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Reconfigurable control of PWA systems with actuator and sensor faults: stability

J. H. Richter*, W. P. M. H. Heemels**, N. van de Wouw**, and J. Lunze*,

Abstract— **A new approach to the reconfigurable control of piecewise affine (PWA) systems after actuator and sensor faults is presented. The approach extends the concept of virtual actuators and virtual sensors from linear to PWA systems on the basis of the fault-hiding principle. The weak fault-hiding goal is introduced as a relaxation of the asymptotic fault-hiding goal. Sufficient linear matrix inequality conditions for the existence of input-to-state stabilizing virtual actuators and sensors are given that lead to a tractable computational algorithm. The stability of the reconfigured closed-loop system is verified. The approach is evaluated using a system of interconnected tanks.**

I. I

In this paper, a new reconfigurable control strategy for piecewise affine (PWA) systems is presented. Reconfigurable control responds to severe component failures that break the control loop by restructuring the control loop on-line [1]. Control reconfiguration is an active fault-tolerant control methodology that uses the estimate $\hat{f}$ of the fault f, which is obtained from a diagnosis component (FDI) (Fig. 1). As opposed to passive fault-tolerant control approaches [2], [3], in reconfigurable control the controller is changed to match the faulty plant once the fault has been isolated [4]. For switched systems, adaptive schemes [5] and output feedback controller redesign have been developed (see for example [6]). For PWA systems, model-predictive control has been used for fault-tolerant control [7].

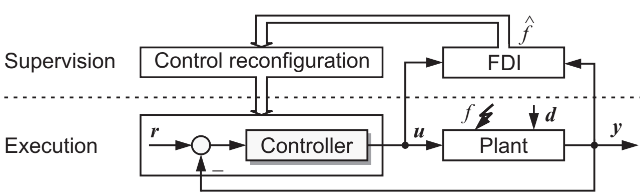

Fig. 1. Active fault-tolerant control scheme.

This approach is based on the idea of keeping the nominal controller in the loop by inserting a reconfiguration block between the faulty plant and the nominal controller when a fault occurs. The reconfiguration block is chosen to hide the fault from the controller and, at the same time, to ensure that the faulty plant controlled by the nominal controller together with the reconfiguration block remains globally input-to-state stable. The fault-hiding approach was previously developed for linear and Hammerstein systems and lead to virtual actuators for the actuator fault case and to virtual sensors for the sensor fault case (see [8]–[12]); until now, the fault-hiding approach was not available for PWA systems. The extension from linear to PWA systems is hard due to the following complexity property: For continuous PWA systems with more than 2 states, the problem of deciding whether or not all system trajectories are bounded is undecidable [13].

* Jan Richter|Jan Lunze, Institute of Automation and Computer Control, Ruhr-Universität Bochum, Universitätsstrasse 150, 44801 Bochum, Germany; email: {richter|lunze}@atp.rub.de

** W.P.M.H. Heemels|N. van de Wouw, Mechanical Engineering, TU Eindhoven, P.O. box 513 5600MB Eindhoven, The Netherlands; email: {m.heemels|n.vandewouw}@tue.nl

The motivation for studying PWA systems is at least twofold. Firstly, PWA systems are receiving wide attention due to the fact that the PWA framework [14] provides a way to describe dynamic systems exhibiting switching between a multitude of linear dynamic regimes, see also [15]. Such switching can be due to piecewise-linear characteristics such as dead-zone, saturation, hysteresis or relays. Secondly, PWA systems may result from piecewise linear approximations of complex nonlinear dynamics.

In this paper, we extend the fault-hiding approach from linear to PWA systems. It is assumed here that the fault isolation task is solved and that the fault model is known [4]. To be precise, we 1) present a reconfiguration block that satisfies the fault-hiding principle for PWA systems after actuator and sensor faults, 2) show a systematic computationally tractable approach to finding stabilizing gains in the reconfiguration block, and 3) demonstrate the feasibility of the approach by means of an example.

This paper is organized as follows. Notations, PWA systems and nonlinear stability concepts are introduced in Section II. Actuator and sensor faults as well as related reconfiguration problems are stated in Section III. The solution to the stable reconfiguration problem is given in Section IV along with a conceptual algorithm. An example illustrates the feasibility of the approach in Section V. The paper concludes in Section VI. Technical proofs are collected in Appendix A.

II. P

A. Notations

Lower case bold letters denote vectors ($\boldsymbol{x}$), capital bold letters denote matrices ($\boldsymbol{A}$), and script capitals denote spaces ($\mathcal{L}$). Systems are denoted by $\Sigma_1, \Sigma_2, \ldots$, where the indices distinguish different systems; the interconnection of two systems through common input/output variables is denoted by (Σ_1, Σ_2). Corresponding dynamical operators are denoted by Ω_1, Ω_2 with the same distinction. The restriction of a system operator with multiple outputs to a specific output $\boldsymbol{y}$ is denoted by Ω_p^y. $\mathbb{R}$ denotes the reals, and $\mathbb{R}_+ := [0, \infty)$. For $1 \leq p \leq \infty$, and for a measurable signal $\boldsymbol{x} : \mathbb{R}_+ \to \mathbb{R}^n$, we say that $\boldsymbol{x} \in \mathcal{L}_p(\mathbb{R}_+, \mathbb{R}^n)$ if $\|\boldsymbol{x}\|_p < \infty$, where $\|\boldsymbol{x}\|_p := \left(\int_{\mathbb{R}_+} \|\boldsymbol{x}(t)\|^p \mathrm{d}t\right)^{1/p}$ and $\|\boldsymbol{x}\|_\infty := \sup_{0 \leq \tau \leq t} |\boldsymbol{x}(\tau)|$. The space of locally integrable functions is denoted by $\mathcal{L}_1^{loc}$. The pseudoinverse of a matrix $\boldsymbol{A}$ satisfying all four Moore-Penrose conditions [16] is denoted by $\boldsymbol{A}^+$. The notation $\boldsymbol{A} \prec 0$ ($\boldsymbol{A} \preceq 0$) means that the matrix $\boldsymbol{A}$ is negative (semi-) definite. Likewise, the symbol $\succ$ ($\succeq$) denotes positive (semi-) definiteness. Linear matrix inequalities (LMI) are understood as linear semi-definite programs. A polyhedron is the intersection of a finite number of half-spaces [17].

The following comparison functions are used to formulate stability definitions for nonlinear systems [18]. A function $F : S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}^n$ containing zero is *positive definite* if $F(\boldsymbol{x}) > 0$ holds for all $\boldsymbol{x} \in S, \boldsymbol{x} \neq \boldsymbol{0}$, and $F(\boldsymbol{0}) = 0$. A *class* $\mathcal{K}$ *function* is a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, strictly increasing, and satisfies $\alpha(0) = 0$. Any function α that satisfies these requirements is said to be in the class $\mathcal{K}$, also denoted by $\alpha \in \mathcal{K}$. A *class* $\mathcal{K}_\infty$ *function* is a function $\alpha \in \mathcal{K}$ that is additionally unbounded, i.e. $\lim_{s\to\infty} \alpha(s) = \infty$. A *class* $\mathcal{KL}$ *function* is a function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\beta(\cdot, t) \in \mathcal{K}_\infty$ for any fixed t, and for each fixed $r \geq 0$, $\beta(r, t) \to 0$ as $t \to \infty$. The notation $\boldsymbol{x} \equiv \boldsymbol{0}$ is an abbreviation for $\forall t > 0 : \ \boldsymbol{x}(t) = \boldsymbol{0}$.

