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Reconstructing the right-hand side of a fractional subdiffusion equation from the final data

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Abstract

In this study, we study an inverse source problem for the time-fractional diffusion equation, where the final data $t = T$ are given. We show that our problem is ill-posed in the sense of Hadamard. Applying a truncation method, we give the regularized solution. Finally, convergence estimates under *a priori* and *a posteriori* parameter choice rules are proved.

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1 Introduction

The last few decades have seen interest in the research of diffusion equation problems; more and more researchers are studying the problem of the fractional diffusion equation. It can model anomalous phenomena in chemical physics, biological cell dynamics, physiology, and finance [1–4]. In this study, we consider the following fractional diffusion equation with Riemann–Liouville derivative:

$$\begin{cases} \partial_t u(x,y,t) = \partial_t^{1-\beta}(u_{xx}(x,y,t) + u_{yy}(x,y,t)) + \Phi(t)f(x,y), \\ \quad (x,y,t) \in \Omega \times (0,T), \\ u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, t \in (0,T], \\ u(x,y,T) = g(x,y), \quad (x,y) \in \Omega, \end{cases} \tag{1.1}$$

with the Riemann–Liouville fractional derivative of order $1-\beta \in (0,1)$ defined by [3, 5] as follows:

$$\partial_t^{1-\beta} u(x,y,t) = \frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t (t-\varsigma)^{\beta-1} u(x,y,\varsigma)\, d\varsigma, \quad t > 0, \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma function and $\Omega = (0,\pi)^2$. The Laplacian operator $\Delta u(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t)$ (with the homogeneous Dirichlet boundary condition) has the nor-

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malized eigenbasis

$$e_k(x) = \left(\frac{2}{\pi}\right)^{\frac{d}{2}} \sin(k_1 x_1) \cdots \sin(k_d x_d) \in \mathbb{R}^d$$

and the eigenvalues $(p^2 + q^2)$. We denote $p = (p_1, p_2, \dots, p_d)$, $x = (x_1, x_2, \dots, x_d)$ and $p^2 = p_1^2 + \cdots + p_d^2$. Similarly, we have $q = (q_1, q_2, \dots, q_d)$, $y = (y_1, y_2, \dots, y_d)$ and $q^2 = q_1^2 + \cdots + q_d^2$. Problem (1.1) has been studied by many authors. The direct problem (or the initial value problem) concerning the first equation of (1.1) seems to come from [6] where the authors introduced the integral formulation of the fractional diffusion equation. These equations have attracted a lot of attention, see, e.g., [7, 8]. Models with a singular kernel at the origin are well-known and arise in the heat conduction and viscoelasticity, etc. [9, 10]. Some works for the linear case can be found in [9, 10]. When we know the final data g, the problem (1.1) is about recovering the initial data $u(x, y, 0)$, called the backward problem, which was studied recently by Yong et al. [11]. Our main aim in this paper is to recover the source function f from the given data $g(x, y)$ and $\Phi(t)$. Here $f(x, y)$ and $\Phi(t)$ describe the spatial distribution of the source and the time evolution pattern, respectively. In this case, our problem is called the inverse source problem (ISP) which is motivated by many practical applications. We can list some applications of (ISP), for example, pollution in the environment [12, 13], dislocation problems [14], biomedical imaging techniques such as the thermo-acoustic tomography [15, 16], electroencephalography/magnetoencephalography (EEG/MEG) problems [17, 18], optical tomography [19]. The inverse source problem for some other fractional diffusion equations has been investigated by many authors and its physical background can be found in [20]; see also the works by Wei et al. [21–23] and Kirane et al. [24, 25]. To the best of our knowledge, there are no results for the inverse source problem (1.1).

Now, we return and discuss more details on the purpose of our paper. It is known that the inverse source problem mentioned above is not well-posed in general, i.e., when a solution exists, it does not depend continuously on the given data. In practice, the exact data (Φ, g) is noised by measured data $(\Phi^\varepsilon, g^\varepsilon)$ with order of $\varepsilon > 0$ as follows:

$$\left\| \Phi^\varepsilon - \Phi \right\|_{\mathcal{L}^\infty(0,T)} \leq \varepsilon, \qquad \left\| g^\varepsilon - g \right\|_{\mathcal{L}^2(\Omega)} \leq \varepsilon, \tag{1.3}$$

where $\|\Phi\|_{\mathcal{L}^\infty(0,T)} = \sup_{0 \leq t \leq T} |\Phi(t)|$ for any $\Phi \in \mathcal{L}^\infty(0, T)$. This problem is ill-posed in the sense of Hadamard, which means that small changes in the observed data can lead to a blow-up of the solution. Hence some regularization methods are required for stable computation of a sought solution. The topic of this paper is to find an approximate solution. Employing some previously suggested ideas, in this study, using the Fourier regularization method, we establish a regularized solution. Under an *a priori* bound assumption of the sought solution and *a priori* parameter choice rule, we give the convergence rate. In practice, an *a priori* bound is difficult to obtain and check, so we need an *a posteriori* parameter choice rule. The strong point of an *a posteriori* parameter choice rule is that it does not depend on the *a priori* bound.

The paper is organized into three sections. In Sect. 2, we present a formula of the source function f and establish some lemmas and theorems which are useful to obtain the next results. The ill-posedness of the inverse source problem is also shown in this section. In

Sect. 3, we apply the Fourier regularization method and give two convergence estimates and two regularization parameter choice rules: an *a priori* parameter choice and an *a posteriori* parameter choice are presented in Sect. 3.1 and Sect. 3.2, respectively.

2 Preliminary results

Definition 2.1 Let $\langle\cdot,\cdot\rangle$ be a scalar product in $\mathcal{L}^2(\Omega)$. The notation $\|\cdot\|_X$ stands for the norm in the Banach space X. We denote by $\mathcal{L}^p(0,T;X)$, $1\le p<\infty$, the Banach space of p-integrable real-valued functions $u:(0,T)\to X$, and set

$$\|u\|_{\mathcal{L}^\infty(0,T;X)} = \operatorname{ess}\sup_{t\in(0,T)}\big\|u(t)\big\|_X, \quad \text{for } p=\infty. \tag{2.1}$$

Definition 2.2 (see [19]) The Mittag-Leffler function is defined by the series

$$E_{\kappa,\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\kappa n+\alpha)}, \quad \kappa>0,\alpha\in\mathbb{R},z\in\mathbb{C}, \tag{2.2}$$

where $\kappa>0$, $\alpha\in\mathbb{R}$ are arbitrary constants.

