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Reconstruction from Vertex-Switching

RICHARD P. STANLEY*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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Let X be a graph with vertices $x_1,..., x_n$. Let X_i be the graph obtained by removing all edges $\{x_i, x_j\}$ of X and inserting all nonedges $\{x_i, x_k\}$. If $n \not\equiv 0 \pmod 4$, then X can be uniquely reconstructed from the unlabeled graphs $X_1,..., X_n$. If $n = 4$ the result is false, while for $n = 4m \geqslant 8$ the result remains open. The proof uses linear algebra and does not explicitly describe the reconstructed graph X.

1. INTRODUCTION

Let X be a graph (with no loops or multiple edges) on the vertex set $\{x_1,..., x_n\}$. The well-known Kelly–Ulam vertex-reconstruction conjecture (see [1] for a survey up to 1977) asks whether X can be uniquely reconstructed from its unlabeled vertex-deleted subgraphs when $n \neq 2$. Here we consider a variation of this problem where vertex-deletion is replaced by *vertex-switching*. More precisely, let X_i be the graph obtained from X by *switching* at vertex x_i [6, 7], i.e., by deleting all edges of X incident to x_i and inserting all possible edges incident to x_i which are not in X.

VERTEX-SWITCHING RECONSTRUCTION PROBLEM. Can X be uniquely reconstructed from the unlabeled graphs $X_1,..., X_n$? In other words, if X' is another graph on the same vertex set and if $X_i \cong X'_i$ for $1 \leqslant i \leqslant n$, then is $X \cong X'$?

1.1. EXAMPLE. Let $X = 4K_1$, the totally disconnected graph on four vertices, and let $X' = C_4$, a cycle of length 4. Then for any i, X_i and X'_i are isomorphic to the star (or claw) $K_{1,3}$. Hence the vertex-switching reconstruction problem has a negative answer in general.

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We shall show, however, that the answer is affirmative when $n \not\equiv 0 \pmod 4$. It remains open for $n = 4m \geqslant 8$. The proof uses a technique from linear algebra introduced in [8] to give an alternative proof of a result of Lovász [3, 4, Sect. 15.17a] on the edge-reconstruction conjecture.

2. Fourier Transforms and Invertible Linear Transformations

Let $\mathbb{Z}_2^k$ denote the additive group of k-tuples of integers modulo 2. If $f: \mathbb{Z}_2^k \to \mathbb{R}$ is a real-valued function, then the *Fourier transform* of f is the function $\hat{f}: \mathbb{Z}_2^k \to \mathbb{R}$ defined by

$$\hat{f}(X) = \sum_{Y} (-1)^{X \cdot Y} f(Y).$$

Here $X \cdot Y$ denotes the dot product, taken modulo 2. Given a nonempty subset $\Gamma \subset \mathbb{Z}_2^k$, also define $\bar{f}: \mathbb{Z}_2^k \to \mathbb{R}$ by

$$\bar{f}(Y) = \sum_{X \in \Gamma + Y} f(X), \tag{1}$$

where $\Gamma + Y = \{Z + Y: Z \in \Gamma\}$. Let $\chi_\Gamma: \mathbb{Z}_2^k \to \mathbb{R}$ denote the characteristic function of Γ, so

$$\hat{\chi}_\Gamma(X) = \sum_{Y \in \Gamma} (-1)^{X \cdot Y}. \tag{2}$$

The following lemma is easily verified (e.g., [2, Lemma 1].)

2.1. Lemma. *The linear transformation $f \mapsto \bar{f}$ (defined on the vector space of all functions $f: \mathbb{Z}_2^k \to \mathbb{R}$) is invertible if and only if $\hat{\chi}_\Gamma(X) \neq 0$ for all $X \in \mathbb{Z}_2^k$.*

Now fix $n \geqslant 1$. Let $\mathscr{G}_n$ denote the set of all graphs on the vertex set $\{x_1, ..., x_n\}$, and let $\mathscr{V}_n$ denote the real vector space of all formal linear combinations $\sum_{X \in \mathscr{G}_n} a_X X$, $a_X \in \mathbb{R}$. (Any field of characteristic 0 will do in place of $\mathbb{R}$.) Thus $\dim \mathscr{V}_n = 2^{\binom{n}{2}}$. Define a linear transformation $\phi: \mathscr{V}_n \to \mathscr{V}_n$ by

$$\phi(X) = X_1 + \cdots + X_n, \tag{3}$$

where $X_1, ..., X_n$ are the *labeled* vertex-switched graphs defined above. We come to our main lemma.

