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Reconstruction of a Variety from the Derived Category and Groups of Autoequivalences*

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Abstract. We consider smooth algebraic varieties with ample either canonical or anticanonical sheaf. We prove that such a variety is uniquely determined by its derived category of coherent sheaves. We also calculate the group of exact autoequivalences for these categories. The technics of ample sequences in Abelian categories is used.

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0. Introduction

Examples of different varieties X exist which have equivalent derived categories $D^b_{coh}(X)$ of coherent sheaves. These kind of equivalences were constructed for Abelian varieties and K3 surfaces by Mukai, Polishchuk and the second author in [7–10]. In [4] we prove equivalence of the derived categories for varieties connected by certain kinds of flops.

Does this mean that $D^b_{coh}(X)$ is a weak invariant of a variety X? In this paper we will show that this is not the case, at least for some types of algebraic varieties.

We prove that a variety X is uniquely determined by its category $D^b_{coh}(X)$, if its anticanonical (Fano case) or canonical (general type case) sheaf is ample.

To reconstruct the variety from the category we use nothing but the graded structure of the category, i.e. we need only to fix the translation functor.

The idea is that for good, in the above sense, varieties we can recognize the skyscraper sheaves of closed points in $D^b_{coh}(X)$. The main tool for this is the Serre functor [3] (see also Section 1), which for $D^b_{coh}(X)$ can be regarded as a categorical incarnation of the canonical sheaf ω_X. In this way we find the variety as a set. Then, one by one, we reconstruct the set of line bundles, Zariski topology and the structural sheaf of rings (for details, see the five steps of the proof of Theorem 2.5).

With respect to the above problem, it is natural to introduce a groupoid with the objects being the categories $D^b_{coh}(X)$ and with the morphisms being equivalences.

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There are two natural questions related to a groupoid: which objects are isomorphic and what is the group of automorphisms of an individual object? The first problem was addressed within the framework of graded categories. To tackle the second one, we need the triangulated structure of the category. In Section 3 we prove that for a smooth algebraic variety with either ample canonical or anticanonical sheaf, the group of exact autoequivalences is the semidirect product of the group of automorphisms of the variety and the Picard group plus translations.

The answers to the above questions for the case of varieties with non-ample and nonantiample canonical sheaves seem to be of considerable interest.

1. Preliminaries

Here we collect some facts related to functors in graded and triangulated categories, with special emphasis on the Serre functor.

In this paper, for simplicity we consider only k-linear additive categories, where k is an arbitrary field.

By definition a **graded category** is a pair $(\mathcal{D}, T_{\mathcal{D}})$ consisting of a category $\mathcal{D}$ and a fixed equivalence functor $T_D : \mathcal{D} \longrightarrow \mathcal{D}$, called a translation functor.

Recall that a **triangulated category** is a graded category with an additional structure: a distinguished class of exact triangles satisfying certain axioms (see [12]).

A functor $F: \mathcal{D} \longrightarrow \mathcal{D}'$ between two graded categories $\mathcal{D}$ and $\mathcal{D}'$ is called **graded** if it commutes with the translation functor. More precisely, a natural isomorphism of functors $t_F: F \circ T_{\mathcal{D}} \xrightarrow{\sim} T_{\mathcal{D}'} \circ F$ is assumed to be fixed.

In the sequel, we omit the subscripts in the notation of translation functors because it is always clear from their position in formulas which category they belong to.

While considering graded functors, we use graded natural transformations. A natural transformation μ between graded functors F and G is called **graded** if the following diagram is commutative:

$$\begin{array}{ccc} F \circ T & \xrightarrow{t_F} & T \circ F \\ \downarrow{\scriptstyle \mu T} & & \downarrow{\scriptstyle T\mu} \\ G \circ T & \xrightarrow{t_G} & T \circ G \end{array}$$

A graded functor $F: \mathcal{D} \longrightarrow \mathcal{D}'$ between triangulated categories is called **exact** if it transforms all exact triangles into exact triangles in the following sense. If $X \to Y \to Z \to TX$ is an exact triangle in $\mathcal{D}$, then one takes $FX \to FY \to FZ \to FTX$ and substitutes in this sequence $FT(X)$ by $TF(X)$ using the natural isomorphism of FT with TF. The result $FX \to FY \to FZ \to TFX$ should be an exact triangle in $\mathcal{D}'$.

A morphism between exact functors is, by definition, a graded natural transformation.

A functor which is isomorphic to an exact functor can be endowed with a structure of a graded functor so that it becomes an exact functor. Indeed, if F is exact, then using the isomorphism $\mu: F \xrightarrow{\sim} G$, one constructs the natural isomorphism

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$t_G\colon GT \xrightarrow{\sim} TG$, $t_G = \mu t_F \mu^{-1}$, which makes G graded. Since any triangle isomorphic to an exact triangle is again exact, G transforms exact triangles into exact ones. The natural transformation μ becomes a graded transformation of exact functors.

Let $F\colon \mathcal{D} \longrightarrow \mathcal{D}'$ be a functor. Suppose we fix a class $\mathcal{C}$ of objects in $\mathcal{D}$ and for any object $X \in \mathcal{C}$ some object X' isomorphic to FX in $\mathcal{D}'$. If, for any $X \in \mathcal{C}$, we additionally fix an isomorphism $FX \xrightarrow{\sim} X'$, then there exists a new functor $G\colon \mathcal{D} \to \mathcal{D}'$, which is isomorphic to F and such that

$$GX = FX, \text{ for } X \notin \mathcal{C}, \qquad GX = X', \text{ for } X \in \mathcal{C}, \tag{1}$$

with the evident action on morphisms.

We shall frequently use this simple fact in the sequel.

