# Recognizing well-dominated graphs is coNP-complete\*

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#### Abstract

A graph G is well-covered if every minimal vertex cover of G is minimum, and a graph G is well-dominated if every minimal dominating set of G is minimum. Studies on well-covered graphs were initiated in [Plummer, JCT 1970], and well-dominated graphs were first introduced in [Finbow, Hartnell and Nowakow, AC 1988]. Well-dominated graphs are well-covered, and both classes have been widely studied in the literature. The recognition of well-covered graphs was proved coNP-complete by [Chvátal and Slater, AODM 1993] and by [Sankaranarayana and Stewart, Networks 1992], but the complexity of recognizing well-dominated graphs has been left open since their introduction. We close this complexity gap by proving that recognizing well-dominated graphs is coNP-complete. This solves a well-known open question (c.f. [Levit and Tankus, DM 2017] and [Gözüpek, Hujdurovic and Milanič, DMTCS 2017]), which was first asked in [Caro, Sebő and Tarsi, JAlg 1996]. Surprisingly, our proof is quite simple, although it was a long-standing open problem. Finally, we show that recognizing well-totally-dominated graphs is coNP-complete, answering a question of [Bahadır, Ekim, and Gözüpek, AMC 2021].

Keywords: well-covered, well-dominated, well-totally-dominated, complexity.

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### 1 Introduction

MINIMUM DOMINATING SET, MAXIMUM INDEPENDENT SET, and MINIMUM VERTEX COVER are some of the most important computational and combinatorial problems, having a number of "real world" relevant applications and appearing in a wide range of natural situations. These problems cannot be solved in polynomial time unless P = NP, since they were proved to be NP-hard in 1972 in the seminal paper of Karp [23]. In spite of this fact, a *minimal* dominating set, and a *maximal* independent set of a graph can be found in polynomial time using a greedy algorithm. Also, the complement of a maximal independent set is a minimal vertex cover, so the same applies for vertex covers.

In 1970, Plummer [27] defined *well-covered* graphs as the class of graphs G where every minimal vertex cover is also a minimum vertex cover. This is equivalent to requesting that all maximal independent sets have the same cardinality. Therefore, well-covered graphs form a natural graph class for which MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER can be solved in polynomial time.

In the 1990's, the problem of recognizing a graph in the class of well-covered graphs, called Well-Coveredness, was independently proved to be coNP-complete by Chvátal and Slater [12] and by Sankaranarayana and Stewart [33]. In addition, structural characterizations or polynomial-time algorithms for recognizing well-covered graphs were studied on claw-free graphs [25, 35], graphs without large cycles [17], block-cactus graphs [31], bipartite graphs [32], graphs with large girth [18],  $P_4$ -sparse graphs [24], planar, chordal, and circular arc graphs [29], bounded degree graphs [10], and perfect graphs of bounded clique size [14]. A survey on well-covered graphs due to Plummer from 1993 can be found in [28]. In addition, Sankaranarayana and Stewart determined the complexity of several problems on well-covered graphs [33]. In 2011, Brown and Hoshino [9] showed that recognizing well-covered graphs is coNP-complete even when restricted to the family of circulant graphs. In 2020, Alves, Couto, Faria, Gravier, Klein, and Souza [1] studied the complexity of the GRAPH SANDWICH **PROBLEM** for the property of being a well-covered graph whose vertex set can be partitioned into kindependent sets and into  $\ell$  cliques for fixed integers k and  $\ell$ , i.e., well-covered graphs that are also  $(k, \ell)$ -graphs. In 2021, Faria and Souza [15] studied the complexity of PROBE PROBLEM for the property of being a  $(k, \ell)$ -graph that is well-covered. Also, a polynomial-time algorithm for recognizing some sparse-dense graphs that are well-covered can be found in [34].

Regarding parameterized complexity, in 2018, Alves, Dabrowski, Faria, Klein, Sau, and Souza proved that determining whether every minimal vertex cover of a given graph G has size k is fixed-parameter tractable with respect to k, but the problem of determining whether every maximal independent set of Ghas size k is coW[2]-hard, considering k as parameter. This last result illustrates that when considering the "wellness" variant of graph problems  $\Pi$ , there can be a leap in terms of complexity when analyzing the resulting well version of  $\Pi$ . Recall that k-INDEPENDENT SET is one of the canonical problems of the class W[1] but its well version is hard for coW[2]. In 2019, Araújo, Costa, Klein, Sampaio, and Souza showed that the problem of determining whether every minimal vertex cover of a graph G has size kadmits a kernel having  $\mathcal{O}(k)$  vertices. In addition, parameterized algorithms for Well-Coveredness considering other structural parameterizations can also be found in [2, 4].

