately, his algorithm is complex and the exponent of the polynomial bounding its running time from above is about 120. Moreover, as far as we know, Thorup's algorithm [7] for recognizing map graphs does not imply a polynomial-time recognition algorithm for hole-free map graphs.

As observed in [2], simple planar graphs are exactly 3-map graphs. Moreover, it is easy to see that maximal planar graphs (i.e., those simple planar graphs to which we can add no more edges without losing planarity) are exactly 3-connected hole-free 3-map graphs; the proof is omitted here. Thus, it is natural to study k-map graphs and hole-free k-map graphs where $k \geq 4$. Thorup's algorithm [7] for recognizing map graphs does not imply a polynomial-time recognition algorithm for k-map graphs or hole-free k-map graphs, because even if we are given a map realizing a map graph, it is not clear that it helps us to find a map with the additional restrictions we want (e.g., a hole-free 4-map). In fact, it is still unknown if k-map graphs (respectively, hole-free k-map graphs) for $k \geq 5$ can be recognized in polynomial time. We note in passing that for every $k \geq 4$, neither the class of k-map graphs nor the class of hole-free k-map graphs can be characterized by forbidden subgraphs or minors (because there are a hole-free 4-map graph G and an edge e in G such that $G - e$ is not a map graph [3]).

We next point out another reason for us to be interested in hole-free 4-map graphs. As a natural extension of planar graphs, 1-planar graphs (i.e., those simple graphs that can be embedded in the plane in such a way that each edge crosses at most one other edge) have been studied extensively in the literature (see [4] and the references therein). It is open whether 1-planar graphs can be recognized in polynomial time. We say that a 1-planar graph G is *triangulated* if it can be embedded in the sphere in such a way that (1) each edge of G crosses at most one other edge and (2) the set of all points of the sphere that lie on no edge of G consists of a finite number of topologically connected regions whose boundaries each consist of points of exactly three edges of G. Then, it is easy to see that triangulated 1-planar graphs are exactly 3-connected hole-free 4-map graphs; the proof is omitted here. In Section 3, we will observe that the problem of recognizing hole-free 4-map graphs can be easily reduced to the problem of recognizing 3-connected hole-free 4-map graphs. Hence, the problem of recognizing triangulated 1-planar graphs is essentially the problem of recognizing hole-free 4-map graphs.

1.2 The New Result

In this paper, we describe a cubic-time algorithm for deciding whether a given graph is a hole-free 4-map graph. Theorem 3.1 in [3] shows that each clique in a map graph can be realized in only four different ways by a map. The basic idea behind our cubic-time algorithm is to figure out the correct way of realizing each maximal clique C of the input graph G in the target map. The correct way of realizing C is found by a case analysis of the neighborhood structure of the countries around C in G. Before the case analysis, certain separators of G are found and used to simplify G so that the case analysis needs to consider only a few cases.

1.3 Organization of the Paper

This paper is organized as follows. Section 2 describes basic definitions and two lemmas about map graphs. Section 3 details how to reduce the recognition problem of hole-free 4-map graphs to its special case where the input graph is 4-connected. Section 4 describes the structure of maximal cliques of 4-connected graphs G in a hole-free 4-map realizing G. Section 5 explains how our algorithm makes progress. Section 6 gives a high-level description of our cubic-time algorithm; the algorithm produces a hole-free 4-map, if one exists. Sections 7 through 9 present the structural results needed to prove the correctness of the algorithm; these sections are the technical core of our paper. We give a time analysis in Section 10, and concluding remarks in Section 11.

2 Preliminaries

Let G be a graph. The *degree* of a vertex v in G is the number of edges incident to v in G. For a $v \in V(G)$, $N_G(v)$ denotes the set of vertices adjacent to v in G. For a $U \subseteq V(G)$, $N_G(U)$ denotes $\bigcup_{u \in U} N_G(u)$. A *path* of G is either a single vertex of G or a connected subgraph H of G such that H is not a cycle and each vertex of H is incident to exactly one or two edges of H. A path is *nontrivial* if it is not a single vertex. A vertex of a nontrivial path P is *internal* if it is incident to exactly two edges of P.

Let $k \geq 1$ be an integer. A *k-cut* of G is a subset U of $V(G)$ with $|U| = k$ whose removal disconnects G. G is *k-connected* if $|V(G)| \geq k$ and G has no i-cut with $i \leq k-1$.

Let $\mathcal{G}$ be a sphere graph (e.g., a map). Two faces of $\mathcal{G}$ *touch* in $\mathcal{G}$ if their boundaries share at least one node of $\mathcal{G}$. Two faces of $\mathcal{G}$ *strongly touch* in $\mathcal{G}$ if their boundaries share at least one edge of $\mathcal{G}$. Two faces of $\mathcal{G}$ *weakly touch* in $\mathcal{G}$ if they touch but do not strongly touch in $\mathcal{G}$.

Let $\mathcal{M}$ be a map. Let $f_1, \ldots, f_k$ be a set of two or more distinct countries of $\mathcal{M}$. Let $f_{i_1}, \ldots, f_{i_k}$ be a permutation of $f_1, \ldots, f_k$. Countries f_1 through f_k *meet at a node* α in $\mathcal{M}$ *in the order* $f_{i_1}, \ldots, f_{i_k}$ if their boundaries all contain α and the countries appear around α in $\mathcal{M}$ in the order $f_{i_1}, \ldots, f_{i_k}$ clockwise. Countries f_1 through f_k *meet at a node* α in $\mathcal{M}$ if they meet at α in $\mathcal{M}$ in some order. Note that when f_1 through f_k meet at a node α in $\mathcal{M}$, α may also appear on the boundary of a country $f \notin \{f_1, \ldots, f_k\}$ or even a hole in $\mathcal{M}$.

The next two lemmas will be useful for analyzing the time complexity of our algorithm.

Lemma 2.1 *[1] For every integer $k \geq 3$, each k-map graph G with $n \geq 3$ vertices has at most $kn - 2k$ edges.*

Lemma 2.2 *For every integer $k \geq 3$, each hole-free k-map graph G with $n \geq 3$ vertices is realized by a hole-free k-map $\mathcal{M}$ with at most $2n - 4$ nodes.*

Proof: Suppose $\mathcal{M}$ is a hole-free k-map realizing G. Since $\mathcal{M}$ is hole-free and its countries are cycle-faces, each node of $\mathcal{M}$ is shared by at least two countries. If some node α of $\mathcal{M}$ is shared by exactly two countries, then we connect the two neighbors of α by a new edge and delete α together with the two edges incident to α. After this change, $\mathcal{M}$ remains a hole-free k-map and G remains the map graph of $\mathcal{M}$. So, we may assume that each node of $\mathcal{M}$ is adjacent to at least 3 other nodes. By Euler's formula, $\mathcal{M}$ has at most $2n - 4$ nodes. ▮

2.1 Marked Graphs and Their Layouts

A *marked graph* is a simple graph in which each edge is either marked or not marked (see Figure 5.1(1) for an example). Note that a marked graph may have no marked edge. Throughout this subsection, G denotes a marked graph. Suppose $U \subseteq V(G)$ and $F \subseteq E(G)$. $G - U - F$ denotes the marked graph obtained from G by deleting the edges in F and the vertices in U together with the edges incident to them. When U or F is empty, we drop it from the notation $G - U - F$. $G[U]$ denotes $G - (V(G) - U)$, the subgraph of G induced by U. A *clique* of G is a set of pairwise adjacent vertices in G. Often times, we identify a clique C of G with $G[C]$. A clique C of G is *maximal* if no clique of G properly contains C. Let $k \geq 1$ be an integer. A *k-clique* of G is a clique C with $|C| = k$. For convenience, we denote a maximal k-clique by MC_k.

Definition 2.3 A *layout* of G is a 4-map $\mathcal{L}$ of G such that

(1) the degree of every node in $\mathcal{L}$ is at most 4, and

(2) for every marked edge $\{u, v\}$ in G, countries u and v strongly touch in $\mathcal{L}$.

$\mathcal{L}$ is *well-formed* if for every edge $\{u, v\}$ in G, the intersection of countries u and v in $\mathcal{L}$ is a single path S of $\mathcal{L}$. (Note: The degree of each 4-node in a hole-free 4-map is 4.)

Note that the path S in Definition 2.3 may be a single node of $\mathcal{L}$. Moreover, if S is not a node, then each internal node of S is incident to exactly two edges of $\mathcal{L}$.

Definition 2.4 If a layout $\mathcal{L}$ of G has no hole, we call it an *atlas* of G.

Since a marked graph may have no marked edge, the problem of recognizing hole-free 4-map graphs is a special case of the problem of deciding whether a given marked graph has an atlas or not. Our goal is to design a cubic-time algorithm for the latter (more general) problem. We prefer to work on marked graphs just for technical reasons.

Throughout the rest of this subsection, fix a $U \subseteq V(G)$ and a layout $\mathcal{L}$ of $G[U]$. A *2-hole* of $\mathcal{L}$ is a hole strongly touched by exactly two countries of $\mathcal{L}$. *Erasing a 2-hole* $\mathcal{H}$ of $\mathcal{L}$ is the operation of modifying $\mathcal{L}$ by extending one of the countries strongly touching $\mathcal{H}$ to completely occupy $\mathcal{H}$. Figure 2.1(1) depicts the operation. (Note: In our figures, we draw a map by projecting one point of the sphere to infinity; we always choose a point that is not on a country's boundary.)

By definition, a 4-node of $\mathcal{L}$ appears on the boundary of exactly four countries. Thus, by Condition (1) in Definition 2.3, no 4-node of $\mathcal{L}$ is on the boundary of a hole. Let $u \in U$ and $v \in U$. A *(u, v)-node* in $\mathcal{L}$ is a 4-node α at which countries u and v together with two other countries x and y meet in $\mathcal{L}$ in the order u, x, v, y. *Erasing (u, v)-node* α in $\mathcal{L}$ is the operation of modifying $\mathcal{L}$ by slightly extending country x so that α appears in the interior of country x (and hence the boundaries of countries u, v, and y no longer contain α). Figure 2.1(2) depicts

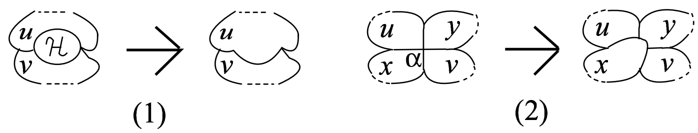

Figure 2.1: Erasing a 2-hole $\mathcal{H}$, and a (u, v)-node α. Dashed curves may intersect.

the operation. Note that after erasing α in $\mathcal{L}$, it is possible (but not always the case) that countries u and v no longer intersect in $\mathcal{L}$.

A (u, v)-*segment* in $\mathcal{L}$ is a nontrivial path S shared by the boundaries of countries u and v in $\mathcal{L}$ such that the degree of each internal node of S in $\mathcal{L}$ is 2 but the degree of each endpoint of S in $\mathcal{L}$ is at least 3. Note that two (u, v)-segments in $\mathcal{L}$ must be disjoint.

An edge $\{u, v\}$ of G is *good* in $\mathcal{L}$ if the intersection of countries u and v in $\mathcal{L}$ is a path of $\mathcal{L}$. An edge that is not good in $\mathcal{L}$ is *bad* in $\mathcal{L}$. Note that $\mathcal{L}$ is well-formed iff every edge of $G[U]$ is good in $\mathcal{L}$.

Definition 2.5 If $\mathcal{M}$ is an atlas of G and U is a subset of $V(G)$, then $\mathcal{M}|_U$ denotes the layout of $G[U]$ obtained from $\mathcal{M}$ by removing all nodes and edges that do not appear on the boundary of any country in U. $\mathcal{L}$ is an *extensible* layout of $G[U]$ if whenever G has an atlas, it has an atlas $\mathcal{M}$ with $\mathcal{L} = \mathcal{M}|_U$. $\mathcal{L}$ is *transformable* to another layout $\mathcal{L}'$ of $G[U]$ if whenever $\mathcal{L}$ is extensible, so is $\mathcal{L}'$.

Literally, a layout of $G[U]$ is extensible iff it can be extended to an atlas of G whenever G has an atlas.

2.2 Figures

Throughout this subsection, G denotes a marked graph and U denotes a subset of $V(G)$. For our arguments of the algorithm's correctness, we need a convenient graphical notation for the possible extensible layouts of $G[U]$. First, as is very natural, we consider two layouts equivalent when they are homeomorphic. But beyond this, we also introduce a convenient graphic notation for partially determined layouts of $G[U]$. In particular, we introduce contractible forests and permutable labels.

Definition 2.6 A *figure* of $G[U]$ is a list[1] $\mathcal{D} = \langle \mathcal{L}, \mathcal{F}, L_1, \ldots, L_k \rangle$, where $\mathcal{L}$ is a layout of $G[U]$, $\mathcal{F}$ is an acyclic subgraph (i.e., a forest) of $\mathcal{L}$, and $L_1, \ldots, L_k$ are disjoint lists of vertices in U. We call $\mathcal{L}$ the *layout* in $\mathcal{D}$, call $\mathcal{F}$ the *contractible forest* in $\mathcal{D}$, and call $L_1, \ldots, L_k$ the *permutable lists* in $\mathcal{D}$. (For an example, see Figure 2.2(1) and the explanation below.)

Intuitively speaking, $\mathcal{L}$ means a temporary layout of $G[U]$ and we can finalize it by contracting zero or more edges in $\mathcal{F}$ and/or permuting the labels of the countries in each L_i $(1 \le i \le k)$.

To illustrate a figure $\mathcal{D}$, we draw $\mathcal{L}$ (a sphere graph), emphasize the contractible forest in bold, and then for each permutable list L_i, we label each country $u \in L_i$ as u^i. The holes are

[1] Throughout this paper, a list is always ordered.

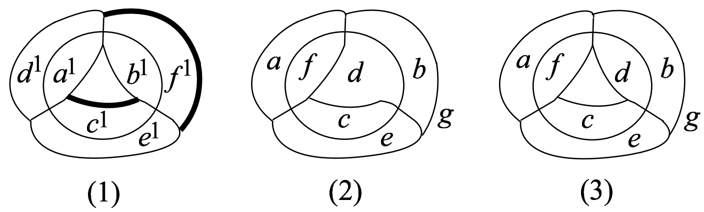

Figure 2.2: (1) A figure of an MC_6 $U = \{a, ..., f\}$ with a single permutable list $\langle a, ..., f\rangle$. (2) A well-formed atlas of an example G. (3) A well-formed 4-map of an example G with one hole.

unlabeled, and should be regarded as "optional" if a contraction could reduce it to a 2-hole. For convenience, a *contractible path* means a connected component of the contractible forest that is a path. Note that a contractible path may be either completely or partially contracted when necessary. In particular, sometimes we may need to contract two or more vertex-disjoint subpaths of a contractible path each to a single node.

Definition 2.7 A figure $\mathcal{D} = \langle \mathcal{L}, \mathcal{F}, L_1, \ldots, L_k\rangle$ of $G[U]$ *displays* a layout $\mathcal{L}'$ of $G[U]$ if $\mathcal{L}'$ can be obtained from $\mathcal{L}$ by:

(1) contracting a set of node-disjoint paths of $\mathcal{F}$ each to a single node,
(2) erasing all resulting 2-holes, and
(3) for each permutable list L_i, selecting a permutation π of L_i and relabeling each country $u \in L_i$ as $\pi(u)$.

We say $\mathcal{D}$ *displays* $G[U]$, or $\mathcal{D}$ is a *display* of $G[U]$, if $\mathcal{D}$ displays an extensible layout of $G[U]$. $\mathcal{D}$ is *transformable* to another figure $\mathcal{D}'$ of $G[U]$ if whenever $\mathcal{D}$ displays $G[U]$, so does $\mathcal{D}'$.

For example, if G has a well-formed atlas and $U = \{a, ..., f\}$ is an MC_6 of G, then Figure 2.2(1) displays $G[U]$. If in addition $V(G) = \{a, \ldots, g\}$, $N_G(g) = \{a, b, e\}$, and $\{b, d\}$ and $\{c, d\}$ are the marked edges of G, then Figure 2.2(2) (respectively, Figure 2.2(3)) is a well-formed atlas (respectively, 4-map) of G.

3 Reduction to the 4-Connected Case

Our goal here is to reduce our algorithmic problem (i.e., the problem of deciding if a given marked graph has an atlas) to its special case where the input marked graph is 4-connected.

Definition 3.1 Let $\mathcal{G}$ be a sphere graph. Let $\mathcal{S}$ be a set of faces of $\mathcal{G}$. The faces in $\mathcal{S}$ *form a cycle-superface* if their union is a topologically connected region and this region's boundary is a cycle of $\mathcal{G}$. The faces in $\mathcal{S}$ *form disjoint cycle-superfaces* of $\mathcal{G}$ if $\mathcal{S}$ can be partitioned into disjoint nonempty subsets $\mathcal{S}_1, \ldots, \mathcal{S}_k$ ($k \geq 2$) such that the faces in each $\mathcal{S}_i$ ($1 \leq i \leq k$) form a cycle-superface $\mathcal{R}_i$ of $\mathcal{G}$ and each pair of cycle-superfaces among $\mathcal{R}_1, \ldots, \mathcal{R}_k$ are disjoint.

Lemma 3.2 *Let $\mathcal{M}$ be a hole-free map, and let G be its map graph. Suppose U is a proper subset of $V(G)$ such that the countries in U form a cycle-superface or disjoint cycle-superfaces of $\mathcal{M}$. Then, $G - U$ is connected.*

Proof: Since the countries in U form a cycle-superface or disjoint cycle-superfaces of $\mathcal{M}$, removing the countries in U from the sphere leaves a topologically connected region. This implies that $G - U$ is connected. ▮

Since each country in a hole-free map is a cycle-face, Lemma 3.2 implies that each hole-free map graph is 2-connected. In the remainder of this section, G denotes a marked graph.

Lemma 3.3 *Suppose G has an atlas $\mathcal{M}$. Let u and v be two distinct vertices of G. Then, the following statements hold:*

1. *$G - \{u, v\}$ is disconnected iff there are at least two (u, v)-segments in $\mathcal{M}$.*
2. *Suppose $G - \{u, v\}$ is disconnected and its connected components are $G_1, \ldots, G_k$. Then for each $i \in \{1, \ldots, k\}$, the marked graph G'_i obtained from $G[V(G_i) \cup \{u, v\}]$ by marking edge $\{u, v\}$ has an atlas. Moreover, given an atlas $\mathcal{M}_i$ for each G'_i, we can easily construct an atlas for G.*

Proof: We first prove Statement 1. If $\{u, v\} \notin E(G)$, then countries u and v are disjoint cycle-faces of $\mathcal{M}$, and hence Lemma 3.2 implies that $G - \{u, v\}$ is connected. Next, suppose that $\{u, v\} \in E(G)$. Let k be the number of (u, v)-segments in $\mathcal{M}$. Consider the following three cases.

Case 1: $k = 0$. We erase all the (u, v)-nodes in $\mathcal{M}$. Then, $\mathcal{M}$ becomes an atlas of $G - \{\{u, v\}\}$ and countries u and v are disjoint cycle-faces of $\mathcal{M}$. So, by Lemma 3.2, $G - \{u, v\}$ is connected.

Case 2: $k = 1$. We erase all the (u, v)-nodes in $\mathcal{M}$. $\mathcal{M}$ remains an atlas of G. Moreover, edge $\{u, v\}$ becomes good in $\mathcal{M}$. So, countries u and v form a cycle-superface of $\mathcal{M}$. By Lemma 3.2, $G - \{u, v\}$ is connected.

Case 3: $k \geq 2$. We erase all the (u, v)-nodes in $\mathcal{M}$. $\mathcal{M}$ remains an atlas of G. Moreover, there are exactly k disjoint holes in $\mathcal{M}|_{\{u,v\}}$. So, removing countries u and v of $\mathcal{M}$ from the sphere leaves exactly k topologically connected regions. Each of these regions forms a connected component of $G - \{u, v\}$. Hence, $G - \{u, v\}$ is disconnected. This completes the proof of Statement 1.

We next prove Statement 2. For each i, let $U_i = V(G_i)$. By Case 3 in the proof of Statement 1, each hole in $\mathcal{M}|_{U_i \cup \{u,v\}}$ is a 2-hole and is touched only by u and v, and hence erasing all the 2-holes of $\mathcal{M}|_{U_i \cup \{u,v\}}$ yields an atlas of G'_i. On the other hand, given an atlas $\mathcal{M}_i$ of each G'_i, we erase all the (u, v)-nodes in $\mathcal{M}_i$. $\mathcal{M}_i$ remains an atlas of G'_i, because edge $\{u, v\}$ is marked in G'_i and so there exists a (u, v)-segment in $\mathcal{M}_i$. Since $G'_i - \{u, v\} = G_i$ is connected, Statement 1 implies that there is exactly one (u, v)-segment in $\mathcal{M}_i$. Thus removing countries u and v of $\mathcal{M}_i$ from the sphere leaves exactly one topologically connected region; let $\mathcal{R}_i$ be the closure of this region. The boundary of $\mathcal{R}_i$ is a cycle of $\mathcal{M}_i$ and can be divided into two nontrivial paths $S_{i,u}$ and $S_{i,v}$ such that $S_{i,u}$ (respectively, $S_{i,v}$) is a portion of the boundary of country u (respectively, v) in $\mathcal{M}$. Now, we obtain an atlas of G as follows. First, put $\mathcal{R}_1, \ldots, \mathcal{R}_k$ on the sphere in such a way that no two of them intersect and each $S_{i,u}$ appears on the upper half of the sphere while each $S_{i,v}$ appears on the lower half. Second, draw country u (respectively, v) to completely occupy the area of the upper (respectively, lower) half of the sphere that is occupied by no $\mathcal{R}_i$. This gives an atlas of G. ▮

Using Statement 2 in Lemma 3.3, we have a linear-time reduction from our algorithmic problem to its special case where the input marked graph is 3-connected.