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B. Piecewise affine systems

In this paper, we consider nominal systems Σ_P that are modeled in PWA state-space form

$$\Sigma_P: \begin{cases} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_i \boldsymbol{x}(t) + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}_c(t) + \boldsymbol{B_d}\boldsymbol{d}(t),\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ & \quad \text{for } \boldsymbol{x} \in \Lambda_i,\ i \in \{1,\dots,p\} \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t),\ z(t) = \boldsymbol{C}_z \boldsymbol{x}(t), \end{cases} \tag{1}$$

with the state $\boldsymbol{x}(t) \in \mathbb{R}^n$, the control input $\boldsymbol{u}_c(t) \in \mathbb{R}^m$, the disturbance $\boldsymbol{d}(t) \in \mathbb{R}^k$, the measured output $\boldsymbol{y}(t) \in \mathbb{R}^r$ and the relevant output $z(t) \in \mathbb{R}^q$ at time $t \in \mathbb{R}_+$. $\boldsymbol{A}_i$, $i \in \{1,\dots,p\}$, is a family of system matrices, $\boldsymbol{b}_i$, $i \in \{1,\dots,p\}$, is a family of affine terms, $\boldsymbol{B}$ is the input matrix, $\boldsymbol{B_d}$ is the disturbance input matrix that defines the structure of disturbance influence, $\boldsymbol{C}$ is the measurement matrix, and $\boldsymbol{C}_z$ is the relevant output matrix. All matrices are of compatible dimensions. Each of the pairwise disjoint sets Λ_i corresponds to a *mode* of the PWA system (1) in the sense that if $\boldsymbol{x}(t) \in \Lambda_i$, then the system is described by the i-th affine system represented by the tuple $(\boldsymbol{A}_i, \boldsymbol{b}_i, \boldsymbol{B}, \boldsymbol{B_d}, \boldsymbol{C}, \boldsymbol{C}_z)$ at time t. The sets Λ_i are here described by polyhedra such that $\forall i \neq j:\ \mathrm{int}(\Lambda_i) \cap \mathrm{int}(\Lambda_j) = \emptyset$ and $\bigcup_{i=1}^p \Lambda_i = \mathbb{R}^n$, and switching is triggered when the state trajectory crosses a boundary hyperplane.

Assumption 1: The right-hand side of the system (1) is assumed to be globally continuous in $\boldsymbol{x}$, $\boldsymbol{u}_c$ and $\boldsymbol{d}$.
However, the PWA system needs not necessarily have a smooth right-hand side. Note that Assumption 1 guarantees that the system (1) is locally Lipschitz-continuous and consequently for any $\boldsymbol{u}_c \in \mathcal{L}_1^{loc}(\mathbb{R}^m)$, $\boldsymbol{d} \in \mathcal{L}_1^{loc}(\mathbb{R}^k)$, and $\boldsymbol{x}_0 \in \mathbb{R}^n$, it has a unique and globally defined solution. Also, sliding modes cannot occur as solutions of the PWA system (1). Associated with the system Σ_P is the dynamical operator $\Omega_P:\ \mathcal{L}_1^{loc}(\mathbb{R}^m) \times \mathcal{L}_1^{loc}(\mathbb{R}^k) \times \mathbb{R}^n \to \mathcal{L}_1^{loc}(\mathbb{R}^r) \times \mathcal{L}_1^{loc}(\mathbb{R}^q)$,

$$(\boldsymbol{y}, z) = \Omega_P(\boldsymbol{u}_c, \boldsymbol{d}, \boldsymbol{x}_0). \tag{2}$$

The control input $\boldsymbol{u}_c$ is generated by a (possibly dynamical, not necessarily PWA) nominal controller Σ_C described by the operator $\Omega_C: \mathcal{L}_1^{loc}(\mathbb{R}^r) \times \mathcal{L}_1^{loc}(\mathbb{R}^r) \times \mathbb{R}^n \to \mathcal{L}_1^{loc}(\mathbb{R}^m)$,

$$\boldsymbol{u}_c = \Omega_C(\boldsymbol{r}, \boldsymbol{y}, \boldsymbol{x}_{c0}), \tag{3}$$

where $\boldsymbol{r}$ denotes the reference input, and $\boldsymbol{x}_{c0}$ is the controller initial state, giving rise to the nominal closed-loop system (1), (3).

C. Lyapunov-like stability notions

We use the following Lyapunov-based stability definitions.

Definition 1 (0-global exponential stability (0-GES)): The system (1) with inputs $\boldsymbol{u}_c, \boldsymbol{d} \equiv \boldsymbol{0}$ is called *0-globally exponentially stable*, if all solutions globally satisfy

$$|\boldsymbol{x}(t)| \leq c|\boldsymbol{x}(0)|\mathrm{e}^{-\lambda t}\ \forall t \geq 0,\ \text{where } c, \lambda > 0.$$

For systems with inputs, the notions of input-to-state stability (ISS) and input-to-output stability (IOS) are useful to characterize the boundedness of solutions $\boldsymbol{x}$ of the system (1) in the presence of inputs. The definition is given with respect to (w.r.t.) the control input $\boldsymbol{u}_c$ for ease of notation.

Definition 2 (Input-to-state stability [19]): The system (1) with $\boldsymbol{d} \equiv \boldsymbol{0}$ is called *input-to-state stable* (ISS) w.r.t the input $\boldsymbol{u}_c$,

$$\text{if } \exists \beta \in \mathcal{KL},\ \gamma \in \mathcal{K}_\infty:\ |\boldsymbol{x}(t)| \leq \beta(|\boldsymbol{x}_0|, t) + \gamma(\|\boldsymbol{u}_c\|_\infty) \tag{4}$$

for all inputs $\boldsymbol{u}_c$, all initial states $\boldsymbol{x}_0$, and all times $t \geq 0$, where $\boldsymbol{x}(t)$ is the solution of (1).

Definition 3 (Input-to-output stability [20]): The system (1) with $\boldsymbol{d} \equiv \boldsymbol{0}$ is called *input-to-output stable* (IOS) w.r.t. the input $\boldsymbol{u}_c$ and the output $\boldsymbol{y}$, if

$$\exists \beta \in \mathcal{KL},\ \gamma \in \mathcal{K}_\infty:\ |\boldsymbol{y}(t)| \leq \beta(|\boldsymbol{x}_0|, t) + \gamma(\|\boldsymbol{u}_c\|_\infty) \tag{5}$$

for all inputs $\boldsymbol{u}_c$, all initial states $\boldsymbol{x}_0$, and all times $t \geq 0$, where $\boldsymbol{y}$ is the output of the system (1).

Assumption 2: The nominal closed-loop system (1), (3) without faults is IOS w.r.t. the input $(\boldsymbol{r}, \boldsymbol{d})$ and the output $(\boldsymbol{x}, \boldsymbol{u}_c)$.

Proposition 1 (IOS of series-interconnected systems [21]):

$$\text{Let the system} \begin{cases} \dot{\boldsymbol{v}}(t) &= \boldsymbol{f}(\boldsymbol{v}(t), \boldsymbol{p}(t), \boldsymbol{u}(t)), \quad \boldsymbol{v}(t) \in \mathbb{R}^s \\ \dot{\boldsymbol{w}}(t) &= \boldsymbol{g}(\boldsymbol{w}(t), \boldsymbol{u}(t)), \qquad \boldsymbol{w}(t) \in \mathbb{R}^n \\ \boldsymbol{p}(t) &= \boldsymbol{h}_1(\boldsymbol{w}(t), \boldsymbol{u}(t)),\ \boldsymbol{q}(t) = \boldsymbol{h}_2(\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{u}) \end{cases} \tag{6}$$

be such that the $\boldsymbol{v}$-subsystem with the input $(\boldsymbol{p}, \boldsymbol{u})$ and output $\boldsymbol{q}$ is IOS, and the $\boldsymbol{w}$-subsystem is IOS w.r.t. the input $\boldsymbol{u}$ and the output $\boldsymbol{p}$. Then, the series connection (6) is IOS w.r.t. the input $\boldsymbol{u}$ and the outputs $\boldsymbol{p}, \boldsymbol{q}$.