Since every maximal independent set is a minimal dominating set, the well-covered graph class is a superclass of the class of graphs whose all minimal dominating sets have the same size. Such graphs were first studied in 1988 by Finbow, Hartnell, and Nowakow [19] and are called *well-dominated* graphs. The structure of well-dominated bipartite graphs and well-dominated graphs with no cycle of length less than 5 were analyzed in [19]. Also, structural characterizations of well-dominated block graphs and unicyclic graphs were presented in [36], and well-dominated chordal graphs were characterized in [29, 37]. In 2011, a characterization of 4-connected, 4-regular, claw-free, well-dominated graphs was given, see [20]. In 2017, Levit and Tankus proved that every well-covered graph without cycles of lengths 4 and 5 is well-dominated [26]. In the same year, Gözüpek, Hujdurovic, Milanič [21] presented a characterization of well-dominated graphs with domination number two and show that well-dominated graphs can be recognized in polynomial time in any class of graphs with bounded domination number. In 2021, Anderson, Kuenzel, and Rall [3] showed that there are exactly eleven

connected, well-dominated, triangle-free graphs whose domination number is at most 3, and Rall [30] gave a complete characterization of nontrivial direct products that are well-dominated.

Well-covered graphs G with no isolated vertices such that every maximal independent set has size  $\frac{|V(G)|}{2}$  form the class of *very well-covered graphs*. Similarly, one can define the *very well-dominated graphs*. Very well-covered graphs and very well-dominated graphs can be recognized in polynomial-time due to their structural characterizations, see [16, 37].

Although well-dominated graphs have been widely studied from 1988 to nowadays, the complexity status of recognizing well-dominated graphs was unknown until this current work. In this paper, we show that recognizing well-dominated graphs is coNP-complete, solving a well-known open question (c.f. [11, 21, 26]). To the best of our knowledge, the first time the recognition of well-dominated graphs was explicitly stated as an open question was in 1996 by Caro, Sebő and Tarsi [11]. In [11], well-dominated graphs are called greedy instances of the MINIMUM DOMINATING SET problem.

Besides that, analogously to well-dominated graphs, in 1997, Hartnell and Rall [22] initiated the study of graphs all whose minimal total dominating sets are of the same size. Such graphs are called *well-totally-dominated* graphs. In 2021, Bahadır, Ekim, and Gözüpek [5] showed among other results that well-totally-dominated graphs having bounded total domination number can be recognized in polynomial time. They left as an open question the complexity of recognizing well-totally-dominated graphs. In this paper, we also answer this question, showing that the recognition of well-totally-dominated graphs is coNP-complete.

We consider the following "wellness" problems related to domination and covering. Well-Coveredness

Instance:	A graph $G = (V, E)$ .
Goal:	Determine whether every minimal vertex cover of $G$ has the same size.
	<i>Note:</i> A vertex cover of $G$ is a subset of $V(G)$ intersecting all edges of $G$ .

Well-Domination

Instance:	A graph $G = (V, E)$ .
Goal:	Determine whether every minimal dominating set of $G$ has the same size.
	<i>Note:</i> A dominating set of G is a subset $S \subseteq V(G)$ such that each vertex
	$v \in V(G) \setminus S$ has a neighbor in $S$ .

Well-Total Domination

Instance:	A graph $G = (V, E)$ .
Goal:	Determine whether whether every minimal total dominating set of $G$ has the
	same size. Note: A total dominating set of G is a subset $S \subseteq V(G)$ such that any vertex of G has a neighbor in S, including vertices of S.

### - Well-Hitting Set

Instance:	An universe set $\mathcal U$ of elements, and a family $\mathcal F$ of subsets of elements of $\mathcal U.$
Goal:	Determine whether every minimal hitting set of $(\mathcal{U}, \mathcal{F})$ has the same size.
	<i>Note:</i> a hitting set of $(\mathcal{U}, \mathcal{F})$ is a subset of $\mathcal{U}$ intersecting every set in $\mathcal{F}$ .