PROPOSITION 1.1. (i) *Let $F\colon \mathcal{D} \longrightarrow \mathcal{D}'$ be a graded functor between graded categories, $G\colon \mathcal{D}' \longrightarrow \mathcal{D}$ its left adjoint, so that the natural transformations are given:*

$$\mathrm{id}_{\mathcal{D}'} \xrightarrow{\alpha} F \circ G, \qquad G \circ F \xrightarrow{\beta} \mathrm{id}_{\mathcal{D}}. \tag{2}$$

Then G can be canonically endowed with the structure of a graded functor, such that (2) *become morphisms of graded functors.*

(ii) *If, in addition, F is an exact functor between triangulated categories, then G also becomes an exact functor.*

Proof. (i) Let us make G graded. By the adjointness of G and F and since $T_{\mathcal{D}}$ and $T_{\mathcal{D}'}$ are equivalences, we have the following sequence of bifunctorial isomorphisms:

$$\begin{aligned}\mathrm{Hom}(GTX,\ Y) &\cong \mathrm{Hom}(TX,\ FY) \cong \mathrm{Hom}(X,\ T^{-1}FY)\\ &\cong \mathrm{Hom}(X,\ FT^{-1}Y) \cong \mathrm{Hom}(GX,\ T^{-1}Y)\\ &\cong \mathrm{Hom}(TGX,\ Y)\end{aligned} \tag{3}$$

for any $X \in \mathcal{D}'$, $Y \in \mathcal{D}$.

By the well known Brown lemma [2], this gives a functorial isomorphism: $t_G\colon GT \xrightarrow{\sim} TG$.

Taking $Y = TGX$ in (3) and carefully tracking the preimage in $\mathrm{Hom}(GTX,\ TGX)$ of id_{TGX} in $\mathrm{Hom}(TGX,\ TGX)$ under the chain of isomorphisms in (3), one obtains a formula for t_G. It is, in fact, canonically given as the composite of the following sequence of natural transformations:

$$GT \xrightarrow{GT\alpha} GTFG \xrightarrow{Gt_F^{-1}G} GFTG \xrightarrow{\beta TG} TG. \tag{4}$$

Here we use morphisms α and β from (2) and the grading isomorphism t_F for F: $t_F\colon FT \xrightarrow{\sim} TF$. To show that, say, α is an isomorphism of graded functors is equiv-

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alent to proving that the diagram

$$\begin{array}{ccccc} T & \xrightarrow{T\alpha} & TFG & \xrightarrow{\iota_F G} & FTG \\ {\scriptstyle \alpha T}\downarrow & & {\scriptstyle \alpha TFG}\,\vdots & & {\scriptstyle \alpha FTG}\,\vdots\,\uparrow{\scriptstyle F\beta TG} \\ FGT & \xrightarrow{FGT\alpha} & FGTFG & \xrightarrow{FG\iota_F G} & FGFTG, \end{array}$$

being considered without dotted arrows, is commutative. One can split it by the dotted arrows into two commutative squares and the loop, the latter being commutative due to the fact that, for adjoint functors, the composite $F \xrightarrow{\alpha F} FGF \xrightarrow{F\beta} F$ equals id_F.

Notice that the inverse morphism to (4) is given by the composition

$$TG \xrightarrow{TGT^{-1}\alpha T} TGT^{-1}FGT \xrightarrow{TGT^{-1}\iota_F T^{-1}GT} TGFT^{-1}GT \xrightarrow{T\beta T^{-1}GT} GT. \tag{5}$$

That can be found in the same way as (4) by putting $Y = GTX$ in (3). It is interesting to note that one needs a great number of commutative diagrams to prove directly, without use of (3), that (4) and (5) are mutually inverse.

(ii) [3] Let $A \xrightarrow{\alpha} B \to C \to TA$ be an exact triangle in $\mathcal{D}'$. We have to show that G transforms this exact triangle into an exact one.

Let us insert the morphism $G(\alpha)\colon GA \to GB$ into an exact triangle:

$$GA \to GB \to Z \to TGA.$$

Applying functor F to it, we obtain an exact triangle

$$FGA \to FGB \to FZ \to TFGA.$$

(Henceforth we make no mention of commutation isomorphisms like $TF \xrightarrow{\sim} FT$). By means of $\mathrm{id} \to FG$, we construct a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ \downarrow & & \downarrow & & & & \downarrow \\ FGA & \longrightarrow & FGB & \longrightarrow & FZ & \longrightarrow & TFGA \end{array}$$

By the axioms of triangulated categories there exists a morphism $\mu\colon C \to FZ$ that completes this commutative diagram. By adjunction, we obtain a morphism $\nu\colon GC \to Z$ that makes the following diagram commutative:

$$\begin{array}{ccccccc} GA & \longrightarrow & GB & \longrightarrow & GC & \longrightarrow & TGA \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \wr\downarrow{\scriptstyle \nu} & & {\scriptstyle \mathrm{id}}\downarrow \\ GA & \longrightarrow & GB & \longrightarrow & Z & \longrightarrow & TGA \end{array}$$

The functors represented by GC and Z are isomorphic via ν in view of the 5-lemma, hence by the Brown lemma, ν is an isomorphism. Therefore the upper triangle is exact. $\square$

If F is a graded autoequivalence in a graded category, then the adjoint functor is its quasi-inverse.

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We may consider a category with objects being graded (or respectively triangulated) categories and morphisms being isomorphism classes of graded (respectively exact) equivalences. The proposition ensures that this category is a groupoid. In particular, the set of isomorphism classes of graded autoequivalences in a graded category or of exact autoequivalences in a triangulated category is a group.

Now we outline the main properties of the Serre functor. Its abstract definition was introduced in [3].

DEFINITION 1.2. Let $\mathcal{D}$ be a k-linear category with finite-dimensional Hom$'s$. A covariant additive functor $S\colon \mathcal{D} \to \mathcal{D}$ is called a **Serre functor** if it is a category equivalence and there are given bi-functorial isomorphisms $\varphi_{A,B}\colon \mathrm{Hom}_{\mathcal{D}}(A,\ B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(B,\ SA)^*$ for any $A, B \in \mathcal{D}$.

Remark. It was postulated in [3] that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(A,\ B) & \xrightarrow{\varphi_{A,B}} & \mathrm{Hom}_{\mathcal{D}}(B,\ SA)^* \\ \wr\downarrow & & \wr\uparrow \\ \mathrm{Hom}_{\mathcal{D}}(SA,\ SB) & \xrightarrow{\varphi_{SA,SB}} & \mathrm{Hom}_{\mathcal{D}}(SB,\ S^2A)^* \end{array}$$

the vertical isomorphisms in this diagram being induced by S. In fact, this can be deduced from bi-functoriality of $\varphi_{A,B}$.