Lemma 2.1 (see [19]) *Let $\kappa>0$ and $\alpha\in\mathbb{R}$. Then one has*

$$E_{\kappa,\alpha}(z)=zE_{\kappa,\kappa+\alpha}(z)+\frac{1}{\Gamma(\alpha)}, \quad z\in\mathbb{C}. \tag{2.3}$$

Lemma 2.2 (see [26]) *Letting $\lambda>0$ and $\beta>0$ with k being and integer from $\mathbb{N}^*$, one has*

$$\begin{aligned}
&\frac{d^k}{dt^k}E_{\beta,1}\big(-\lambda t^\beta\big) = -\lambda t^{\beta-k}E_{\beta,\beta-k+1}\big(-\lambda t^\beta\big), \quad t>0,\\
&\frac{d}{dt}\big(tE_{\beta,2}\big(-\lambda t^\beta\big)\big) = E_{\beta,1}\big(-\lambda t^\beta\big), \quad t>0.
\end{aligned} \tag{2.4}$$

Lemma 2.3 (see [3]) *Let $0<\beta_0<\beta_1<1$. Then there exist positive constants $\widetilde{\Lambda}_1,\widetilde{\Lambda}_2,\widetilde{\Lambda}_3$ depending only on β_0,β_1 such that for all $\beta\in[\beta_0,\beta_1]$,*

$$\frac{\widetilde{\Lambda}_1}{1+z}\le E_{\beta,1}(-z)\le\frac{\widetilde{\Lambda}_2}{1+z}, \qquad E_{\beta,\kappa}(-z)\le\frac{\widetilde{\Lambda}_3}{1+z}, \quad \textit{for all } z\ge0,\kappa\in\mathbb{R}. \tag{2.5}$$

Lemma 2.4 (see [3]) *Let $0<\beta<1$ and $\lambda,a>0$. Then*

(i) $\partial_t(E_\beta(-\lambda t^\beta)) = -\lambda t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)$, *for $t\ge0$,*
(ii) $\partial_t(t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)) = t^{\beta-2}E_{\beta,\beta-1}(-\lambda t^\beta)$, *for $t\ge0$,*
(iii) $\int_0^\infty e^{-st}E_\alpha(-at^\alpha)=\frac{s^{\alpha-1}}{s^\alpha+a}$, *for $\Re(s)>a^{\frac{1}{\alpha}}$.*

Lemma 2.5 *Let $0<\beta<1$, for $p,q>0$ and denote $\widetilde{\lambda}_{pq}=p^2+q^2$, then*

$$\frac{\widetilde{\mathcal{M}}_\beta^{\dagger}(\widetilde{\Lambda}_1)}{\widetilde{\lambda}_{pq}}\le\int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta\big)\,d\varsigma\le\frac{\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)}{\widetilde{\lambda}_{pq}}, \tag{2.6}$$

where

$$\widetilde{M}_\beta^{\dagger}(\widetilde{\Lambda}_1)=\frac{2\widetilde{\Lambda}_1T}{1+2T^\beta}, \qquad \widetilde{M}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)=\frac{\widetilde{\Lambda}_2T^{1-\beta}}{1-\beta}. \tag{2.7}$$

Proof From Lemma 2.3, we have

$$\begin{aligned}\int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^{\beta}\big)\,d\varsigma &\le \int_0^T \frac{\widetilde{\Lambda}_2\,ds}{1+\widetilde{\lambda}_{pq}(T-\varsigma)^{\beta}} \\ &\le \frac{\widetilde{\Lambda}_2}{\widetilde{\lambda}_{pq}}\int_0^T \frac{d\varsigma}{(T-\varsigma)^{\beta}} = \frac{1}{\widetilde{\lambda}_{pq}}\bigg(\frac{\widetilde{\Lambda}_2 T^{1-\beta}}{1-\beta}\bigg). \end{aligned} \tag{2.8}$$

Next, because of inequality $(T-\varsigma)^{\beta} \le T^{\beta}$ valid for any $\varsigma \in [0,T]$ and Lemma 2.3, one has

$$\begin{aligned}\int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^{\beta}\big)d\varsigma &\ge \widetilde{\Lambda}_1\int_0^T \frac{d\varsigma}{1+\widetilde{\lambda}_{pq}(T-\varsigma)^{\beta}} \ge \frac{\widetilde{\Lambda}_1}{\widetilde{\lambda}_{pq}}\int_0^T \frac{d\varsigma}{\frac{1}{2}+(T-\varsigma)^{\beta}} \\ &\ge \frac{\widetilde{\Lambda}_1}{\widetilde{\lambda}_{pq}}\int_0^T \frac{d\varsigma}{\frac{1}{2}+T^{\beta}} = \frac{1}{\widetilde{\lambda}_{pq}}\bigg(\frac{2\widetilde{\Lambda}_1 T}{1+2T^{\beta}}\bigg). \end{aligned} \tag{2.9}$$

Combining (2.9) and (2.8), we conclude that

$$\frac{1}{\widetilde{\lambda}_{pq}}\bigg(\frac{\widetilde{\Lambda}_1 2T}{1+2T^{\beta}}\bigg) \le \int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^{\beta}\big)d\varsigma \le \frac{1}{\widetilde{\lambda}_{pq}}\bigg(\frac{\widetilde{\Lambda}_2 T^{1-\beta}}{1-\beta}\bigg). \tag{2.10}$$

Denoting $\widetilde{M}^{\dagger}_{\beta}(\widetilde{\Lambda}_1) = (\frac{\widetilde{\Lambda}_1 2T}{1+2T^{\beta}}), \widetilde{M}^{\dagger\dagger}_{\beta}(\widetilde{\Lambda}_2) = (\frac{\widetilde{\Lambda}_2 T^{1-\beta}}{1-\beta})$ shows that (2.6) holds. □

Lemma 2.6 *Suppose that there exist positive constants such that* $m_0 \le |\Phi^{\varepsilon}(t)| \le M_0$, $\forall t \in [0,T]$. *By choosing* $\varepsilon \in (0, \frac{m_0}{2})$, *we have*

$$\frac{m_0}{2} \le \big|\Phi(t)\big| \le \mathcal{P}(m_0, M_0), \tag{2.11}$$

where $\mathcal{P}(m_0,M_0) = M_0 + \frac{m_0}{2}$.

Proof First of all, we have

$$\begin{aligned}\big|\Phi^{\varepsilon}(t)\big| &\le \big|\Phi(t)\big| + \big|\Phi^{\varepsilon}(t)-\Phi(t)\big| \le \big|\Phi(t)\big| + \sup_{t\in[0,T]}\big|\Phi^{\varepsilon}(t)-\Phi(t)\big| \\ &\le \big|\Phi(t)\big| + \big\|\Phi^{\varepsilon}-\Phi\big\|_{\mathcal{L}^{\infty}(0,T)} \le \big|\Phi(t)\big| + \varepsilon. \end{aligned} \tag{2.12}$$

From (2.12), we know that

$$\big|\Phi(t)\big| \ge \big|\Phi^{\varepsilon}(t)\big| - \varepsilon \ge m_0 - \varepsilon \ge \frac{m_0}{2}. \tag{2.13}$$

Similarly, we get

$$\big|\Phi(t)\big| \le M_0 + \varepsilon < M_0 + \frac{m_0}{2}. \tag{2.14}$$

Denoting $\mathcal{P}(m_0,M_0) = M_0 + \frac{m_0}{2}$, and combining (2.13) with (2.14) leads to (2.11). □