2.2. Lemma. *The linear transformation ϕ is invertible if and only if $n \not\equiv 0 \pmod 4$.*

Proof. Identify $\mathscr{G}_n$ with $\mathbb{Z}_2^{\binom{n}{2}}$ in the obvious way, viz., order the $\binom{n}{2}$ pairs

$e_1, e_2, \ldots, e_{\binom{n}{2}}$ of distinct vertices and let $X \in \mathscr{G}_n$ correspond to the characteristic vector of its edge set. For $1 \leqslant i \leqslant n$, let $C_i \in \mathscr{G}_n = \mathbb{Z}_2^{\binom{n}{2}}$ be the star $K_{1,n-1}$ with center x_i (so as an element of $\mathbb{Z}_2^{\binom{n}{2}}$, C_i is the characteristic vector of the set of pairs of vertices which contain x_i). Set $\Gamma = \{C_1, \ldots, C_n\}$. Identify $f \in \mathscr{V}_n$ with the function $f\colon \mathbb{Z}_2^{\binom{n}{2}} \to \mathbb{R}$ given by $f = \sum_X f(X)X$. In particular, $\chi_\Gamma = C_1 + \cdots + C_n$. Then $\phi f = \hat{f}$, the transform of f based on translates of Γ as defined in (1). Hence by Lemma 2.1, ϕ if invertible if and only if

$$\hat{\chi}_\Gamma(X) = \sum_{Y \in \Gamma} (-1)^{X \cdot Y} \neq 0 \qquad \text{for all } X \in \mathbb{Z}_2^{\binom{n}{2}} = \mathscr{G}_n.$$

If $Y = C_i \in \Gamma$, then $(-1)^{X \cdot Y} = (-1)^{d_i}$, where d_i is the degree of the vertex x_i of X. Therefore $\hat{\chi}_\Gamma(X) = (-1)^{d_1} + \cdots + (-1)^{d_n}$. If n is odd, then $\hat{\chi}_\Gamma(X)$ is odd and hence nonzero. If $n \equiv 2 \pmod 4$, then $\hat{\chi}_\Gamma(X) \equiv 2 \pmod 4$ since $d_1 + \cdots + d_n$ is even, so again $\hat{\chi}_\Gamma(X) \neq 0$. If $n \equiv 0 \pmod 4$ then one can easily construct an X for which $(-1)^{d_1} + \cdots + (-1)^{d_n} = 0$, e.g., take X to be a disjoint union of $n/4$ edges and $n/2$ vertices. From this the proof follows. ∎

Let us remark (as pointed out by J. Kahn) that it is very easy to show directly that ϕ is invertible when n is odd. Namely, one easily sees that $\phi\phi^t \equiv nI \pmod 2$, where t denotes transpose with the respect to the basis $\mathscr{G}_n$ of $\mathscr{V}_n$, and I denotes the identity transformation.

3. Unlabeling

The symmetric group $\mathfrak{S}_n$ of all permutations of $\{x_1, \ldots, x_n\}$ acts on $\mathscr{G}_n$ by permuting vertices, and hence acts on $\mathscr{V}_n$ by $w \cdot \Sigma a_X X = \Sigma a_X (w \cdot X)$. Given $f \in \mathscr{V}_n$, define $[f] \in \mathscr{V}_n$ by

$$[f] = \sum_{w \in \mathfrak{S}_n} w \cdot f.$$

The map $f \to [f]$ is a linear transformation on $\mathscr{V}_n$. If $X \in \mathscr{G}_n$ then one can regard $[X]$ as the *unlabeling* of X, and we clearly have

$$[X] = [X'] \Leftrightarrow X \cong X'. \tag{4}$$

We now come to the main result of this paper.