Corollary 3.4 *Suppose G has an atlas. Then, G is 3-connected iff G has a well-formed atlas.*

Proof: By Statement 1 in Lemma 3.3, the "if" part is obvious. For the other direction, suppose G is 3-connected. Let $\mathcal{M}$ be an atlas of G. If no edge of G is bad in $\mathcal{M}$, then $\mathcal{M}$ is well-formed and we are done. So, suppose that some edge $\{u, v\}$ is bad in $\mathcal{M}$. Since G is 3-connected, Statement 1 in Lemma 3.3 implies that there is at most one (u, v)-segment in $\mathcal{M}$. If there is no (u, v)-segment in $\mathcal{M}$, we erase all but one (u, v)-nodes in $\mathcal{M}$; otherwise, we erase all the (u, v)-nodes in $\mathcal{M}$. In both cases, $\mathcal{M}$ remains an atlas of G and edge $\{u, v\}$ becomes good in $\mathcal{M}$ while no good edge becomes bad in $\mathcal{M}$. Consequently, we can make all bad edges good in $\mathcal{M}$. ∎

Lemma 3.5 *Suppose G has a well-formed atlas $\mathcal{M}$. Let $C = \{a, b, c\}$ be a set of three distinct vertices in G. Then, the following statements hold:*

1. *Suppose C is not a clique in G. Then, $G - C$ is connected.*
2. *Suppose C is a clique in G. Then, $G - C$ is disconnected if and only if (i) the countries in C do not meet at a node in $\mathcal{M}$ and (ii) each pair of countries in C strongly touch in $\mathcal{M}$.*
3. *Suppose $G - C$ is disconnected. Then, (i) $G - C$ has exactly two connected components G_1 and G_2, and (ii) both G'_1 and G'_2 have a well-formed atlas, where G'_1 (respectively, G'_2) is the marked graph obtained from $G[V(G_1) \cup C]$ (respectively, $G[V(G_2) \cup C]$) by marking the edges in $E(G[C])$. Moreover, given a well-formed atlas for G'_1 and another for G'_2, we can easily construct one for G.*

Proof: To prove Statement 1, suppose that C is not a clique. For each edge $\{u, v\} \in E(G[C])$, if countries u and v weakly touch in $\mathcal{M}$, then we erase the (u, v)-node in $\mathcal{M}$. Now, $\mathcal{M}$ is an atlas of a subgraph of G and countries in C form a cycle-superface or disjoint cycle-superfaces of $\mathcal{M}$. By Lemma 3.2, $G - C$ is connected.

To prove Statement 2, suppose that C is a clique. Since $\mathcal{M}$ is hole-free, the "if" part is clear. To prove the "only if" part, suppose that (i) or (ii) in Statement 2 does not hold. In case (i) is false, a, b and c meet at a node in $\mathcal{M}$, and the well-formedness of $\mathcal{M}$ ensures that countries a, b and c form a cycle-superface of $\mathcal{M}$; so, by Lemma 3.2, $G - C$ is connected. Otherwise, suppose (i) is true and (ii) is false. For each edge $\{u, v\} \in E(G[C])$, if countries u and v weakly touch in $\mathcal{M}$, then we erase the (u, v)-node to get atlas $\mathcal{M}'$. Obviously, $\mathcal{M}'$ is an atlas of a subgraph of G and countries in C form a cycle-superface or disjoint cycle-superfaces of $\mathcal{M}'$. By Lemma 3.2, $G - C$ is connected.

Next, we prove Statement 3. Since $G - C$ is disconnected, (i) and (ii) in Statement 2 hold. By this, there are exactly two holes $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{M}|_C$ and they are disjoint. For $i \in \{1, 2\}$, let U_i be the set of countries that occupy $\mathcal{H}_i$ in atlas $\mathcal{M}$. Each $G[U_i]$ is a connected component of G. Let G'_i be the marked graph obtained from $G[U_i \cup C]$ by marking the edges in $E(G[C])$. There is a unique hole in $\mathcal{M}|_{U_1 \cup C}$ and it is (strongly) touched only by the countries of C. So, modifying $\mathcal{M}|_{U_1 \cup C}$ by extending country a to completely occupy its unique hole yields a well-formed atlas of G'_1. Similarly, we can obtain a well-formed atlas of G'_2.

On the other hand, suppose we are given a well-formed atlas $\mathcal{M}_1$ for G'_1 and another $\mathcal{M}_2$ for G'_2. Let $i \in \{1, 2\}$. Since the edges in $E(G[C])$ are marked in G'_i, each pair of countries of C strongly touch in $\mathcal{M}_i$. Note that $G'_i - C$ is connected. Then by Statement 2 and the well-formedness of $\mathcal{M}_i$, the countries in C share a 3-node α_i in $\mathcal{M}_i$. Let D_i be a disk in the sphere such that α_i is an interior point of D_i and no country other than a, b, c intersects D_i. To obtain a well-formed atlas of G, we remove each D_i from the sphere to obtain a connected region $\mathcal{R}_i$, and then glue $\mathcal{R}_1$ and $\mathcal{R}_2$ together by identifying countries a, b, c in $\mathcal{R}_1$ with those in $\mathcal{R}_2$, respectively. ▮

By the lemmas in this section, we now have:

Lemma 3.6 *There is a linear-time reduction from the problem of deciding whether a given marked graph has an atlas, to its special case where the input graph is 4-connected.*

4 Maximal Cliques in Hole-Free 4-Map Graphs

Throughout this section, $\mathcal{M}$ denotes a well-formed atlas and G denotes its map graph. It is known [3] that for every integer $k \geq 3$, each k-map graph has no clique of size larger than $\lfloor 3k/2 \rfloor$. So, G has no 7-clique.

By Theorem 3.1 in [3], we can classify the layout $\mathcal{M}|_C$ of each maximal clique C of G with $4 \leq |C| \leq 6$ into four types as follows:

Pizzas: *There is a node α in $\mathcal{M}$ at which the countries in C meet.* (See Figure 4.1(1).) This is possible only when $|C| \leq 4$, because $\mathcal{M}$ is a 4-map. We say that C is a *pizza* in $\mathcal{M}$. Since $\mathcal{M}$ is well-formed, α must be unique. So, we call α the *center* of C in $\mathcal{M}$. Note that no country in $V(G) - C$ contains α as a boundary node.

Rice-balls: *No node of $\mathcal{M}$ is shared by more than two countries in C.* (See Figure 4.1(2).) This is possible only when $|C| \leq 4$ (as observed in [3]). We say that C is a *rice-ball* in $\mathcal{M}$.

Hamantaschen: *There are exactly three nodes in $\mathcal{M}$ each of which is shared by exactly four countries in C.* This is possible only when $|C| = 6$ (as observed in [3]). We say that C is a *hamantasch* in $\mathcal{M}$. Figure 2.2(1) displays C.

Pizzas-with-crust: *C is not a pizza, rice-ball, or hamantasch in $\mathcal{M}$.* (See Figures 4.1(3), (4), and (5).) Then, there is at least one node α in $\mathcal{M}$ at which exactly $|C| - 1$ countries in C meet (as shown in [3]). This is possible only when $|C| \leq 5$, because $\mathcal{M}$ is a 4-map. We say that C is a *pizza-with-crust* in $\mathcal{M}$. Since $\mathcal{M}$ is well-formed, α must be unique if $|C| = 5$. So, when $|C| = 5$, we call α the *center* of C in $\mathcal{M}$, and call the country in C not containing α the *crust* of C in $\mathcal{M}$.

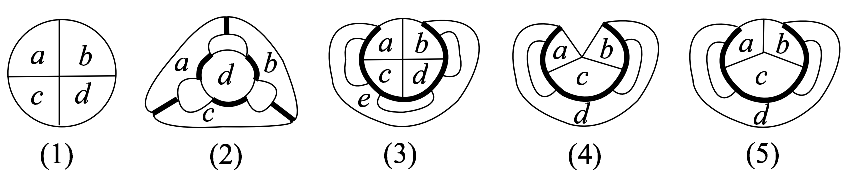

Figure 4.1: Well-formed layouts of maximal cliques.

Lemma 4.1 *Suppose G is 4-connected and $|V(G)| \geq 7$. Then, G has no 6-clique.*

Proof: For a contradiction, assume that G has an MC_6 C. Then, it must be a hamantasch and Figure 2.2(1) displays $\mathcal{M}|_C$. After Figure 2.2(1) is modified by contracting the two paths in the contractible forest each to a single node and erasing all resulting 2-holes, it still displays $\mathcal{M}|_C$ because G is 4-connected. However, the modification yields a layout of C without holes, a contradiction against the assumption that $|V(G)| \geq 7$. ∎

Lemma 4.2 *[3] A map graph with n vertices has at most $27n$ maximal cliques.*

5 Making Progress

Throughout this section, let G be the input marked graph. To find an atlas for G, our algorithm may "make progress" by producing one or more smaller marked graphs, so that finding an atlas for G is reduced to finding an atlas for each of these smaller graphs. Here we define the graph features that our algorithm may identify in order to make progress; subsequent sections show how to make progress for each.

Lemma 3.6 shows that the algorithm can always make progress when G is not 4-connected. So, in the remainder of this section, we assume that G is 4-connected. Then, by Corollary 3.4, it suffices to look for a well-formed atlas realizing G.

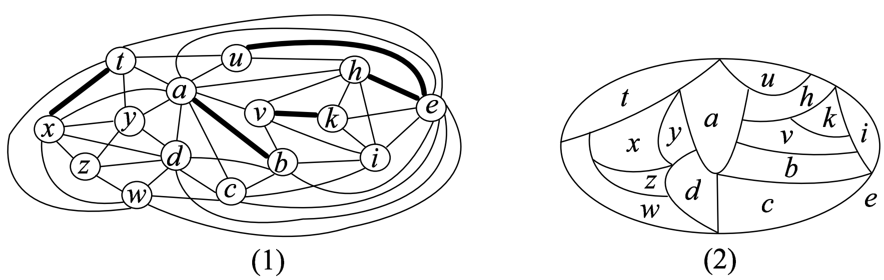

Figure 5.1: (1) A marked graph G (whose marked edges are shown in bold). (2) A well-formed atlas of G.

Definition 5.1 A *correct 4-pizza* in G is a list $\langle a, b, c, d \rangle$ of four countries in G such that if G has a well-formed atlas, then it has one in which countries a, b, c, d meet at a 4-node in this order. (For example, in the marked graph in Figure 5.1(1), $\langle a, b, c, d \rangle$ is a correct 4-pizza as can be seen from Figure 5.1(2).) *Removing a correct 4-pizza* $\langle a, b, c, d \rangle$ from G is the operation of modifying G as follows: Delete edge $\{a, c\}$ from G and mark edges $\{b, d\}$, $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{d, a\}$.

Lemma 5.2 *Let G' be the marked graph obtained from G by removing a correct 4-pizza $\langle a, b, c, d \rangle$. Then, G' has a well-formed atlas if G has one. Moreover, given a well-formed atlas for G', we can easily construct one for G.*

Proof: Suppose $\mathcal{M}$ is a well-formed atlas of G in which countries a, b, c, d meet at a node α in this order. After erasing the (a, c)-node α in $\mathcal{M}$, we obtain a well-formed atlas of G' in which countries a, b, and d meet at a 3-node and countries b, c, and d meet at another 3-node. Thus, by Statement 2 in Lemma 3.5, both $G' - \{a, b, d\}$ and $G' - \{b, c, d\}$ are connected.

Let $\mathcal{M}'$ be a well-formed atlas of G'. Since $G' - \{a, b, d\}$ is connected and the edges $\{a, b\}$, $\{a, d\}$, and $\{b, d\}$ are marked in G', countries a, b, and d must meet at a 3-node α_1 in $\mathcal{M}'$ according to Statement 2 in Lemma 3.5. Similarly, countries b, c, and d must meet at a 3-node

α_2 in $\mathcal{M}'$. Thus, the intersection of countries b and d in $\mathcal{M}'$ is a nontrivial path S between α_1 and α_2 in $\mathcal{M}'$. We modify $\mathcal{M}'$ by contracting S to a single node, obtaining a well-formed atlas of G. ∎

In some of our reductions we will discover that an induced subgraph of G has a well-formed extensible layout in which there are several correct 4-pizzas. In those situations we may remove all the 4-pizzas at once. This is because that if G has a well-formed atlas, then the graph obtained from G by removing a correct 4-pizza still has a well-formed atlas (and therefore is 3-connected) and Lemma 5.2 can be applied further.

To see a particular type of correct 4-pizza in G, consider an extensible layout of an MC_5 C in G. As pointed out in Section 4, each extensible layout of C is a pizza-with-crust. The center of this pizza-with-crust motivates the following definition.

Definition 5.3 A *correct center* of an MC_5 C is a list $\langle a, b, c, d \rangle$ of four countries in C, such that C has a well-formed extensible layout in which countries a, b, c, d meet at a 4-node in this order. (For example, in the marked graph in Figure 5.1(1), $\{a, \ldots, e\}$ is an MC_5 and $\langle a, b, c, d \rangle$ is a correct center of the MC_5 as can be seen from Figure 5.1(2).) The unique country in $C - \{a, b, c, d\}$ is the corresponding *correct crust* of C.

Fact 5.4 *Let C be an MC_5 in G. Then, every correct center of C is a correct 4-pizza in G.*

Note that C may have multiple correct centers, each from a different extensible layout.

Besides the k-cuts mentioned above, we also consider the more specialized separators introduced below in Definition 5.7. Section 7 will show how the algorithm may make progress as long as G contains one of these.

Definition 5.5 Edges $\{a, b\}$ and $\{x, y\}$ in G are *crossable* if they are both unmarked and $\{a, b, x, y\}$ is an MC_4 in G. For an edge $\{a, b\}$, if $\{a, b\}$ is unmarked, then let $\mathcal{E}[a, b]$ denote the set of all edges $\{x, y\}$ crossable with $\{a, b\}$; otherwise, let $\mathcal{E}[a, b]$ be the empty set. (For example, in the marked graph in Figure 5.1(1), $\{a, e\}$ and $\{t, u\}$ are crossable but $\{h, i\}$ and $\{k, v\}$ are not. Moreover, $\mathcal{E}[a, e] = \{\{t, u\}, \{h, u\}\}$.)

Note that if G has an atlas where countries a, x, b, y meet at a 4-node *in this order*, then either they are part of an MC_5, or $\{a, b\}$ and $\{x, y\}$ are crossable. This is because $\{a, x, b, y\}$ has to be a 4-clique which can be either maximal or not.

Fact 5.6 *If $\{a, b\}$ is an edge and $G - \{a, b\}$ has a 3-clique $\{c, d, e\}$, then at most one edge of that 3-clique is in $\mathcal{E}[a, b]$.*

Proof: Two edges would imply two MC_4's, sitting inside the 5-clique $\{a, b, c, d, e\}$. ∎

Definition 5.7 We define the following separators in the marked graph G:

1. A *separating edge* of G is an edge $\{a, b\}$ such that $G - \{a, b\} - \mathcal{E}[a, b]$ is disconnected. (For example, in the marked graph in Figure 5.1(1), $\{a, e\}$ is a separating edge.)
2. An *induced 4-cycle* of G is a set C of four vertices in G such that $G[C]$ is a cycle of G. A *separating 4-cycle* of G is an induced 4-cycle C of G such that $G - C$ is disconnected. (For example, in the marked graph in Figure 5.1(1), $\{a, d, w, t\}$ is a separating 4-cycle.)

3. A *separating triple* of G is a list $\langle a, b, c\rangle$ of three vertices in G such that $C = \{a, b, c\}$ is a clique in G and $G - C - \mathcal{E}[a, b]$ is disconnected. (For example, in the marked graph in Figure 5.1(1), $\langle h, i, v\rangle$ is a separating triple.)
4. A *separating quadruple* of G is a list $\langle a, b, c, d\rangle$ of four vertices in G such that (i) $\{a, b, c, d\}$ is an induced 4-cycle of G and (ii) $G - \{a, b, c, d\} - \mathcal{E}[a, b]$ is disconnected. (For example, in the marked graph in Figure 5.1(1), $\langle h, i, b, a\rangle$ is a separating quadruple.)
5. A *separating triangle* of G is a list $\langle a, b, c\rangle$ of three vertices in G such that (i) $C = \{a, b, c\}$ is a clique in G and (ii) $G' = G - C - (\mathcal{E}[a, b] \cup \mathcal{E}[a, c])$ is disconnected. If in addition, G' has a connected component consisting of a single vertex, then $\langle a, b, c\rangle$ is a *strongly separating triangle* of G. (For example, in the marked graph in Figure 5.1(1), $\langle x, a, d\rangle$ is a strongly separating triangle.)

6 Sketch of the Algorithm

Throughout the rest of this paper, G denotes the input marked graph. By Lemma 3.6, we may assume that G is 4-connected. Then, by Corollary 3.4, it suffices to look for a well-formed atlas realizing G. Moreover, if $|V(G)| \leq 8$, our algorithm will solve the problem by exhaustive search. So, we further assume that $|V(G)| \geq 9$. For ease of describing our algorithm, we further make the following assumption:

Assumption 1 *G has a well-formed atlas $\mathcal{M}$.*

When G really has a well-formed atlas, our algorithm will output one with at most $2|V(G)| - 4$ nodes (cf. Lemma 2.2). On the other hand, when G has no atlas indeed, our algorithm will either finish without giving an atlas (e.g., this may happen when the input graph has too many maximal cliques), or finish with an invalid atlas (because of Assumption 1).

Given G, our algorithm searches it for a separating edge (cf. Lemma 7.2), separating 4-cycle (cf. Lemma 7.5), separating triple (cf. Lemma 7.7), separating quadruple (cf. Lemma 7.9), strongly separating triangle (cf. Lemma 7.18), or separating triangle (cf. Lemma 7.19), *in this order*. In each case, as the lemmas show, the algorithm makes progress by either (1) removing a correct 4-pizza or (2) reducing the problem for G to the problems for certain marked graphs smaller than G whose total size is that of G plus a constant.

If none of the above separators exists in G, then G has no 6-clique (cf. Lemma 4.1) and the algorithm searches G for an MC_5 or MC_4, *in this order*. If an MC_5 C is found, it tries to find an extensible layout of C by doing a case-analysis based on the neighborhood of C in G (cf. Section 8). The absence of the above separators guarantees that only a few cases need to be analyzed. The case-analysis either yields an extensible layout of C whose center is then removed to make progress, or produces a marked graph G' smaller than G such that finding a well-formed atlas for G can be reduced to finding a well-formed atlas for G'.

If no MC_5 but an MC_4 is found in G, the algorithm scans all MC_4's of G in an arbitrary order. While scanning an MC_4 C, it decides whether C has a rice-ball layout (cf. Lemma 9.1). If C has a rice-ball layout, the algorithm quits the scanning and makes progress by removing a correct 4-pizza obtained from the rice-ball layout of C. On the other hand, if no rice-ball is found after scanning all MC_4's, the algorithm scans all MC_4's of G in an arbitrary order, once again. But this time, while scanning an MC_4 C, it decides whether C has a non-pizza layout, by doing a case-analysis based on the neighborhood of C in G (cf. Section 9.2). The analysis consists of only a few cases due to the absence of the above separators. If C has a non-pizza layout, the algorithm quits the scanning and makes progress by removing a correct 4-pizza obtained from the layout of C. Otherwise, all MC_4's are pizzas; the algorithm finds their centers (cf. Section 9.3), and removes all of them so that G no longer has an MC_4.

If neither MC_5 nor MC_4 is found in G, then this is a base case. Since each map graph without 4-cliques is planar [3], G must be planar, or else we reject. When G is planar, then it has a unique planar embedding because G is 4-connected (for lack of 3-cuts). We claim that G has a well-formed atlas if and only if all its faces are triangles. The "if" direction is obvious because the planar dual of G is an atlas of G, which is well-formed by the 4-connectivity of G and the absence of 4-cliques in G. Conversely, suppose G has a well-formed atlas $\mathcal{M}$. Since $\mathcal{M}$ has no k-node for $k > 3$, all adjacent pairs of countries strongly touch in $\mathcal{M}$, and so the 3-nodes and boundaries in $\mathcal{M}$ define a 3-regular planar graph G', whose dual is G. So, it suffices for the algorithm to check that G is planar and has a 3-regular dual; if so, it returns the dual as an atlas.