Lemma 2.7 *Let* $\Phi : [0,T] \to \mathbb{R}^{+}$ *be a continuous function, then we have*

$$\frac{m_0 \widetilde{M}^{\dagger}_{\beta}(\widetilde{\Lambda}_1)}{2\widetilde{\lambda}_{pq}} \le \int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^{\beta}\big)\Phi(\varsigma)\,d\varsigma \le \frac{\mathcal{P}(m_0,M_0)\widetilde{M}^{\dagger\dagger}_{\beta}(\widetilde{\Lambda}_2)}{\widetilde{\lambda}_{pq}}. \tag{2.15}$$

2.1 The formula of source term *f*

In this section, we introduce the mild solution of the following initial value problem:

$$\begin{cases} \partial_t u(x,y,t) = \partial_t^{1-\beta}(u_{xx}(x,y,t) + u_{yy}(x,y,t)) + \mathbf{H}(x,y,t), \\ \quad (x,y,t) \in \Omega \times (0,T), \\ u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, t \in (0,T], \\ u(x,y,0) = a(x,y), \quad (x,y) \in \Omega. \end{cases} \tag{2.16}$$

We use the separation of variables to obtain the solution of (2.16). Suppose that the exact u is defined by Fourier series

$$u(x,y,t) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} u_{pq}(t)\mathcal{S}_{pq}(x,y) \tag{2.17}$$

in which $\mathcal{S}_{pq}(x,y) = \sin(px)\sin(qy)$. Taking the Laplace transform of (2.16) we obtain that

$$\varsigma\widetilde{u_{pq}}(\varsigma) - \widetilde{a_{pq}} = -\widetilde{\lambda}_{pq}\varsigma^{1-\beta}\widetilde{u_{pq}}(\varsigma) + \widetilde{\mathbf{H}_{pq}}(\varsigma), \quad \Re(\varsigma) > 0, \tag{2.18}$$

in which $\widetilde{v}$ is the Laplace transform of function v, and so

$$\widetilde{u_{pq}}(s) = \frac{\varsigma^{\beta-1}}{\varsigma^{\beta} + \widetilde{\lambda}_{pq}}\widetilde{a_{pq}} + \frac{\varsigma^{\beta-1}}{\varsigma^{\beta} + \widetilde{\lambda}_{pq}}\widetilde{\mathbf{H}_{pq}}(\varsigma), \tag{2.19}$$

where $\mathbf{H}_{pq}(t) = \langle \mathbf{H}(\cdot,\cdot,t), \mathcal{S}_{pq}(x,y)\rangle$. It follows from Lemma 2.3 and the uniqueness of Laplace transform that

$$u_{pq}(t) = E_{\beta,1}\bigl(-\widetilde{\lambda}_{pq}t^{\beta}\bigr)a_{pq} + \int_0^t E_{\beta,1}\bigl(-\widetilde{\lambda}_{pq}(t-\varsigma)^{\beta}\bigr)\mathbf{H}_{pq}(\varsigma)\,d\varsigma. \tag{2.20}$$

By letting $t = T$ in the last equality, recalling $a_{pq} = 0$ and $\mathbf{H}_{pq}(\varsigma) = \Phi(\varsigma)f_{pq}$, we have

$$u_{pq}(T) = g_{pq} = f_{pq}\int_0^T E_{\beta,1}\bigl(-\widetilde{\lambda}_{pq}(T-\varsigma)^{\beta}\bigr)\Phi(\varsigma)\,d\varsigma. \tag{2.21}$$

We get the formula of the source function f as

$$f(x,y) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{g_{pq}\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^{\beta})\Phi(\varsigma)\,d\varsigma}, \tag{2.22}$$

where

$$\begin{aligned} a_{pq} &= \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} a(x,y)\mathcal{S}_{pq}(x,y)\,dx\,dy, \\ g_{pq} &= \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} g(x,y)\mathcal{S}_{pq}(x,y)\,dx\,dy, \\ \mathbf{H}_{pq}(s) &= \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} \mathbf{H}(x,y,s)\mathcal{S}_{pq}(x,y)\,dx\,dy, \end{aligned} \tag{2.23}$$

and where we note that $E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^{\beta}) > 0$ and $\Phi(\varsigma) > 0$ for $0 \le \varsigma \le T$.

Theorem 2.1 *Let $\Phi : [0,T] \to \mathbb{R}$ be as in Lemma 2.7. Then the solution $(u(x,y,t), f(x,y))$ of Problem (1.1) is unique.*

Proof Let f_1 and f_2 be the source functions corresponding to the final values g_1 and g_2, respectively. Suppose that $g_1 = g_2$. Then we prove that $f_1 = f_2$. Using the fact that $E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \geq 0$ for $\varsigma \leq T$. Using Lemma 2.2, we have

$$\begin{aligned} \int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta\big)\Phi(\varsigma)\,d\varsigma &\geq \frac{m_0}{2}\int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta\big)\,d\varsigma \\ &= \frac{m_0}{2} T E_{\beta,2}\big(-\widetilde{\lambda}_{pq}T^\beta\big) > 0. \end{aligned} \tag{2.24}$$

Therefore, we get

$$f_1(x,y) - f_2(x,y) = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{(g_{pq}^1 - g_{pq}^2)\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma} = 0. \tag{2.25}$$

The proof is complete. □

Theorem 2.2 *The inverse source problem is ill-posed.*

Proof To illustrate the ill-posedness of our problem, the relevant counterexample is indicated. Choose the input final data $g_{mn}(x,y) = \frac{\mathcal{S}_{mn}(x,y)}{\sqrt{mn}}$. By (2.22), the source term corresponding to g_{mn} is

$$\begin{aligned} f_{mn}(x,y) &= \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{g_{mn}\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma} \\ &= \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \frac{\frac{\mathcal{S}_{mn}(x,y)}{\sqrt{mn}}\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma} \\ &= \frac{\mathcal{S}_{mn}(x,y)}{\sqrt{mn}}\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{mn}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma. \end{aligned} \tag{2.26}$$

Assume that input final data $g(x,y) = 0$. By (2.22), the source term corresponding to g is $f(x,y) = 0$. The error in $\mathcal{L}^2(\Omega)$ norm between two input final data is

$$\|g_{mn} - g\|_{\mathcal{L}^2(\Omega)} = \left\|\frac{\mathcal{S}_{mn}(x,y)}{\sqrt{mn}}\right\|_{\mathcal{L}^2(\Omega)} = \frac{1}{\sqrt{mn}} \to 0 \quad \text{when } m,n \to \infty. \tag{2.27}$$