3.1. Theorem. *Let* $n \not\equiv 0 \pmod 4$. *Suppose* $X, X' \in \mathscr{G}_n$ *and* $X_i \cong X'_i$ *for* $1 \leqslant i \leqslant n$. *Then* $X \cong X'$.

Proof. Let $\phi: \mathscr{V}_n \to \mathscr{V}_n$ be given by (3). For any $X \in \mathscr{G}_n$, $w \in \mathfrak{S}_n$, and $1 \leqslant i \leqslant n$, we have $(w \cdot X)_i = w \cdot X_{w^{-1}(i)}$. It follows that

$$\begin{aligned}\phi[X] &= \sum_w \sum_{i=1}^{n} (w \cdot X)_i \\ &= \sum_w \sum_{i=1}^{n} w \cdot X_{w^{-1}(i)} \\ &= \sum_w w \cdot \sum_{i=1}^{n} X_i \qquad \text{(since } w \text{ permutes the vertices)} \\ &= [\phi X].\end{aligned}$$

Now suppose that X, $X' \in \mathscr{G}_n$ and that $X_i \cong X'_i$ for $1 \leqslant i \leqslant n$. Thus

$$\begin{aligned}\phi[X] &= [\phi X] \\ &= [X_1 + \cdots + X_n] \\ &= [X_1] + \cdots + [X_n] \\ &= [X'_i] + \cdots + [X'_n] \qquad \text{(by (4))} \\ &= \phi[X'].\end{aligned}$$

When $n \not\equiv 0 \pmod 4$ then by Lemma 2.2 we get $[X] = [X']$. Thus $X \cong X'$ by (4), as desired. ∎

Let us make two remarks concerning the case $n \equiv 0 \pmod 4$.

(a) Every edge of X will appear in $n-2$ of the switches $X_1, \ldots, X_n$, while every nonedge of X will appear as an edge in two of $X_1, \ldots, X_n$. Hence if X has q edges then the total number of edges in $X_1, \ldots, X_n$ is given by

$$q(n-2) + 2(\tbinom{n}{2} - q) = n^2 - n + (n-4)\, q.$$

Thus q can be reconstructed when $n \neq 4$. Moreover, if vertex x_i has degree d_i, then X_i has $q + n - 1 - 2d_i$ edges, so that the degree sequence of X can be reconstructed when $n \neq 4$. Thus Example 1.1 is rather special, since $4K_1$ and C_4 have a different number of edges.

(b) Example 1.1 also has the remarkable property that there is a permutation w of $\{1, 2, 3, 4\}$ for which the *labeled* graphs X_i and $X'_{w(i)}$ are identical, for $1 \leqslant i \leqslant 4$. Such a phenomenon cannot hapen when $n \neq 4$, because if $e = \{x_i, x_j\}$ is an edge of X then it appears in exactly $n-2$ of the labeled graphs $X_1, \ldots, X_n$, while if e is a nonedge then it appears in two of $X_1, \ldots, X_n$. Hence when $n - 2 \neq 2$ it can be determined which pairs e are edges of X.

4. VARIATIONS

Many variations of Theorem 3.1 can be established using essentially the same proof. We state one such result here, leaving the reader to verify that the proof carries over from before.

Let $1 \leqslant i \leqslant n-1$, and let $\mathcal{T}_i$ denote the set of i-element subsets of $\{1,..., n\}$, so $\mathcal{T}_i$ has $\binom{n}{i}$ elements. Given a graph $X \in \mathcal{G}_n$ and a subset T of $\{1,..., n\}$, let X_T denote the graph obtained by switching successively at all vertices x_i for $i \in T$. (X_T is independent of the order in which the switches are applied.) Define a linear transformation $\phi_i: \mathcal{V}_n \to \mathcal{V}_n$ by

$$\phi_i(X) = \sum_{T \in \mathcal{T}_i} X_T. \tag{5}$$

4.1. THEOREM. *Let X, $X' \in \mathcal{G}_n$, and suppose there is a permutation w: $\mathcal{T}_i \to \mathcal{T}_i$ for which $X_T \cong X'_{w \cdot T}$ for all $T \in \mathcal{T}_i$. (Here if $T = \{a_1,..., a_i\}$ then $w \cdot T = \{w \cdot a_1,..., w \cdot a_i\}$.) If ϕ_i is invertible, then $X \cong X'$.*