PROPOSITION 1.3. *Any autoequivalence* $\Phi\colon \mathcal{D} \longrightarrow \mathcal{D}$ *commutes with a Serre functor, i.e. there exists a natural graded isomorphism of functors* $\Phi \circ S \xrightarrow{\sim} S \circ \Phi$.

Proof. For any pair of objects A, B in $\mathcal{D}$ we have a system of natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}(\Phi A,\ \Phi SB) &\cong \mathrm{Hom}(A,\ SB) \cong \mathrm{Hom}(B,\ A)^* \\ &\cong \mathrm{Hom}(\Phi B,\ \Phi A)^* \cong \mathrm{Hom}(\Phi A,\ S\Phi B). \end{aligned} \tag{6}$$

Since Φ is an equivalence, the essential image of Φ covers the whole $\mathcal{D}$, i.e. up to isomorphism, any object can be presented as ΦA for some A. This means that (6) gives rise to an isomorphism, of the contravariant functors represented by objects ΦSB and $S\Phi B$. By the Brown lemma [2], morphisms between representable functors are in one-to-one correspondence with those between the representation objects. This yields an isomorphism $\Phi SB \xrightarrow{\sim} S\Phi B$, which is, in fact, natural with respect to B. □

PROPOSITION 1.4. (i) *Any Serre functor in a graded category is graded.* (ii) *A Serre functor in a triangulated category is exact.*

Proof. (i) This follows from the previous proposition.

(ii) The fact that a Serre functor takes exact triangles into exact ones is proved in [3].

PROPOSITION 1.5 [3]. *Any two Serre functors are connected by a canonical graded functorial isomorphism, which commutes with the bifunctorial isomorphisms $\phi_{A,B}$ in the definition of Serre functor.*

Proof. Let S and S' be two Serre functors in a category $\mathcal{D}$. Then, for any object A in $\mathcal{D}$ we have natural isomorphisms:

$$\mathrm{Hom}(A,\ A) \cong \mathrm{Hom}(A,\ SA)^* \cong \mathrm{Hom}(SA,\ S'A)$$

Taking the image of the identity morphism id_A with respect to this identification, we obtain a morphism $SA \to S'A$, which, in fact, gives a graded functorial isomorphism $S \xrightarrow{\sim} S'$, which commutes with $\phi_{A,B}$. □

Thus, a Serre functor in a category $\mathcal{D}$, if it exists, is unique up to a graded natural isomorphism. By definition, it is intrinsically related to the structure of the category. We shall use this later to reconstruct a variety from its derived category and to find the group of exact autoequivalences for algebraic varieties with ample or antiample canonical sheaves.

2. Reconstruction of a Variety from the Derived Category of Coherent Sheaves

In this Section we show that a variety X can be uniquely reconstructed from the derived category of coherent sheaves on it, provided X is smooth and has ample or antiample canonical sheaf. We need only grading from the category, i.e. fixed translation functor.

Roughly, the reconstruction proceeds as follows. First, by means of the Serre functor, we distinguish the skyscraper sheaves of closed points in the variety. Then we find the invertible sheaves and use them to define the Zariski topology and the structure sheaf of the variety.

Let $\mathcal{D}$ be a k-linear category. Denote by $S_{\mathcal{D}}$ the Serre functor in $\mathcal{D}$ (for the case it exists).

Let X be a smooth algebraic variety, $n = \dim X$, $\mathcal{D} = D^b_{coh}(X)$ the derived category of coherent sheaves on X and ω_X the canonical sheaf. Then the functor

$$(\cdot) \otimes \omega_X[n] \tag{7}$$

is the Serre functor in $\mathcal{D}$, in view of the Serre–Grothendieck duality:

$$\mathrm{Ext}^i(F,\ G) = \mathrm{Ext}^{n-i}(G,\ F \otimes \omega_X)^*$$

for any pair F, G of coherent sheaves on X ([5, 11]).

For derived categories the translation functor we consider is always the usual shift of grading.

For a closed point $x \in X$, we denote by $k(x)$ the residue field of this point.

We use the standard notations for the iterated action of the translation functor on an object P: $P[i] := T^i P,\ i \in \mathbb{Z}$, and for the composition of the functors Hom and T:

$$\mathrm{Hom}^i(P,\ Q) = \mathrm{Ext}^i(P, Q) := \mathrm{Hom}(P,\ Q[i]).$$

DEFINITION 2.1. An object $P \in \mathcal{D}$ is called a point object of codimension s, if

(i) $S_{\mathcal{D}}(P) \simeq P[s]$,
(ii) $\mathrm{Hom}^{<0}(P,\ P) = 0$,
(iii) $\mathrm{Hom}^0(P,\ P) = k(P)$,

with $k(P)$ being a field (which is automatically a finite extension of the field k).

PROPOSITION 2.2. *Let X be a smooth algebraic variety of dimension n with the ample canonical or anticanonical sheaf. Then an object $P \in D^b_{coh}(X)$ is a point object, iff $P \cong \mathcal{O}_x[r]$, $r \in \mathbb{Z}$, is isomorphic (up to translation) to the skyscraper sheaf of a closed point $x \in X$.*

Remark. Since X has an ample invertible sheaf it is projective.

Proof. Any skyscraper sheaf of a closed point obviously satisfies the properties of a point object of the same codimension as the dimension of the variety.

Suppose now that for some object $P \in D^b_{coh}(X)$, properties (i)–(iii) of Definition 2.1 are verified.