The error in $\mathcal{L}^2$ norm between two corresponding source terms is

$$\begin{aligned} \|f_{mn} - f\|^2_{\mathcal{L}^2(\Omega)} &= \left(\frac{\pi^2}{4}\right)\left\|\frac{\mathcal{S}_{mn}(x,y)}{\sqrt{mn}\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{mn}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma}\right\|^2_{\mathcal{L}^2(\Omega)} \\ &= \left(\frac{\pi^2}{4}\right)\frac{1}{mn}\left(\int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{mn}(T-\varsigma)^\beta\big)\Phi(\varsigma)\,d\varsigma\right)^{-2} \\ &\geq \left(\frac{\pi^2}{4}\right)\frac{1}{mn}\frac{(\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{mn}(T-\varsigma)^\beta)\,d\varsigma)^{-2}}{|\mathcal{P}(m_0,M_0)|} \end{aligned}$$

$$= \left(\frac{\pi^2}{4}\right)\frac{1}{\mathcal{P}(m_0,M_0)\widetilde{M}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)}\frac{\widetilde{\lambda}_{mn}^2}{mn} \to \infty \quad \text{as } m,n \to \infty. \tag{2.28}$$

Hence, from (2.27) and (2.28), there is in general no stability result. Thus, problem (1.1) is ill-posed in the Hadamard sense in $\mathcal{L}^2$ norm. □

2.2 Conditional stability of source term *f*

Theorem 2.3 *For $\mathcal{M} > 0$ such that $\|f\|_{H^\delta(\Omega)} \le \mathcal{M}$,*

$$\|f\|_{\mathcal{L}^2(\Omega)} \le \widetilde{C}(\delta,\mathcal{M})\|g\|_{\mathcal{L}^2(\Omega)}^{\frac{\delta}{\delta+1}}, \tag{2.29}$$

in which

$$\widetilde{C}(\delta,\mathcal{M}) = \left(\frac{2}{m_0(\widetilde{M}_\beta^{\dagger}(\widetilde{\Lambda}_1))}\right)^{\frac{\delta}{\delta+1}}\mathcal{M}^{\frac{1}{\delta+1}}. \tag{2.30}$$

Proof From (2.22), combining Hölder inequality and Lemma 2.7, we obtain

$$\begin{aligned}
&\|f\|_{\mathcal{L}^2(\Omega)}^2 \\
&\quad\le \left(\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{|\langle g_{pq},\mathcal{S}_{pq}(x,y)\rangle|^2}{|\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma|^{2\delta+2}}\right)^{\frac{1}{\delta+1}}\left(\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\big|\langle g_{pq},\mathcal{S}_{pq}(x,y)\rangle\big|^2\right)^{\frac{\delta}{\delta+1}} \\
&\quad\le \left(\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{|\langle f_{pq},\mathcal{S}_{pq}(x,y)\rangle|^2}{|\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma|^{2\delta}}\right)^{\frac{1}{\delta+1}}\|g\|_{\mathcal{L}^2(\Omega)}^{\frac{2\delta}{\delta+1}} \\
&\quad\le \left(\frac{\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\widetilde{\lambda}_{pq}^{2\delta}|\langle f_{pq},\mathcal{S}_{pq}(x,y)\rangle|^2}{|\frac{m_0}{2}|^{2\delta}(\widetilde{M}_\beta^{\dagger}(\widetilde{\Lambda}_1))^{2\delta}}\right)^{\frac{1}{\delta+1}}\|g\|_{\mathcal{L}^2(\Omega)}^{\frac{2\delta}{\delta+1}}.
\end{aligned} \tag{2.31}$$

From (2.31), we conclude that

$$\|f\|_{\mathcal{L}^2(\Omega)}^2 \le \left(\frac{2}{|m_0|(\widetilde{M}_\beta^{\dagger}(\widetilde{\Lambda}_1))}\right)^{\frac{2\delta}{\delta+1}}\|f\|_{H^\delta(\Omega)}^{\frac{2}{\delta+1}}\|g\|_{\mathcal{L}^2(\Omega)}^{\frac{2\delta}{\delta+1}}. \tag{2.32}$$

□

3 Fourier truncation regularization and error estimate

In this section, we eliminate all the components of large p,q from the solution where the positive integer $\mathcal{N}_{\mathcal{TR}}$ plays the role of the regularization parameter. The nature of our regularization method is just eliminating all high frequencies from the solution and considering instead of (2.22) only for $\widetilde{\lambda}_{pq} = p^2 + q^2$, where $\mathcal{N}_{\mathcal{TR}}$ is a suitable positive constant. We note that the $\mathcal{N}_{\mathcal{TR}}$ constant will be selected appropriately as a formal parameter. This regularization method is quite simple and convenient to handle some of the problems posed. The current article is devoted to establishing such an approach for problem (1.1). Let

$$\overline{B}_\varepsilon := \{p,q \in \mathbb{N}^*, \widetilde{\lambda}_{pq} = p^2 + q^2 \le \mathcal{N}_{\mathcal{TR}}\}, \tag{3.1}$$

$$\underline{B}_\varepsilon := \{p,q \in \mathbb{N}^*, \widetilde{\lambda}_{pq} = p^2 + q^2 > \mathcal{N}_{\mathcal{TR}}\}. \tag{3.2}$$

We have the regularized solution as follows:

$$f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y) = \sum_{p,q\in \overline{B}_\varepsilon} \frac{g_{pq}^\varepsilon \mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi^\varepsilon(\varsigma)\,d\varsigma}. \tag{3.3}$$

Next, we show error estimation for $\|f(x,y)-f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)\|_{\mathcal{L}^2(\Omega)}$ and give convergence rate under a suitable choice for the regularization parameter.

3.1 Convergence estimate under an a priori regularization parameter choice rule

Theorem 3.1 *Let $f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)$ be the regularized solution of Problem (1.1) with observed data $g^\varepsilon(x,y)$ and let $f(x,y)$ be the exact solution of Problem (1.1) with exact data $g(x,y)$. By choosing parameter regularization $\mathcal{N}_{\mathcal{TR}} = [\zeta]$, where $[\zeta]$ denotes the largest integer less than or equal to ζ,*

- *If $0 < \delta \le 1$ then, by choosing $\zeta = (\frac{\mathcal{M}}{\varepsilon})^{\frac{1}{\delta+1}}$, we obtain*

$$\big\|f(x,y)-f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)\big\|_{\mathcal{L}^2(\Omega)} \le \varepsilon^{\frac{\delta}{\delta+1}}\mathcal{M}^{\frac{1}{\delta+1}}\mathcal{D}(m_0,\beta,f,\Lambda_1). \tag{3.4}$$

- *If $\delta > 1$, then, by choosing $\zeta = (\frac{\mathcal{M}}{\varepsilon})^{\frac{1}{2}}$, we obtain*

$$\big\|f(x,y)-f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)\big\|_{\mathcal{L}^2(\Omega)} \le \varepsilon^{\frac{1}{2}}\mathcal{M}^{\frac{1}{2}}\mathcal{D}(m_0,\beta,f,\Lambda_1), \tag{3.5}$$

where $\mathcal{D}(m_0,\beta,f,\Lambda_1) = \frac{\pi}{2}(1+\max\{\frac{1}{|m_0|\widetilde{\mathcal{M}}_\beta^\dagger(\Lambda_1)}, \frac{\|f\|_{\mathcal{L}^2(\Omega)}}{|m_0|}\})$.