The invertibility of ϕ_i can (in principle) be checked as in the proof of Lemma 2.2. In particular, define the Krawtchouk polynomial (see [5, p. 130])

$$p_i^n(v) = \sum_{l=0}^{i} (-1)^l \binom{v}{l}\binom{n-v}{i-l}.$$

We then have the following result:

4.2. THEOREM. *Fix $1 \leqslant i \leqslant n-1$. Then the linear transformation ϕ_i is invertible if and only if the Krawtchouk polynomial $p_i^n(v)$ has no* even *integer zeros v in the interval* $[0, n]$.

Proof. Arguing as in the proof of Lemma 2.2, we see that ϕ_i is invertible if and only if for all $X \in \mathcal{G}_n$, we have

$$\sum (-1)^{d_{j_1} \cdots + d_{j_i}} \neq 0, \tag{6}$$

where the sum ranges over all sequences $1 \leqslant j_1 < \cdots < j_i \leqslant n$ and d_j is the degree of vertex x_j of X. The sum on the left-hand side of (6) is just the coefficient of x^{n-i} in the polynomial $\prod_{j=1}^{n}(x + (-1)^{d_j}) = (1-x)^v(1+x)^{n-v}$, where X has v vertices of odd degree. This coefficient is just $p_i^n(v)$. Clearly v is even and $0 \leqslant v \leqslant n$. Since by suitable choice of X we can let v achieve any even value in $[0, n]$, the proof follows. ∎

4.3. EXAMPLE. It is convenient to set $u = n - 2v$, so $u \equiv n \pmod 4$ and $-n \leqslant u \leqslant n$.

(a) $p_1^n(v) = n - 2v = u$. Hence ϕ_1 is invertible if and only if $n \not\equiv 0 \pmod 4$. Of course this is just Lemma 2.2.

(b) $p_2^n(v) = \frac{1}{2}(u^2 - n)$, so ϕ_2 is invertible if and only if n is not the square of an integer $\equiv 0,1 \pmod 4$.

(c) $p_3^n(v) = \frac{1}{6}u(u^2 - 3n + 2)$, so ϕ_3 is invertible if and only if $n \not\equiv 0 \pmod 4$ and $3n - 2$ is not the square of an integer $\equiv 1,2 \pmod 4$.

Note. For any $f\colon \mathbb{Z}_2^n \to \mathbb{R}$, define $\bar{f}\colon \mathbb{Z}_2^n \to \mathbb{R}$ by

$$\bar{f}(Y) = \sum_{X \in \Gamma + Y} f(X),$$

where Γ consists of all vectors in $\mathbb{Z}_2^n$ of Hamming weight (number of 1's) equal to i. In [2, Ex. 2.3] it is shown that the linear transformation $f \overset{\psi_i}{\mapsto} \bar{f}$ is invertible if and only if $p_n^i(v)$ has no integer zeros in $[0, n]$. In fact, it can be proved directly that if ψ_i is invertible then ϕ_i (as defined by (5)) is also invertible. We omit the details.

5. Open Problems

Two problems obviously suggest themselves at this point:

(a) Is Theorem 3.1 true when $n = 4m \geqslant 8$?

(b) Is there a proof of Theorem 3.1 which explicitly describes the reconstructed graph X?

A solution of sorts to (b) is due to N. Alon and D. Coppersmith. Alon has pointed out that when n is odd, if we look at the n^2 graphs obtained by switching $X_1, \dots, X_n$ at each of their vertices separately, then X is the only (unlabeled) graph that occurs an odd number of times. Coppersmith has found a similar but more complicated argument for the case $n \equiv 2 \pmod 4$. There are related questions involving computational complexity. For instance, is there a polynomial-time algorithm for obtaining X from $X_1, \dots, X_n$, assuming that X can be uniquely reconstructed?

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