The following proposition, which is central to most of the subsequent proofs, summarizes prior results on the incremental stability and ISS of continuous PWA systems [22], [23].[1]

Proposition 2 (PWA ISS and incremental stability): Consider the PWA system (1) with the right-hand side $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}_c, \boldsymbol{d}) := \boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i + \boldsymbol{B}\boldsymbol{u}_c + \boldsymbol{B_d}\boldsymbol{d}$, and suppose that Assumption 1 holds. If there exists a matrix $\boldsymbol{X} \in \mathbb{R}^{n\times n}$, $\boldsymbol{X} = \boldsymbol{X}^T > 0$ that satisfies the LMI

$$\boldsymbol{X}\boldsymbol{A}_i + \boldsymbol{A}_i^T\boldsymbol{X} \prec 0,\ i = 1,\dots,p, \tag{7}$$

then the system (1) is 0-GES for $\boldsymbol{u}_c = \boldsymbol{d} = \boldsymbol{0}$, globally ISS w.r.t. $(\boldsymbol{u}_c, \boldsymbol{d})$, and for any two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, it holds that

$$\begin{aligned}(\boldsymbol{x}_1 - \boldsymbol{x}_2)^T\boldsymbol{X}\,(\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{u}_c, \boldsymbol{d}) - \boldsymbol{f}(\boldsymbol{x}_2, \boldsymbol{u}_c, \boldsymbol{d}))& \\ \leq -\beta(\boldsymbol{x}_1 - \boldsymbol{x}_2)^T\boldsymbol{X}(\boldsymbol{x}_1 - \boldsymbol{x}_2).&\end{aligned} \tag{8}$$

That is, the system is quadratically incrementally stable. The number $\beta > 0$ depends only on the matrix $\boldsymbol{X}$.

III. R

A. Fault model

Faults f in actuators and sensors are modeled by changed input and output maps. In other words, the nominal system (1) (Fig. 2a)) is changed to the faulty PWA system

$$\Sigma_{Pf}: \begin{cases} \dot{\boldsymbol{x}}_f(t) &= \boldsymbol{A}_i\boldsymbol{x}_f(t) + \boldsymbol{b}_{f,i} + \boldsymbol{B}_f\boldsymbol{u}_f(t) + \boldsymbol{B_d}\boldsymbol{d}(t) \\ & \quad \text{for } \boldsymbol{x}_f \in \Lambda_i,\ i \in \{1,\dots,p\},\ \boldsymbol{x}_f(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}_f(t) &= \boldsymbol{C}_f\boldsymbol{x}_f(t),\ z_f(t) = \boldsymbol{C}_z\boldsymbol{x}_f(t), \end{cases} \tag{9}$$

where neither $\boldsymbol{A}_i, \boldsymbol{B_d}, \boldsymbol{C}_z$ nor the dimensions of any signals or matrices change (Fig. 2b)). In case of actuator faults, columns of $\boldsymbol{B}_f$ that correspond to faulty or failed actuators are scaled or set to zero, respectively. It is assumed that $\boldsymbol{B_d}$ is unaffected by actuator faults, and that the faulty system satisfies Assumption 1. Blockage of actuators in given positions is modeled by means of a changed affine term $\boldsymbol{b}_{f,i}$. Sensor faults are modeled in the same way by means of a changed measurement matrix $\boldsymbol{C}_f$. To compare the faulty and nominal systems, the faulty system starts from the nominal initial state $\boldsymbol{x}_0$.

Associated with the faulty plant is the operator $\Omega_{Pf}:\ \mathcal{L}_1^{loc}(\mathbb{R}^m) \times \mathcal{L}_1^{loc}(\mathbb{R}^k) \times \mathbb{R}^n \to \mathcal{L}_1^{loc}(\mathbb{R}^r) \times \mathcal{L}_1^{loc}(\mathbb{R}^q)$,

$$(\boldsymbol{y}_f, z_f) = \Omega_{Pf}(\boldsymbol{u}_f, \boldsymbol{d}, \boldsymbol{x}_0). \tag{10}$$

B. Fault-hiding approach

After the fault, the nominal controller (3) with $\boldsymbol{y} = \boldsymbol{y}_f$ and $\boldsymbol{u}_c = \boldsymbol{u}_f$ is generally not suitable for controlling the faulty plant (9). In particular, in the case of component failures, the loop is partially open. It is assumed that a fault diagnosis component provides the model (9). The reconfiguration problem now consists in finding a new controller Σ_{Cr} based on the faulty model (9) such that the reconfigured closed-loop system $(\Sigma_{Pf}, \Sigma_{Cr})$ shown in Fig. 2b) meets the original control goals as closely as possible. The new controller

[1] Lyapunov-characterizations of incremental stability were first presented in [24] and ISS results for locally Lipschitz systems were published in [25].

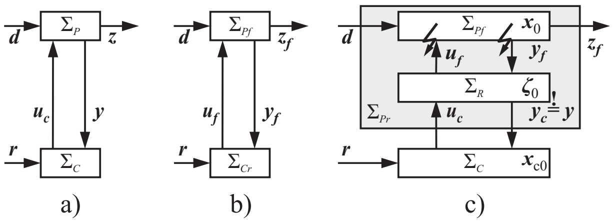

Fig. 2. a) Nominal closed-loop system, b) reconfigured closed-loop system with new controller, c) reconfigured closed-loop system for fault-hiding.

may use all available control input signals, also those ignored by the nominal controller.

This reconfiguration problem is equivalent to a closed-loop model-matching problem, which we will, however, not solve directly. Instead, we impose a special structure on the new controller Σ_{Cr}, which is factored into the original nominal controller Σ_C and a so-called reconfiguration block Σ_R: $\Sigma_{Cr} = (\Sigma_C, \Sigma_R)$. This approach offers the following advantages: 1) If the nominal controller is a human operator, e.g. a pilot, then the fault-hiding approach reduces the difficulty linked to dealing with a faulty system. Namely, it may help reducing training efforts for large numbers of fault scenarios and stress during fault situations. 2) If the controller is automatic and the fault affects small parts of the plant only, then large parts of the nominal controller are still valid and should be kept instead of performing a complete redesign, which may be costly and time-consuming. The fault-hiding strategy allows for minimum-invasive alterations of the loop by hiding the fault from the nominal controller.

The reconfiguration approach adopted here, therefore, consists in augmenting the closed loop by means of a dynamical reconfiguration block Σ_R described by the operator $\Omega_R : \mathcal{L}_1^{loc}(\mathbb{R}^m) \times \mathcal{L}_1^{loc}(\mathbb{R}^r) \times \mathbb{R}^n \rightarrow \mathcal{L}_1^{loc}(\mathbb{R}^m) \times \mathcal{L}_1^{loc}(\mathbb{R}^r)$,

$$(\boldsymbol{u}_f, \boldsymbol{y}_c) = \Omega_R(\boldsymbol{u}_c, \boldsymbol{y}_f, \boldsymbol{\zeta}_0), \tag{11}$$

whose realization is deferred to Section IV-A.