Proof of Theorem 3.1 Using (2.22) and (3.3) and the triangle inequality, we have

$$\begin{aligned}
&f(x,y)-f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)\\
&\quad= \sum_{p=1}^\infty\sum_{q=1}^\infty \frac{g_{pq}\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma}\\
&\qquad- \sum_{p,q\in \overline{B}_\varepsilon} \frac{g_{pq}^\varepsilon\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi^\varepsilon(\varsigma)\,d\varsigma}\\
&\quad= \sum_{p=1}^\infty\sum_{q=1}^\infty \frac{g_{pq}\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma}\\
&\qquad- \sum_{p,q\in \overline{B}_\varepsilon} \frac{g_{pq}\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma}\\
&\qquad+ \sum_{p,q\in \overline{B}_\varepsilon} \frac{g_{pq}\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(s)\,ds}\\
&\qquad- \sum_{p,q\in \overline{B}_\varepsilon} \frac{g_{pq}^\varepsilon\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi^\varepsilon(\varsigma)\,d\varsigma}.
\end{aligned} \tag{3.6}$$

Using (3.6), we can write

$$f(x,y)-f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)$$

$$
\begin{aligned}
&= \underbrace{\sum_{p,q\in \underline{B}_\varepsilon} \frac{g_{pq}\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)d\varsigma}}_{:=\mathcal{A}_1} + \underbrace{\sum_{p,q\in \overline{B}_\varepsilon} \frac{(g_{pq}(x)-g^\varepsilon_{pq}(x))\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi^\varepsilon(\varsigma)d\varsigma}}_{:=\mathcal{A}_2} \\
&\quad + \underbrace{\sum_{p,q\in \overline{B}_\varepsilon} \frac{g_{pq}\mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)d\varsigma} \times \sum_{p,q\in \overline{B}_\varepsilon} \frac{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)(\Phi^\varepsilon(\varsigma)-\Phi(\varsigma))d\varsigma}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi^\varepsilon(\varsigma)d\varsigma}}_{:=\mathcal{A}_3}.
\end{aligned}
\tag{3.7}
$$

Step 1. Firstly, we have the following estimate:

$$
\begin{aligned}
\|\mathcal{A}_1\|^2_{\mathcal{L}^2(\Omega)} &= \frac{\pi^2}{4} \sum_{p,q\in \underline{B}_\varepsilon} \frac{|\langle g_{pq}, \mathcal{S}_{pq}(x,y)\rangle|^2}{|\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma|^2} \\
&= \frac{\pi^2}{4} \sum_{p,q\in \underline{B}_\varepsilon} \bigl|\langle f_{pq}, \mathcal{S}_{pq}(x,y)\rangle\bigr|^2 \\
&\le \frac{\pi^2}{4} \sum_{p,q\in \underline{B}_\varepsilon} (1+\widetilde{\lambda}_{pq})^{-2\delta}(1+\widetilde{\lambda}_{pq})^{2\delta}\bigl|\langle f_{pq}, \mathcal{S}_{pq}(x,y)\rangle\bigr|^2 \\
&\le \frac{\pi^2}{4}(1+\mathcal{N}_{\mathcal{TR}})^{-2\delta}\mathcal{M}^2.
\end{aligned}
\tag{3.8}
$$

Hence, we obtain

$$
\|\mathcal{A}_1\|_{\mathcal{L}^2(\Omega)} \le \frac{\pi}{2}(1+\mathcal{N}_{\mathcal{TR}})^{-\delta}\mathcal{M}. \tag{3.9}
$$

Step 2. The term $\|\mathcal{A}_2\|_{\mathcal{L}^2(\Omega)}$ is bounded by

$$
\begin{aligned}
\|\mathcal{A}_2\|^2_{\mathcal{L}^2(\Omega)} &\le \frac{\pi^2}{4} \sum_{p,q\in \overline{B}_\varepsilon} \frac{|g_{pq}(x)-g^\varepsilon_{pq}(x)|^2}{|\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi^\varepsilon(\varsigma)\,d\varsigma|^2} \\
&\le \frac{\pi^2}{4} \sum_{p,q\in \overline{B}_\varepsilon} \frac{\widetilde{\lambda}^2_{pq}|g_{pq}-g^\varepsilon_{pq}|^2}{|m_0|^2(\widetilde{\mathcal{M}}^\dagger_\beta(\widetilde{\Lambda}_1))^2} \\
&\le \frac{\pi^2}{4} \sup_{1\le p\le \mathcal{N}_{\mathcal{TR}}} \sup_{1\le q\le \mathcal{N}_{\mathcal{TR}}} \frac{\widetilde{\lambda}^2_{pq}}{|m_0|^2(\widetilde{\mathcal{M}}^\dagger_\beta(\widetilde{\Lambda}_1))^2} \sum_{p,q\in \overline{B}_\varepsilon} \bigl|g_{pq}-g^\varepsilon_{pq}\bigr|^2 \\
&\le \frac{\pi^2\widetilde{\lambda}^2_{pq}}{4|m_0|^2(\widetilde{\mathcal{M}}^\dagger_\beta(\Lambda_1))^2} \bigl\|g-g^\varepsilon\bigr\|^2_{\mathcal{L}^2(\Omega)} \le \frac{\pi^2{\mathcal{N}_{\mathcal{TR}}}^2\varepsilon^2}{4|m_0|^2(\widetilde{\mathcal{M}}^\dagger_\beta(\Lambda_1))^2}.
\end{aligned}
\tag{3.10}
$$

Therefore,

$$
\|\mathcal{A}_2\|_{\mathcal{L}(\Omega)} \le \frac{\varepsilon\pi\mathcal{N}_{\mathcal{TR}}}{2|m_0|\widetilde{\mathcal{M}}^\dagger_\beta(\Lambda_1)}. \tag{3.11}
$$

Step 3. The term $\|\mathcal{A}_3\|_{\mathcal{L}^2(\Omega)}$ can be estimated as follows:

$$
\|\mathcal{A}_3\|^2_{\mathcal{L}^2(\Omega)} \le \frac{\pi^2}{4}\left(\sum_{p,q\in \overline{B}_\varepsilon} \left|\frac{|\langle g_{pq}, \mathcal{S}_{pq}(x,y)\rangle|}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi^\varepsilon(\varsigma)\,d\varsigma}\right|^2\right)
$$

$$\times \left(\sum_{p,q \in \overline{B}_\varepsilon} \left| \frac{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)(\Phi^\varepsilon(\varsigma)-\Phi(\varsigma))\,d\varsigma}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi^\varepsilon(\varsigma)\,d\varsigma} \right|^2 \right). \tag{3.12}$$