In all the recursive cases, the smaller graphs that we generate have total size at most the size of G plus a constant, and we spend quadratic time on generating them. A simple argument (cf. Section 10) shows that the overall time is cubic.

7 Advanced Separations

In this section we prove the necessary properties of the separators in Definition 5.7. Figures 7.1(1), (2), and (3) help understand the proofs in Sections 7.1, 7.2, and 7.3, respectively.

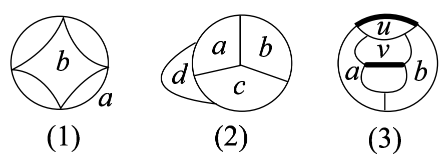

Figure 7.1: Three figures for Sections 7.1, 7.2, and 7.3, respectively.

7.1 Separating Edges

By the 4-connectivity of G, if $\{a, b\}$ is a separating edge of G, then $\mathcal{E}[a, b] \neq \emptyset$ and hence $\{a, b\}$ is an unmarked edge of G.

Definition 7.1 A *shrinkable segment* in $\mathcal{M}$ is a (u, v)-segment S in $\mathcal{M}$ such that (i) $\{u, v\}$ is an unmarked edge in G, (ii) both the two endpoints α and β of S are 3-nodes, and (iii) the two countries a and b such that u, v, a meet at α and u, v, b meet at β are distinct and adjacent in G. We call a and b the *ending countries* of S.

In the next two results, we show a close relationship between separating edges and shrinkable segments.

Lemma 7.2 *Assume that G has a separating edge $\{a, b\}$. Let $G' = G - \{a, b\} - \mathcal{E}[a, b]$. Then, for every $\{x, y\} \in E(G)$ such that x and y belong to different connected components of G', $\langle a, x, b, y\rangle$ is a correct 4-pizza in G.*

Proof: Let $\mathcal{M}'$ be the atlas of G obtained from $\mathcal{M}$ by contracting those shrinkable segments whose ending countries are a and b. All edges except $\{a, b\}$ are good in $\mathcal{M}'$.

First, we claim that for every $\{u, v\} \in E(G)$ such that u and v belong to different connected components of G', there is a node α at which countries u, a, v, b meet in $\mathcal{M}'$ in this order. Toward a contradiction, assume that such a node does not exist in $\mathcal{M}'$. By the definition of G', $\{u, v\}$ is in $\mathcal{E}[a, b]$. There is no country $w \in V(G) - \{a, b, u, v\}$ adjacent to both u and v; otherwise, w would connect u and v in G' (by Fact 5.6). So, by the absence of holes in $\mathcal{M}'$, the intersection of countries u and v in $\mathcal{M}'$ must be a nontrivial path S in $\mathcal{M}'$ and neither endpoint of S appears on the boundary of a country other than a and b in $\mathcal{M}'$. At least one endpoint of S is not on the boundary of country a in $\mathcal{M}'$; otherwise, since the edges $\{a, u\}$ and $\{a, v\}$ are still good in $\mathcal{M}'$, countries a, u, and v together would have to occupy the whole sphere, a contradiction. Similarly, at least one endpoint of S is not on the boundary of country b in $\mathcal{M}'$. Thus, both endpoints of S are 3-nodes. In summary, countries u, v, a meet at one endpoint of S in $\mathcal{M}'$ while countries u, v, b meet at the other endpoint of S in $\mathcal{M}'$. Therefore, S would be a shrinkable segment with ending countries a and b in $\mathcal{M}'$, a contradiction.

Second, we claim that there is no (a, b)-segment in $\mathcal{M}'$. Toward a contradiction, assume that an (a, b)-segment S exists in $\mathcal{M}'$. By the first claim, there is an (a, b)-node α in $\mathcal{M}'$. Note that α is not on S. Let x and y be the two countries of $V(G) - \{a, b\}$ that meet at α. Since $\mathcal{M}'$ has no hole and G is a 4-connected graph with at least nine vertices, there is a country $z \in V(G) - \{a, b, x, y\}$ that touches either x or y in $\mathcal{M}'$. If z touches x (respectively, y) in $\mathcal{M}'$, then z is not reachable from y (respectively, x) in $G - \{a, b, x\}$ (respectively, $G - \{a, b, y\}$), contradicting the 4-connectivity of G.

Third, we claim that there are at least two (a, b)-nodes in $\mathcal{M}'$. Toward a contradiction, assume that there is at most one (a, b)-node in $\mathcal{M}'$. Then, there is a unique (a, b)-node β in $\mathcal{M}'$, because countries a and b are adjacent but there is no (a, b)-segment in $\mathcal{M}'$. So, by the first claim, $\mathcal{E}[a, b]$ would have at most one edge, namely, the edge $\{x, y\}$ such that countries a, x, b, y meet at β in $\mathcal{M}'$. Hence, erasing the (x, y)-node β in $\mathcal{M}'$ results in an atlas $\mathcal{M}''$ of $G - \{\{x, y\}\}$ such that countries a and b form a cycle-superface of $\mathcal{M}''$. Thus, by Lemma 3.2, $G - \{a, b\} - \{\{x, y\}\}$ is connected. Now, since $\mathcal{E}[a, b] \subseteq \{\{x, y\}\}$, $G' = G - \{a, b\} - \mathcal{E}[a, b]$ would be connected too, a contradiction.

Let ℓ be the number of (a, b)-nodes in $\mathcal{M}'$. Since $\ell \geq 2$ and there is no (a, b)-segment in $\mathcal{M}'$, atlas $\mathcal{M}'$ has a cyclic sequence of (a, b)-nodes $\beta_0, \ldots, \beta_{\ell-1}$. These nodes alternate with ℓ 2-holes in $\mathcal{M}'|_{\{a,b\}}$; Figure 7.1(1) displays $\mathcal{M}'|_{\{a,b\}}$ when $\ell = 4$.

For each $j \in \{0, 1, \ldots, \ell - 1\}$, let x_j and y_j be the countries such that a, x_j, b, y_j meet at β_j in $\mathcal{M}'$. Clearly, $\{a, b, x_j, y_j\}$ is a 4-clique of G. We claim that $\{a, b, x_j, y_j\}$ is an MC_4 of G; otherwise to form a containing 5-clique would force $\ell \leq 3$ and $\mathcal{E}[a, b] = \emptyset$, contradicting the disconnectivity of G'. So, each β_j corresponds to an edge $\{x_j, y_j\}$ in $\mathcal{E}[a, b]$. Moreover, for each hole $\mathcal{H}$ of $\mathcal{M}'|_{\{a,b\}}$, the countries occupying $\mathcal{H}$ in atlas $\mathcal{M}'$ form a connected component of G'.

Now consider a particular edge $\{x_j, y_j\}$ of G. To show that $\langle a, x_j, b, y_j \rangle$ is a correct 4-pizza in G, we must find a well-formed atlas of G in which countries a, x_j, b, y_j meet at a node in this order. This is easy to do: we simply erase all (a, b)-nodes in $\mathcal{M}'$ except β_j, and the resulting atlas is a well-formed atlas of G. ∎

Corollary 7.3 *Let $\{a, b\}$ be an edge of G. Then, $\{a, b\}$ is a separating edge iff the following conditions hold:*

1. *There is a shrinkable segment in $\mathcal{M}$ with ending countries a and b.*
2. *Countries a and b weakly touch in $\mathcal{M}$, and no MC_5 of G contains both the two countries in $V(G) - \{a, b\}$ that meet at the (a, b)-node in atlas $\mathcal{M}$.*

Proof: The "only if" part is obvious from the proof of Lemma 7.2. To prove the "if" part, suppose that Conditions 1 and 2 hold. Let $\mathcal{M}'$ be the atlas of G obtained from $\mathcal{M}$ by contracting a shrinkable segment with ending countries a and b to a single node α. Besides α, there is exactly one (a, b)-node β in $\mathcal{M}'$, inherited from $\mathcal{M}$. Now, $\mathcal{M}'|_{\{a,b\}}$ has exactly two holes $\mathcal{H}_0$ and $\mathcal{H}_1$. Let Z_0 (respectively, Z_1) be the set of countries of $V(G) - \{a, b\}$ occupying $\mathcal{H}_0$ (respectively, $\mathcal{H}_1$) in atlas $\mathcal{M}'$. Let $x \in Z_0$ and $y \in Z_1$ be the two countries that meet at α in $\mathcal{M}'$. Similarly, let $x' \in Z_0$ and $y' \in Z_1$ be the two countries that meet at β in $\mathcal{M}'$. By $\mathcal{M}'$, edges $\{x, y\}$ and $\{x', y'\}$ are not marked in G and they are all the edges connecting countries of Z_0 to those of Z_1. Now, since no MC_5 of G contains both x' and y' (by Condition 2), no MC_5 of G contains both x and y either. Therefore, both edges $\{x, y\}$ and $\{x', y'\}$ belong to $\mathcal{E}[a, b]$, and no connected component of $G - \{a, b\} - \mathcal{E}[a, b]$ contains both the countries of Z_0 and those of Z_1. In other words, $\{a, b\}$ is a separating edge of G. ∎

7.2 Separating 4-Cycles

Since $\mathcal{M}$ is hole-free, the following fact is clear.

Fact 7.4 *Let C be an induced 4-cycle of G. If for each edge $\{u, v\}$ of $G[C]$, countries u and v strongly touch in an atlas of G, then C is a separating 4-cycle of G.*

Lemma 7.5 *Suppose $C = \{a, b, c, d\}$ is a separating 4-cycle of G. Let the edges of $G[C]$ be $\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}$. Then, $G - C$ has exactly two connected components G_1 and G_2; and for each $i \in \{1, 2\}$, the marked graph G'_i obtained from $G[V(G_i) \cup C]$ by adding edge $\{a, c\}$ and marking edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, a\}$, $\{a, c\}$ has a well-formed atlas. Moreover, given a well-formed atlas for G'_1 and another for G'_2, we can easily construct one for G.*

Proof: Since $G[C]$ is a cycle and $\mathcal{M}$ is well-formed, there are exactly two holes $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{M}|_C$. For $j \in \{1, 2\}$, let U_j be the set of countries that occupy $\mathcal{H}_j$ in atlas $\mathcal{M}$. Clearly, the countries in U_j are connected together in $G - C$. By this and the assumption that $G - C$ is disconnected, both $G[U_1]$ and $G[U_2]$ are connected components of $G - C$ and $G - C$ has no other connected component. So, $\mathcal{H}_1$ and $\mathcal{H}_2$ must be disjoint. Thus, for each edge $\{u, v\}$ in $G[C]$, countries u and v strongly touch in $\mathcal{M}$.

For each $j \in \{1, 2\}$, there is a unique hole in $\mathcal{M}|_{U_j \cup C}$ and it may be (strongly) touched only by the countries of C. So, modifying $\mathcal{M}|_{U_j \cup C}$ by extending country a to cover its unique hole yields a well-formed atlas of G'_j in which countries a, b, and c meet at a 3-node and countries a, c, and d meet at a 3-node. So, by Statement 2 in Lemma 3.5, both $G'_j - \{a, b, c\}$ and $G'_j - \{a, c, d\}$ are connected.

Conversely, suppose we are given an atlas $\mathcal{M}_j$ for each G'_j. Since $G'_j - \{a, b, c\}$ is connected and the three edges $\{a, b\}$, $\{a, c\}$, and $\{c, b\}$ are marked in G'_j, countries a, b, and c meet at a 3-node in $\mathcal{M}_j$, by Statement 2 in Lemma 3.5. Similarly, countries a, c, and d must meet at a 3-node in $\mathcal{M}_j$. Thus, by the well-formedness of $\mathcal{M}_j$, Figure 7.1(2) displays $\mathcal{M}_j|_C$. By the figure, we can modify $\mathcal{M}_j$ by drawing a new edge that starts at the middle point of the (a, b)-segment, crosses the interior of country a, and ends at the middle point of the (a, d)-segment;

let $\mathcal{M}'_j$ be the resulting map. In map $\mathcal{M}'_j$, countries a and c no longer touch, and there is a hole $\mathcal{H}_j$ strongly touching all of countries a, b, c, d. Now, to obtain a well-formed atlas of G, we remove each $\mathcal{H}_j$ from the sphere to obtain a connected region $\mathcal{R}_j$, and then glue $\mathcal{R}_1$ and $\mathcal{R}_2$ together by identifying countries a, b, c, d in $\mathcal{R}_1$ with those in $\mathcal{R}_2$, respectively. ∎

7.3 Separating Triples

Since $\mathcal{M}$ is hole-free, the following fact is clear.

Fact 7.6 *Let $C = \{a, b, c\}$ be a 3-clique of G. If the following three conditions hold, then $\langle a, b, c\rangle$ is a separating triple of G:*

1. *Countries in C do not meet at a node in $\mathcal{M}$.*
2. *If countries a and b weakly touch in $\mathcal{M}$, then no MC_5 of G contains both the two countries in $V(G) - C$ that meet at the (a, b)-node in atlas $\mathcal{M}$.*
3. *Countries c and a strongly touch in $\mathcal{M}$, and so do countries c and b.*

By the 4-connectivity of G, if $\langle a, b, c\rangle$ is a separating triple of G, then $\mathcal{E}[a, b] \neq \emptyset$ and hence $\{a, b\}$ is an unmarked edge of G.

Lemma 7.7 *Suppose G has no separating edge but has a separating triple $\langle a, b, c\rangle$. Let $C = \{a, b, c\}$ and $G' = G - C - \mathcal{E}[a, b]$. Then, G' has exactly two connected components G_1 and G_2 and exactly one edge $\{u, v\} \in E$ connects G_1 to G_2 in $G - C$. Moreover, $\langle a, u, b, v\rangle$ is a correct 4-pizza in G.*

Proof: Since G is 4-connected, $G - C$ is connected. So $\mathcal{E}[a, b]$ is non-empty to disconnect G', and we may choose $\{u, v\} \in \mathcal{E}[a, b]$ such that u belongs to a connected component G_1 of G' and v belongs to another different connected component G_2 of G'. By definition of $\mathcal{E}[a, b]$, $\{a, b, u, v\}$ is an MC_4 in G.

We claim that countries u and v do not strongly touch in $\mathcal{M}$. Assume, on the contrary, that a (u, v)-segment S exists in $\mathcal{M}$. Since $\mathcal{M}$ is hole-free and $|V(G)| \geq 9$, there are countries w_1, w_2 in $V(G) - \{u, v\}$ such that one endpoint of S is on the boundary of w_1 and the other is on the boundary of w_2. If w_1 were neither a nor b, then by Fact 5.6, w_1 would connect u and v in G'. Thus $w_1 \in \{a, b\}$, and similarly $w_2 \in \{a, b\}$. By the well-formedness of $\mathcal{M}$ and the fact that $|V(G)| \geq 9$, we can verify that there is no way for country a (respectively, b) to have both endpoints of S on its boundary. So, both endpoints of S are 3-nodes in $\mathcal{M}$. Moreover, one endpoint of S is on the boundary of country a and the other is on the boundary of country b. In summary, S is a shrinkable segment in $\mathcal{M}$ with ending countries a and b. Thus, if countries a and b weakly touch in $\mathcal{M}$, then by Corollary 7.3, $\{a, b\}$ would be a separating edge of G, a contradiction. On the other hand, if countries a and b strongly touch in $\mathcal{M}$, then Figure 7.1(3) displays $\mathcal{M}|_{\{a,b,u,v\}}$. Since $|V(G)| \geq 9$, at least one of the two contractible paths in Figure 7.1(3) should be fixed to be no longer contractible. This together with Statement 2 in Lemma 3.5 implies that at least one of $\{a, b, v\}$ and $\{a, b, u\}$ would be a 3-cut of G, a contradiction. Therefore, the claim holds.

By the claim, countries u and v weakly touch in $\mathcal{M}$. Let α be the unique node at which countries u and v meet in $\mathcal{M}$. Then, since $\mathcal{M}$ has no hole, there are two distinct countries $w_1, w_2 \in V(G) - \{u, v\}$ such that countries u, w_1, v, w_2 meet at α in $\mathcal{M}$ in this order. As before, we can show that $\{w_1, w_2\} = \{a, b\}$. Thus, by the well-formedness of $\mathcal{M}$, $\langle a, u, b, v\rangle$ is a correct 4-pizza in G.

The discussions above actually prove that for every pair of adjacent countries x and y of G that belong to different connected components of G', countries a, x, b, y must meet at a 4-node in $\mathcal{M}$ in this order. Since α is the unique node at which countries a and b meet in $\mathcal{M}$, (u, v) is the unique pair of adjacent countries of G that belong to different connected components of G'. We now claim that G' has only two connected components G_1 and G_2. Assume, on the contrary, that G' has a connected component G_3 other than G_1 and G_2. Then, there exists a country $w \in V(G) - (C \cup V(G_3))$ which touches some country w' of G_3 in $\mathcal{M}$; otherwise, G_3 would be a connected component of $G-C$, a contradiction. But now, (w, w') would be another pair (than (u, v)) of adjacent countries of G that belong to different connected components of G', a contradiction. Thus, the connected components of G' are only G_1 and G_2, and $\{u, v\}$ is the unique edge connecting G_1 to G_2 in $G - C$. ∎

7.4 Separating Quadruples

Since $\mathcal{M}$ is hole-free, the following fact is clear.

Fact 7.8 *Let $\{a, b, c, d\}$ be an induced 4-cycle of G. If the following two conditions hold, then $\langle a, b, c, d\rangle$ is a separating quadruple of G:*

1. *If countries a and b weakly touch in $\mathcal{M}$, then no MC_5 of G contains both the two countries in $V(G) - C$ that meet at the (a, b)-node in atlas $\mathcal{M}$.*
2. *Countries b and c strongly touch in $\mathcal{M}$, so do countries c and d, and so do countries d and a.*

Note that among the facts used in the proof of Lemma 7.7, only the fact that $G - C$ is connected is related to C. So, we can modify the proof of Lemma 7.7 to prove the following:

Lemma 7.9 *Suppose G has neither separating edge nor separating 4-cycle, but has a separating quadruple $\langle a, b, c, d\rangle$. Let $C = \{a, b, c, d\}$. Then, $G - C - \mathcal{E}[a, b]$ has exactly two connected components G_1 and G_2 and exactly one edge $\{u, v\} \in E(G)$ connects G_1 to G_2 in $G - C$. Moreover, $\langle a, u, b, v\rangle$ is a correct 4-pizza in G.*

7.5 Separating Triangles

Since $\mathcal{M}$ is hole-free, the following fact is clear.

Fact 7.10 *Let $C = \{a, b, c\}$ be a 3-clique of G. If the following three conditions hold, then $\langle a, b, c\rangle$ is a separating triangle of G:*

1. *Countries in C do not meet at a node in $\mathcal{M}$.*
2. *If countries a and b (respectively, countries a and c) weakly touch in $\mathcal{M}$, then no MC_5 of G contains both the two countries in $V(G) - C$ that meet at the (a, b)-node (respectively, (a, c)-node) in atlas $\mathcal{M}$.*
3. *Countries b and c strongly touch in $\mathcal{M}$.*

The results in Sections 7.1 through 7.4 allow our algorithm to simplify G whenever it contains a separating edge, triple, or quadruple. In this subsection, we consider how to make progress when G has no such separators. So, throughout this subsection, we assume:

Assumption 2 *G does not have a separating edge, triple, or quadruple.*

Suppose G has a separating triangle $\langle a, b, c\rangle$. By Assumption 2 and the 4-connectivity of G, both $\mathcal{E}[a, b]$ and $\mathcal{E}[a, c]$ are nonempty and hence both $\{a, b\}$ and $\{a, c\}$ are unmarked edges of G. Let C and G' be as described in Definition 5.7(5). Our goal is to show that using C and G', our algorithm can proceed by finding correct 4-pizzas in G.

Claim 7.11 *If $\{u, v\}$ is an edge in $G - C$ but not in G', then $a \in N_G(u) \cap N_G(v)$. Also, countries u, v, b, and c cannot meet at a 4-node in a well-formed atlas of G.*

Proof: Since $\{u, v\} \in \mathcal{E}[a, b] \cup \mathcal{E}[a, c]$, either $\{a, b, u, v\}$ or $\{a, c, u, v\}$ is an MC_4 of G. In both cases, $a \in N_G(u) \cap N_G(v)$. For the last part, such a 4-node would imply a 5-clique containing the MC_4, contradicting its maximality. ∎

Claim 7.12 *For every connected component K of G', (i) $C \subseteq N_G(V(K))$ and (ii) G' has another connected component J such that $V(K) \cap N_G(V(J)) \neq \emptyset$.*

Proof: For (i), let $S = C \cap N_G(V(K))$. Since $G - C$ is connected, some edge $\{x, y\} \in \mathcal{E}[a, b] \cup \mathcal{E}[a, c]$ connects K to an outside vertex. So, $\{a, b, x, y\}$ or $\{a, c, x, y\}$ is an MC_4 of G. Hence, $\{a, b\} \subseteq S$ or $\{a, c\} \subseteq S$. If $|S| = 2$, then S would be a separating edge of G, separating K from the rest. Thus, $S = C$.