Together with the faulty plant (9), the reconfiguration block (11) forms the *reconfigured plant* $\Sigma_{Pr} = (\Sigma_{Pf}, \Sigma_R)$ described by the operator $\Omega_{Pr} : \mathcal{L}_1^{loc}(\mathbb{R}^m) \times \mathcal{L}_1^{loc}(\mathbb{R}^k) \times \mathbb{R}^n \rightarrow \mathcal{L}_1^{loc}(\mathbb{R}^r) \times \mathcal{L}_1^{loc}(\mathbb{R}^q)$

$$(\boldsymbol{y}_c, \boldsymbol{z}_f) = \Omega_{Pr}(\boldsymbol{u}_c, \boldsymbol{d}, \boldsymbol{x}_0, \boldsymbol{\zeta}_0) \tag{12}$$

to which the nominal controller (3) is connected by means of the signal pair $(\boldsymbol{u}_c, \boldsymbol{y}_c)$ (see Fig. 2c).

From a control point of view, it is of interest to at least recover the nominal closed-loop stability for the reconfigured closed-loop system (3), (9), (11).

Problem 1 (Stability recovery): Consider the nominal controller (3) and the faulty PWA system (9). Find a reconfiguration block (11) such that $\{(\Sigma_P, \Sigma_C)$ ISS w.r.t. $\boldsymbol{r}, \boldsymbol{d}\} \Rightarrow \{(\Sigma_{Pr}, \Sigma_C)$ ISS w.r.t. $\boldsymbol{r}, \boldsymbol{d}\}$.

The following additional goal makes sure that the original controller "sees" the fault-free plant behavior when attached to the reconfigured plant. It implies that the nominal controller remains part of the overall reconfigured closed loop and will be used to solve Problem 1.

Definition 4 (Weak fault-hiding goal): The reconfigured plant Ω_{Pr} meets the weak fault-hiding goal, if

$$\forall \boldsymbol{x}_0,\ \exists \boldsymbol{\zeta}_0 : \Omega_{Pr}^{\boldsymbol{y}_c}(\cdot, \cdot, \boldsymbol{x}_0, \boldsymbol{\zeta}_0) - \Omega_P^{\boldsymbol{y}}(\cdot, \cdot, \boldsymbol{x}_0) = \boldsymbol{0}.$$

We speak of weak fault-hiding because the initial state $\boldsymbol{\zeta}_0$ depends on $\boldsymbol{x}_0$. The weak fault-hiding goal is a relaxation of the asymptotic fault-hiding goal (where in addition, $\lim_{t\rightarrow\infty} \Omega_{Pr}^{\boldsymbol{y}_c}(\cdot, \cdot, \boldsymbol{x}_0, \boldsymbol{\zeta}_0) - \Omega_P^{\boldsymbol{y}}(\cdot, \cdot, \boldsymbol{x}_0) = \boldsymbol{0}$) and the strict fault-hiding goal (where $\boldsymbol{\zeta}_0$ must not depend on $\boldsymbol{x}_0$) [26].

IV. R M

A. Reconfiguration block

The reconfiguration operator (11) is realized for actuator and sensor faults by the combination of a state observer

$$\Sigma_O : \begin{cases} \dot{\hat{\boldsymbol{x}}}_f(t) &= \boldsymbol{A}_{\delta,i}\hat{\boldsymbol{x}}_f(t) + \boldsymbol{b}_{f,i} + \boldsymbol{B}_f\boldsymbol{u}_f(t) + \boldsymbol{L}\boldsymbol{y}_f(t) \\ &\quad \text{for } \hat{\boldsymbol{x}}_f \in \Lambda_i,\ i \in \{1,\ldots,p\},\ \hat{\boldsymbol{x}}_f(0) = \hat{\boldsymbol{x}}_{f,0} \\ \boldsymbol{A}_{\delta,i} &:= \boldsymbol{A}_i - \boldsymbol{L}\boldsymbol{C}_f. \end{cases} \tag{13}$$

and a so-called virtual actuator

$$\Sigma_V : \begin{cases} \dot{\tilde{\boldsymbol{x}}}(t) &= \boldsymbol{A}_j\tilde{\boldsymbol{x}}(t) + \boldsymbol{b}_j + \boldsymbol{B}\boldsymbol{u}_c(t),\ \tilde{\boldsymbol{x}}(0) = \hat{\boldsymbol{x}}_{f,0} \\ &\quad \text{for } \tilde{\boldsymbol{x}} \in \Lambda_j,\ j \in \{1,\ldots,p\} \\ \boldsymbol{y}_c(t) &= \boldsymbol{C}\tilde{\boldsymbol{x}}(t) \\ \boldsymbol{u}_f(t) &= \boldsymbol{M}\boldsymbol{x}_\Delta(t) + \boldsymbol{N}_j\boldsymbol{u}_c(t) + \boldsymbol{B}_f^+\boldsymbol{b}_{\Delta,j} \end{cases} \tag{14}$$

with the definitions

$$\begin{aligned} &\boldsymbol{x}_\Delta(t) := \tilde{\boldsymbol{x}}(t) - \hat{\boldsymbol{x}}_f(t),\ \boldsymbol{e}(t) := \hat{\boldsymbol{x}}_f(t) - \boldsymbol{x}_f(t) \\ &\boldsymbol{A}_{\Delta,j} := \boldsymbol{A}_j - \boldsymbol{B}_f\boldsymbol{M},\ \boldsymbol{B}_{\Delta,j} := \boldsymbol{B} - \boldsymbol{B}_f\boldsymbol{N}_j,\ \boldsymbol{b}_{\Delta,j} := \boldsymbol{b}_j - \boldsymbol{b}_{f,j}. \end{aligned} \tag{15}$$

The reconfiguration block $\Sigma_R = (\Sigma_O, \Sigma_V)$ contains an observer (13) for the faulty plant with the state $\hat{\boldsymbol{x}}_f$. The virtual actuator (14) contains a reference model for the desired plant behavior with the state $\tilde{\boldsymbol{x}}$, and a feedback and feedforward structure to shape the behavior of the faulty plant. The PWA virtual actuator Σ_V together with the state observer Σ_O realizes the reconfiguration block (11) with $\boldsymbol{\zeta}_0 = (\hat{\boldsymbol{x}}_{f,0}^T,\ \tilde{\boldsymbol{x}}_0^T)^T$ (Fig. 3). Both subsystems are initialized at equal values at reconfiguration time $t = 0$. The affine input $\boldsymbol{B}_f^+\boldsymbol{b}_{\Delta,j}$ compensates the difference in the affine terms between the nominal and the faulty plant, which arises, for example, from blocking actuators. The compensation is successful if and only if

$$\boldsymbol{b}_{\Delta,j} \in \operatorname{im} \boldsymbol{B}_f.$$

The observer error subsystem Σ_e and difference system Σ_Δ are governed by the following differential equations, which are easily obtained from the definitions of $\boldsymbol{e}$ and $\boldsymbol{x}_\Delta$:

$$\Sigma_e :\ \dot{\boldsymbol{e}}(t) = \boldsymbol{k}(\boldsymbol{x}_f(t) + \boldsymbol{e}(t)) - \boldsymbol{k}(\boldsymbol{x}_f(t)) - \boldsymbol{B_d}\boldsymbol{d}(t) \tag{16a}$$

$$\text{where } \boldsymbol{k}(\boldsymbol{\xi}) := \boldsymbol{A}_{\delta,i}\boldsymbol{\xi} + \boldsymbol{b}_{f,i},\ \boldsymbol{\xi} \in \Lambda_i,\ i \in \{1,\ldots,p\}, \tag{16b}$$

$$\begin{aligned} \Sigma_\Delta :\ \dot{\boldsymbol{x}}_\Delta(t) &= \boldsymbol{k}_\Delta(\tilde{\boldsymbol{x}}(t)) - \boldsymbol{k}_\Delta(\tilde{\boldsymbol{x}}(t) - \boldsymbol{x}_\Delta(t)) + \boldsymbol{L}\boldsymbol{C}_f\boldsymbol{e}(t), \\ &+ \boldsymbol{B}_{\Delta,j}\boldsymbol{u}_c(t),\ \tilde{\boldsymbol{x}} \in \Lambda_j,\ j \in \{1,\ldots,p\} \end{aligned} \tag{16c}$$

$$\text{where } \boldsymbol{k}_\Delta(\boldsymbol{\eta}) := \boldsymbol{A}_{\Delta,j}\boldsymbol{\eta} + \boldsymbol{b}_{f,j},\ \boldsymbol{\eta} \in \Lambda_j,\ j \in \{1,\ldots,p\}. \tag{16d}$$