#### Well-Set Cover

Instance:	An universe set $\mathcal{U}$ of elements, and a family $\mathcal{F}$ of subsets of elements of $\mathcal{U}$ .
Goal:	Determine whether every minimal set cover of $(\mathcal{U}, \mathcal{F})$ has the same size.
	Note: a set cover of $(\mathcal{U},\mathcal{F})$ is a subset $\mathcal S$ of $\mathcal F$ such that every element of $\mathcal U$ is
	contained in at least one set in $\mathcal{S}$ .

Well-Hitting-Set Cover

Instance:	An universe set $\mathcal{U}$ of elements, and a family $\mathcal{F}$ of subsets of elements of $\mathcal{U}$ .
Goal:	Determine whether every minimal set cover and every minimal hitting set of
	$(\mathcal{U},\mathcal{F})$ has the same size.

In this paper, we see that all these problems are coNP-complete.

### 2 Computational Complexity

First, observe that all problems studied in this work are in coNP, since any pair of minimal solutions having different sizes certifies *no*-instances for them. In addition, the minimality of solutions for such problems can easily checked in polynomial time. A similar general observation is contained in [11]. Thus, we focus on coNP-hardness in this section.

Any VERTEX COVER instance can be interpreted as a HITTING SET instance or a SET COVER instance. Therefore, the following corollary holds as a consequence of the coNP-completeness of Well-COVEREDNESS [12, 33].

Corollary 1. Well-HITTING SET and Well-SET COVER are coNP-complete.

Recall that, for n > 1, any *n*-vertex connected bipartite graph *G* is well-dominated if and only if every minimal dominating set of *G* has size  $\frac{n}{2}$ , because any maximal independent set is also a minimal dominating set. Thus, bipartite well-dominated graphs are very well-dominated and so recognized in polynomial time, since they are either a  $C_4$  or the corona product of a connected graph with a  $K_1$ , see [37]. Contrastingly, the next result shows that Well-Total Domination on bipartite graphs is unlikely to be polynomial-time solvable.

**Theorem 2.** Well-TOTAL DOMINATION on bipartite graphs is coNP-complete.

PROOF. Let  $H = (\mathcal{U}, \mathcal{F})$  be a hypergraph  $(\mathcal{F} \subseteq 2^{\mathcal{U}})$  such that each hitting set has at least two element. Then define G = (V, E) with

$$V = \{s,t\} \cup \{v_u \mid u \in \mathcal{U}\} \cup \{w_F \mid F \in \mathcal{F}\},\$$
  
$$E = \{\{s,t\}\} \cup \{\{s,v_u\} \mid u \in U\} \cup \{\{v_u, w_F\} \mid u \in F \text{ and } F \in \mathcal{F}\}$$

Since t has only the neighbor s, vertex s has to be in each total dominating set.

Let Z be a minimal hitting set of H. Define  $D = \{s\} \cup \{v_z \mid z \in Z\}$ . We want to show that D is a minimal total dominating set. As  $s \in D$ , the vertices in  $\{t\} \cup \{v_u \mid u \in \mathcal{U}\}$  are dominated. Since Z is not empty, vertex s is also dominated. Let  $F \in \mathcal{F}$ . Then there exists some  $u \in Z \cap F$ . This implies there exists a vertex  $v_u \in D \cap N(w_F)$ . Therefore, D is a total dominating set of G. As mentioned before, s has to be in D for total domination reasons. Assume there exists a  $v_u \in D$  such that  $D \setminus \{v_u\}$  is a total dominating set of G. This implies that for each  $w_F \in N(v_u)$ , there exists a  $v_{y_F} \in (N(w_F) \cap D) \setminus \{v_u\}$ . Hence, for each  $F \in \mathcal{F}$  with  $u \in F$ , there exists a  $y_F \in F \cap Z$  such that  $y_F \neq u$ . This contradicts the minimality of Z.

Let D be a minimal total dominating set of G. As mentioned before,  $s \in D$ . For each  $F \in \mathcal{F}$ ,  $N(w_F) \subseteq \{v_u \mid u \in \mathcal{U}\} \subseteq N(s)$  holds. Therefore,  $w_F$  is in no minimal total dominating set. But since  $w_F$  is not dominated by s, there has to be at least one element in  $D \cap \{v_u \mid u \in \mathcal{U}\}$ , implying that t cannot be in a minimal total dominating set. Therefore, each minimal total dominating set of G is a subset of  $\{s\} \cup \{v_u \mid u \in \mathcal{U}\}$ . Define  $Z = \{u \mid v_u \in D\}$ . As D is a total dominating set, for each  $F \in \mathcal{F}$ ,  $w_F$  is dominated. Thus, Z is a hitting set of H. Assume that Z is not minimal. This implies the existence of a  $u \in Z$  such that for each  $F \in \mathcal{F}$  having  $u \in F$ , there exists a  $z_F \in (F \cap Z) \setminus \{u\}$ .