From (3.12), we get

$$\begin{aligned}
&\|\mathcal{A}_3\|^2_{\mathcal{L}^2(\Omega)} \\
&\quad\le \frac{\pi^2}{4} \left(\sum_{p,q \in \overline{B}_\varepsilon} \frac{|\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)(\Phi^\varepsilon(\varsigma)-\Phi(\varsigma))\,d\varsigma|^2}{|\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi^\varepsilon(\varsigma)\,d\varsigma|^2} \right) \\
&\qquad\times \left(\sum_{p,q \in \overline{B}_\varepsilon} \frac{|\langle g_{pq}, \mathcal{S}_{pq}(x,y)\rangle|^2}{|\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma|^2} \right) \\
&\quad\le \frac{\pi^2}{4} \frac{\|\Phi^\varepsilon - \Phi\|^2_{\mathcal{L}^\infty(0,T)}}{|m_0|^2} \sum_{p=1}^\infty \sum_{q=1}^\infty \frac{|\langle g_{pq}, \mathcal{S}_{pq}(x,y)\rangle|^2}{|\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)\Phi(\varsigma)\,d\varsigma|^2} \\
&\quad= \frac{\pi^2}{4} \frac{\|\Phi^\varepsilon - \Phi\|^2_{\mathcal{L}^\infty(0,T)}}{|m_0|^2} \|f\|^2_{\mathcal{L}^2(\Omega)} \le \frac{\pi^2 \varepsilon^2 \|f\|^2_{\mathcal{L}^2(\Omega)}}{4|m_0|^2}.
\end{aligned} \tag{3.13}$$

From (3.12), we conclude that

$$\|\mathcal{A}_3\|_{\mathcal{L}^2(\Omega)} \le \frac{\pi \varepsilon \|f\|_{\mathcal{L}^2(\Omega)}}{2|m_0|}. \tag{3.14}$$

Combining (3.8), (3.11), and (3.12) yields

$$\begin{aligned}
&\left\| f(x,y) - f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y) \right\|_{\mathcal{L}^2(\Omega)} \\
&\quad\le \frac{\pi \mathcal{M}}{2(1+\mathcal{N}_{\mathcal{TR}})^\delta} + \frac{\pi \varepsilon \mathcal{N}_{\mathcal{TR}}}{2|m_0| \widetilde{\mathcal{M}}^\dagger_\beta(\Lambda_1)} + \frac{\pi \varepsilon \|f\|_{\mathcal{L}^2(\Omega)}}{2|m_0|}.
\end{aligned} \tag{3.15}$$

Using the fact that $\mathcal{N}_{\mathcal{TR}} \le \zeta \le \mathcal{N}_{\mathcal{TR}} + 1$ gives

$$\begin{aligned}
&\left\| f(x,y) - f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y) \right\|_{\mathcal{L}^2(\Omega)} \\
&\quad\le \frac{\pi}{2} \left[\zeta^{-\delta} \mathcal{M} + \varepsilon \zeta \max\left\{ \frac{1}{|m_0| \widetilde{\mathcal{M}}^\dagger_\beta(\Lambda_1)}, \frac{\|f\|_{\mathcal{L}^2(\Omega)}}{|m_0|} \right\} \right] &(3.16)\\
&\quad\le \varepsilon^{\frac{\delta}{\delta+1}} \mathcal{M}^{\frac{1}{\delta+1}} \frac{\pi}{2} \left(1 + \max\left\{ \frac{1}{|m_0| \widetilde{\mathcal{M}}^\dagger_\beta(\Lambda_1)}, \frac{\|f\|_{\mathcal{L}^2(\Omega)}}{|m_0|} \right\} \right). &(3.17)
\end{aligned}$$

□

3.2 Convergence estimate under an a posteriori regularization parameter choice rule

In this subsection, by using the discrepancy principle, we consider an a posteriori regularization parameter choice. Define

$$\mathcal{R}_{\mathcal{N}_{\mathcal{TR}}} g^\varepsilon = \sum_{p,q \in \overline{B}_\varepsilon} g^\varepsilon_{pq} \mathcal{S}_{pq}(x,y). \tag{3.18}$$

Because of the discrepancy principle, we take $\mathbb{K} = \mathbb{K}(\varepsilon, g^\varepsilon)$ as the solution of

$$\left\| (I - \mathcal{R}_{\mathcal{N}_{\mathcal{TR}}}) g^\varepsilon \right\|_{\mathcal{L}^2(\Omega)} \le \tau \varepsilon \le \left\| (I - \mathcal{R}_{\mathcal{N}_{\mathcal{TR}}-1}) g^\varepsilon \right\|_{\mathcal{L}^2(\Omega)}, \quad \tau > 1. \tag{3.19}$$

For this choice rule, we get an upper bound estimate for the $\mathcal{N}_{\mathcal{TR}}$ in the following lemma.

Lemma 3.1 *We have*

$$\mathcal{N}_{\mathcal{TR}} \le \left(\frac{\pi \widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)\mathcal{P}(m_0,M_0)\mathcal{M}}{2(\tau-1)\varepsilon}\right)^{\frac{1}{1+\delta}}. \tag{3.20}$$

Proof From $\|g^\varepsilon - g\|_{\mathcal{L}^2(\Omega)} \le \varepsilon$ and (3.19), we have

$$\begin{aligned}\|\mathcal{R}_{\mathcal{N}_{\mathcal{TR}}-1}g - g\|_{\mathcal{L}^2(\Omega)} &= \big\|(\mathcal{R}_{\mathcal{N}_{\mathcal{TR}}-1} - I)g^\varepsilon - (I - \mathcal{R}_{\mathcal{N}_{\mathcal{TR}}-1})\big(g - g^\varepsilon\big)\big\|_{\mathcal{L}^2(\Omega)} \\ &\ge \big\|(\mathcal{R}_{\mathcal{N}_{\mathcal{TR}}-1} - I)g^\varepsilon\big\|_{\mathcal{L}^2(\Omega)} - \big\|(I - \mathcal{R}_{\mathcal{N}_{\mathcal{TR}}-1})\big(g - g^\varepsilon\big)\big\|_{\mathcal{L}^2(\Omega)} \\ &\ge (\tau - 1)\varepsilon. \end{aligned} \tag{3.21}$$

On the other hand, for $\widetilde{\lambda}_{pq} \ge \mathcal{N}_{\mathcal{TR}}$, we obtain

$$\begin{aligned}&\int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta\big)\Phi(\varsigma)\,d\varsigma \\ &\quad\le \big|\mathcal{P}(m_0,M_0)\big|\int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta\big)\Phi(\varsigma)\,d\varsigma \\ &\quad= \big|\mathcal{P}(m_0,M_0)\big|\frac{\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)}{\widetilde{\lambda}_{pq}} \le \frac{|\mathcal{P}(m_0,M_0)|\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)}{\mathcal{N}_{\mathcal{TR}}}. \end{aligned} \tag{3.22}$$