The virtual actuator may be interpreted as an approach to matching the reconfigured plant behavior to the nominal plant behavior. The reference model for $\tilde{\boldsymbol{x}}$ is central to this idea. The common feedback gains $\boldsymbol{L} \in \mathbb{R}^{n\times r}$ and $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ will be designed to simultaneously stabilize the observer error $\boldsymbol{e}$ as well as the difference system state $\boldsymbol{x}_\Delta$ for all modes. The feedforward gains $\boldsymbol{N}_j \in \mathbb{R}^{m\times m}$ represent additional freedom available to achieve additional objectives. Due to space limitations, their systematic use is not discussed here.

Remark 1: The PWA virtual actuator is a generalization of the linear virtual actuator [8]–[11], [27]. Although it appears to be quite different from its linear counterpart, the basic idea of *predicting the difference between the state trajectories of the nominal and faulty plant starting from the same initial state, and using that predicted difference for state feedback*, is the same. In the linear case, that difference can be lumped into a single difference state due to the superposition principle. In PWA systems, superposition does not hold any longer, hence both systems must be independently predicted. This fact renders the initialization problem more difficult.

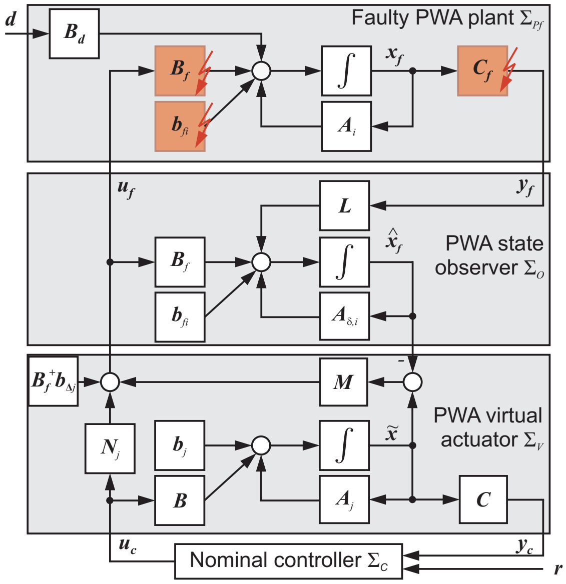

Fig. 3. PWA reconfiguration block in the closed-loop system.

B. Weak fault-hiding

It is shown that the reconfiguration block (13), (14) with the faulty plant (9) satisfies the weak fault-hiding goal.

Theorem 1 (Weak fault-hiding): Consider the faulty PWA system (9). The reconfigured plant (9), (13), (14) achieves the weak fault-hiding goal.

Proof: The model of the reconfigured plant is given by the equations (subscript j defined by $\tilde{\boldsymbol{x}} \in \Lambda_j$)

$$\begin{pmatrix} \dot{\tilde{\boldsymbol{x}}}(t) \\ \dot{\boldsymbol{e}}(t) \\ \dot{\boldsymbol{x}}_\Delta(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_j \tilde{\boldsymbol{x}}(t) + \boldsymbol{b}_j \\ \boldsymbol{k}\left(\tilde{\boldsymbol{x}}(t) - \boldsymbol{x}_\Delta(t)\right) - \boldsymbol{k}\left(\tilde{\boldsymbol{x}}(t) - \boldsymbol{x}_\Delta(t) - \boldsymbol{e}(t)\right) \\ \boldsymbol{k}_\Delta(\tilde{\boldsymbol{x}}(t)) - \boldsymbol{k}_\Delta(\tilde{\boldsymbol{x}}(t) - \boldsymbol{x}_\Delta(t)) + \boldsymbol{L}\boldsymbol{C}_f \boldsymbol{e}(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{B}^T & \boldsymbol{0} & \boldsymbol{B}_{\Delta,j}^T \end{pmatrix}^T \boldsymbol{u}_c(t) - \begin{pmatrix} \boldsymbol{0} & \boldsymbol{B}_d^T & \boldsymbol{0} \end{pmatrix}^T \boldsymbol{d}(t), \tag{17a}$$

$$\boldsymbol{y}_c(t) = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{x}}(t) \\ \boldsymbol{e}(t) \\ \boldsymbol{x}_\Delta(t) \end{pmatrix}, \quad \begin{pmatrix} \tilde{\boldsymbol{x}}(0) \\ \boldsymbol{e}(0) \\ \boldsymbol{x}_\Delta(0) \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{x}}_{f,0} \\ \hat{\boldsymbol{x}}_{f,0} - \boldsymbol{x}_0 \\ \boldsymbol{0} \end{pmatrix} \tag{17b}$$

This model shows that the dynamical equation for the reference state $\tilde{\boldsymbol{x}}$ is decoupled from the observer error $\boldsymbol{e}$ and the difference state $\boldsymbol{x}_\Delta$. Moreover, the output $\boldsymbol{y}_c$ depends only on the $\tilde{\boldsymbol{x}}$. Fault hiding is achieved by setting $\hat{\boldsymbol{x}}_{f,0} = \boldsymbol{x}_0$. ∎

The latter matching initialization $\hat{\boldsymbol{x}}_{f,0} = \boldsymbol{x}_0$ is practically not achievable, because $\boldsymbol{x}_0$ is unmeasurable. However, if the observer is also run with the nominal PWA model (1) before the fault, and the diagnosis delay and reconfiguration time are small, then approximately matching initialization is achievable, and the caused system trajectory offset is small.

C. Stability

In this section, we provide sufficient conditions for the global ISS properties of the observer error and difference system dynamics (Theorems 2, 3), and show that these conditions also imply the global ISS of the reconfigured closed-loop system (Theorem 4).

Theorem 2 (Observer error ISS): Consider the faulty PWA system (9), and suppose that Assumption 1 holds. If there exist matrices $\boldsymbol{X}_s \in \mathbb{R}^{n\times n}$ and $\boldsymbol{Y}_s \in \mathbb{R}^{n\times r}$ that satisfy the linear matrix inequalities

$$\boldsymbol{X}_s = \boldsymbol{X}_s^T > 0 \tag{18a}$$

$$\boldsymbol{X}_s \boldsymbol{A}_i + \boldsymbol{A}_i^T \boldsymbol{X}_s - \boldsymbol{Y}_s \boldsymbol{C}_f - \boldsymbol{C}_f^T \boldsymbol{Y}_s^T < 0, \ i = 1, \ldots, p, \tag{18b}$$

then the system (13) with $\boldsymbol{L} := \boldsymbol{X}_s^{-1}\boldsymbol{Y}_s$ is an observer for the faulty system (9) with 0-GES error dynamics for $\boldsymbol{d} \equiv \boldsymbol{0}$. The observer error (15) satisfies the dynamics (16a), (16b) whose solutions satisfy the relation

$$|\boldsymbol{e}(t)| \le c\mathrm{e}^{-a(t-t_0)}|\boldsymbol{e}(t_0)|, \tag{19}$$

where the real numbers $c > 0$ and $a > 0$ depend only on $\boldsymbol{X}_s$ and $\boldsymbol{Y}_s$. Furthermore the observer error dynamics (16a), (16b) are ISS w.r.t. the disturbance input $\boldsymbol{d}$.