For (ii), if on the contrary $V(K) \cap N_G(V(J)) = \emptyset$ for every J, then K would be a component of $G - C$, contradicting the 4-connectivity of G. ∎

Claim 7.13 *Let $Z \subseteq V(G) - C$. Suppose a subset $\{u, v, w\}$ of Z is a 3-clique of G such that u and v belong to different connected components of $G'[Z]$. Then, the following hold:*

1. *Either (i) $C \subseteq N_G(u)$ and $\{C \cap N_G(v), C \cap N_G(w)\} = \{\{a, b\}, \{a, c\}\}$ or (ii) $C \subseteq N_G(v)$ and $\{C \cap N_G(u), C \cap N_G(w)\} = \{\{a, b\}, \{a, c\}\}$.*
2. *There is no $x \in Z - \{u, v, w\}$ with $\{u, v, w\} \subseteq N_G(x)$.*

Proof: Since u and v are disconnected in $G'[Z]$, at least two of the edges in $G[\{u, v, w\}]$ are not in G'. Claim 7.11 applied to these edges implies $\{u, v, w\} \subseteq N_G(a)$. On the other hand, by Fact 5.6 each of $\mathcal{E}[a, b]$ and $\mathcal{E}[a, c]$ contains at most one edge of $G[\{u, v, w\}]$. So, exactly two edges of $G[\{u, v, w\}]$ are not in G', and either edge $\{u, w\}$ or $\{v, w\}$ remains in G'.

We suppose $\{v, w\}$ remains; the other case is similar (by swapping u and v). We also suppose $\{u, v\} \in \mathcal{E}[a, b]$ and $\{u, w\} \in \mathcal{E}[a, c]$, the other case is similar (by swapping b and c). Then $\{a, b, c\} \subseteq N_G(u)$, $\{a, b\} \subseteq N_G(v)$, and $\{a, c\} \subseteq N_G(w)$. On the other hand, G cannot have the edge $\{v, c\}$ (respectively, $\{w, b\}$), since this edge would imply a 5-clique containing the MC_4 $\{a, b, u, v\}$ (respectively, $\{a, c, u, w\}$). So, the first assertion holds.

For the second assertion, suppose on the contrary there is an $x \in Z - \{u, v, w\}$ with $\{u, v, w\} \subseteq N_G(x)$. As above, we suppose that both $\{a, b, u, v\}$ and $\{a, c, u, w\}$ are MC_4's of G. Then neither $\{a, b\}$ nor $\{a, c\}$ is a subset of $N_G(x)$, since otherwise x would extend one of these MC_4's to a 5-clique. But then the edges from x to u and v would all survive in $G'[Z]$, contradicting the disconnection of u and v. ∎

Claim 7.14 *Suppose countries a, b, c meet at a node in some well-formed atlas $\mathcal{M}$ of G. Then, for every connected component K of G', there is no node β in $\mathcal{M}$ at which two countries x and y of K together with two countries w and z of $V(G) - V(K)$ meet in the order x, w, y, z.*

Proof: Since countries a, b, c meet at a node α in $\mathcal{M}$, Figure 7.2 displays $\mathcal{M}|_C$. Node α is either a 3-node or a 4-node in $\mathcal{M}$. If α is a 3-node in $\mathcal{M}$, then each pair of countries in C strongly touch (i.e., the contractible path in Figure 7.2 should be fixed to be no longer contractible); otherwise, the contractible path in Figure 7.2 should be contracted to a single node. In either case, let $\mathcal{P}$ be the set of all nodes γ in $\mathcal{M}$ such that exactly two countries in C (together with some country(s) not in C) meet at γ. Note that $\alpha \notin \mathcal{P}$.

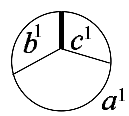

Figure 7.2: A possible display of $G[\{a, b, c\}]$.

Assume, on the contrary, that for some connected component K of G', some node β in $\mathcal{M}$ satisfies the condition in the claim. Then, by Claim 7.13(2), $C \cap \{w, z\} \neq \emptyset$. By Figure 7.2, $\beta \notin \{\alpha\} \cup \mathcal{P}$ no matter whether the contractible path in the figure should be contracted or not; so, $|C \cap \{w, z\}| \leq 1$. Hence, $|C \cap \{w, z\}| = 1$. In turn, $C \cap \{w, z\} = \{a\}$; otherwise, by Claim 7.11, $\{x, y, a, w, z\}$ would be a 5-clique of G, a contradiction. We assume that $w = a$; the other case is similar (by replacing z with w). Now, by Claim 7.13(1), $\{C \cap N_G(x), C \cap N_G(y)\} = \{\{a, b\}, \{a, c\}\}$ and $C \subseteq N_G(z)$. We assume that $C \cap N_G(x) = \{a, b\}$ and $C \cap N_G(y) = \{a, c\}$; the other case is similar (by swapping x and y). In summary, Figure 7.3(1) or (2) displays $\mathcal{M}|_{\{a,b,c,x,y,z\}}$.

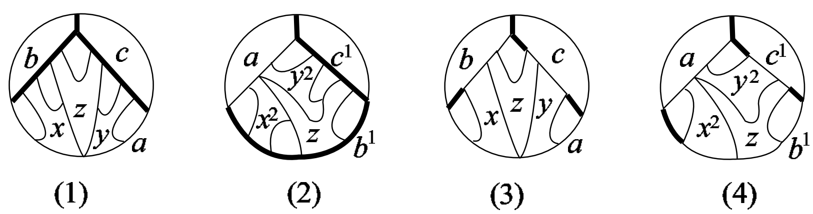

Figure 7.3: Possible displays of $G[\{a, b, c, x, y, z\}]$.

Let γ be the endpoint of the unique (x, z)-segment other than β in $\mathcal{M}$. There is no $f \in V(G) - \{a, b, c, x, y, z\}$ with $\{x, z\} \subseteq N_G(f)$; otherwise, by Claim 7.13(1), $C \cap N_G(f) = \{a, c\}$ which is impossible by Figures 7.3(1) and (2) (even if we contract a set of vertex-disjoint paths of the contractible forests). In turn, no country $f \in V(G) - \{a, b, c, x, y, z\}$ has node γ on its boundary in $\mathcal{M}$. Neither country a nor c has node γ on its boundary in $\mathcal{M}$ either, because $\{x, c\} \notin E(G)$ and $\mathcal{M}$ is well-formed. Thus, the absence of holes in $\mathcal{M}$ implies that γ is a 3-node on the boundary of country b in $\mathcal{M}$. Similarly, there is no $f \in V(G) - \{a, b, c, x, y, z\}$ with $\{y, z\} \subseteq N_G(f)$, and the endpoint of the unique (y, z)-segment other than β in $\mathcal{M}$ is a 3-node on the boundary of country c in $\mathcal{M}$. Thus, Figures 7.3(1) and (2) are transformable to Figures 7.3(3) and (4), respectively. Figures 7.3(3) and (4) together with Fact 7.6 imply that $\langle a, z, b \rangle$ or $\langle a, z, b^1 \rangle$ would be a separating triple of G (separating x from y), a contradiction. So, the claim holds. ∎

If the countries in C meet at a node in $\mathcal{M}$, then Figure 7.4(1), (2), or (3) displays $G[C]$; otherwise, Figure 7.4(4) displays $G[C]$. However, we can show that the countries in C in fact cannot meet at a node in $\mathcal{M}$, and hence Figure 7.4(4) is the only possible display of $G[C]$ (we will further show that Figure 7.4(5) displays $G[C]$ if $\langle a, b, c\rangle$ is a separating triangle of G).

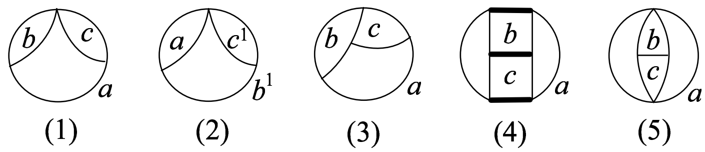

Figure 7.4: Possible displays of a separating triangle $\langle a, b, c\rangle$.

Lemma 7.15 *Figure 7.4(1) does not display* $G[C]$.

Proof: Assume, on the contrary, that G has a well-formed atlas $\mathcal{M}$ such that Figure 7.4(1) displays $\mathcal{M}|_C$. Let α be the node in $\mathcal{M}|_C$ at which countries a, b, and c meet. Let $\alpha_{a,b}$ (respectively, $\alpha_{a,c}$) be the endpoint of the (a, b)-segment (respectively, (a, c)-segment) other than α in $\mathcal{M}$. There must exist a $d \in V(G) - C$ such that countries a, b, d, c meet at α in $\mathcal{M}$. By the well-formedness of $\mathcal{M}$, α is the unique node shared by countries a and d, and hence $\{a, d\}$ is an unmarked edge in G. Let G'_d be the connected component of G' containing d. Let K be a connected component of G' other than G'_d such that some country u of G'_d touches some country v of K in $\mathcal{M}$; K exists by Claim 7.12.

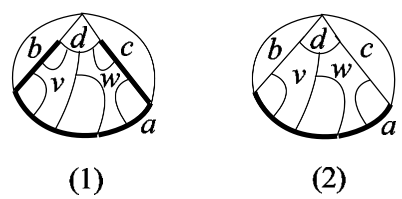

Figure 7.5: Possible displays of $G[\{a, b, c, d, v, w\}]$.

We claim that country a touches some country of $G'_d - \{d\}$ in $\mathcal{M}$. Assume, on the contrary, that the claim is false. Clearly, $\{a, b, u, v\}$ or $\{a, c, u, v\}$ is an MC_4 of G. Since no country of $G'_d - \{d\}$ touches a in $\mathcal{M}$, $u = d$. That is, $\{a, b, d, v\}$ or $\{a, c, d, v\}$ is an MC_4 of G. Since $C \subseteq N_G(d)$, we have $|N_G(v) \cap \{b, c\}| = 1$; otherwise, $\{a, b, c, d, v\}$ would be a 5-clique of G. We assume that $N_G(v) \cap \{b, c\} = \{b\}$; the other case is similar (by swapping b and c). Then, since country v cannot touch country c in $\mathcal{M}$ and $\mathcal{M}$ has no hole, there is a node β in $\mathcal{M}$ at which countries v, d and some $w \in V(G) - \{a, b, c, d, v\}$ meet. By Claim 7.13, $C \cap N_G(w) = \{a, c\}$ and there is no $x \in V(G) - \{a, b, c, d, v, w\}$ such that $\{d, v, w\} \subseteq N_G(x)$. In turn, no country $x \in V(G) - \{a, b, c, d, v, w\}$ has node β on its boundary. No country in C has node β on its boundary either, because $\{b, w\} \notin E(G)$, $\{c, v\} \notin E(G)$ and $\mathcal{M}$ is well-formed. So, the absence of holes in $\mathcal{M}$ implies that β is a 3-node in $\mathcal{M}$. Now, we see that Figure 7.5(1) displays $\mathcal{M}|_{\{a,b,c,d,v,w\}}$. There is no $x \in V(G) - \{a, b, c, d, v, w\}$ with $\{d, v\} \subseteq N_G(x)$; otherwise, $C \cap N_G(x) = \{a, c\}$ by Claim 7.13(1), which is impossible by Figure 7.5(1) (even if we completely or partially contract the contractible path). This together with the absence of holes in $\mathcal{M}$ and the well-formedness of $\mathcal{M}$ implies that the endpoint of the unique (v, d)-segment other than β in $\mathcal{M}$ must be a 3-node on the boundary of country b in

$\mathcal{M}$. Similarly, since edge $\{v,w\}$ remains in G' (because neither $\{a,b,v,w\}$ nor $\{a,c,v,w\}$ is an MC_4 of G), there is no $x \in V(G) - \{a,b,c,d,v,w\}$ with $\{d,w\} \subseteq N_G(x)$ and the endpoint of the unique (w,d)-segment other than β in $\mathcal{M}$ must be a 3-node on the boundary of country c in $\mathcal{M}$. Thus, Figure 7.5(1) is transformable to Figure 7.5(2). Figure 7.5(2) together with Fact 7.8 implies that $\langle b,c,w,v\rangle$ would be a separating quadruple of G (separating d from the rest), a contradiction. So, the claim holds.

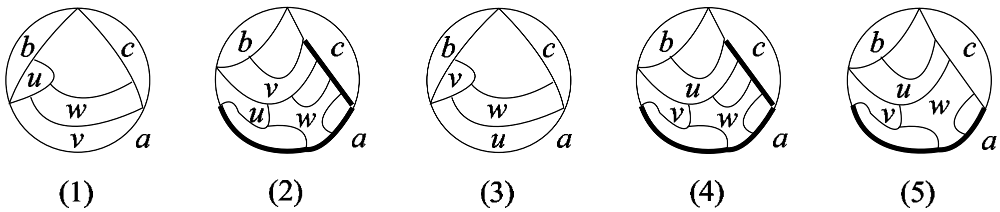

Figure 7.6: Possible displays of $G[\{a,b,c,u,v,w\}]$.

By the above claim, Claim 7.14 and the fact (Claim 7.12) that $C \subseteq N_G(V(K))$, it follows that $\alpha_{a,b}$ or $\alpha_{a,c}$ is shared by G'_d and K in $\mathcal{M}$. Suppose $\alpha_{a,b}$ is shared by G'_d and K; the other case is similar (by swapping b and c). Let u (respectively, v) be the country of G'_d (respectively, K) that has node $\alpha_{a,b}$ on its boundary. Then, countries u and v strongly touch in $\mathcal{M}$. Let S be the (u,v)-segment in $\mathcal{M}$. One endpoint of S is $\alpha_{a,b}$. Let β be the other endpoint of S. Neither country d nor c has node β on its boundary in $\mathcal{M}$; otherwise, $\{u,v,a,b,d\}$ or $\{u,v,a,b,c\}$ would be a 5-clique of G (and hence edge $\{u,v\}$ would remain in G'). By the well-formedness of $\mathcal{M}$, neither country a nor b has node β on its boundary in $\mathcal{M}$. In turn, since $\mathcal{M}$ has no hole, there is a country $w \in V(G) - \{a,b,c,d,u,v\}$ that has node β on its boundary in $\mathcal{M}$. By Claim 7.13(2) and the fact that no country in C has node β on its boundary in $\mathcal{M}$, it follows that β is a 3-node. Moreover, by Claim 7.13(1), $C \cap N_G(w) = \{a,c\}$ and either (i) $C \subseteq N_G(v)$ and $C \cap N_G(u) = \{a,b\}$ or (ii) $C \subseteq N_G(u)$ and $C \cap N_G(v) = \{a,b\}$. In case (i) holds, Figure 7.6(1) or (2) displays $\mathcal{M}|_{\{a,b,c,u,v,w\}}$. However, Figure 7.6(2) contradicts Claim 7.14 (because d,u,w belong to G'_d while v belongs to K), and Figure 7.6(1) together with Fact 7.8 implies that $\langle b,c,w,u\rangle$ would be a separating quadruple (separating d from v), a contradiction. So, (ii) holds and only Figure 7.6(3) or (4) can possibly display $\mathcal{M}|_{\{a,b,c,u,v,w\}}$. However, Figure 7.6(3) together with Fact 7.8 implies that $\langle b,c,w,v\rangle$ would be a separating quadruple (separating d from u), a contradiction. Thus, only Figure 7.6(4) can possibly display $\mathcal{M}|_{\{a,b,c,u,v,w\}}$. There is no $f \in V(G) - \{a,b,c,u,v,w\}$ with $\{u,w\} \subseteq N_G(f)$; otherwise, by Claim 7.13(1), $C \cap N_G(f) = \{a,b\}$ which is impossible by Figure 7.6(4) (even if we completely or partially contract the contractible path). This together with the absence of holes in $\mathcal{M}$ and the well-formedness of $\mathcal{M}$ implies that the endpoint of the (u,w)-segment other than β in $\mathcal{M}$ must be a 3-node on the boundary of country c. Now, Figure 7.6(4) is transformable to Figure 7.6(5). However, Figure 7.6(5) together with Fact 7.6 implies that $\langle a,u,c\rangle$ would be a separating triple of G (separating b from v), a contradiction. This completes the proof. ∎

Lemma 7.16 *Figure 7.4(2) does not display $G[C]$.*

Proof: Assume, on the contrary, that G has a well-formed atlas $\mathcal{M}$ such that Figure 7.4(2) displays $\mathcal{M}|_C$. We assume that $\langle b^1,c^1\rangle = \langle b,c\rangle$ in the figure; the other case is similar (by swapping b and c). Define nodes α and $\alpha_{a,b}$, country d and G'_d as in the proof of Lemma 7.15.

By the well-formedness of $\mathcal{M}$, country d meets b only at α and $\{b, d\}$ is not a marked edge in G. Let $\alpha_{b,c}$ be the endpoint of the (b, c)-segment other than α in $\mathcal{M}$.

We claim that some country of $G'_d - \{d\}$ touches country b in $\mathcal{M}$. Assume, on the contrary, that no country of $G'_d - \{d\}$ touches country b. Let K be a connected component of G' other than G'_d such that some country u of G'_d touches some country v of K in $\mathcal{M}$. By Claim 7.12, such K exists. Clearly, $\{a, b, u, v\}$ or $\{a, c, u, v\}$ is an MC_4 of G.

Case 1: $u \neq d$. Then, countries u and b do not touch in $\mathcal{M}$; hence, $C \cap N_G(u) = \{a, c\}$ and $\{a, c, u, v\}$ is an MC_4 of G. Moreover, there is no $w \in V(G) - \{a, b, c, u, v\}$ with $\{u, v\} \subseteq N_G(w)$; otherwise, since $C \cap N_G(u) = \{a, c\}$, we would have $C \subseteq N_G(v)$ and $C \cap N_G(w) = \{a, b\}$ by Claim 7.13(1), and in turn w would be a country of $G'_d - \{d\}$ that touches country b in $\mathcal{M}$, a contradiction. So, by Figure 7.4(2) and the absence of holes in $\mathcal{M}$, countries u and v strongly touch in $\mathcal{M}$ and both endpoints of the unique (u, v)-segment S in $\mathcal{M}$ are 3-nodes one of which is on the boundary of country a and the other is on the boundary of country c in $\mathcal{M}$. In turn, S is a shrinkable segment in $\mathcal{M}$. Moreover, no MC_5 of G contains both b and d. Consequently, by Corollary 7.3, $\{a, c\}$ would be a separating edge of G (indeed, G'_d is a connected component of $G - \{a, c\} - \mathcal{E}[a, c]$), a contradiction.

Case 2: $u = d$. Then $\{a, b, d, v\}$ or $\{a, c, d, v\}$ is an MC_4 of G. Since $\{a, b, c\} \subseteq N_G(d)$, we have $|N_G(v) \cap \{b, c\}| = 1$; otherwise, $\{a, b, c, d, v\}$ would be a 5-clique of G. So we have two sub-cases.

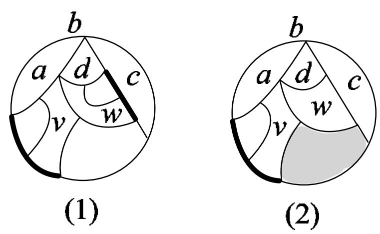

Figure 7.7: Possible displays of $G[\{a, b, c, d, v, w\}]$.

Case 2.1: $N_G(v) \cap \{b, c\} = \{b\}$. Then $C \cap N_G(v) = \{a, b\}$ and $\{a, b, d, v\}$ is an MC_4 of G. Moreover, since country v cannot touch country c in $\mathcal{M}$ and $\mathcal{M}$ has no hole, there is a node in $\mathcal{M}$ at which countries v, d and some $w \in V(G) - \{a, b, c, d, v\}$ meet. By Claim 7.13, $C \cap N_G(w) = \{a, c\}$ and there is no $x \in V(G) - \{a, b, c, d, v, w\}$ such that $\{d, v, w\} \subseteq N_G(x)$. Now, we see that Figure 7.7(1) displays $\mathcal{M}|_{\{a,b,c,d,v,w\}}$. There is no $x \in V(G) - \{a, b, c, d, v, w\}$ with $\{d, w\} \subseteq N_G(x)$; otherwise, $C \cap N_G(x) = \{a, b\}$ by Claim 7.13(1), which is impossible by Figure 7.7(1) (even if we completely or partially contract the two contractible paths). This together with the absence of holes in $\mathcal{M}$ implies that Figure 7.7(1) is transformable to Figure 7.7(2). By Figure 7.7(2) and Fact 7.8, $\langle b, v, w, c\rangle$ would be a separating quadruple of G (separating a from those occupying the shaded hole of $\mathcal{M}|_{\{a,b,c,d,v,w\}}$ in atlas $\mathcal{M}$), a contradiction.

Case 2.2: $N_G(v) \cap \{b, c\} = \{c\}$. If there is a $w \in V(G) - \{a, b, c, d, v\}$ with $\{d, v\} \subseteq N_G(w)$, then similarly to Case 2.1 (by swapping v and w), we can prove that $\langle b, w, v, c\rangle$ would be a separating quadruple of G, a contradiction. Otherwise, countries d and v strongly touch in $\mathcal{M}$, and both endpoints of the unique (d, v)-segment S in $\mathcal{M}$ are 3-nodes one of which is on the boundary of country a and the other is on the boundary of country c in $\mathcal{M}$; by this, S is a shrinkable segment in $\mathcal{M}$, d constitutes a connected component of $G - \{a, c\} - \mathcal{E}[a, c]$, and

$\{a, c\}$ would be a separating edge of G, a contradiction.