This implies that

$$\begin{aligned}&\|\mathcal{R}_{\mathcal{N}_{\mathcal{TR}}-1}g - g\|^2_{\mathcal{L}^2(\Omega)} \\ &\quad= \frac{\pi^2}{4}\sum_{p,q\in\underline{B}_\varepsilon}\big|\langle g(\cdot,\cdot),\mathcal{S}_{pq}(x,y)\rangle\big|^2 \\ &\quad= \frac{\pi^2}{4}\sum_{p,q\in\underline{B}_\varepsilon}\bigg|\int_0^T E_{\beta,1}\big(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta\big)\Phi(\varsigma)\,d\varsigma\langle f_{pq},\mathcal{S}_{pq}(x,y)\rangle\bigg|^2 \\ &\quad\le \frac{\pi^2}{4}\frac{|\mathcal{P}(m_0,M_0)|^2(\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2))^2}{(\mathcal{N}_{\mathcal{TR}})^2}\sum_{p,q\in\underline{B}_\varepsilon}\big|\langle f_{pq},\mathcal{S}_{pq}(x,y)\rangle\big|^2 \\ &\quad\le \frac{\pi^2}{4}\frac{|\mathcal{P}(m_0,M_0)|^2(\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2))^2}{(\mathcal{N}_{\mathcal{TR}})^2}\sum_{p,q\in\underline{B}_\varepsilon}(1+\widetilde{\lambda}_{pq})^{-2\delta}(1+\widetilde{\lambda}_{pq})^{2\delta}\big|\langle f_{pq},\mathcal{S}_{pq}(x,y)\rangle\big|^2 \\ &\quad\le \frac{\pi^2}{4}\frac{|\mathcal{P}(m_0,M_0)|^2(\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2))^2}{(\mathcal{N}_{\mathcal{TR}})^2(\mathcal{N}_{\mathcal{TR}})^{2\delta}}\sum_{p,q\in\underline{B}_\varepsilon}(1+\widetilde{\lambda}_{pq})^{2\delta}\big|\langle f_{pq},\mathcal{S}_{pq}(x,y)\rangle\big|^2 \\ &\quad\le \frac{\pi^2}{4}\frac{|\mathcal{P}(m_0,M_0)|^2(\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2))^2\|f\|^2_{H^\delta(\Omega)}}{(\mathcal{N}_{\mathcal{TR}})^{2+2\delta}} \\ &\quad\le \frac{\pi^2}{4}\frac{|\mathcal{P}(m_0,M_0)|^2(\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2))^2\mathcal{M}^2}{(\mathcal{N}_{\mathcal{TR}})^{2+2\delta}}. \end{aligned} \tag{3.23}$$

Hence, we conclude that

$$\|F_{\mathcal{N}-1}g-g\|_{\mathcal{L}^2(\Omega)} \le \frac{\pi |\mathcal{P}(m_0,M_0)|\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)\mathcal{M}}{2(\mathcal{N}_{\mathcal{TR}})^{1+\delta}}. \tag{3.24}$$

Combining (3.21) and (3.24), we conclude that

$$(\tau-1)\varepsilon \le \frac{\pi |\mathcal{P}(m_0,M_0)|\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)\mathcal{M}}{2(\mathcal{N}_{\mathcal{TR}})^{1+\delta}}. \tag{3.25}$$

From (3.25), so we can obtain

$$\mathcal{N}_{\mathcal{TR}} \le \left(\frac{\pi |\mathcal{P}(m_0,M_0)|\widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)\mathcal{M}}{2(\tau-1)\varepsilon}\right)^{\frac{1}{1+\delta}}. \tag{3.26}$$

□

Next we present an error estimate for the approximate solution of problem (1.1).

Theorem 3.2 *Let $f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}$ and f be as in Theorem 3.1. Then we obtain*

$$\left\|f(x,y)-f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)\right\|_{\mathcal{L}^2(\Omega)} \le \varepsilon^{\frac{\delta}{\delta+1}}\mathcal{M}^{\frac{1}{\delta+1}}\left[Q_1+Q_2(\tau+1)^{\frac{\delta}{\delta+1}}\right], \tag{3.27}$$

where

$$\begin{aligned} Q_1 &= \left(\frac{\pi \widetilde{\mathcal{M}}_\beta^{\dagger\dagger}(\widetilde{\Lambda}_2)|\mathcal{P}(m_0,M_0)|}{2(\tau-1)|m_0|^{\delta+1}}\right)^{\frac{1}{\delta+1}} \max\left\{\|f\|_{\mathcal{L}^2(\Omega)}, \frac{\pi}{2\widetilde{M}_\beta^{\dagger}(\widetilde{\Lambda}_1)}\right\}, \\ Q_2 &= \frac{1}{|\frac{m_0}{2}|^{\frac{\delta}{\delta+1}}(\widetilde{M}_\beta^{\dagger}(\widetilde{\Lambda}_1))^{\frac{\delta}{\delta+1}}}. \end{aligned} \tag{3.28}$$

Proof Using the triangle inequality, we have

$$\begin{aligned} &\left\|f(x,y)-f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)\right\|_{\mathcal{L}^2(\Omega)} \\ &\quad \le \left\|f(x,y)-f^{\mathcal{N}_{\mathcal{TR}}}(x,y)\right\|_{\mathcal{L}^2(\Omega)} + \left\|f^{\mathcal{N}_{TR}}(x,y)-f_\varepsilon^{\mathcal{N}_{TR}}(x,y)\right\|_{\mathcal{L}^2(\Omega)}. \end{aligned} \tag{3.29}$$

We split the proof into two steps.

Step 1. Estimate for $\|f(\cdot,\cdot)-f^{\mathcal{N}_{\mathcal{TR}}}(\cdot,\cdot)\|_{\mathcal{L}^2(\Omega)}$. We obtain it as

$$\begin{aligned} &\left\|f(x,y)-f^{\mathcal{N}_{\mathcal{TR}}}(x,y)\right\|_{H^\delta(\Omega)} \\ &\quad \le \left\|\sum_{p=\mathcal{N}_{\mathcal{TR}}+1}^{\infty}\sum_{q=\mathcal{N}_{\mathcal{TR}}+1}^{\infty}\langle f_{pq},\mathcal{S}_{pq}(x,y)\rangle\right\| \\ &\quad = \left(\sum_{p=\mathcal{N}_{\mathcal{TR}}+1}^{\infty}\sum_{q=\mathcal{N}_{\mathcal{TR}}+1}^{\infty}(1+\widetilde{\lambda}_{pq})^{2\delta}\left|\langle f(\cdot,\cdot),\mathcal{S}_{pq}(x,y)\rangle\right|^2\right)^{1/2} \le \mathcal{M}. \end{aligned} \tag{3.30}$$