Proof: See Appendix A. ∎

Theorem 3 (Difference system ISS): Consider the faulty PWA system (9) and suppose that Assumption 1 holds. If there exist matrices $\boldsymbol{X}_a \in \mathbb{R}^{n\times n}$ and $\boldsymbol{Y}_a \in \mathbb{R}^{m\times n}$ that satisfy the linear matrix inequalities

$$\boldsymbol{X}_a = \boldsymbol{X}_a^T > 0 \tag{20a}$$

$$\boldsymbol{A}_i \boldsymbol{X}_a + \boldsymbol{X}_a \boldsymbol{A}_i^T - \boldsymbol{B}_f \boldsymbol{Y}_a - \boldsymbol{Y}_a^T \boldsymbol{B}_f^T < 0, \ i = 1, \ldots, p, \tag{20b}$$

then the difference system (16c), (16d) of the virtual actuator (14) with $\boldsymbol{M} := \boldsymbol{Y}_a \boldsymbol{X}_a^{-1}$ is 0-GES for $\boldsymbol{u}_c, \boldsymbol{e} \equiv \boldsymbol{0}$. Therefore, all solutions of the unforced (i.e. $\boldsymbol{u}_c, \boldsymbol{e} \equiv \boldsymbol{0}$) difference system (16c), (16d) satisfy the relation

$$|\boldsymbol{x}_\Delta(t)| \le c\mathrm{e}^{-a(t-t_0)}|\boldsymbol{x}_\Delta(t_0)|, \tag{21}$$

where the real numbers $c > 0$ and $a > 0$ depend only on $\boldsymbol{X}_a$ and $\boldsymbol{Y}_a$. Furthermore, the difference system (16c), (16d) is ISS w.r.t. the input $(\boldsymbol{u}_c(t), \boldsymbol{e}(t))$.

Proof: See Appendix A. ∎

Theorem 4 (Closed-loop stability): Suppose that all assumptions of Theorems 2 and 3 hold. Then, the reconfigured closed-loop system consisting of the controller (3), the faulty PWA system (9), the PWA observer (13), and the PWA virtual actuator (14) is globally ISS w.r.t. the input $(\boldsymbol{r}, \boldsymbol{d})$.

Proof: See Appendix A. ∎

Remark 2: The obtained stability results are valid whether or not the reconfiguration block initial state satisfies $\hat{\boldsymbol{x}}_{f,0} = \boldsymbol{x}_0, \tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0$, as becomes clear from the proof of Theorem 4 while observing that $\hat{\boldsymbol{x}}_{f,0}$ only affects $\boldsymbol{e}(0)$ and $\boldsymbol{x}_\Delta(0)$.

D. Reconfiguration algorithm

The design procedure for the reconfiguration block is summarized in Algorithm 1. Steps 1-4 describe the nominal closed-loop operation before any faults occur. Once faults are detected in step 4, the actual observer and virtual actuator design proceeds in steps 5-10, which is the gains calculation phase. After completed gain calculations, the reconfigured closed-loop system is run in step 11.

Algorithm 1 Synthesis of stabilizing PWA virtual actuator

Require: PWA model $\boldsymbol{A}_i$, $\boldsymbol{b}_i$, $\boldsymbol{B}$, $\boldsymbol{C}$, $i \in \{1, \ldots, p\}$

1: Initialize the nominal closed-loop system (1), (3), (13), (14), with $\boldsymbol{C}_f = \boldsymbol{C}$, $\boldsymbol{B}_f = \boldsymbol{B}$, $\boldsymbol{b}_{f,i} = \boldsymbol{b}_i$, $\boldsymbol{L} = \boldsymbol{0}$, $\boldsymbol{M} = \boldsymbol{0}$, $\boldsymbol{N}_i = \boldsymbol{I}$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$, $\boldsymbol{x}_c(0) = \boldsymbol{x}_{c0}$, $\hat{\boldsymbol{x}}_f(0) = \boldsymbol{x}_0$, $\tilde{\boldsymbol{x}}(0) = \boldsymbol{x}_0$.
2: **repeat**
3: Run nominal closed-loop system
4: **until** actuator or sensor fault f detected and isolated
5: Construct fault model $\boldsymbol{b}_{f,i}$, $\boldsymbol{B}_f$, $\boldsymbol{C}_f$ and update the PWA observer (13) and virtual actuator (14)
6: Solve LMI (18) and (20) for $\boldsymbol{X}_s$, $\boldsymbol{Y}_s$, $\boldsymbol{X}_a$, $\boldsymbol{Y}_a$
7: Compute $\boldsymbol{L} = \boldsymbol{X}_s^{-1}\boldsymbol{Y}_s$ and $\boldsymbol{M} = \boldsymbol{Y}_a \boldsymbol{X}_a^{-1}$
8: Update PWA observer (13) with $\boldsymbol{L}$
9: Wait for observer convergence for specified time interval
10: Update PWA virtual actuator (14) with $\boldsymbol{M}$ and initialize $\tilde{\boldsymbol{x}}(t_r) = \hat{\boldsymbol{x}}_f(t_r)$
11: Run reconfigured closed-loop system (3), (9), (13), (14)

Result: Globally ISS reconfigured closed-loop system.

If the LMI (18), (20) are infeasible, a stabilising reconfiguration block might still exist, but cannot be found with the described sufficient (but not necessary) LMI conditions.

V. E

A successful application of Algorithm 1 to the model of a two-tanks system is presented in this section. The plant consists of tanks T_1, T_2 with levels h_1, h_2 interconnected by valves u_L, u_U, where T_1 is filled via pump u_P (Fig. 4). With state $\boldsymbol{x} = (h_1, h_2)^T$ and input vector $\boldsymbol{u} = (u_P, u_L, u_U)^T$, the plant is described by the model (1) with $p = 22$ and

$$\boldsymbol{B} = 10^{-3}\begin{pmatrix} 8.1 & -3.2 & -3.4 \\ 0 & 3.2 & 3.4 \end{pmatrix},\ \boldsymbol{B_d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \boldsymbol{B}_f = 10^{-3}\begin{pmatrix} 8.1 & 0 & -0.68 \\ 0 & 0 & 0.68 \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \boldsymbol{C}_f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and controlled by two affine decentralized controllers

$$\begin{pmatrix} u_P(t) \\ u_L(t) \\ u_U(t) \end{pmatrix} = \begin{pmatrix} 50 \cdot (r_1(t) - y_1(t)) + 4 \cdot \int_0^t (r_1(\tau) - y_1(\tau))\mathrm{d}\tau \\ 50 \cdot (r_2(t) - y_2(t)) + 4 \cdot \int_0^t (r_2(\tau) - y_2(\tau))\mathrm{d}\tau \\ 0.8 \end{pmatrix}.$$

The controlled quantities are the levels h_1, h_2, for which the control aims (i) stability, and (ii) regulation to a given setpoint, are formulated. In the process, an abrupt failure of the level sensor for h_1 ($f_1 : y_{f,1}(t > t_{f1}) = 0$) at time $t_{f1} = 60$ s and a failure of the lower valve and gain reduction for the upper valve ($f_2 : u_{f,L}(t > t_{f2}) = 0, u_{f,U}(t > t_{f2}) = 0.2u_U(t > t_{f2})$) at fault time $t_{f2} = 80$ s. The plant is excited by reference steps of 0.35 m for the level h_1 at time $t = 30$ s and of 0.03 m at time $t = 100$ s for the level h_2. The steps drive the process through a large operating regime, and thus realistically describe a startup procedure. A leak of tank T_1 represents a disturbance d. Note that the fault breaks the loop at several points and the reconfiguration method must change the control loop structure to meet the control objectives.