Therefore, the claim holds: $G'_d - \{d\}$ touches b.

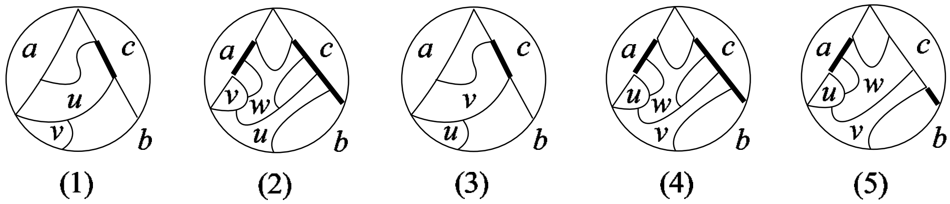

Figure 7.8: Possible displays of $G[\{a, b, c, u, v\}]$ or $G[\{a, b, c, u, v, w\}]$.

By the above claim, Claim 7.14, and the fact that $C \subseteq N_G(V(K))$, it follows that $\alpha_{a,b}$ or $\alpha_{b,c}$ is shared by G'_d and K in $\mathcal{M}$. By Claim 7.11, $\alpha_{b,c}$ cannot be shared by G'_d and K. So, $\alpha_{a,b}$ is shared by G'_d and K. Let u (respectively, v) be the country of G'_d (respectively, K) that has node $\alpha_{a,b}$ on its boundary. Then, countries u and v strongly touch in $\mathcal{M}$. One endpoint of the unique (u, v)-segment S in $\mathcal{M}$ is $\alpha_{a,b}$. Let β be the other endpoint of S. Neither country d nor c has node β on its boundary in $\mathcal{M}$; otherwise, $\{u, v, a, b, d\}$ or $\{u, v, a, b, c\}$ would be a 5-clique of G (and hence edge $\{u, v\}$ would remain in G'). By the well-formedness of $\mathcal{M}$, neither country a nor b has node β on its boundary in $\mathcal{M}$. So, there is a country $w \in V(G) - \{a, b, c, d, u, v\}$ that has node β on its boundary in $\mathcal{M}$. Moreover, by Claim 7.13(2) and the fact that no country in C has node β on its boundary in $\mathcal{M}$ (because $\mathcal{M}$ is well-formed and $\{v, c\} \notin E(G)$), it follows that β is a 3-node. Now, by Claim 7.13(1), $C \cap N_G(w) = \{a, c\}$ and either (i) $C \subseteq N_G(u)$ and $C \cap N_G(v) = \{a, b\}$ or (ii) $C \subseteq N_G(v)$ and $C \cap N_G(u) = \{a, b\}$. In case (i) holds, Figure 7.8(1) displays $\mathcal{M}|_{\{a,b,c,u,v\}}$ or Figure 7.8(2) displays $\mathcal{M}|_{\{a,b,c,u,v,w\}}$. However, Figure 7.8(2) contradicts Claim 7.14 (because u and d belong to G'_d while v and w belong to K), and Figure 7.8(1) gives no way for country w to touch all of countries v, a, c in $\mathcal{M}$ (even if we completely or partially contract the contractible path), a contradiction. So, (ii) holds and Figure 7.8(3) displays $\mathcal{M}|_{\{a,b,c,u,v\}}$ or Figure 7.8(4) displays $\mathcal{M}|_{\{a,b,c,u,v,w\}}$. However, Figure 7.8(3) contradicts Claim 7.14 (because d and u belong to G'_d while v belongs to K). So, only Figure 7.8(4) can possibly display $\mathcal{M}|_{\{a,b,c,u,v,w\}}$. There is no $f \in V(G) - \{a, b, c, u, v, w\}$ with $\{v, w\} \subseteq N_G(f)$; otherwise, by Claim 7.13(1), $C \cap N_G(f) = \{a, b\}$ which is impossible by Figure 7.8(4) (even if we completely or partially contract the two contractible paths). By this and the absence of holes in $\mathcal{M}$, Figure 7.8(4) is transformable to Figure 7.8(5). However, Figure 7.8(5) together with Fact 7.8 implies that $\langle b, u, w, c\rangle$ would be a separating quadruple of G (separating a from v), a contradiction. This completes the proof. ∎

Lemma 7.17 *Figure 7.4(3) does not display $G[C]$.*

Proof: Assume, on the contrary, that G has a well-formed atlas $\mathcal{M}$ such that Figure 7.4(3) displays $\mathcal{M}|_C$. Define nodes α, $\alpha_{a,b}$ and $\alpha_{a,c}$ as in the proof of Lemma 7.15. Let $\alpha_{b,c}$ be the endpoint of the (b, c)-segment other than α in $\mathcal{M}$. Let d be a country in $V(G) - \{a, b, c\}$ that has node $\alpha_{b,c}$ on its boundary in $\mathcal{M}$. Let G'_d be the connected component of G' containing d. Let K be a connected component of G' other than G'_d such that some country u of G'_d touches some country v of K in $\mathcal{M}$; K exists by Claim 7.12.

By Claims 7.11, 7.12(i) and 7.14, it follows that $\alpha_{a,b}$ or $\alpha_{a,c}$ is shared by G'_1 and G'_2 in $\mathcal{M}$. We assume that $\alpha_{a,b}$ is shared by G'_1 and G'_2 in $\mathcal{M}$; the other case is similar (by swapping b and c). Let u (respectively, v) be the country of G'_d (respectively, K) that has node $\alpha_{a,b}$

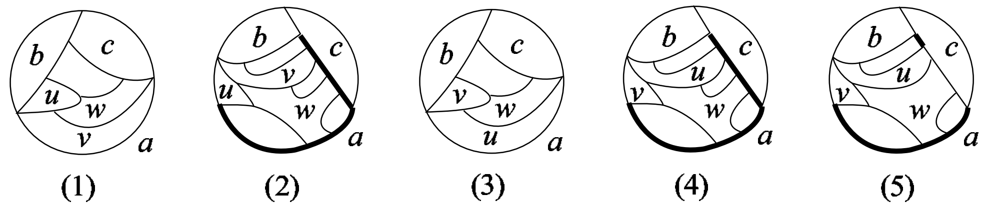

Figure 7.9: Possible displays of $G[\{a, b, c, u, v, w\}]$.

on its boundary in $\mathcal{M}$. Similarly to the proof of Lemma 7.15, we can prove that there is a country $w \in V(G) - \{a, b, c, u, v\}$ such that only Figures 7.9(1) through (4) can possibly display $\mathcal{M}|_{\{a,b,c,u,v,w\}}$. However, Figure 7.9(1) together with Fact 7.5 implies that $\{b, u, w, c\}$ would be a separating 4-cycle of G (separating v from d), a contradiction. Similarly, Figure 7.9(3) together with Fact 7.5 implies that $\langle b, v, w, c\rangle$ would be a separating 4-cycle of G (separating u from d), a contradiction. Also, Figure 7.9(2) contradicts Claim 7.14 (because d, u, w belong to G'_d while v belongs to K). So, only Figure 7.9(4) can possibly display $\mathcal{M}|_{\{a,b,c,u,v,w\}}$. Now, there is no $f \in V(G) - \{a, b, c, u, v, w\}$ with $\{u, w\} \subseteq N_G(f)$; otherwise, by Claim 7.13, $C \cap N_G(f) = \{a, b\}$ which is impossible by Figure 7.9(4) (even if we completely or partially contract the contractible path). By this, Figure 7.9(4) is transformable to Figure 7.9(5). By Figure 7.9(5) and Fact 7.8, $\langle b, v, w, c\rangle$ would be a separating quadruple of G (separating a from u), a contradiction. This completes the proof. ∎

By Lemmas 7.15, 7.16 and 7.17, only Figure 7.4(4) can display $G[C]$.

Lemma 7.18 *Suppose $\langle a,b,c\rangle$ is a strongly separating triangle of G. Let d be the vertex that constitutes a connected component of G'. Then, $C \subseteq N_G(d)$ and d has exactly two neighbors x,y in graph $G-C$. Moreover, either (i) $N_G(x)\cap\{b,c\}=\{b\}$ and $N_G(y)\cap\{b,c\}=\{c\}$, or (ii) $N_G(x)\cap\{b,c\}=\{c\}$ and $N_G(y)\cap\{b,c\}=\{b\}$. Furthermore, if (i) (respectively, (ii)) holds, then both $\langle a,d,b,x\rangle$ and $\langle a,d,c,y\rangle$ (respectively, both $\langle a,d,b,y\rangle$ and $\langle a,d,c,x\rangle$) are correct 4-pizzas in G.*

Proof: Figure 7.4(4) displays $\mathcal{M}|_C$. Let $\mathcal{H}_1$ be one hole of $\mathcal{M}|_C$, and $\mathcal{H}_2$ be the other. Let Z_1 (respectively, Z_2) be the set of countries in $V(G)-C$ that occupy hole $\mathcal{H}_1$ (respectively, $\mathcal{H}_2$) in atlas $\mathcal{M}$. Let $\alpha_{a,b}$ be the node at which countries a and b together with some country(s) of Z_1 meet in $\mathcal{M}$. Define nodes $\alpha_{a,c}$ and $\alpha_{b,c}$ similarly.

First, we observe that $C \subseteq N_G(V(K))$ for every connected component K of $G'[Z_1]$. If $V(K)=Z_1$, then this is clear from Figure 7.4(4). Otherwise $G'[Z_1]$ has some other component K' adjacent to K in $G[Z_1]$, and now our argument resembles that for Claim 7.12(i). That is, let $S=C\cap N_G(V(K))$. Since an edge between K and K' is absent in G', S contains either $\{a,b\}$ or $\{a,c\}$. Toward a contradiction, assume $S=\{a,b\}$; the $\{a,c\}$ case is similar (by swapping b and c). Then, in case K is also a connected component of G', it is clear that $\{a,b\}$ would be a separating edge in G (separating K from K'), a contradiction. In case K is not a connected component of G', Figure 7.4(4) ensures that there is exactly one edge $\{x_1,x_2\}\in E(G)$ with $x_1\in V(K)$ and $x_2\in Z_2$; moreover, the four countries a,x_1,b,x_2 must meet at node $\alpha_{a,b}$ in atlas $\mathcal{M}$ in this order (so, the (a,b)-segment in the layout in Figure 7.4(4) should be contracted to a single node). If $\{a,x_1,b,x_2\}$ is an MC_4 of G, then K would be a connected component of G', a contradiction. Otherwise, there is a 5-clique C' with $\{a,x_1,b,x_2\}\subseteq C'$. By Figure 7.4(4), the country $x_3\in C'-\{a,x_1,b,x_2\}$ cannot have node $\alpha_{a,b}$ on its boundary and hence has to touch country c in order to touch both x_1 and x_2 in $\mathcal{M}$. Moreover, since $\{x_1,c\}\notin E(G)$, x_3 must belong to Z_1 or else x_3 could not touch x_1 in $\mathcal{M}$. Therefore, $\{x_1,x_3\}$ remains an edge in $G'[Z_1]$. This implies that $\{x_1,x_3\}\subseteq V(K)$ and $C\subseteq N_G(V(K))$, contradicting the assumption that $S=\{a,b\}$. So, $S=C$.

Similarly, we have $C\subseteq N_G(V(K))$ for every connected component K of $G'[Z_2]$.

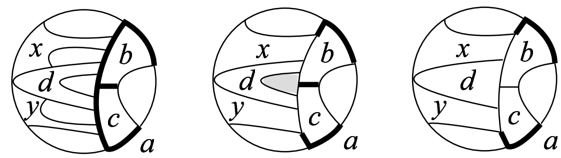

Figure 7.10: Possible displays of $G[\{a,b,c,d,x,y\}]$.

We assume that $d\in Z_1$; the other case is similar (by swapping Z_1 and Z_2). We want to prove that $Z_1=\{d\}$. Toward a contradiction, assume that $Z_1\neq\{d\}$. Then, since $\mathcal{M}$ has no hole, there is a connected component K of $G'[Z_1]$ with $V(K)\cap N_G(d)\neq\emptyset$. First, we claim that d and some country of K must meet at $\alpha_{a,b}$, $\alpha_{a,c}$, or $\alpha_{b,c}$. Assume, on the contrary, that the claim does not hold. Then, since $C\subseteq N_G(V(K))\cap N_G(d)$ by the above observation, there must exist a node β in $\mathcal{M}$ at which two countries x and y of K together with d and some $u\in C$ meet in the order x, d, y, u. Claim 7.13 ensures that either (i) $C\cap N_G(x)=\{a,b\}$ and $C\cap N_G(y)=\{a,c\}$ or (ii) $C\cap N_G(x)=\{a,c\}$ and $C\cap N_G(y)=\{a,b\}$. In either case, we have $u=a$. We assume that (i) holds; the other case is similar (by swapping b and c). Then,

Figure 7.10(1) displays $\mathcal{M}|_{\{a,b,c,d,x,y\}}$. There is no $u \in Z_1 - \{d, x, y\}$ with $\{x, d\} \subseteq N_G(u)$; otherwise, by Claim 7.13(1), $C \cap N_G(u) = \{a, c\}$ which is impossible by Figure 7.10(1) (even if we contract a set of vertex-disjoint paths of the contractible forest). This together with the well-formedness of $\mathcal{M}$ and the absence of edge $\{x, c\}$ in G implies that the endpoint of the unique (x, d)-segment other than β in $\mathcal{M}$ must be a 3-node on the boundary of country b. Similarly, there is no $u \in Z_1 - \{d, x, y\}$ with $\{y, d\} \subseteq N_G(u)$, and the endpoint of the unique (y, d)-segment other than β in $\mathcal{M}$ must be a 3-node on the boundary of country c. So, Figure 7.10(1) is transformable to Figure 7.10(2). By Figure 7.10(2), if there were countries in $\mathcal{M}$ occupying the shaded hole of the layout in Figure 7.10, then none of these countries could touch country a in $\mathcal{M}$ and hence they together with d would fall into the same connected component of $G'[Z_1]$, a contradiction. This together with Claim 7.11 implies that Figure 7.10(2) is transformable to Figure 7.10(3). However, Figure 7.10(3) together with Fact 7.8 implies that $\langle x, y, c, b\rangle$ would be a separating quadruple of G (separating d from a), a contradiction. So, the claim holds: d meets K at $\alpha_{a,b}$, $\alpha_{a,c}$, or $\alpha_{b,c}$.

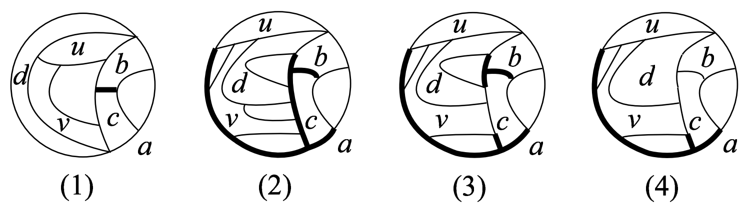

Figure 7.11: Possible displays of $G[\{a, b, c, d, u, v\}]$.

Next, we use the above claim to get a contradiction. By the above claim, d and a country u of K must meet at $\alpha_{a,b}$, $\alpha_{a,c}$, or $\alpha_{b,c}$ in $\mathcal{M}$. By Claim 7.11, d and u cannot meet at $\alpha_{b,c}$. So, they meet at $\alpha_{a,b}$ or $\alpha_{a,c}$. We assume that they meet at $\alpha_{a,b}$; the other case is similar (by swapping b and c). Then, countries d and u in $\mathcal{M}$ strongly touch in $\mathcal{M}$. One endpoint of the unique (d, u)-segment S in $\mathcal{M}$ is $\alpha_{a,b}$. Let β be the other endpoint of S. Since $\mathcal{M}$ is well-formed, neither country a nor b has node β on its boundary in $\mathcal{M}$. Moreover, country c cannot have node β on its boundary in $\mathcal{M}$; otherwise, $\{a, b, c, d, u\}$ would be a 5-clique of G. On the other hand, by the absence of holes in $\mathcal{M}$, it is impossible that only countries u and d meet at β. So, there is a country $v \in Z_1 - \{d, u\}$ such that countries u, d, v meet at β in $\mathcal{M}$; β must be a 3-node by Claim 7.13(2). Now, by Claim 7.13(1), $C \cap N_G(v) = \{a, c\}$. Thus, Figure 7.11(1) or (2) displays $\mathcal{M}|_{\{a,b,c,d,u,v\}}$. Actually, Figure 7.11(1) does not display $\mathcal{M}|_{\{a,b,c,d,u,v\}}$ or else Fact 7.6 would imply that $\langle b, c, a\rangle$ is a separating triple of G (separating countries in Z_1 from countries in Z_2), a contradiction. So, only Figure 7.11(2) can possibly display $\mathcal{M}|_{\{a,b,c,d,u,v\}}$. There is no $w \in Z_1 - \{d, u, v\}$ with $\{d, v\} \subseteq N_G(w)$; otherwise, by Claim 7.13(1), $C \cap N_G(w) = \{a, b\}$ which is impossible by Figure 7.11(2) (even if we contract a set of vertex-disjoint paths of the contractible forest). By this, Figure 7.11(2) is transformable to Figure 7.11(3). By Claim 7.11 and the fact that $\langle b, c, d\rangle$ is not a separating triple of G, each pair of countries in $\{b, c, d\}$ must strongly touch in $\mathcal{M}$ (cf. Fact 7.6). So, Figure 7.11(3) is further transformable to Figure 7.11(4). Figure 7.11(4) and the absence of edge $\{u, c\}$ in G together with Fact 7.8 implies that $\langle u, b, c, v\rangle$ would be a separating quadruple of G (separating d from the rest), a contradiction. This completes the proof that $Z_1 = \{d\}$.

Now, $Z_1 = \{d\}$. Thus, by Claim 7.11 and Assumption 2 (G has no separating triple), Figure 7.4(5) displays $\mathcal{M}|_C$. By the figure, d and some country $x \in Z_2$ meet at the (a, b)-node in atlas $\mathcal{M}$; d and some country $y \in Z_2$ meet at the (a,c)-node in atlas $\mathcal{M}$. Since d constitutes a connected component of G', $x \neq y$, $N_G(x) \cap \{b,c\} = \{b\}$, and $N_G(y) \cap \{b,c\} = \{c\}$. By Figure 7.4(5), only x and y can be the neighbors of d in graph $G - C$, and both $\langle a,d,b,x\rangle$ and $\langle a,d,c,y\rangle$ are correct 4-pizzas in G. This completes the proof of Lemma 7.18. ∎

Lemma 7.19 *Suppose there is no strongly separating triangle of G. Then, G' has exactly two connected components G_1 and G_2, and exactly two edges $\{u,v\},\{x,y\} \in E(G)$ connect G_1 to G_2 in graph $G - C$. Moreover, either (i) both $\{a,b,u,v\}$ and $\{a,c,x,y\}$ are MC$_4$'s of G, or (ii) both $\{a,b,x,y\}$ and $\{a,c,u,v\}$ are MC$_4$'s of G. Furthermore, if (i) (respectively, (ii)) holds, then both $\langle a,u,b,v\rangle$ and $\langle a,x,c,y\rangle$ (respectively, both $\langle a,x,b,y\rangle$ and $\langle a,u,c,v\rangle$) are correct 4-pizzas in G.*

Proof: Define sets Z_1 and Z_2 and points $\alpha_{a,b}$, $\alpha_{a,c}$, and $\alpha_{b,c}$ as in Lemma 7.18. As in the proof of Lemma 7.18, we observe that $C \subseteq N_G(V(K))$ for every connected component K of $G'[Z_1]$ or $G'[Z_2]$.

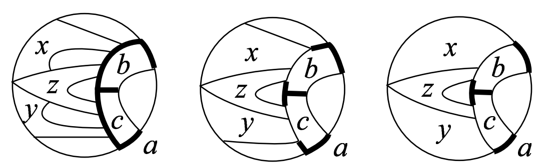

Figure 7.12: Possible displays of $G[\{a,b,c,x,y,z\}]$.