Let $\mathcal{H}^i$ be cohomology sheaves of P. It immediately follows from (i) that $s = n$ and $\mathcal{H}^i \otimes \omega_X = \mathcal{H}^i$. Since ω_X is either an ample or antiample sheaf, we conclude that $\mathcal{H}^i$ are finite length sheaves, i.e. their supports consist of isolated closed points. Sheaves with the support in different points are homologically orthogonal, therefore any such object decomposes into the direct sum of those which have the support of all cohomology sheaves in a single point. By (iii), the object P is indecomposable, hence all $\mathcal{H}^i$ have their support in one single point. Now consider the spectral sequence which calculates $\mathrm{Hom}^m(P,\ P)$ by $\mathrm{Ext}^i(\mathcal{H}^j,\ \mathcal{H}^k)$:

$$E_2^{p,q} = \bigoplus_{k-j=q} \mathrm{Ext}^p(\mathcal{H}^j,\ \mathcal{H}^k) \Longrightarrow \mathrm{Hom}^{p+q}(P,\ P).$$

Note that for any two finite length sheaves which have the same single point as their support, there exists a nontrivial homomorphism from one to the other, which sends the generators of the first one to the socle of the second.

Considering $\mathrm{Hom}^0(\mathcal{H}^j,\ \mathcal{H}^k)$ with minimal $k - j$, we observe that this nontrivial space survives at E_∞, hence by (ii) $k - j = 0$. This means that all but one cohomology sheaves are trivial. Moreover, (iii) implies that this sheaf is a skyscraper. This concludes the proof. □

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Now, having the skyscrapers, we are able to reconstruct the invertible sheaves.

DEFINITION 2.3. An object $L \in \mathcal{D}$ is called invertible if, for any point object $P \in \mathcal{D}$, there exists $s \in \mathbb{Z}$ such that

(i) $\mathrm{Hom}^s(L, P) = k(P)$,

(ii) $\mathrm{Hom}^i(L, P) = 0, \quad \text{for } i \neq s$.

PROPOSITION 2.4 *Let X be a smooth irreducible algebraic variety. Assume that all point objects have the form $\mathcal{O}_x[s]$ for some $x \in X, s \in \mathbb{Z}$. Then an object $L \in \mathcal{D}$ is invertible, iff $L \cong \mathcal{L}[t]$ for some invertible sheaf $\mathcal{L}$ on X, $t \in \mathbb{Z}$.*

Proof. For an invertible sheaf $\mathcal{L}$ we have

$$\mathrm{Hom}(\mathcal{L}, \mathcal{O}_x) = k(x), \qquad \mathrm{Ext}^i(\mathcal{L}, \mathcal{O}_x) = 0, \quad \text{if } i \neq 0.$$

Therefore, if $L = \mathcal{L}[s]$, then it is an invertible object.

Now let $\mathcal{H}^i$ be the cohomology sheaves for an invertible object L. Consider the spectral sequence that calculates $\mathrm{Hom}^{\cdot}(L, \mathcal{O}_x)$ for a point $x \in X$ by means of $\mathrm{Hom}^i(\mathcal{H}^j, \mathcal{O}_x)$:

$$E_2^{p,q} = \mathrm{Hom}^p(\mathcal{H}^q, \mathcal{O}_x) \Longrightarrow \mathrm{Ext}^{p-q}(L, \mathcal{O}_x).$$

Let $\mathcal{H}^{q_0}$ be the nontrivial cohomology sheaf with maximal index. Then for any closed point $x \in X$ from the support of $\mathcal{H}^{q_0}$, $\mathrm{Hom}(\mathcal{H}^{q_0}, \mathcal{O}_x) \neq 0$. But both $\mathrm{Hom}(\mathcal{H}^{q_0}, \mathcal{O}_x)$ and $\mathrm{Ext}^1(\mathcal{H}^{q_0}, \mathcal{O}_x)$ are intact by differentials of the spectral sequence. Therefore, in view of the definition of an invertible object, we conclude that for *any* point x from the support of $\mathcal{H}^{q_0}$

(a) $\mathrm{Hom}(\mathcal{H}^{q_0}, \mathcal{O}_x) = k(x)$,

(b) $\mathrm{Ext}^1(\mathcal{H}^{q_0}, \mathcal{O}_x) = 0$.

Since X is smooth and irreducible, it follows from (b) that the $\mathcal{H}^{q_0}$ is locally free on X, while (a) implies it is invertible.

It follows that $\mathrm{Ext}^i(\mathcal{H}^{q_0}, \mathcal{O}_x) = 0$ for $i > 0$. Hence, $\mathrm{Hom}(\mathcal{H}^{q_0-1}, \mathcal{O}_x)$ are intact by differentials of the spectral sequence. This means that $\mathrm{Hom}(\mathcal{H}^{q_0-1}, \mathcal{O}_x) = 0$, for any $x \in X$, i.e. $\mathcal{H}^{q_0-1} = 0$. Repeating this argument for $\mathcal{H}^q$ with smaller q, we easily see that all $\mathcal{H}^q$, except $q = q_0$, are trivial. This proves the proposition. □

Now we are ready to prove the reconstruction theorem. Invertible sheaves help us to 'glue' points together.

THEOREM 2.5. *Let X be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. If $\mathcal{D} = D^b_{coh}(X)$ is equivalent as a graded category to $D^b_{coh}(X')$ for some other smooth algebraic variety X', then X is isomorphic to X'.*

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Note that this theorem is stronger than just a reconstruction for a variety with an ample canonical or anticanonical sheaf from its derived category: since X' might not have an ample canonical or anticanonical sheaf, the situation is not symmetric with respect to X and X'.

We divide the proof in several steps, so that the reconstruction procedure will be transparent.

Proof. During the proof, while saying that two isomorphism classes of objects, one in $D^b_{coh}(X)$ and the other one in $D^b_{coh}(X')$, are equal, we mean that the former is taken to the latter by the primary equivalence $D^b_{coh}(X) \xrightarrow{\sim} D^b_{coh}(X')$.