Fig. 5 shows the behavior of the successfully reconfigured plant, the discrete modes of Σ_{Pf}, Σ_O, and Σ_V, as well as $\boldsymbol{d}(t)$, $\boldsymbol{e}(t)$, and $\boldsymbol{x}_\Delta(t)$. Times $t < t_{f1}$ correspond to steps 1-3 of Algorithm 1. The application of steps 4-11 of Algorithm 1 result in the matrices

$$\boldsymbol{L} = \begin{pmatrix} 0 & 83.9 \\ 0 & 5.4 \end{pmatrix},\ \boldsymbol{M} = \begin{pmatrix} 60 & 61.4 \\ 0 & 0 \\ -3.9 & 700.5 \end{pmatrix},\ \boldsymbol{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4.3 & 5 \end{pmatrix}. \quad (22)$$

The matrix $\boldsymbol{N}$ may be arbitrary according to Algorithm 1, and is designed to recover the equilibrium using a synthesis method that will be described in a future publication. The observer was augmented for unknown-input observation to obtain an unbiased state estimate in spite of the disturbance. The figure shows that the virtual actuator successfully keeps $\boldsymbol{x}_\Delta$ small.

VI. C

In this paper a new approach to reconfigurable control of piecewise affine systems was presented that works by placing a reconfiguration block between the faulty plant and the nominal controller. The idea generalizes fault-hiding-based ideas known from the linear framework to piecewise affine systems (Theorem 1). It was shown how the gains of the reconfiguration block may be determined as a feasible solution to a set of linear matrix inequalities (Theorems 2,

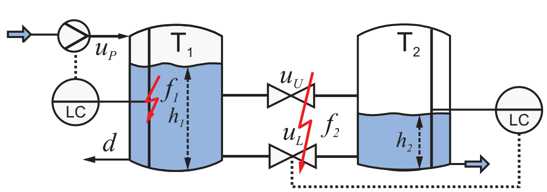

Fig. 4. Two-tank system with nominal control loops.

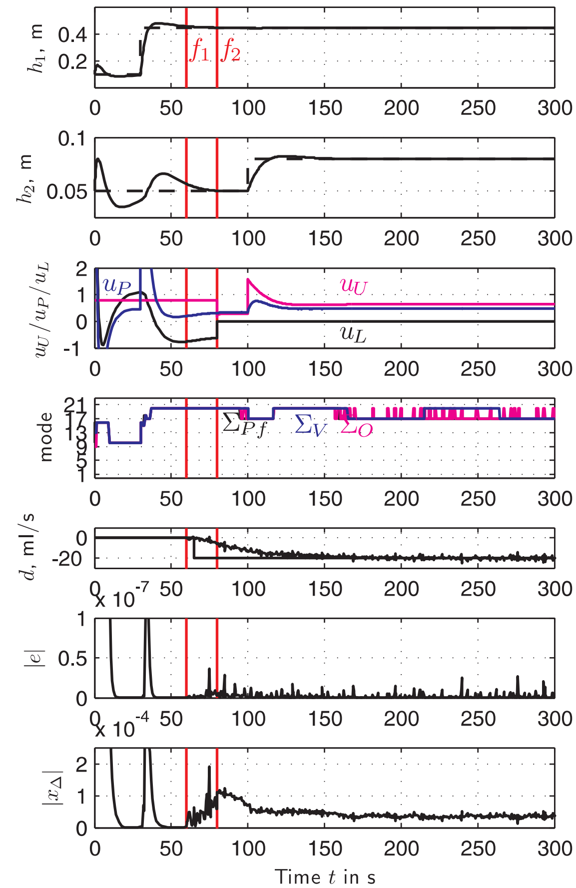

Fig. 5. Control reconfiguration to pump u_P after valve faults.

3) in order to obtain an input-to-state stable reconfigured closed-loop system (Theorem 4). The complexity of the problem is the same as that of general PWA observation and stabilization problems, and sufficient but not necessary conditions have been derived.

The approach bears potential for further extension by explicitly incorporating performance recovery and disturbance rejection goals in addition to the stabilization goal.

VII. A

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A A. P

Proof of Theorem 2: The case without disturbance is equivalent to the case considered in [28]. To show ISS of the error dynamics (16a), (16b) w.r.t. the disturbance input $\boldsymbol{d}$, construct an ISS-Lyapunov function $V(\boldsymbol{e}) = \frac{1}{2}\boldsymbol{e}^T\boldsymbol{X}\boldsymbol{e}$, and use Proposition 2 to obtain

$$\dot{V}(\boldsymbol{e}) = \boldsymbol{e}^T\boldsymbol{X}\dot{\boldsymbol{e}} = \boldsymbol{e}^T\boldsymbol{X}(\boldsymbol{k}(\boldsymbol{x}_f + \boldsymbol{e}) - \boldsymbol{k}(\boldsymbol{x}_f) - \boldsymbol{B}_d\boldsymbol{d})$$
$$\le -a\boldsymbol{e}^T\boldsymbol{X}\boldsymbol{e} - \boldsymbol{e}^T\boldsymbol{X}\boldsymbol{B}_d\boldsymbol{d},\ a > 0.$$

This is already a Lyapunov characterization of ISS [25]. ∎

Proof of Theorem 3: By Proposition 2, the difference system (16c), (16d) is exponentially stable for $\boldsymbol{u}_c(t) = \boldsymbol{e}(t) = \boldsymbol{0}$ and ISS w.r.t. its inputs $\boldsymbol{u}_c$ and $\boldsymbol{e}$ if the condition

$$\tilde{\boldsymbol{X}}(\boldsymbol{A}_i - \boldsymbol{B}_f\boldsymbol{M}) + (\boldsymbol{A}_i - \boldsymbol{B}_f\boldsymbol{M})^T\tilde{\boldsymbol{X}} < 0,\ \tilde{\boldsymbol{X}} = \tilde{\boldsymbol{X}}^T > 0$$

is satisfied for all $i = 1, \ldots, p$, which is equivalent to Condition (20) after pre-and postmultiplication with $\tilde{\boldsymbol{X}}^{-1}$, reordering, and a linearizing change of variables $\boldsymbol{X}_a = \tilde{\boldsymbol{X}}^{-1}$ as well as $\boldsymbol{Y}_a = \boldsymbol{M}\boldsymbol{X}_a$. The exponential decay rate of the initial state follows from