We claim that for every connected component K of $G'[Z_1]$, there is no node β in $\mathcal{M}$ at which two countries x and y of K together with two countries w and z of $(C \cup Z_1) - V(K)$ meet in the order x, w, y, z. Assume, on the contrary, that such β exists in $\mathcal{M}$. Then, by Claim 7.13(2) with $Z = Z_1$, we have $C \cap \{w,z\} \neq \emptyset$. By Figure 7.4(4), $\beta \notin \{\alpha_{a,b}, \alpha_{a,c}, \alpha_{b,c}\}$ and hence $|C \cap \{w,z\}| \leq 1$. So, $|C \cap \{w,z\}| = 1$. Thus, $C \cap \{w,z\} = \{a\}$; otherwise, by Claim 7.11, $\{x,y,a,w,z\}$ would be a 5-clique of G, a contradiction. We assume that $w = a$; the other case is similar (by replacing z with w). Now, by Claim 7.13(1), $\{C \cap N_G(x), C \cap N_G(y)\} = \{\{a,b\},\{a,c\}\}$ and $C \subseteq N_G(z)$. We assume that $C \cap N_G(x) = \{a,b\}$ and $C \cap N_G(y) = \{a,c\}$; the other case is similar (by swapping x and y). In summary, Figure 7.12(1) displays $G[\{a,b,c,x,y,z\}]$. There is no $f \in Z_1 - \{x,y,z\}$ with $\{x,z\} \subseteq N_G(f)$; otherwise, by Claim 7.13(1), $C \cap N_G(f) = \{a,c\}$ which is impossible by Figure 7.12(1) (even if we contract a set of vertex-disjoint paths of the contractible forest). This together with the well-formedness of $\mathcal{M}$ and the absence of edge $\{x,c\}$ in G implies that the endpoint of the unique (x,z)-segment other than β in $\mathcal{M}$ is a 3-node on the boundary of country b in $\mathcal{M}$. Similarly, the endpoint of the unique (y,z)-segment other than β in $\mathcal{M}$ is a 3-node on the boundary of country c in $\mathcal{M}$. So, Figure 7.12(1) is transformable to Figure 7.12(2). Figure 7.12(2) is further transformable to Figure 7.12(3), because (i) $\langle a,b,x\rangle$ and $\langle a,c,y\rangle$ are not separating triples of G and (ii) both $\{a,b,x,z\}$ and $\{a,c,y,z\}$ are MC$_4$'s of G. By Figure 7.12(3) and the fact that both $\{a,b,x,z\}$ and $\{a,c,y,z\}$ are MC$_4$'s of G, $\langle a,b,z\rangle$ would be a strongly separating triangle of G (separating x from the rest), a contradiction. So, the claim holds.

Next, we claim that $G'[Z_1]$ is connected. Assume, on the contrary, that $G'[Z_1]$ is disconnected. Then, since $\mathcal{M}$ has no hole, there are two distinct connected components K and K'

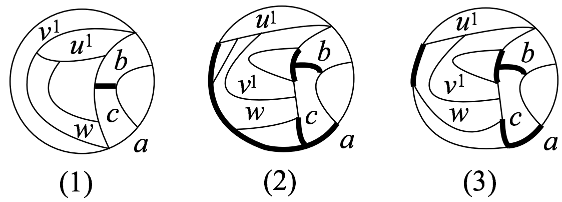

Figure 7.13: Possible displays of $G[\{a, b, c, u, v, w\}]$.

of $G'[Z_1]$ such that $V(K) \cap N_G(V(K')) \neq \emptyset$. Since $C \subseteq N_G(V(K))$ and $C \subseteq N_G(V(K'))$, some country u of K and some country v of K' have to meet at $\alpha_{a,b}$, $\alpha_{a,c}$ or $\alpha_{b,c}$ in $\mathcal{M}$, by the claim of the previous paragraph and Figure 7.4(4). By Claim 7.11, u and v cannot meet at $\alpha_{b,c}$ in $\mathcal{M}$. We assume that u and v meet at $\alpha_{a,b}$ in $\mathcal{M}$; the other case is similar (by swapping b and c). Similarly to the proof of Lemma 7.18 (by replacing d there with v^1), we can prove that there is a country $w \in Z_1 - \{u, v\}$ such that only Figure 7.13(1) or (2) can possibly display $\mathcal{M}|_{\{a,b,c,u,v,w\}}$. Actually, Figure 7.13(1) does not display it or else $\langle a, u^1, w\rangle$ would be a strongly separating triangle of G (separating v^1 from the rest). So, only Figure 7.13(2) can possibly display it. Since $\langle a, w, u^1\rangle$ is not a separating triple of G, Fact 7.6 implies that Figure 7.13(2) is transformable to Figure 7.13(3). By Figure 7.13(3), $\langle a, w, v^1\rangle$ would be a strongly separating triangle of G (separating u^1 from the rest), a contradiction. So, the claim holds. Similarly, we can prove that $G'[Z_2]$ is connected.

Since both $G'[Z_1]$ and $G'[Z_2]$ are connected, both have to be connected components of G' (or else G' would be connected), and G' has no other connected component. So, by Claim 7.11, the figure obtained from Figure 7.4(4) by contracting the bold (b, c)-segment to a single node does not display $\mathcal{M}|_C$. Thus, the bold (a, b)-segment in Figure 7.4(4) should be contracted to a single node; otherwise, $\langle a, c, b\rangle$ would be a separating triple of G (separating countries of Z_1 from countries of Z_2), by Fact 7.6. Similarly, the bold (a, c)-segment in Figure 7.4(4) should be contracted to a single node. Hence, Figure 7.4(5) displays $\mathcal{M}|_C$. By the figure, a unique country $u \in Z_1$ and a unique country $v \in Z_2$ meet at the (a, b)-node in atlas $\mathcal{M}$; and a unique country $x \in Z_1$ and a unique country $y \in Z_2$ meet at the (a, c)-node in atlas $\mathcal{M}$. Since both $G'[Z_1]$ and $G'[Z_2]$ are connected components of G', both $\{a, b, u, v\}$ and $\{a, c, x, y\}$ are MC_4's of G. Moreover, by Figure 7.4(5), both $\langle a, u, b, v\rangle$ and $\langle a, x, c, y\rangle$ are correct 4-pizzas in G, and only $\{u, v\}$ and $\{x, y\}$ can be the edges connecting $G'[Z_1]$ to $G'[Z_2]$ in graph $G - C$. ▮

By the above reductions, our algorithm may make progress whenever G has a separating edge, quadruple, or triangle. Hereafter we assume that all such reductions have been made:

Assumption 3 *G does not have a separating edge, quadruple, or triangle.*

In fact Assumption 3 implies the 4-connectivity of G (by Lemma 3.5(1)) and Assumption 2.

8 Removing Maximal 5-Cliques

We assume that G has an MC_5; our goal of this section is to show how to remove MC_5's from G. The idea behind the removal of an MC_5 C from G is to try to find and remove a correct center P of C. By Fact 5.4, we make progress after removing P. After removing P, the resulting G may no longer satisfy Assumption 3; in that case, the algorithm must therefore reapply the reductions of the previous sections before considering another MC_5. Also, not unexpectedly, our search for a correct center of C may fail. In this case, we will be able to decompose G into smaller graphs to make progress.

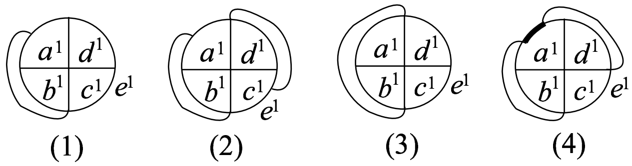

Figure 8.1: Possible displays of MC_5 $\{a, b, c, d, e\}$.

Throughout this section, let $C = \{a, \ldots, e\}$ be an MC_5 of G. We argue that one of Figures 8.1(1) through (4) must display $\mathcal{M}|_C$ as follows. First, C is a pizza-with-crust in $\mathcal{M}$. Suppose the four non-crust countries a^1, b^1, c^1, d^1 meet at a 4-node α in $\mathcal{M}$ in this order. Let $\beta_{a,b}$ be the endpoint of the (a^1, b^1)-segment other than α in $\mathcal{M}$. Define $\beta_{b,c}$, $\beta_{c,d}$, and $\beta_{d,a}$ similarly. Let k be the number of nodes among $\beta_{a,b}$, $\beta_{b,c}$, $\beta_{c,d}$, $\beta_{d,a}$ that are shared by the crust e^1 of C and another country of $V(G) - C$ in $\mathcal{M}$. Since $\mathcal{M}$ is well-formed, $k \leq 2$. On the other hand, since $C \neq V(G)$ and G has no separating triangle, we have $k \geq 1$ (otherwise, by Fact 7.10, at least one of $\langle e^1, a^1, b^1 \rangle$, $\langle e^1, a^1, d^1 \rangle$, $\langle e^1, d^1, c^1 \rangle$, and $\langle e^1, c^1, b^1 \rangle$ would be a separating triangle of G). If $k = 1$, then Fact 7.10 implies that Figure 8.1(1) displays $\mathcal{M}|_C$. If $k = 2$, then Fact 7.10 implies that Figure 8.1(2), (3) or (4) displays $\mathcal{M}|_C$.

For a positive integer k, two maximal cliques C' and C'' are *k-sharing* if $|C' \cap C''| = k$.

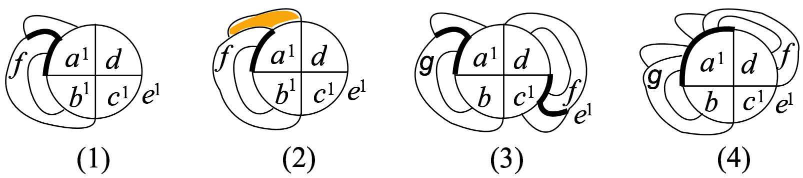

Figure 8.2: Possible displays of 4-sharing MC_5's.

C is 4-sharing with at most two other MC_5's C' of G (and so is every MC_5 of G); this is because the center of C' must be a 3-node bordering a hole in $\mathcal{M}|_C$, and there are at most two such nodes in the possible displays of Figure 8.1. We claim that at least one MC_5 of G is 4-sharing with two other MC_5's of G. Toward a contradiction, assume that the claim does not hold. When C is 4-sharing with no MC_5 of G, none of Figures 8.1(1) through (4) displays $\mathcal{M}|_C$ or else either $V(G)$ would equal C or at least one of $\langle e^1, a^1, b^1 \rangle$, $\langle e^1, c^1, d^1 \rangle$, and $\langle e^1, a^1, d^1 \rangle$ would be a separating triangle of G, a contradiction. So, consider the case where C is 4-sharing with exactly one MC_5, say $C_1 = \{a^1, b^1, c^1, e^1, f\}$, of G. In this case, by Assumption 3 (G has no separating triangle), Figures 8.1(2) and (4) are transformable to Figure 8.1(1). By Figures 8.1(1) and (3), only Figure 8.2(1) or (2) can possibly display $\mathcal{M}|_{\{a,\ldots,f\}}$. Actually, Figure 8.2(2) does not display $\mathcal{M}|_{\{a,\ldots,f\}}$; otherwise, since C_1 is 4-sharing with no MC_5 of G other than C, there is no $g \in V(G) - \{a, \ldots, f\}$ with $\{a^1, b^1, e^1, f\} \subseteq N_G(g)$ and Fact 7.10 implies that $\langle a^1, f, e^1 \rangle$ would be a separating triangle of G (separating d from those occupying the shaded hole of $\mathcal{M}|_{\{a,\ldots,f\}}$ in atlas $\mathcal{M}$), a contradiction. Similarly, Figure 8.2(1) does not display $\mathcal{M}|_{\{a,\ldots,f\}}$; otherwise, since $|V(G)| \geq 9$, Fact 7.6 implies that $\langle a^1, f, b^1 \rangle$ or $\langle a^1, f, e^1 \rangle$ would be a separating triple of G, a contradiction. Therefore, the claim holds.

By the above claim, if G has an MC_5, then it has an MC_5 that is 4-sharing with two other

MC$_5$'s of G. By our assumption, C is an arbitrary MC$_5$ of G and hence we can assume that C is 4-sharing with two other MC$_5$'s, say $C_1 = \{a, c, d, e, f\}$ and $C_2 = \{a, b, c, e, g\}$, of G. Let $U = C \cup \{f, g\}$. We show how to find a correct center of C below. First, we observe the following simple but useful fact (which is clear from Figures 8.1(1) through (4)).

Fact 8.1 *Let W be a subset of an MC$_5$ C' of G with $|W| \geq 3$. If all the edges of $G[W]$ are marked in G or $G - C'$ has a vertex x with $W = C' \cap N_G(x)$, then W contains all correct crusts of C'. In particular, if C' and C'' are MC$_5$'s with $|C' \cap C''| \geq 3$, then both their crusts are in the intersection.*

$\{f, g\}$ is not an edge in G; otherwise, only Figure 8.1(3) or (4) can possibly display $\mathcal{M}|_C$, but after drawing countries f and g in the two figures, we see that the 4-connectedness of G would force $V(G)$ to equal U, contradicting the assumption that $|V(G)| \geq 9$. So, only Figure 8.2(3) or (4) can possibly display $\mathcal{M}|_U$. By the figures, a correct center of C can be found from a correct crust immediately. So, it suffices to find out which one of a, c, and e is a correct crust of C.

Let k be the number of vertices $v \in \{a, c, e\}$ such that $N_G(v) \subseteq U$. We have $k \leq 1$; otherwise, no matter which of Figures 8.2(3) and (4) displays $\mathcal{M}|_U$, the 4-connectedness of G would force $V(G)$ to equal U, contradicting the assumption that $|V(G)| \geq 9$. First, consider the case where $k = 0$. In this case, only Figure 8.2(3) displays $\mathcal{M}|_U$. Moreover, by this figure, there is a (unique) country $h \in V(G) - U$ with $\{a^1, b, e^1, g\} \subseteq N_G(h)$ or else Fact 7.6 would imply that $\langle a^1, g, b\rangle$ or $\langle a^1, g, e^1\rangle$ is a separating triple of G, a contradiction. Similarly, there is a unique country $i \in V(G) - U$ with $\{c^1, d, e^1, f\} \subseteq N_G(i)$. So by Fact 8.1, the unique country in $N_G(h) \cap N_G(i)$ is a correct crust of C.

Now, we may assume that $k = 1$. We may further assume that c is the unique $u \in \{a, c, e\}$ such that $N_G(u) \subseteq U$; the other cases are similar (by swapping and relabeling). For each of Figures 8.2(3) and (4), we want to figure out which of countries a^1, c^1, e^1 in the figure can actually be c. If Figure 8.2(4) displays $\mathcal{M}|_U$, then neither a^1 nor e^1 can be c or else the 4-connectedness of G would force $V(G)$ to be U, a contradiction. So, in Figure 8.2(4), $c^1 = c$. Similarly, if Figure 8.2(3) displays $\mathcal{M}|_U$, e^1 cannot be c or else both $\{a^1, b, g\}$ and $\{c^1, d, f\}$ would be 3-cuts of G, a contradiction. So, if Figure 8.2(3) displays $\mathcal{M}|_U$, either $a^1 = c$ in Figure 8.2(3) (and hence $N_G(\{b, c, g\}) \subseteq U$), or $c^1 = c$ in Figure 8.2(3) (and hence $N_G(\{c, d, f\}) \subseteq U$). No matter which of Figures 8.2(3) and (4) displays $\mathcal{M}|_U$, if there is a $u \in \{a, e\}$ such that $\{u, d\}$ or $\{u, b\}$ is a marked edge in G, then the unique country in $\{a, e\} - \{u\}$ is a correct crust of C. So, we may assume that none of $\{a, d\}$, $\{e, d\}$, $\{a, b\}$, and $\{e, b\}$ is a marked edge in G. It remains to consider three cases as follows.

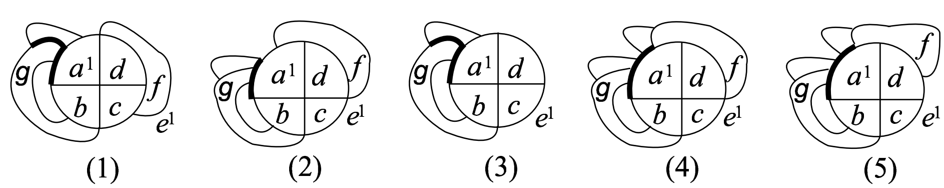

Figure 8.3: (1) A possible display of $G[U]$ in Case 1. (2) Another possible display of $G[U]$ in Case 1. (3) A display of $G'[\{a, \ldots, e, g\}]$ in Case 1.1. (4) A display of $G[U]$ in Case 3.1. (5) A display of $G'[U]$ in Case 3.1.

Case 1: $N_G(\{c,d,f\}) \subseteq U$. Then, Figures 8.2(3) and (4) are transformable to Figures 8.3(1) and (2), respectively.

Case 1.1: Edge $\{c,f\}$ is not marked in G. Then, Figure 8.3(1) is transformable to Figure 8.3(2), and hence Figure 8.3(2) displays $\mathcal{M}|_U$. Let G' be the marked graph obtained from $G - \{f\}$ by marking the following edges: $\{b,c\}$, $\{c,d\}$, $\{a,e\}$, $\{a,d\}$, $\{e,d\}$. By Figure 8.3(2), we can obtain a well-formed atlas $\mathcal{M}'$ of G' from $\mathcal{M}$ by extending country e^1 to completely occupy country f. Figure 8.3(3) displays $\mathcal{M}'|_{\{a,\ldots,e,g\}}$. On the other hand, we claim that every well-formed atlas $\mathcal{M}''$ of G' can be used to construct a well-formed atlas of G. To see this, first note that by Fact 8.1, the crust of C in $\mathcal{M}''$ must be either a or e. Suppose the crust is e; the other case is similar (by swapping a and e). Then, since edges $\{b,c\}$ and $\{c,d\}$ are marked in G', the center of C in $\mathcal{M}''$ must be $\langle a,b,c,d\rangle$. Moreover, since $N_{G'}(\{d\}) \subseteq C$, the four countries a, c, d, and e must be related in $\mathcal{M}''$ as shown in Figure 8.3(3). Thus, we can assign a suitable sub-region of e to f to obtain an atlas of G. This establishes the claim.

Case 1.2: Edge $\{c,f\}$ is marked in G. Then, only Figure 8.3(1) displays $\mathcal{M}|_U$. By the figure, at most one of edges $\{a,f\}$ and $\{e,f\}$ is marked in G. Moreover, if $\{a,f\}$ is marked in G, then a is a correct crust of C. Similarly, if $\{e,f\}$ is marked in G, then e is a correct crust of C. So, it remains to consider the case where neither $\{a,f\}$ nor $\{e,f\}$ is a marked edge in G. In this case, it suffices to construct a marked graph G' as in Case 1.1.

Case 2: $N_G(\{b,c,g\}) \subseteq U$. Similar to Case 1, after relabeling.

Case 3: Neither $N_G(\{b,c,g\}) \subseteq U$ nor $N_G(\{c,d,f\}) \subseteq U$. Then as argued above, Figure 8.2(4) displays $G[U]$. We consider three sub-cases as follows:

Case 3.1: There is no $v \in V(G) - U$ such that $d \in N_G(v)$ and $N_G(v) \cap \{a,e\} \neq \emptyset$. Then, Figure 8.3(4) displays $\mathcal{M}|_U$ by the 4-connectedness of G. By the figure, $N_G(d) = C \cup \{f\}$. Let G' be the marked graph obtained from $G - \{\{c,f\}\}$ by marking the following edges: $\{b,c\}$, $\{c,d\}$, $\{a,d\}$, $\{e,d\}$, $\{a,f\}$, $\{e,f\}$, $\{d,f\}$. By Figure 8.3(4), we can obtain a well-formed atlas $\mathcal{M}'$ of G' by erasing the (c,f)-node in $\mathcal{M}$. Figure 8.3(5) displays $\mathcal{M}'|_{\{a,\ldots,g\}}$. By Figure 8.3(5) and Lemma 3.5, both $G' - \{a,d,f\}$ and $G' - \{e,d,f\}$ are connected. We claim that every well-formed atlas $\mathcal{M}''$ of G' can be used to construct a well-formed atlas of G. To see this, first note that by Fact 8.1, the crust of C in $\mathcal{M}''$ must be either a or e. We assume that the crust is e; the other case is similar (by swapping e and a). Then, since $\{b,c\}$ and $\{c,d\}$ are marked edges in G', the center of C in $\mathcal{M}''$ must be $\langle a,b,c,d\rangle$. Moreover, since $G' - \{a,d,f\}$ is connected, the marked edges $\{a,d\}$, $\{d,f\}$ and $\{f,a\}$ of G' force countries a, d and f to meet at a 3-node in $\mathcal{M}''$. For a similar reason, countries e, d and f meet at a 3-node in $\mathcal{M}''$. Now, since $N_{G'}(d) = C \cup \{f\}$, the four countries c, d, e, and f must be related in $\mathcal{M}''$ as shown in Figure 8.3(5). Thus, to obtain a well-formed atlas of G, it suffices to modify $\mathcal{M}''$ by contracting the (e,d)-segment to a single node.

Case 3.2: No $v \in V(G) - U$ satisfies $b \in N_G(v)$ and $N_G(v) \cap \{a,e\} \neq \emptyset$. Similar to Case 3.1.