Step 1. Denote $\mathcal{P}_D$ the set of isomorphism classes of the point objects in $\mathcal{D}$, $\mathcal{P}_X$ the set of isomorphism classes of objects in $D^b_{coh}(X)$

$$\mathcal{P}_X := \left\{ \mathcal{O}_x[k] \,\middle|\, x \in X, k \in \mathbb{Z} \right\}.$$

By Proposition 2.2, $\mathcal{P}_D \cong \mathcal{P}_X$. Obviously, $\mathcal{P}_{X'} \subset \mathcal{P}_D$. Suppose that there is an object $P \in \mathcal{P}_D$,which is not contained in $\mathcal{P}_{X'}$. Since $\mathcal{P}_D \cong \mathcal{P}_X$, any two objects in $\mathcal{P}_D$ either are homologically mutually orthogonal or belong to a common orbit with respect to the translation functor. It follows that $P \in D^b_{coh}(X')$ is orthogonal to any skyscraper sheaf $\mathcal{O}_{x'}, x' \in X'$. Hence, P is zero. Therefore, $\mathcal{P}_{X'} = \mathcal{P}_D = \mathcal{P}_X$.

Step 2. Denote by $\mathcal{L}_D$ the set of isomorphism classes of invertible objects in D, $\mathcal{L}_X$ the set of isomorphism classes of objects in $D^b_{coh}(X)$ defined by

$$\mathcal{L}_X := \left\{ L[k] \,\middle|\, L \text{ being an invertible sheaf on } X, k \in \mathbb{Z} \right\}.$$

By Step 1, both varieties X and X' satisfy the assumptions of Proposition 2.4. It follows that $\mathcal{L}_X = \mathcal{L}_D = \mathcal{L}_{X'}$.

Step 3. Let us fix some invertible object L_0 in $\mathcal{D}$ which is an invertible sheaf on X. By Step 2, L_0 can be regarded, up to translation, as an invertible sheaf on X'. Moreover, changing, if necessary, the equivalence $D^b_{coh}(X) \simeq D^b_{coh}(X')$ by the translation functor, we can assume that L_0, regarded as an object on X', is a genuine invertible sheaf. (Formally speaking, L_0 is taken by the equivalence $D^b_{coh}(X) \xrightarrow{\sim} D^b_{coh}(X')$ to an object which is isomorphic to an invertible sheaf on X'. But as was explained in Section 1 (formula (1)) we can adjust this equivalence so that it takes L_0 into the invertible sheaf on X'.)

Obviously, by Step 1, the set $p_D \subset \mathcal{P}_D$

$$p_D := \left\{ P \in \mathcal{P}_D \,\middle|\, \mathrm{Hom}(L_0,\ P) = k(P) \right\}$$

coincides with both sets $p_X = \{\mathcal{O}_x, x \in X\}$ and $p_{X'} = \{\mathcal{O}_{x'}, x' \in X'\}$. This gives us a pointwise identification of X with X'.

Step 4. Now let l_X (resp., $l_{X'}$) be the subset in $\mathcal{L}_D$ of isomorphism classes of invertible sheaves on X (resp., on X').

They can be recognized from the graded category structure in D as follows:

$$l_{X'} = l_X = l_D := \left\{ L \in \mathcal{L}_D \,\middle|\, \mathrm{Hom}(L,\ P) = k(P) \text{ for any } P \in p_D \right\}.$$

For $\alpha \in \mathrm{Hom}(L_1,\ L_2)$, where $L_1, L_2 \in l_D$, and $P \in p_D$, denote by α_P^* the induced morphism:

$$\alpha_P^* : \mathrm{Hom}(L_2,\ P) \longrightarrow \mathrm{Hom}(L_1,\ P)$$

and by U_α the subset of those objects P in p_D for which $\alpha_P^* \neq 0$. By [6], any algebraic variety has an ample system of invertible sheaves. This means that U_α, where α runs over all elements in $\mathrm{Hom}(L_1,\ L_2)$ and L_1 and L_2 run over all elements in l_D,constitute a basis for the Zariski topologies on both X and X'. It follows that the topologies on X and X' coincide.

Step 5. Since codimensions of all point objects are equal to the dimensions of X and of X', we have $\dim X = \dim X'$. Then, formula (7) for the Serre functor $S_{\mathcal{D}} = S$ shows that the operations of twisting by the canonical sheaf on X and on X' induce equal transformations on the set l_D.

Let $L_i = S^i L_0[-ni]$. Then $\{L_i\}$ is the orbit of L_0 with respect to twisting by the canonical sheaf on X. Changing, if necessary, the equivalence $D^b_{coh}(X) \xrightarrow{\sim} D^b_{coh}(X')$, we can assume that $\{L_i\}$ is the orbit of L_0 with respect to twisting by the canonical sheaf on X' too.

Since the canonical sheaf ω_X is either ample or antiample, the set of all U_α, where α runs over all elements in $\mathrm{Hom}(L_i,\ L_j)$, $i, j \in \mathbb{Z}$, is the basis of the Zariski topology on X, hence, by Step 4, on X'. This means that the canonical sheaf on X' is also ample or, respectively, antiample (see [6]).

For all pairs (i, j) there are natural isomorphisms:

$$\begin{aligned}\mathrm{Hom}(L_i,\ L_j) &\cong \mathrm{Hom}(S^i L_0[-ni],\ S^j L_0[-nj]) \\ &\cong \mathrm{Hom}(L_0,\ S^{j-i} L_0[-n(j-i)]) \cong \mathrm{Hom}(L_0,\ L_{j-i}).\end{aligned}$$

They induce a ring structure in the graded algebra A over k with graded components $A_i = \mathrm{Hom}(L_0,\ L_i)$.

This algebra, being defined intrinsically by the graded category structure, is isomorphic to the coordinate algebra B of the canonical sheaf for X, i.e. to the algebra with graded components $B_i = \mathrm{Hom}_X(\mathcal{O}_X, \omega_X^{\otimes i})$.

Indeed, $L_i = L_0 \otimes \omega_X^{\otimes i}$, the isomorphism being given by tensoring by L_0. It is a ring homomorphism, because the functor of tensoring by L_0 commutes with the Serre functor by Proposition 1.3.

The same is true for the coordinate algebra B' of the canonical sheaf for X'. Eventually, we obtain an isomorphism $B \xrightarrow{\sim} B'$ of the canonical algebras on X and X'. Since the canonical sheaves on both X and X' are ample (or antiample), both varieties can be obtained by projectivization from the canonical algebras $X \xrightarrow{\sim} \mathbf{Proj} B \xrightarrow{\sim} \mathbf{Proj} B' \xrightarrow{\sim} X'$ This gives a biregular isomorphism between X and X' as algebraic varieties. □

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3. Group of Exact Autoequivalences

It was explained in Section 1 that the set of isomorphism classes of exact autoequivalences in a triangulated category $\mathcal{D}$ is a group. We denote this group by $\mathrm{Aut}\mathcal{D}$.