From (3.19), we get

$$\left\|Af(x,y)-Af^{\mathcal{N}_{\mathcal{TR}}}(x,y)\right\|_{\mathcal{L}^2(\Omega)}$$

$$
\begin{aligned}
&\le \big\| (I - \mathcal{R}_{\mathcal{N}_{\mathcal{TR}}}) g \big\|_{\mathcal{L}^2(\Omega)} \\
&\le \big\| (I - \mathcal{R}_{N_{TR}}) g^{\epsilon} + (I - \mathcal{R}_{\mathcal{N}_{\mathcal{TR}}}) \big(g - g^{\epsilon} \big) \big\|_{\mathcal{L}^2(\Omega)} \\
&\le \big\| (I - \mathcal{R}_{\mathcal{N}_{\mathcal{TR}}}) g^{\varepsilon} \big\|_{\mathcal{L}^2(\Omega)} + \big\| (I - \mathcal{R}_{\mathcal{N}_{\mathcal{TR}}}) \big(g - g^{\varepsilon} \big) \big\|_{\mathcal{L}^2(\Omega)} \\
&\le (\tau + 1)\varepsilon.
\end{aligned}
\tag{3.31}
$$

Therefore, we have

$$
\big\| f(\cdot,\cdot) - f^{\mathcal{N}_{\mathcal{TR}}}(\cdot,\cdot) \big\|_{\mathcal{L}^2(\Omega)} \le Q_2 \big((\tau + 1)\varepsilon \big)^{\frac{\delta}{\delta+1}}.
\tag{3.32}
$$

Step 2. Estimate for $\|f^{\mathcal{N}_{\mathcal{TR}}}(\cdot,\cdot) - f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(\cdot,\cdot)\|_{\mathcal{L}^2(\Omega)}$. We obtain it from

$$
\begin{aligned}
&f^{\mathcal{N}_{\mathcal{TR}}}(x,y) - f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y) \\
&\quad = \sum_{p,q \in \overline{B}_\varepsilon} \frac{g_{pq} \mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \Phi(\varsigma)\, d\varsigma - \sum_{p,q \in \overline{B}_\varepsilon} \frac{g_{pq}^\varepsilon \mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \Phi^\varepsilon(\varsigma)\, d\varsigma}} \\
&\quad \le \sum_{p,q \in \overline{B}_\varepsilon} \frac{g_{pq} \mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \Phi(\varsigma)\, d\varsigma} - \sum_{p,q \in \overline{B}_\varepsilon} \frac{g_{pq} \mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \Phi^\varepsilon(\varsigma)\, d\varsigma} \\
&\qquad + \sum_{p,q \in \overline{B}_\varepsilon} \frac{(g_{pq} - g_{pq}^\varepsilon) \mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \Phi^\varepsilon(\varsigma)\, d\varsigma}.
\end{aligned}
\tag{3.33}
$$

From (3.33), we know that

$$
\begin{aligned}
&f^{\mathcal{N}_{\mathcal{TR}}}(x,y) - f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y) \\
&\quad \le \underbrace{\sum_{p,q \in \overline{B}_\varepsilon} \frac{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta)(\Phi(\varsigma) - \Phi^\varepsilon(\varsigma)) d\varsigma}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \Phi^\varepsilon(\varsigma) d\varsigma} \times \sum_{p,q \in \overline{B}_\varepsilon} \frac{g_{pq} \mathcal{S}_{pq}(x,y)}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \Phi(\varsigma) d\varsigma}}_{:=\mathcal{A}_3} \\
&\qquad + \underbrace{\sum_{p,q \in \overline{B}_\varepsilon} \frac{\langle g_{pq} - g_{pq}^\varepsilon, \mathcal{S}_{pq}(x,y) \rangle}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \Phi^\varepsilon(\varsigma) d\varsigma}}_{:=\mathcal{A}_4}.
\end{aligned}
\tag{3.34}
$$

From (3.33), it is easy to check that

$$
\|\mathcal{A}_3\|_{\mathcal{L}^2(\Omega)} \le \frac{\varepsilon \|f\|_{\mathcal{L}^2(\Omega)}}{m_0}.
\tag{3.35}
$$

Our estimate of $\mathcal{A}_4$ is based on Lemma 2.7, and we obtain

$$
\begin{aligned}
\|\mathcal{A}_4\|_{\mathcal{L}^2(\Omega)}^2 &= \frac{\pi^2}{4} \sum_{p,q \in \overline{B}_\varepsilon} \left| \frac{\langle g_{pq}(x,y) - g_{pq}^\varepsilon(x,y), \mathcal{S}_{pq}(x,y) \rangle}{\int_0^T E_{\beta,1}(-\widetilde{\lambda}_{pq}(T-\varsigma)^\beta) \Phi^\varepsilon(\varsigma)\, d\varsigma} \right|^2 \\
&\le \frac{\pi^2}{4} \sum_{p,q \in \overline{B}_\varepsilon} \frac{\widetilde{\lambda}_{pq}^2 |\langle g_{pq}(x,y) - g_{pq}^\varepsilon(x,y), \mathcal{S}_{pq}(x,y) \rangle|^2}{|m_0|^2 (\widetilde{M}_\beta^\dagger(\widetilde{\Lambda}_1))^2}
\end{aligned}
$$

$$\leq \frac{\pi^2}{4}\frac{\varepsilon^2(\mathcal{N}_{\mathcal{TR}})^2}{m_0^2(\widetilde{M}_\beta^\dagger(\widetilde{\Lambda}_1))^2}. \tag{3.36}$$

Hence, we get

$$\|\mathcal{A}_4\|_{\mathcal{L}^2(\Omega)} \leq \frac{\pi\varepsilon\mathcal{N}_{\mathcal{TR}}}{2m_0\widetilde{M}_\beta^\dagger(\widetilde{\Lambda}_1)}. \tag{3.37}$$

From the above observations, we deduce that

$$\left\|f^{\mathcal{N}_{\mathcal{TR}}}(x,y) - f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)\right\|_{\mathcal{L}^2(\Omega)} \leq \frac{\varepsilon\mathcal{N}_{\mathcal{TR}}}{m_0}\max\left\{\|f\|_{\mathcal{L}^2(\Omega)}, \frac{\pi}{2\widetilde{M}_\beta^\dagger(\widetilde{\Lambda}_1)}\right\}. \tag{3.38}$$

Substituting (3.26) into (3.38), we obtain

$$\left\|f^{\mathcal{N}_{\mathcal{TR}}}(x,y) - f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)\right\|_{\mathcal{L}^2(\Omega)} \leq \varepsilon^{\frac{\delta}{\delta+1}}\mathcal{M}^{\frac{1}{\delta+1}}Q_1. \tag{3.39}$$

Combining *Steps* 1 and 2, we obtain the final estimate as follows:

$$\left\|f(x,y) - f_\varepsilon^{\mathcal{N}_{\mathcal{TR}}}(x,y)\right\|_{\mathcal{L}^2(\Omega)} \leq \varepsilon^{\frac{\delta}{\delta+1}}\mathcal{M}^{\frac{1}{\delta+1}}\left[Q_1 + Q_2(\tau+1)^{\frac{\delta}{\delta+1}}\right], \tag{3.40}$$

in which Q_1 depends on $m_0, M_0, \Lambda_1, \Lambda_2, \delta, f, \tau, \beta$, and Q_2 depends on $m_0, \delta, \beta, \Lambda_1$ defined in (3.28). This completes the proof. □

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