This implies that for each  $w_F \in N(v_u)$ , there exists a  $v_{z_F} \in (N(w_F) \cap D) \setminus \{v_u\}$ . This contradicts the minimality of D.

Therefore, for each minimal hitting set Z of H, there exists a minimal total dominating set  $D_Z$  of G with  $|D_Z| = |Z| + 1$ . Conversely, for each minimal total dominating set D of G, there exists a minimal hitting set  $Z_D$  of H with  $|Z_D| = |D| - 1$ . Thus,  $H = (\mathcal{U}, \mathcal{F})$  is a yes-instance of Well-HITTING SET if and only if G is a yes-instance of Well-TOTAL DOMINATION. Therefore, by Corollary 1 the claim holds.

The argument would also hold if H is a VERTEX COVER instance. In this case, all vertices in  $\{w_F \mid F \in \mathcal{F}\}$  would have degree 2 and  $G[V \setminus \{w_F \mid F \in \mathcal{F}\}]$  is a tree. Therefore, G would be a 2-degenerate graph. If we would define  $\{s\} \cup \{v_u \mid u \in \mathcal{U}\}$  as a clique, it would not change the argument, either. This modification turns G into a split graph. Hence, the following holds.

**Corollary 3.** Well-Total Domination is coNP-complete on split or 2-degenerate bipartite graphs.

This result on split graphs is interesting insofar, as Well-DOMINATION on chordal graphs is solvable in polynomial time, see [37]. This indicates that Well-Total Domination tends to be a more difficult problem to solve than Well-DOMINATION. However, next result shows that on general graphs the Well-DOMINATION problem is also coNP-complete.

Theorem 4. Well-DOMINATION is coNP-complete.

**PROOF.** Let  $(\mathcal{U}, \mathcal{F})$  be an instance of Well-HITTING SET, where  $\mathcal{U}$  is the universe set and  $\mathcal{F}$  is a family of subsets of  $\mathcal{U}$ . Let k be the size of a minimal hitting set of  $(\mathcal{U}, \mathcal{F})$ .

Note that  $(\mathcal{U}, \mathcal{F})$  is a *yes*-instance of a Well-Hitting Set if and only if every minimal hitting set of  $(\mathcal{U}, \mathcal{F})$  has size k. Since a minimal hitting set of  $(\mathcal{U}, \mathcal{F})$  can be obtained in polynomial time, without loss of generality, we assume that k is given together with  $(\mathcal{U}, \mathcal{F})$  and we are asked if every minimal hitting set of  $(\mathcal{U}, \mathcal{F})$  has size k.

From  $(\mathcal{U}, \mathcal{F})$  we construct a graph *G* as follows:

1. Define

$$V(G) = \{r\} \cup U \cup F_1, F_2 \cup \ldots \cup F_{k-1},$$

where U is a vertex set of size  $|\mathcal{U}|$  such that each  $v_u \in U$  represents a distinct element u of the universe  $\mathcal{U}$ , and each  $F_i$  is a vertex set of size  $|\mathcal{F}|$  where each vertex  $v_j^i$  of  $F_i$  represents a set  $S_i \in \mathcal{F}$ .

- 2. Define U and each vertex set  $F_i$  as cliques of G.
- 3. Add an edge between a vertex  $v_j^i \in F_i$  and a vertex  $v_u \in U$  if  $u \in S_j$ . Note that the subgraph  $G[U \cup F_i]$  is isomorphic to the bipartite incidence graph of  $(\mathcal{U}, \mathcal{F})$ .
- 4. Add an edge between r and each vertex in U.

This completes the construction of G.