The problem of reconstructing of a variety from its derived category is closely related to the problem of computing the group of exact autoequivalences for $D^b_{coh}(X)$. For ample canonical or anticanonical sheaf we have the following.

THEOREM 3.1. *Let X be a smooth irreducible projective variety with an ample canonical or anticanonical sheaf. Then the group of isomorphism classes of exact autoequivalences $D^b_{coh}(X) \to D^b_{coh}(X)$ is generated by the automorphisms of the variety, the twists by invertible sheaves and the translations.*

Proof. Assume for definiteness that the canonical sheaf is ample. Choose an autoequivalence F. Since the class of invertible objects is defined intrinsically with respect to the graded structure of the category, it is preserved by any autoequivalence. Moreover, the set of isomorphism classes of invertible objects is transitive with respect to the action of the subgroup $\mathrm{Aut}D^b_{coh}(X)$ generated by translations and twists. Indeed, by Propositions 2.2 and .4, all invertible objects in $D^b_{coh}(X)$ are invertible sheaves up to translations. Any invertible sheaf can be obtained from the trivial sheaf $\mathcal{O}$ by applying the functor of tensoring with this invertible sheaf. Therefore, using twists with invertible sheaves and translations we can assume that our functor F takes $\mathcal{O}$ to $\mathcal{O}$. It follows that F takes any tensor power $\omega_X^{\otimes i}$ of the canonical sheaf into itself, because, by Proposition 1.3, it commutes with the Serre functor.

Therefore, our functor induces an automorphism of the graded coordinate algebra A of the canonical sheaf, i.e. algebra with graded components $A_i = \mathrm{Hom}(\mathcal{O}, \omega_X^{\otimes i}) = \mathrm{H}^0(\omega_X^{\otimes i})$.

Any graded automorphism of the canonical algebra induces an automorphism of the variety. Adjusting our functor F by an autoequivalence induced by an automorphism of the variety we can assume that the automorphism of the canonical algebra induces the trivial automorphism of the variety.

Such an automorphism is actually a scaling, i.e. it takes an element $a \in \mathrm{H}^0(\omega_X^{\otimes i})$ to $\lambda^i a$, for some fixed scalar λ. Indeed, the graded ideal generated by any element $a \in \mathrm{H}^0(\omega_X^{\otimes i})$ is stable with respect to the automorphism. It follows that a is multiplied by a scalar. Then the linear operator in the graded component $\mathrm{H}^0(\omega_X^{\otimes i})$ induced by the automorphism should be scalar, say λ_i. Since our automorphism is an algebra homomorphism, it follows that $\lambda_i = \lambda^i$, for $\lambda = \lambda_1$ (in case $\mathrm{H}^0(\omega_X) = 0$, i.e. when λ_1 is not defined, we may substitute in the above argument Serre functor by a sufficient jth power of it such that $\mathrm{H}^0(\omega_X^{\otimes j}) \neq 0$ and, respectively, the canonical algebra by the corresponding Veronese subalgebra) .

To kill the scaling of the canonical algebra, we substitute functor F by an isomorphic one. For this we take the subclass $\mathcal{C}$ of objects in $\mathcal{D}$ consisting of powers

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of the canonical sheaf $\mathcal{C} = \{\omega_X^{\otimes i}\}$, $i \in \mathbb{Z}$. As in Section 1, for any object C in $\mathcal{C}$ we need to choose an isomorphism of its image with some other object C'. Our functor preserves all objects from $\mathcal{C}$. We choose $C' = C$ for any C in $\mathcal{C}$ and the nontrivial isomorphism: if $C = \omega_X^{\otimes i}$, then the isomorphism is $\lambda^{-i} \cdot \mathrm{id}_C$.

Then the new functor constructed by formula (1) induces the trivial automorphism of the canonical ring.

Thus we have a functor, which takes the trivial invertible sheaf and any power of the canonical sheaf to themselves and preserves the homomorphisms between all these sheaves. Let us now show that such a functor is isomorphic to the identity functor.

First, our functor takes pure sheaves to objects, isomorphic to pure sheaves, because such objects can be characterized as the objects $\mathcal{G}$ in $D^b_{coh}(X)$ which have trivial $\mathrm{Hom}^k(\omega_X^{\otimes i}, \mathcal{G})$, for $k \neq 0$ and for sufficiently negative i. Again we can substitute our functor by an isomorphic one which takes sheaves to pure sheaves. By Serre theorem [11], the Abelian category of pure sheaves is equivalent to the category of graded finitely generated modules over the canonical algebra A modulo the subcategory of finite-dimensional modules. The equivalence takes a sheaf $\mathcal{G}$ into a module $\mathcal{M}(\mathcal{G})$ with graded components $\mathcal{M}_i(\mathcal{G}) = \mathrm{Hom}(\omega_X^{\otimes -i}, \ \mathcal{G})$. Our functor F gives the isomorphisms:

$$\mathrm{Hom}(\omega_X^{\otimes -i}, \ \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(F(\omega_X^{\otimes -i}), \ F(\mathcal{G})) \cong \mathrm{Hom}(\omega_X^{\otimes -i}, \ F(\mathcal{G})).$$

Since F induces trivial action on the canonical algebra, these isomorphisms form an isomorphism of A-modules $\mathcal{M}(\mathcal{G}) \xrightarrow{\sim} \mathcal{M}(F(\mathcal{G}))$.

It is natural with respect to $\mathcal{G}$. Hence, we obtain an isomorphism of functors $\mathcal{M} \xrightarrow{\sim} \mathcal{M} \circ F$.

Since modulo the subcategory of finite-dimensional modules $\mathcal{M}$ is an equivalence, we have a functorial isomorphism $\mathrm{id} \xrightarrow{\sim} F$ on the subcategory of coherent sheaves.