Case 3.3: There are countries h and i in $V(G) - U$ such that $d \in N_G(h)$, $N_G(h) \cap \{a,e\} \neq \emptyset$, $b \in N_G(i)$, and $N_G(i) \cap \{a,e\} \neq \emptyset$. By Figure 8.2(4), no country of $V(G) - U$ can touch both b and d in $\mathcal{M}$. So, h and i are distinct countries. Moreover, if $|N_G(h) \cap \{a,e\}| = 1$ (respectively, $|N_G(i) \cap \{a,e\}| = 1$), then the unique country in $\{a,e\} - N_G(h)$ (respectively, $\{a,e\} - N_G(i)$) must be a correct crust and we are done. So, we assume that $\{a,e\} \subseteq N_G(h)$ and $\{a,e\} \subseteq N_G(i)$. Then, by Figure 8.2(4), $\{a,d,e,f,h\}$ and $\{a,b,e,g,i\}$ are MC_5's in G. Let $U_h = U \cup \{h\}$. If $\{g,h\}$ were an edge in G, then by Figure 8.2(4), after drawing country h in $\mathcal{M}|_U$, we see that the 4-connectedness of G would force $V(G)$ to equal U_h, contradicting the assumption that $|V(G)| \geq 9$. So, $\{g, h\} \notin E(G)$. Similarly, $\{f, i\} \notin E(G)$. Then, Figure 8.4(1) or (5) displays $\mathcal{M}|_{U_h}$. If edge $\{d, h\}$ is marked in G or $N_G(d) - U_h \neq \emptyset$, Figure 8.4(5) displays $\mathcal{M}|_{U_h}$; otherwise, Figure 8.4(5) is transformable to Figure 8.4(1). So, we can decide which of Figures 8.4(1) and (5) displays $\mathcal{M}|_{U_h}$.

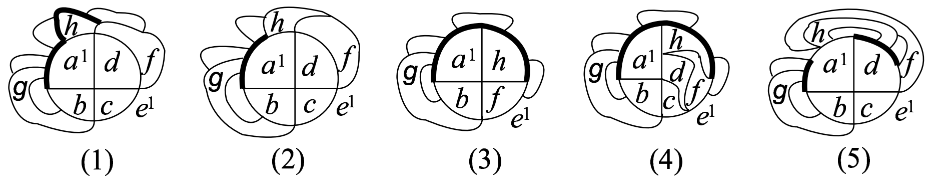

Figure 8.4: (1) A display of $\mathcal{M}|_{U_h}$ in Case 3.3.1. (2) A display of $\mathcal{M}|_{U_h}$ in Case 3.3.1.1. (3) A display of $G'[\{a, b, e, \ldots, h\}]$ in Case 3.3.1.2. (4) Splitting countries f and h in Figure 8.4(3) into four countries. (5) A display of $\mathcal{M}|_{U_h}$ in Case 3.3.2.

Case 3.3.1: Figure 8.4(1) displays $\mathcal{M}|_{U_h}$. We further distinguish two cases as follows.

Case 3.3.1.1: There is no $v \in V(G) - U_h$ *such that* $f \in N_G(v)$ *and* $\{a, e\} \cap N_G(v) \neq \emptyset$. Then, Figure 8.4(2) displays $\mathcal{M}|_{U_h}$ and so $N_G(f) \subseteq U_h$. By the figure, if there is a $w \in \{a, e\}$ such that edge $\{w, f\}$ is marked in G, then w is a correct crust of C. So, we may assume that neither edge $\{a, f\}$ nor $\{e, f\}$ is marked in G. Let G' be the marked graph obtained from $G - \{f\}$ by marking the following edges: $\{b, c\}$, $\{c, d\}$, $\{a, d\}$, $\{e, d\}$, $\{a, h\}$, $\{e, h\}$, $\{d, h\}$. By Figure 8.4(2), we can obtain a well-formed atlas $\mathcal{M}'$ of G' from $\mathcal{M}$ by erasing the (c, f)-node and further extending country h to completely occupy f. Indeed, by renaming country f in Figure 8.3(5) as h, we obtain a figure displaying $\mathcal{M}'|_{\{a,\ldots,e,g,h\}}$. Moreover, similarly to Case 3.1, we can prove that every well-formed atlas of G' can be used to construct one of G.

Case 3.3.1.2: There is a $j \in V(G) - U_h$ *such that* $f \in N_G(j)$ *and* $\{a, e\} \cap N_G(j) \neq \emptyset$. If $\{a, e\} \not\subseteq N_G(j)$, then by Figure 8.4(1), the unique country in $\{a, e\} \cap N_G(j)$ is a correct crust of C and we are done. So, we assume that $\{a, e\} \subseteq N_G(j)$. Then, by Figure 8.4(1), $h \in N_G(j)$. Recall that $\{f, i\} \notin E(G)$. So, $j \neq i$. By Figure 8.4(1), if there is a $w \in \{a, e\}$ such that $\{w, c\}$ is a marked edge in G, then w is a correct crust of C. So, we may assume that neither $\{a, c\}$ nor $\{e, c\}$ is a marked edge in G. Let G' be the graph obtained from $G - \{c, d\}$ by adding the three edges $\{g, f\}$, $\{b, f\}$, and $\{h, b\}$ and further marking the two edges $\{b, f\}$ and $\{f, h\}$. By Figure 8.4(1), we can obtain a well-formed atlas $\mathcal{M}'$ of G' from $\mathcal{M}$ by (i) erasing the (d, e^1)-node, (ii) erasing the (a^1, f)-node, (iii) extending country f to completely occupy country c, and (iv) extending country h to completely occupy country d. Indeed, Figure 8.4(3) displays $\mathcal{M}'|_{\{a,e,b,f,g,h\}}$. We claim that every well-formed atlas $\mathcal{M}''$ of G' can be used to construct a well-formed atlas of G. To see this, first note that G' contains the MC_5's $C' = \{a, e, b, f, h\}$, $C'_1 = \{a, e, b, f, g\}$, $C'_2 = \{a, e, f, h, j\}$, and $C'_3 = \{a, e, b, g, i\}$. By these MC_5's and Fact 8.1, the crust of C' in $\mathcal{M}''$ must be a or e. Moreover, the marked edges $\{b, f\}$ and $\{f, h\}$ together ensure that countries b and h do not appear consecutively around the center of C' in $\mathcal{M}''$. We assume that the crust of C' in $\mathcal{M}''$ is e; the other case is similar (by swapping e and a). Then, the center of C' in $\mathcal{M}''$ is $\langle a, b, f, h \rangle$. Because of this, countries a, b, f, g cannot meet at a 4-node in $\mathcal{M}''$ and hence the crust of C'_1 in $\mathcal{M}''$ cannot be e. On the other hand, by Fact 8.1 and the existence of MC_5's C', C'_1, C'_2, C'_3 in G', the crust of C'_1 in $\mathcal{M}''$ must be a or e. Thus, the crust of C'_1 in $\mathcal{M}''$ is a. Therefore, the centers of C' and C'_1 are as shown in Figure 8.4(3). From this, the claim follows immediately (see Figure 8.4(4)).

Case 3.3.2: Figure 8.4(5) displays $\mathcal{M}|_{U_h}$. In this case, we check if there is a $v \in V(G) - U_h$ such that $d \in N_G(v)$ and $N_G(v) \cap \{a, e\} \neq \emptyset$. If such v exists, then by Figure 8.4(5), $|N_G(v) \cap \{a, e\}| = 1$ and the unique country in $\{a, e\} - N_G(v)$ is a correct crust of C. If no such v exists, then by Figure 8.4(5) and the 4-connectedness of G, we have $N_G(\{d, f, h\}) \subseteq U_h$ and so Figure 8.4(5) is transformable to a figure $\mathcal{D}$, where $\mathcal{D}$ is obtained from Figure 8.4(5) by extending country h to completely occupy the two holes touched by h. By figure $\mathcal{D}$, if there is a $w \in \{a, e\}$ such that edge $\{w, f\}$ is marked in G, then w is a correct crust of C. Similarly, if there is a $w \in \{a, e\}$ such that edge $\{w, h\}$ is marked in G, then the unique country in $\{a, e\} - \{w\}$ is a correct crust of C. So, we may assume that none of the edges $\{a, f\}$, $\{e, f\}$, $\{a, h\}$ and $\{e, h\}$ are marked in G. Let G' be the marked graph obtained from $G - \{f, h\}$ by marking the following edges: $\{b, c\}$, $\{c, d\}$, $\{a, e\}$, $\{a, d\}$, $\{e, d\}$. By figure $\mathcal{D}$, we can obtain a well-formed atlas $\mathcal{M}'$ of G' from $\mathcal{M}$ by extending country e^1 to completely occupy countries f and h. On the other hand, as in Case 1.1, we can prove that every well-formed atlas of G' can be used to construct a well-formed atlas of G.

9 Removing Maximal 4-Cliques

Throughout this section, we assume that G has no MC_5. We further assume that G has an MC_4; our goal of this section is to show how to remove MC_4's from G. The idea behind the removal of an MC_4 C from G is to try to find and remove a correct 4-pizza via constructing an extensible layout of C. After the removal of a correct 4-pizza, the resulting G may be not 4-connected and may have a separating 4-cycle, edge, triple, quadruple, or triangle. To restore Assumption 3, the algorithm reapplies the reductions in Sections 3 and 7 to the resulting G.

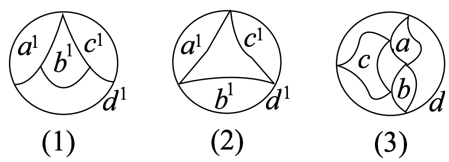

Figure 9.1: Possible displays of MC_4 $\{a, b, c, d\}$.

Suppose $C = \{a, b, c, d\}$ is an MC_4 of G; using Fact 7.10 and the assumption $|V(G)| > 8$, we find that only Figure 9.1(1), (2) or (3) can possibly display $\mathcal{M}|_C$. Note that these are a pizza, a pizza-with-crust, and a rice-ball, respectively. Obviously, if G has a marked edge between two vertices of C, then Figure 9.1(3) does not display $\mathcal{M}|_C$ (i.e., C has no extensible rice-ball layout).

9.1 Finding Rice-Balls

Let $C = \{a, b, c, d\}$ be an MC_4 of G such that no two vertices of C are connected by a marked edge in G. We want to decide whether C has an extensible rice-ball layout (i.e., whether Figure 9.1(3) displays $\mathcal{M}|_C$). For a subset W of C, let $\mathcal{E}[W]$ be the set of unmarked edges $\{u, v\} \in E(G)$ such that $u \notin W$, $v \notin W$, and some MC_4 of G consists of u, v, and two vertices in W. Note that when W consists of only two countries x and y, it holds that $\mathcal{E}[W] = \mathcal{E}[x, y]$ (cf. Definition 5.5).

Let $G' = G - C - \mathcal{E}[C]$. A *3-subset* of C is a subset S of C with $|S| = 3$. For each 3-subset S of C, let $V_S = \cup_K V(K)$, where K ranges over all connected components K of G' with

$C \cap N_G(V(K)) = S$.

Lemma 9.1 *Figure 9.1(3) displays $\mathcal{M}|_C$ iff the following statements hold:*

1. *$V_{\{a,b,c\}}$, $V_{\{a,b,d\}}$, $V_{\{a,c,d\}}$, and $V_{\{b,c,d\}}$ each are nonempty, and they together form a partition of $V(G) - C$.*
2. *For every two distinct 3-subsets S and T of C, $V_S \cap N_G(V_T)$ consists of a unique country y, $V_T \cap N_G(V_S)$ consists of a unique country z, and $\langle y, x_1, z, x_2 \rangle$ is a correct 4-pizza in G, where $S \cap T = \{x_1, x_2\}$.*
3. *For every 3-subset S of C, the following hold:*
   - *(a) $G - V_S$ is connected.*
   - *(b) $G[V_S]$ is connected.*
   - *(c) $G'[V_S]$ is a collection of connected components of G'.*

Proof: For the "only if" direction, suppose that Figure 9.1(3) displays $\mathcal{M}|_C$. Then, $\mathcal{M}|_C$ has four holes, and each hole is touched by exactly three countries of C. For each 3-subset S of C, let $\mathcal{H}_S$ be the hole touched by the countries of S, and let Z_S be the countries of $V(G) - C$ that occupy $\mathcal{H}_S$ in atlas $\mathcal{M}$. We want to prove that for each 3-subset S of C, $Z_S = V_S$. To this end, first observe that for each connected component K of G', there is a 3-subset S of C with $V(K) \subseteq Z_S$ and $C \cap N_G(V(K)) \subseteq S$. This is because Figure 9.1(3) implies that for each pair (u, v) of countries in C, exactly two countries $x, y \in V(G) - C$ meet at the (u, v)-node in $\mathcal{M}$ but the edge $\{x, y\} \in E(G)$ is absent in G'. We claim that $C \cap N_G(V(K)) = S$ indeed. Toward a contradiction, assume that G' has a connected component K with $|C \cap N_G(V(K))| \le 2$. Let $W = C \cap N_G(V(K))$. If $|W| \le 1$, then K would be a connected component of $G - W$, a contradiction. If $|W| = 2$, then K is a connected component of $G - W - \mathcal{E}[W]$, and the vertices of W define a separating edge of G, a contradiction. So, the claim holds. This claim together with the above observation and Figure 9.1(3), implies that $Z_S = V_S$ for each 3-subset S of C. So, by Figure 9.1(3), Statements 1 through 3 hold.

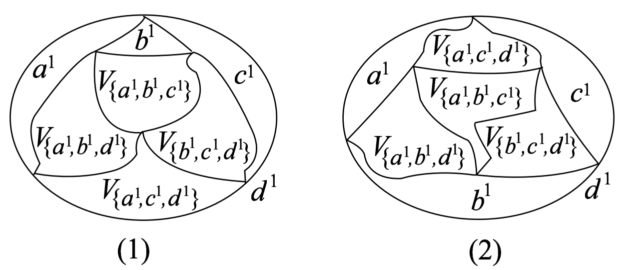

Figure 9.2: Possible atlases of G.

For the "if" direction, suppose that Statements 1 through 3 hold. We first prove that Figure 9.1(1) does not display $\mathcal{M}|_C$. Toward a contradiction, assume that Figure 9.1(1) displays $\mathcal{M}|_C$. Let S be a 3-subset of C. We claim that there is no 4-node in $\mathcal{M}$ at which two countries $u, v \in V_S$ together with two countries $x, y \in V(G) - V_S$ meet in the order u, x, v, y. This claim holds; otherwise, $\{x, y\} \not\subseteq C$ by Figure 9.1(1), so x or y belongs to V_T for some 3-subset T of C other than S, and hence $\{u, v\}$ would be a subset of $V_S \cap N_G(V_T)$, contradicting Statement 2. By this claim and Statement 3b, the countries of V_S form a cycle-superface of $\mathcal{M}$ (otherwise, $\mathcal{M}|_{V_S}$ has at least two holes and they are disjoint, contradicting Statement 3a). Thus, by Statements 1 and 2, Figure 9.2(1) displays $\mathcal{M}$. By this figure, there is a 4-node α

in $\mathcal{M}$ such that for each 3-subset S of C, exactly one country $v_S \in V_S$ has node α on its boundary. Since the countries v_S meet at α but no two of them belong to the same connected component of G' because of Statement 3c, we have $C \cap N_G(v_S) = S$. So, by Figure 9.2(1) and the 4-connectedness of G, each V_S would equal $\{v_S\}$, contradicting the assumption that $|V(G)| \geq 9$. Therefore, Figure 9.1(1) does not display $\mathcal{M}|_C$.

We next prove that Figure 9.1(2) does not display $\mathcal{M}|_C$. Toward a contradiction, assume that Figure 9.1(2) displays $\mathcal{M}|_C$. As in the last paragraph, we can claim that the countries of each V_S form a cycle-superface of $\mathcal{M}$. Thus, by Statements 1 and 2, Figure 9.2(2) displays $\mathcal{M}$. By this figure, there is a 4-node α in $\mathcal{M}$ at which country a^1, some $u \in V_{\{a^1,b^1,c^1\}}$, some $v \in V_{\{a^1,b^1,d^1\}}$, and some $w \in V_{\{a^1,c^1,d^1\}}$ meet. Since u, v and w meet at α but no two of them belong to the same connected component of G' because of Statement 3c, we have $C \cap N_G(u) = \{a^1, b^1, c^1\}$, $C \cap N_G(v) = \{a^1, b^1, d^1\}$, and $C \cap N_G(w) = \{a^1, c^1, d^1\}$. So, by Figure 9.2(2), countries v, a^1, b^1, d^1 meet at a node in $\mathcal{M}$, and countries w, a^1, c^1, d^1 meet at a node in $\mathcal{M}$. Thus, $V_{\{a^1,b^1,d^1\}} = \{v\}$ or else $\langle u, b^1, v\rangle$ would be a separating triple of G by Fact 7.6, a contradiction. Similarly, $V_{\{a^1,c^1,d^1\}} = \{w\}$. In a similar way, we can also prove that $|V_{\{b^1,c^1,d^1\}}| = 1$. Now, by Figure 9.2(2) and the 4-connectedness of G, we have $V_{\{a^1,b^1,c^1\}} = \{u\}$. In summary, $|V(G)| = 8$, a contradiction. Therefore, Figure 9.1(2) does not display $\mathcal{M}|_C$.

Since both Figures 9.1(1) and (2) do not display $\mathcal{M}|_C$, only Figure 9.1(3) can display $\mathcal{M}|_C$. This completes the proof. ∎

Since it is easy to check whether Statements 1 through 3 hold, we can easily decide whether C has an extensible "rice-ball" layout. Once we know that C has an extensible "rice-ball" layout, then by Statement 2, we can easily find and then remove six correct 4-pizzas from G. By examining all the MC_4's in G, our algorithm can either find one that is a rice-ball, and thus make progress; or else it can establish that none of the MC_4's is a rice-ball.

9.2 Distinguishing Pizzas and non-Pizzas

By the previous discussion, we now suppose that our algorithm reaches a point where none of the MC_4's has a rice-ball layout. Then all the remaining MC_4's are either pizzas or pizza-with-crusts. Specifically, we have:

Corollary 9.2 *For every MC_4 C of G, either Figure 9.1(1) or (2) displays $\mathcal{M}|_C$. Consequently, if the countries of C do not meet at a 4-node in atlas $\mathcal{M}$, then C has a 3-subset S such that the countries of S pairwise weakly touch in $\mathcal{M}$ and one of the two holes of $\mathcal{M}|_S$ is completely occupied by the unique country of $C - S$ in atlas $\mathcal{M}$.*

Let $C = \{a, b, c, d\}$ be an MC_4 of G. Our goal in this section is to give a linear-time decision procedure to decide which of Figures 9.1(1) and (2) displays $\mathcal{M}|_C$. Moreover, the procedure always chooses Figure 9.1(2) when both are possible. Whenever we arrive at the conclusion that Figure 9.1(2) displays $\mathcal{M}|_C$, we will have identified d^1 and therefore we immediately make progress by removing three correct 4-pizzas (cf. Statement 2 in Claim 9.4) from G. When Figure 9.1(1) (the pizza) displays $\mathcal{M}|_C$, we do nothing with this MC_4 C and proceed to consider other MC_4's; this may eventually lead to a situation where all MC_4's in G have to be pizzas, as considered in Section 9.3.

Claim 9.3 *If Figure 9.1(2) displays $\mathcal{M}|_C$, then the following hold:*

1. *C is 3-sharing with exactly three MC_4's C_1, C_2 and C_3 of G.*
2. *$C_1 \cap C_2 \cap C_3$ consists of a unique country; this country belongs to C and is adjacent to no country of $V(G) - (C \cup C_1 \cup C_2 \cup C_3)$ in graph G.*

Proof: Suppose Figure 9.1(2) displays $\mathcal{M}|_C$. Let w_{a^1,b^1} be the country in $V(G)-C$ such that d^1 and w_{a^1,b^1} meet at the (a^1, b^1)-node in $\mathcal{M}$. Define w_{a^1,c^1} and w_{b^1,c^1} similarly. Countries w_{a^1,b^1}, w_{a^1,c^1} and w_{b^1,c^1} are distinct or else G would have an MC$_5$. Let $C_1 = \{a^1, b^1, d^1, w_{a^1,b^1}\}$, $C_2 = \{a^1, c^1, d^1, w_{a^1,c^1}\}$, and $C_3 = \{b^1, c^1, d^1, w_{b^1,c^1}\}$. Obviously, C_1 through C_3 are 3-sharing with C, and they together satisfy Statement 2. To finish the proof of the claim, it remains to show that no MC$_4$ C_4 of G other than C_1, C_2, C_3 is 3-sharing with C. For a contradiction, assume that such C_4 exists in G. Then, by Figure 9.1(2), $C_4 \cap C = \{a^1, b^1, c^1\}$. Let x be the unique country in $C_4 - C$. Since countries a^1, b^1, c^1 pairwise weakly touch in $\mathcal{M}$ (according to Figure 9.1(2)), Corollary 9.2 (applied to MC$_4$ C_4) implies that one of the two holes of $\mathcal{M}|_{a^1,b^1,c^1}$ is completely occupied by country x in atlas $\mathcal{M}$. However, by Figure 9.1(2), one hole of $\mathcal{M}|_{a^1,b^1,c^1}$ is completely occupied by country d^1 in atlas $\mathcal{M}$, and the other hole is partly occupied by countries w_{a^1,b^1}, w_{a^1,c^1} and w_{b^1,c^1} in atlas $\mathcal{M}$; so, neither hole of $\mathcal{M}|_{a^1,b^1,c^1}$ could be completely occupied by country x in atlas $\mathcal{M}$, a contradiction. ∎

Whether Statements 1 and 2 in Claim 9.3 hold can be checked in linear time. So, we assume that Statements 1 and 2 in Claim 9.3 hold; otherwise, Figure 9.1(2) does not display $\mathcal{M}|_C$ (and we are done).