Now, we argue that every minimal hitting set of  $(\mathcal{U}, \mathcal{F})$  has size k if and only if G is well dominated. Notice that G has a clique cover C of size k formed by the cliques  $(U \cup \{r\}), F_1, \ldots, F_{k-2}$ , and  $F_{k-1}$ . Also, any minimal dominating set of G that contains at least one vertex per clique of C, by minimality, must contain exactly one vertex per clique (hence, it has size k). By taking r and one vertex per clique  $F_i$ , we know that G has a minimal dominating set of size k. Besides, by construction, any minimal dominating set of G contains either r or some vertex of U. Therefore, any minimal dominating set of G having size different from k must contain no vertex of  $F_1 \cup \ldots \cup F_{k-1} \cup \{r\}$ , but contain a subset of vertices of U that dominate each  $F_i$ . Since  $G[U \cup F_i]$  is isomorphic to the bipartite incidence graph of  $(\mathcal{U}, \mathcal{F})$ , it holds that G has a minimal dominating set of G having size different from k (G is not well-dominated) if and only if  $(\mathcal{U}, \mathcal{F})$  has a minimal hitting set of size different than k (so,  $(\mathcal{U}, \mathcal{F})$  is not a *yes*-instance of WELL-HITTING SET). Finally, recall that DOMINATING SET and TOTAL DOMINATING SET instances G can be interpreted as HITTING SET OF SET COVER instances where the elements of  $\mathcal{U}$  are the vertices of G and the family  $\mathcal{F}$  is formed by the close neighborhood or the open neighborhood of the vertices of G, respectively. Also, from the incidence bipartite graph of the resulting instance  $(\mathcal{U}, \mathcal{F})$ , it is easy to see that the existence of a hitting set of size k in  $(\mathcal{U}, \mathcal{F})$  also implies the existence of a set cover having the same size in  $(\mathcal{U}, \mathcal{F})$ , and vice versa. Hence, the following corollary holds as a consequence of the coNP-completeness of Well-DOMINATION and Well-TOTAL DOMINATION.

Corollary 5. Well-HITTING-SET COVER is coNP-complete.

Corollary 5 completes a P versus coNP-complete dichotomy regarding well domination problems on bipartite instances. Let B be a bipartite graph having vertex set bipartition  $V(B) = V_{\mathcal{U}} \cup V_{\mathcal{F}}$ . If the question is if all minimal subsets of  $V_{\mathcal{U}}$  that dominate  $V_{\mathcal{F}}$  have the same size, we are dealing with Well-HITTING SET. The converse would be Well-SET COVER. Also, one could consider, at the same time, all minimal subsets of  $V_{\mathcal{U}}$  that dominate  $V_{\mathcal{F}}$  and all minimal subsets of  $V_{\mathcal{F}}$  that dominate  $V_{\mathcal{U}}$ having all of them the same size, which is precisely the Well-HITTING-SET COVER problem. However, if asked about all minimal subsets of  $V_{\mathcal{U}} \cup V_{\mathcal{F}}$  that dominate  $V_{\mathcal{U}} \cup V_{\mathcal{F}}$  having the same size then we are dealing with Well-DOMINATION on bipartite graphs, which is polynomial-time solvable, as previously discussed. Therefore, the hardness result of Corollary 5 is tight concerning these constraints.

### 3 Conclusions

We have shown that well variations of a number of combinatorial properties is complete for the complexity class coNP. One algorithmic interpretation of the well variations is that this defines a graph class where a natural greedy strategy always finds the optimum. One could actually relax this requirement and ask, say, in the case of MAXIMUM INDEPENDENT SET, when the greedy heuristic that always picks a vertex of smallest degree next achieves, say, a factor-2 approximation. (Recall that in general, MAXIMUM INDEPENDENT SET does not allow polynomial-time constant-factor approximation algorithms.) Alas, determining if a graph can be approximated by a factor of two in this way is again a problem complete for coNP, as shown in [8]. It might be interesting to ask similar questions for MINIMUM DOMINATING SET, for instance, seeing the flavor of results in this paper. In 1996, Caro, Sebő and Tarsi [11] had also pointed as a potentially interesting direction the study of instances where a greedy algorithm always guarantees to provide a good approximation of the optimal goal.

There is also a further combinatorial way of looking at and possibly generalizing the questions discussed in this paper. For instance, one could interpret the Well-Domination problem as asking to decide, for a given graph G, if  $\gamma(G) = \Gamma(G)$ , i.e., if the lower and upper domination numbers of G coincide. Likewise, Well-Coveredness aks if  $\iota(G) = \alpha(G)$ , i.e., if the independent domination number and the independence number of G coincide. In this spirit, one could ask similar decision questions for other parameters of the famous domination chain, as introduced in [13]. For (a survey of) more recent computational results, we refer to [6, 7]. As it is known that all these parameters can differ arbitrarily, the approximation questions discussed in the previous paragraph can be asked analogously for any pair of parameters of the domination chain, or similarly for other graph parameters where inequality relations are known.

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