Our system of objects $\{\omega_X^{\otimes i}\}$ has some nice properties with respect to the Abelian category of coherent sheaves on X which allow us to extend the natural transformation $id \to F$, from the core of the t-structure to a natural isomorphism in the whole derived category. It was done in Proposition A.3 of the Appendix.

This finishes the proof of the theorem. □

In the hypothesis of Theorem 3.1 the group $\mathrm{Aut} D^b_{coh}(X)$ is the semi-direct product of its subgroups $G_1 = \mathrm{Pic} X \oplus \mathbb{Z}$ and $G_2 = \mathrm{Aut} X$, $\mathbb{Z}$ being generated by the translation functor:

$$\mathrm{Aut} D^b_{coh}(X) \cong \mathrm{Aut} X \ltimes (\mathrm{Pic} X \oplus \mathbb{Z}).$$

Indeed, during the course of the proof of the theorem, we in fact showed that any element from $\mathrm{Aut} D^b_{coh}(X)$ could be decomposed as $g = g_1 g_2$ with $g_1 \in G_1$ and $g_2 \in G_2$. The subgroups G_1 and G_2 meet trivially in G, because the elements from the latter take the structure sheaf $\mathcal{O}$ to itself, while those from the former do not.

Group G_1 is obviously preserved by conjugation by elements from $G_1$and G_2, hence normal in G.

Appendix

This appendix is devoted to describing the conditions under which one can extend to the whole category a natural isomorphism between the identity functor and an exact autoequivalence in the bounded derived category $D^b(\mathcal{A})$, provided one has such an isomorphism in an Abelian category $\mathcal{A}$ (or even in a smaller subcategory, see the proposition below).

To find our way through the technical details, we need a sequence of objects in the Abelian category with some remarkable properties. For the case when the sequence consists of powers of an invertible sheaf, these properties result from ampleness of this sheaf. For this reason, we postulate them under the name of ampleness.

DEFINITION A.1. Let $\mathcal{A}$ be an Abelian category. We call a sequence of objects $\{P_i\}$, $i \in \mathbb{Z}_{\leqslant 0}$, ample if for every object $X \in \mathcal{A}$, there exists N such that for all $i < N$ the following conditions hold:

(a) the canonical morphism $\mathrm{Hom}(P_i, X) \otimes P_i \longrightarrow X$ is surjective,
(b) $\mathrm{Ext}^j(P_i, X) = 0$ for any $j \neq 0$,
(c) $\mathrm{Hom}(X, P_i) = 0$.

Denote by $\mathcal{D}^b(\mathcal{A})$ the bounded derived category of $\mathcal{A}$. Let us consider $\mathcal{A}$ as a full subcategory $j: \mathcal{A} \hookrightarrow \mathcal{D}^b(\mathcal{A})$ in $\mathcal{D}^b(\mathcal{A})$ in the usual way. We also consider a full subcategory $q: \mathcal{C} \hookrightarrow \mathcal{D}^b(\mathcal{A})$ with $Ob\mathcal{C} = \{P_i\}_{i\in\mathbb{Z}_{\leqslant 0}}$. We shall show that if there exists an exact autoequivalence $F: \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{D}^b(\mathcal{A})$ and an isomorphism of its restriction to $\mathcal{C}$ with the identity functor $\mathrm{id}_{\mathcal{C}}$, then this isomorphism can be uniquely extended to an isomorphism of F with the identity functor $\mathrm{id}_{\mathcal{D}^b(\mathcal{A})}$ in the whole $\mathcal{D}^b(\mathcal{A})$.

The idea is in reducing the number of nonzero cohomologies for an object by killing the highest one by means of a surjective morphism from $\oplus P_i$ for sufficiently negative i.

In the proof, we shall repeatedly use the following lemma (see [1]).

LEMMA A.2. *Let g be a morphism from Y to Y' and suppose that these objects are included into the following two exact triangles:*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1]
\end{array}$$

If $v'gu = 0$, then there exist morphisms $f: X \to X'$ and $h: Z \to Z'$ such that the triple (f, g, h) is a morphism of triangles.

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If, in addition, $\mathrm{Hom}(X[1], Z') = 0$, *then the morphisms* f *and* h, *making commutative the first and, respectively, the second square of the diagram, are unique.*

PROPOSITION A.3. *Let* $\mathcal{A}$ *be an Abelian category possessing an ample sequence* $\{P_i\}$ *and let* $F\colon \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{D}^b(\mathcal{A})$ *be an exact autoequivalence. Suppose there exists an isomorphism* $f\colon q \xrightarrow{\sim} F|_{\mathcal{C}}$ *(where* $q\colon \mathcal{C} \hookrightarrow \mathcal{D}^b(\mathcal{A})$ *is the natural embedding). Then this isomorphism can be uniquely extended to an isomorphism* $\mathrm{id} \xrightarrow{\sim} F$ *in the whole* $\mathcal{D}^b(\mathcal{A})$.

Proof. Note that in view of the condition (b) from Definition A.1, $X \in \mathcal{D}^b(\mathcal{A})$ is isomorphic to an object in $\mathcal{A}$ iff $\mathrm{Hom}^j(P_i, X) = 0$ for $j \neq 0$ and $i \ll 0$.

This allows us to 'extract' the Abelian subcategory $\mathcal{A}$ from $\mathcal{D}^b(\mathcal{A})$ by means of the sequence $\{P_i\}$. Then using surjective coverings $\mathrm{Hom}(P_i, X) \otimes P_i \longrightarrow X$ by the standard technique from the theory of Abelian categories, one can extend f to an isomorphism (which we denote by the same letter) $f\colon j \xrightarrow{\sim} F|_{\mathcal{A}}$, where j stands for the natural embedding $j\colon \mathcal{A} \hookrightarrow \mathcal{D}^b(\mathcal{A})$. We skip the details of this part of the proof because we don't need it in the main body of the paper.