Claim 9.4 *If Figure 9.1(2) displays $\mathcal{M}|_C$, then the following hold:*

1. *Country d^1 in Figure 9.1(2) must be the unique country in $C_1 \cap C_2 \cap C_3$.*
2. *For every $C_i \in \{C_1, C_2, C_3\}$, $\langle u, v, w, x\rangle$ is a correct 4-pizza in G, where $\{u, v, w, x\} = C_i$, $\{u\} = C_1 \cap C_2 \cap C_3$, and $w \notin C$.*

Proof: Suppose Figure 9.1(2) displays $\mathcal{M}|_C$. For a contradiction, assume that Statement 1 in the claim is false. Then, exactly one of a^1, b^1, and c^1 in Figure 9.1(2) is the unique country in $C_1 \cap C_2 \cap C_3$. We assume that a^1 in Figure 9.1(2) is the unique country in $C_1 \cap C_2 \cap C_3$; the other two cases are similar (e.g., when b^1 in Figure 9.1(2) is the unique country in $C_1 \cap C_2 \cap C_3$, it suffices to swap a^1 and b^1 in the proof). Then, there are countries $x, y, z \in V(G) - C$ such that $C_1 = \{a^1, b^1, c^1, x\}$, $C_2 = \{a^1, b^1, d^1, y\}$, and $C_3 = \{a^1, c^1, d^1, z\}$. Since countries a^1, b^1, c^1 pairwise weakly touch in $\mathcal{M}$ (according to Figure 9.1(2)), Corollary 9.2 (applied to MC$_4$ C_1) implies that one of the two holes of $\mathcal{M}|_{a^1,b^1,c^1}$ is completely occupied by country x in atlas $\mathcal{M}$. However, by Figure 9.1(2), one hole of $\mathcal{M}|_{a^1,b^1,c^1}$ is completely occupied by country d^1 in atlas $\mathcal{M}$, and the other hole is partly occupied by countries y and z; so, neither hole of $\mathcal{M}|_{a^1,b^1,c^1}$ could be completely occupied by country x in atlas $\mathcal{M}$, a contradiction. So, Statement 1 holds. Statement 2 follows from Statement 1 immediately. ∎

We assume that d is the unique country in $C_1 \cap C_2 \cap C_3$; the other cases are similar (e.g., when a is the unique country in $C_1 \cap C_2 \cap C_3$, it suffices to modify Figures 9.1(1) and (2) by swapping countries d^1 and a^1, and to modify the following discussions by swapping a and d and swapping a^1 and d^1). Then, by Claim 9.4, $d^1 = d$ in Figure 9.1(2). Let $C_1 = \{a, b, d, e\}$, $C_2 = \{a, c, d, f\}$ and $C_3 = \{b, c, d, g\}$. Note that e, f, g are distinct (otherwise, G would have a 5-clique). Let $U = \{a, b, \dots, g\}$.

Recall that we want to distinguish Figures 9.1(1) and (2). If Figure 9.1(1) displays $\mathcal{M}|_C$, then it remains so even after we set $d^1 = d$ in it (because we still have the freedom to permute countries a, b, c). So, we may assume that $d^1 = d$ in Figure 9.1(1).

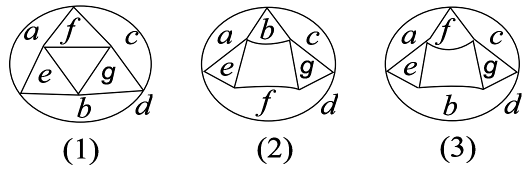

Figure 9.3: (1) A possible display of $G[U]$ when $\{e, f, g\}$ is a clique. (2) A possible display of $G[U]$ when $\{e, g\} \notin E(G)$. (3) Another possible display of $G[U]$ when $\{e, g\} \notin E(G)$.

To distinguish Figures 9.1(1) and (2), first consider the case where $\{e, f, g\}$ is a clique of G. In this case, $C_4 = \{e, f, g, d\}$, $C_5 = \{a, d, e, f\}$, $C_6 = \{c, d, f, g\}$, and $C_7 = \{b, d, e, g\}$ are MC_4's of G; we can claim that Figure 9.1(2) does not display $\mathcal{M}|_C$. For a contradiction, assume that Figure 9.1(2) displays $\mathcal{M}|_C$. Then, Corollary 9.2 (applied to MC_4's C_5, C_6, C_7) implies that Figure 9.3(1) displays $\mathcal{M}|_U$. However, by the figure, C_4 is a rice-ball, a contradiction. Thus, Figure 9.1(2) does not display $\mathcal{M}|_C$, and we are done.

So, in the sequel, we assume that $\{e, f, g\}$ is not a clique of G. In case Figure 9.1(1) displays $\mathcal{M}|_C$, a simple inspection shows that one country in $\{e, f, g\}$ (the one adjacent to a^1, c^1, d) is adjacent to the other two. So we assume that only one edge is missing among $\{e, f, g\}$, for otherwise Figure 9.1(2) must display $\mathcal{M}|_C$. We suppose the absent edge is $\{e, g\}$; the other two cases are similar (e.g., when the absent edge is $\{e, f\}$, it suffices to modify the following discussions by swapping g and f and swapping a and b). Then, $\{a, d, e, f\}$ and $\{c, d, f, g\}$ are MC_4's in G. Moreover, Corollary 9.2 (applied to these two MC_4's and C_1 through C_3) implies that Figure 9.1(1) (respectively, Figure 9.1(2)) displays $\mathcal{M}|_C$ iff Figure 9.3(2) (respectively, Figure 9.3(3)) displays $\mathcal{M}|_U$. Figure 9.3(3) does not display $\mathcal{M}|_U$ if $\{d, f\}$ is a marked edge. Also, if $\{d, b\}$ is a marked edge, then Figure 9.3(2) does not display $\mathcal{M}|_U$ and so Figure 9.3(3) displays $\mathcal{M}|_U$. Thus, we may assume that neither $\{d, b\}$ nor $\{d, f\}$ is a marked edge.

To distinguish Figures 9.3(2) and (3), we perform the following three steps in turn:

Step 1. We check whether at least one of the edges $\{a, b\}$, $\{c, b\}$, $\{e, f\}$, and $\{g, f\}$ is marked in G. If at least one of these edges is marked in G, then Figure 9.3(3) does not display $\mathcal{M}|_U$ and our task of distinguishing Figures 9.3(2) and (3) is done.

Step 2. We check whether at least one of the edges $\{a, f\}$, $\{c, f\}$, $\{e, b\}$, and $\{g, b\}$ is marked in G. If at least one of these edges is marked in G, then Figure 9.3(2) does not display $\mathcal{M}|_U$ and our task of distinguishing Figures 9.3(2) and (3) is done.

Step 3. We do a case-analysis as follows: (Comment: During the case-analysis, once we reach the conclusion that one of Figures 9.3(2) and (3) does not display $\mathcal{M}|_U$, or the conclusion that Figure 9.3(3) displays $\mathcal{M}|_U$, then we quit the case-analysis immediately because our task of distinguishing Figures 9.3(2) and (3) is done.)

Case 1: There is no $h \in V(G) - U$ with $\{a, b, e\} \subseteq N_G(h)$ or there is no $i \in V(G) - U$ with $\{b, c, g\} \subseteq N_G(i)$. Then, Figure 9.3(2) does not display $\mathcal{M}|_U$. Note that whether h and i exist can be decided in $O(1)$ time (assuming that G's adjacency matrix is available), because $|N_G(a)| = |N_G(c)| \leq 6$ by Figures 9.3(2) and (3).

Case 2: There are $h \in V(G) - U$ and $i \in V(G) - U$ such that $\{a, b, e\} \subseteq N_G(h)$ and $\{b, c, g\} \subseteq N_G(i)$. Then, if $f \notin N_G(h)$ or $f \notin N_G(i)$, Figure 9.3(3) does not display $\mathcal{M}|_U$. So, we may assume that $f \in N_G(h)$ and $f \in N_G(i)$. Let $\alpha_{e,f}$ and $\alpha_{g,f}$ be the endpoints of the path shared by country f and the hole of the layout in Figure 9.3(2), where $\alpha_{e,f}$ (respectively,

$\alpha_{g,f}$) is on the boundary of country e (respectively, g). Similarly, let $\beta_{e,b}$ and $\beta_{g,b}$ be the endpoints of the path shared by country b and the hole in the layout in Figure 9.3(3), where $\beta_{e,b}$ (respectively, $\beta_{g,b}$) is on the boundary of country e (respectively, g). If Figure 9.3(2) displays $\mathcal{M}|_U$, then Corollary 9.2 (applied to MC$_4$'s $\{a, e, f, h\}$ and $\{c, f, g, i\}$) implies that $\alpha_{e,f}$ is the unique (h, f)-node in $\mathcal{M}$ and $\alpha_{g,f}$ is the unique (i, f)-node in $\mathcal{M}$; so, $h \neq i$ (by the well-formedness of $\mathcal{M}$) and $N_G(f) \not\subseteq U \cup \{h, i\}$ (by the absence of holes in $\mathcal{M}$). Similarly, if Figure 9.3(3) displays $\mathcal{M}|_U$, then Corollary 9.2 (applied to MC$_4$'s $\{a, e, f, h\}$ and $\{c, f, g, i\}$) implies that $\beta_{e,b}$ is the unique (h, b)-node in $\mathcal{M}$ and $\beta_{g,b}$ is the unique (i, b)-node in $\mathcal{M}$; so, $h \neq i$ (by the well-formedness of $\mathcal{M}$) and $N_G(b) \not\subseteq U \cup \{h, i\}$ (by the absence of holes in $\mathcal{M}$). Thus, we always have $h \neq i$. Moreover, if $N_G(f) \subseteq U \cup \{h, i\}$, then Figure 9.3(2) does not display $\mathcal{M}|_U$. Similarly, if $N_G(b) \subseteq U \cup \{h, i\}$, then Figure 9.3(3) does not display $\mathcal{M}|_U$. So, we may assume that $N_G(f) \not\subseteq U \cup \{h, i\}$ and $N_G(b) \not\subseteq U \cup \{h, i\}$. Let $W = U \cup \{h, i\}$.

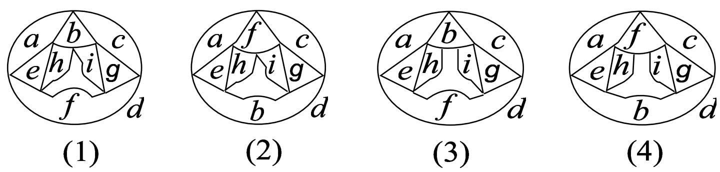

Figure 9.4: (1) A possible layout of $G[W]$ when $\{h, i\} \in E(G)$. (2) Another possible layout of $G[W]$ when $\{h, i\} \in E(G)$. (3) A possible layout of $G[W]$ when $\{h, i\} \notin E(G)$. (4) Another possible layout of $G[W]$ when $\{h, i\} \notin E(G)$.

Case 2.1: $\{h, i\} \in E(G)$. Then, Corollary 9.2 (applied to MC$_4$'s $\{a, e, f, h\}$, $\{a, e, b, h\}$, $\{c, b, g, i\}$, and $\{c, f, g, i\}$), Assumption 3 (the absence of separating triangles in G), and Fact 7.10 (applied to 3-cliques $\{b, h, i\}$ and $\{f, h, i\}$) together imply that Figure 9.4(1) (respectively, Figure 9.4(2)) displays $\mathcal{M}|_W$ iff Figure 9.3(2) (respectively, Figure 9.3(3)) displays $\mathcal{M}_U$. By Figures 9.4(1) and (2), $|N_G(e)| = |N_G(g)| = 6$; let j be the country in $N_G(e) - W$ and k be the country in $N_G(g) - W$. In case j or k is not adjacent to f in G, Figure 9.4(1) does not display $\mathcal{M}|_W$. Similarly, in case j or k is not adjacent to b in G, Figure 9.4(2) does not display $\mathcal{M}|_W$. So, we may further assume that j and k are adjacent to both f and b in G. Then, no matter which of Figures 9.4(1) and (2) displays $\mathcal{M}|_W$, we must have $j = k$ and $V(G) = W \cup \{j\}$. Now, Figure 9.4(2) displays $\mathcal{M}|_W$ (and hence Figure 9.3(3) displays $\mathcal{M}|_U$) only if none of $\{a, b\}$, $\{b, c\}$, $\{b, h\}$, $\{b, i\}$, $\{e, f\}$, $\{f, g\}$, $\{f, j\}$ is a marked edge in G. On the other hand, if none of these edges is marked in G, then Figure 9.4(1) is transformable to Figure 9.4(2) and hence Figure 9.4(2) displays $\mathcal{M}|_W$ (and so Figure 9.3(3) displays $\mathcal{M}|_U$).

Case 2.2: $\{h, i\} \notin E(G)$. Then, by Corollary 9.2, Figure 9.4(3) (respectively, Figure 9.4(4)) displays $\mathcal{M}|_W$ iff Figure 9.3(2) (respectively, Figure 9.3(3)) displays $\mathcal{M}_U$. Now, observe a resemblance between Figure 9.3(2) and Figure 9.4(3), and a resemblance between Figure 9.3(3) and Figure 9.4(4). We want to iterate the above three steps to distinguish Figures 9.4(3) and (4). To this end, first observe that the above three steps are independent of country d and edge $\{a, c\}$. Moreover, the above three steps can be viewed as a procedure $CA(a, e, b, c, g, f)$ where the input parameters are countries of G related as in Figure 9.3(2) or (3) except for the possible absence of edge $\{a, c\}$. Thus, to distinguish Figures 9.4(3) and (4), it suffices to set $U = W$ and recursively call $CA(g, i, f, e, h, b)$. (Comment: U is treated as a global variable.)

There can be a linear number of subsequent calls of procedure CA. Each call takes $O(1)$ time, so the overall time is linear.

9.3 Removing Pizzas

By the discussions in the last two subsections, we may assume that for every MC_4 $C = \{a, b, c, d\}$ of G, only Figure 9.1(1) displays $\mathcal{M}|_C$. That is, the four countries of every MC_4 of G meet at a node in $\mathcal{M}$.

Fix an MC_4 $C = \{a, b, c, d\}$ of G. C is 3-sharing with no MC_4 C' of G because otherwise, C' would have a non-pizza layout. By Figure 9.1(1), there are distinct countries e, f, g and h in $V(G) - C$ such that $C \cap N_G(e) = \{a^1, b^1\}$, $C \cap N_G(f) = \{b^1, c^1\}$, $C \cap N_G(g) = \{c^1, d^1\}$ and $C \cap N_G(h) = \{d^1, a^1\}$, because $\mathcal{M}$ has no hole. On the other hand, the existence of the countries e, f, g and h ensures that the countries of C have to meet at a node in $\mathcal{M}$ in the order w, x, y, z, where $\{w, x\} = C \cap N_G(e)$, $\{x, y\} = C \cap N_G(f)$, $\{y, z\} = C \cap N_G(g)$ and $\{z, w\} = C \cap N_G(h)$. Thus, by finding out countries e, f, g and h, we can find and remove a correct 4-pizza from G.

By this method we may identify a correct 4-pizza for every MC_4 in G. Since these 4-pizzas all exist in every well-formed atlas of G, we may remove them all in one step by the remarks after Lemma 5.2.

10 Time Analysis

Let n and m be the number of vertices and edges in the input graph G, respectively. Suppose this is not a base case; that is, $n \geq 9$ and G has a 4-clique. Then we will show that the algorithm can always make progress in $O(n^2)$ time. In each case, the time needed to produce the subproblems from G dominates the time needed to recover a solution from the subproblem solutions, so we ignore the latter.

By Lemma 2.1 (with $k = 4$) G has $m = O(n)$ edges and arboricity $\alpha(G) = O(1)$, so we can list its $O(n)$ maximal cliques in linear time [5]. From the listed MC_4's, we can precompute the sets $\mathcal{E}[a, b]$ for all unmarked edges $\{a, b\}$, again in linear time.

We claim that testing the existence of a separating triangle takes $O(n^2)$ time. Since G has $O(n)$ maximal cliques and no 7-clique, it has $O(n)$ 3-cliques and these can be found in linear time. For each 3-clique C, it takes $O(n)$ time to test whether some (ordered) list of the vertices in C is a separating triangle. So, the claim holds. A similar analysis applies for finding a 3-cut (by Lemma 3.5(1)), a separating edge, or a separating triple.

In order to detect separating quadruples, we use an algorithm of Chiba and Nishizeki [5] which implicitly lists all 4-cycles of G in $O(m \cdot \alpha(G)) = O(n)$ time. The algorithm produces a list of triples (u_i, v_i, S_i) with the following properties:

1. u_i and v_i are non-adjacent vertices of G.
2. S_i is a set of vertices adjacent to both u_i and v_i.
3. Every induced 4-cycle in G occurs as $\langle u_i, x, v_i, y\rangle$ for some choice of i and $x, y \in S_i$.

In particular, the sum of all $|S_i|$ is $O(n)$.

We claim that testing the existence of a separating quadruple takes $O(n^2)$ time. It suffices to show the following: for each triple (u_i, v_i, S_i), we can test whether there is a separating quadruple $\langle u_i, x, v_i, y\rangle$ or $\langle v_i, x, u_i, y\rangle$ (with $x, y \in S_i$) in time $O(|S_i|n)$. By similarity, it suffices to show how to find those quadruples starting with u_i.

For x in S_i, let $G^x = G - \{u_i, v_i, x\} - \mathcal{E}[u_i, x]$. In linear time we may compute G^x and identify the set S^x of all cut vertices in G^x. Now there is a separating quadruple of the form

$\langle u_i, x, v_i, y\rangle$ precisely if S^x contains some y which is in S but not adjacent to x. By repeating this for every $x \in S_i$, we have the required time bound.

A similar analysis applies for finding separating 4-cycles in $O(n^2)$ time.

The case analysis for eliminating an MC_5 in Section 8 may be executed in $O(n)$ time. In particular, we may identify an MC_5 4-sharing with two other MC_5's in $O(n)$ time as follows. First, for each MC_5 C_i and for each $S \subseteq C_i$ with $|S| = 4$, create a pair (S, i). Next, bucket-sort all the pairs, and use the result to count the number of 4-sharing MC_5's with each C_i.

When the graph has no MC_5 but still has some MC_4's, we make progress in at most $O(n^2)$ time as follows. First, we list the $O(n)$ MC_4's in some arbitrary order. For each one, we test the conditions of Lemma 9.1 in $O(n)$ time; if we find such an MC_4, then we remove the identified 4-pizzas and we are done. Otherwise, we go through the list again, this time applying the linear time decision procedure of Section 9.2; if we determine that some MC_4 is a non-pizza, then we remove the identified 4-pizzas and we are done again. Otherwise, we have established that all the MC_4's are pizzas, and so we can remove a 4-pizza for each MC_4 by the method in Section 9.3.

Finally, if the algorithm reaches a base case, our graph G either has at most 8 vertices, or no 4-clique. In the former case we solve the problem exhaustively in $O(1)$ time. Otherwise, G should be planar; we finish in linear time [6], as described in Section 6.

Let $N = n + m$ be the *size* of our input graph, and let $T(N)$ be the maximum running time of the algorithm on any input of size N. We claim that there is a constant c such that $T(N) \le cN^3$. The claim is clearly true for the base cases, as argued above. In all other cases, the algorithm makes progress in c_1N^2 time for some constant c_1. That is, the algorithm produces one or more smaller marked graphs whose total size is larger than that of G by a constant c_2; the problem for G is reduced to solving the problem for each of these smaller instances. More precisely, there are integers $n_1, \ldots, n_\ell \in \{1, \ldots, N-1\}$ such that $\sum_{i=1}^{\ell} n_i \le N + c_2$ and $T(N) \le \sum_{i=1}^{\ell} T(n_i) + c_1N^2$. We prove our claim by induction. For small N ($N < c_2^2$), our claim is true simply by choosing c large enough. For larger N, we have $T(N) \le \sum_{i=1}^{\ell} cn_i^3 + c_1N^2$ by the inductive hypothesis. Note that $\sum_{i=1}^{\ell} cn_i^3$ is maximized when $\ell = 2$, $n_1 = N - 1$ and $n_2 = c_2 + 1$. Hence, by choosing c large enough ($c_1 + 2$ suffices), we have $T(N) \le cN^3$.

11 Concluding Remarks

Our algorithm is complex. We would like to find a faster algorithm, with simpler arguments. Perhaps such a simplification is possible using some of Thorup's ideas. It would be interesting to produce succinct certificates in the case that G has no desired map; here "succinct" means that we can check them asymptotically faster than we can run our decision algorithm.

The authors [2] claimed an algorithm for recognizing 4-map graphs (possibly with holes), and subsequently produced a proof manuscript which is quite long even compared to the present argument. We believe that the result is correct, but we prefer to pursue simpler arguments rather than attempting to publish it as it stands.

Naturally, we are interested in polynomial-time algorithms for recognizing (hole-free or not) k-map graphs with $k \ge 5$. In view of the complication of our algorithm for hole-free 4-map graphs, however, new insights seem necessary in order to make progress in this direction.

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