Proposition 2. We now show that the difference system is ISS w.r.t. $\boldsymbol{u}_c$ and $\boldsymbol{e}$. Consider the quadratic function $V(\boldsymbol{x}_\Delta) = \frac{1}{2}\boldsymbol{x}_\Delta^T \boldsymbol{P}\boldsymbol{x}_\Delta$. Using Proposition 2, its derivative along solutions of (16c), (16d) satisfies

$$\begin{aligned}\dot{V}(\boldsymbol{x}_\Delta) &= \boldsymbol{x}_\Delta^T \boldsymbol{P}\left(\boldsymbol{k}_\Delta(\tilde{\boldsymbol{x}}) - \boldsymbol{k}_\Delta(\tilde{\boldsymbol{x}} - \boldsymbol{x}_\Delta) + \boldsymbol{L}\boldsymbol{C}_f\boldsymbol{e} + \boldsymbol{B}_{\Delta,j}\boldsymbol{u}_c\right)\\ &\le -a\boldsymbol{x}_\Delta^T\boldsymbol{P}\boldsymbol{x}_\Delta + \boldsymbol{x}_\Delta^T\boldsymbol{P}\boldsymbol{L}\boldsymbol{C}_f\boldsymbol{e} + \boldsymbol{x}_\Delta^T\boldsymbol{P}\boldsymbol{B}_{\Delta,j}\boldsymbol{u}_c\\ &\le -a\boldsymbol{x}_\Delta^T\boldsymbol{P}\boldsymbol{x}_\Delta + \boldsymbol{x}_\Delta^T\boldsymbol{P}\boldsymbol{L}\boldsymbol{C}_f\boldsymbol{e} + \boldsymbol{x}_\Delta^T\boldsymbol{P}\tilde{\boldsymbol{B}}_\Delta\boldsymbol{u}_c\end{aligned}$$

where $\tilde{\boldsymbol{B}}_\Delta$ is a constant matrix that satisfies the condition $\forall i$: $\|\boldsymbol{B}_{\Delta,j}\| \le \|\tilde{\boldsymbol{B}}_\Delta\|$, and some constant $a > 0$. This is a Lyapunov characterization of ISS [25], and the difference system (16c), (16d) is ISS w.r.t. the input $(\boldsymbol{u}_c, \boldsymbol{e})$. ■

Proof of Theorem 4: Closed-loop stability is established by using Theorem 1, nominal closed-loop IOS (Assumption 2) and the established stability properties of the observer error and difference systems together with Proposition 1. The equivalent closed-loop block diagram is shown in Fig. 6.

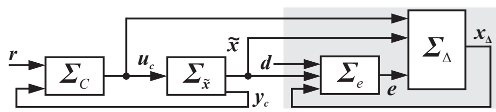

Fig. 6. Transformed closed-loop system (3), (9), (13), (14).

First, consider the feedback connection $(\Sigma_e, \Sigma_\Delta)$ (shaded block in Fig. 6), to which the signals $\boldsymbol{u}_c$, $\boldsymbol{d}$, and $\tilde{\boldsymbol{x}}$ are external inputs. Normally, to conclude ISS of feedback connections, a small-gain argument is needed. In this case, however, Σ_e is exponentially stable for arbitrary interconnection inputs $\tilde{\boldsymbol{x}}$, $\boldsymbol{x}_\Delta$, which we use as follows. From Theorem 2, the error dynamics are 0-GES for arbitrary inputs $\tilde{\boldsymbol{x}}$ and $\boldsymbol{x}_\Delta$ and ISS w.r.t. the disturbance $\boldsymbol{d}$. In other words, we have that $|\boldsymbol{e}(t)| \le \beta_e(|\boldsymbol{e}(0)|, t) + \gamma_d(\|\boldsymbol{d}\|_\infty)$ for some $\beta_e \in \mathcal{KL}$, $\gamma_d \in \mathcal{K}_\infty$. The difference system is globally exponentially stable with $\boldsymbol{u}_c = \boldsymbol{e} = \boldsymbol{0}$ for arbitrary $\tilde{\boldsymbol{x}}$ and ISS w.r.t. $\boldsymbol{u}_c$ and $\boldsymbol{e}$, hence we have $|\boldsymbol{x}_\Delta(t)| \le \beta_\Delta(|\boldsymbol{x}_\Delta(0)|, t) + \gamma_u(\|\boldsymbol{u}_c\|_\infty) + \gamma_e(\|\boldsymbol{e}\|_\infty)$ for some $\beta_\Delta \in \mathcal{KL}$, $\gamma_u, \gamma_e \in \mathcal{K}_\infty$. Observing that $|(\boldsymbol{e}(t)^T, \boldsymbol{x}_\Delta(t)^T)^T| \le |\boldsymbol{e}(t)| + |\boldsymbol{x}_\Delta(t)|$, and inserting the above relations, the interconnection satisfies the relation

$$\begin{aligned}|(\boldsymbol{e}(t), \boldsymbol{x}_\Delta(t))^T| &\le \beta_e(|\boldsymbol{e}(0)|, t) + \beta_\Delta(|\boldsymbol{x}_\Delta(0)|, t) + \gamma_d(\|\boldsymbol{d}\|_\infty)\\ &\quad + \gamma_u(\|\boldsymbol{u}_c\|_\infty) + \gamma_e(\|\boldsymbol{e}\|_\infty)\\ &\le \beta_e(|\boldsymbol{e}(0)|, t) + \beta_\Delta(|\boldsymbol{x}_\Delta(0)|, t) + \gamma_e\left[\beta_e(|\boldsymbol{e}(0)|, t)\right.\\ &\quad \left.+\gamma_d(\|\boldsymbol{d}\|_\infty)\right] + \gamma_d(\|\boldsymbol{d}\|_\infty) + \gamma_u(\|\boldsymbol{u}_c\|_\infty)\\ &\le \beta_e(|\boldsymbol{e}(0)|, t) + \gamma_e(\beta_e(|\boldsymbol{e}(0)|, t)) + \beta_\Delta(|\boldsymbol{x}_\Delta(0)|, t)\\ &\quad + \gamma_e(\gamma_d(\|\boldsymbol{d}\|_\infty)) + \gamma_d(\|\boldsymbol{d}\|_\infty) + \gamma_u(\|\boldsymbol{u}_c\|_\infty),\end{aligned}$$

where we have used the fact that $\|\boldsymbol{e}\|_\infty \le \beta_e(|\boldsymbol{e}(0)|, t) + \gamma_d(\|\boldsymbol{d}\|_\infty)$. We have obtained an ISS-characterization for the interconnected system $(\Sigma_e, \Sigma_\Delta)$, where clearly $[\beta_e(|\boldsymbol{e}(0)|, t) + \gamma_e(\beta_e(|\boldsymbol{e}(0)|, t)) + \beta_\Delta(|\boldsymbol{x}_\Delta(0)|, t)] \in \mathcal{KL}$ and $[\gamma_e(\gamma_d(\|\boldsymbol{d}\|_\infty)) + \gamma_d(\|\boldsymbol{d}\|_\infty) + \gamma_u(\|\boldsymbol{u}_c\|_\infty)] \in \mathcal{K}_\infty$ hold. We conclude that the subsystem $(\Sigma_e, \Sigma_\Delta)$ is ISS w.r.t. the input $(\boldsymbol{u}_c, \boldsymbol{d}, \tilde{\boldsymbol{x}})$, hence also IOS for the output $(\boldsymbol{e}, \boldsymbol{x}_\Delta)$. The system $(\Sigma_{\tilde{x}}, \Sigma_C)$ is IOS for the input $(\boldsymbol{r}, \boldsymbol{d})$ and the output $(\boldsymbol{u}_c, \tilde{\boldsymbol{x}})$ by Assumption 2. The series connection $((\Sigma_{\tilde{x}}, \Sigma_C), (\Sigma_e, \Sigma_\Delta))$ representing the reconfigured closed-loop system is IOS w.r.t. the input $(\boldsymbol{r}, \boldsymbol{d})$ and the output $(\boldsymbol{e}, \boldsymbol{x}_\Delta)$ by Proposition 1. ■

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