Let us define $f_{X[n]}\colon X[n] \longrightarrow F(X[n]) \cong F(X)[n]$ for $X \in \mathcal{A}$ by $f_{X[n]} = f_X[n]$. It is not difficult to show that, for any X and Y in $\mathcal{A}$ and for any $u \in \mathrm{Ext}^k(X, Y)$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y[k] \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f_{Y[k]}} \\ F(X) & \xrightarrow{F(u)} & F(Y)[k] \end{array} \tag{8}$$

is commutative. Indeed, since any element $u \in \mathrm{Ext}^k(X, Y)$ can be represented as the Yoneda composition $u = u_1 \dots u_k$ of elements $u_i \in \mathrm{Ext}^1(Z_i, Z_{i+1})$ for some objects Z_i, with $Z_1 = X, Z_{k+1} = Y$, then we can restrict ourselves to the case $u \in \mathrm{Ext}^1(X, Y)$. Consider the following diagram:

$$\begin{array}{ccccccc} Y & \longrightarrow & Z & \xrightarrow{p} & X & \xrightarrow{u} & Y[1] \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f_Z} & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle f_Y[1]} \\ F(Y) & \longrightarrow & F(Z) & \xrightarrow{F(p)} & F(X) & \xrightarrow{F(u)} & F(Y)[1] \end{array}$$

By axioms of triangulated categories, there exists a morphism $h\colon X \to F(X)$ such that (f_Y, f_Z, h) is a morphism of triangles. On the other hand, since $\mathrm{Hom}(Y[1], F(X)) = 0$, by the lemma above h is a unique morphism such that $F(p) \circ f_Z = h \circ p$. As $F(p) f_Z = f_X p$, we conclude that $h = f_X$. This implies commutativity of the diagram (8) for $k = 1$.

We shall prove by induction over n the following statement. Consider the full subcategory $j_n : \mathcal{D}_n \hookrightarrow \mathcal{D}^b(\mathcal{A})$ in $\mathcal{D}^b(\mathcal{A})$ generated by objects which have nontrivial cohomology in a (nonfixed) segment of length n. Then there is a unique extension of f to a natural functorial isomorphism $f_n : j_n \longrightarrow F|_{\mathcal{D}_n}$.

Above, we have completed the first, $n = 1$, step of the induction.

Now take the step $n = a$, $a \geqslant 1$, for granted . Let X be an object in $\mathcal{D}_{a+1}$ and suppose, for definiteness, that its cohomology objects $\mathcal{H}^p(X)$ are nontrivial only for $p \in [-a, 0]$. Take P_i from the given ample sequence with sufficiently negative i such that

$$\begin{aligned} &\text{(a) } \mathrm{Hom}^j(P_i,\ \mathcal{H}^p(X)) = 0 \text{ for all } p \text{ and for } j \neq 0,\\ &\text{(b) there exists a surjective morphism } u\colon P_i^{\oplus k} \longrightarrow \mathcal{H}^0(X),\\ &\text{(c) } \mathrm{Hom}(\mathcal{H}^0(X),\ P_i) = 0. \end{aligned} \tag{9}$$

Note that in view of condition (a) and the standard spectral sequence $\mathrm{Hom}(P_i,\ X) \xrightarrow{\sim} \mathrm{Hom}(P_i,\ \mathcal{H}^0(X))$. This means that we can find a morphism $\upsilon\colon P_i^{\oplus k} \longrightarrow X$ such that the composition of υ with the canonical morphism $X \longrightarrow \mathcal{H}^0(X)$ coincides with u. Consider an exact triangle

$$Y[-1] \longrightarrow P_i^{\oplus k} \stackrel{\upsilon}{\longrightarrow} X \longrightarrow Y.$$

Denote by f_i the morphism f_Z for $Z = P_i^{\oplus k}$. Since Y belongs to $\mathcal{D}_a$ by the induction hypothesis, the isomorphism f_Y is already defined and the right-hand square of the diagram

$$\begin{array}{ccccccc} P_i^{\oplus k} & \stackrel{\upsilon}{\longrightarrow} & X & \longrightarrow & Y & \longrightarrow & P_i^{\oplus k}[1] \\ \downarrow{\scriptstyle f_i} & & \vdots{\scriptstyle f_X} & & \downarrow{\scriptstyle f_Y} & & \downarrow{\scriptstyle f_i[1]} \\ F(P_i^{\oplus k}) & \stackrel{F(\upsilon)}{\longrightarrow} & F(X) & \longrightarrow & F(Y) & \longrightarrow & F(P_i^{\oplus k})[1] \end{array} \tag{10}$$

is commutative.

Further, we have the following sequence of isomorphisms:

$$\mathrm{Hom}(X,\ F(P_i^{\oplus k})) \cong \mathrm{Hom}(X,\ P_i^{\oplus k}) \cong \mathrm{Hom}(\mathcal{H}^0(X),\ P_i^{\oplus k}) = 0.$$

Hence, applying Lemma A.2 to g equal f_Y, we obtain a unique morphism $f_X\colon X \longrightarrow F(X)$ that preserves the commutativity of the above diagram.

It is clear from the definition that f_X is an isomorphism if so are f_i and f_Y. For the sequel, we need to show that f_X does not depend on the choice of i and u. Suppose we are given two surjective morphisms $u_1 : P_{i_1}^{\oplus k_1} \longrightarrow \mathcal{H}^0(X)$ and $u_2 : P_{i_2}^{\oplus k_2} \longrightarrow \mathcal{H}^0(X)$, where i_1 and i_2 are sufficiently negative to satisfy conditions (a), (b) and (c).Then we can find sufficiently negative j and surjective morphisms w_1, w_2 such that the following diagram commutes:

$$\begin{array}{ccc} P_j^{\oplus l} & \stackrel{w_2}{\longrightarrow} & P_{i_2}^{\oplus k_2} \\ \downarrow{\scriptstyle w_1} & & \downarrow{\scriptstyle u_2} \\ P_{i_1}^{\oplus k_1} & \stackrel{u_1}{\longrightarrow} & \mathcal{H}^0(X). \end{array}$$

Denote by $v_1 : P_{i_1}^{\oplus k_1} \longrightarrow X$, $v_2 : P_{i_2}^{\oplus k_2} \longrightarrow X$ the morphisms corresponding to u_1 and u_2. Since $\mathrm{Hom}(P_j,\ X) \xrightarrow{\sim} \mathrm{Hom}(P_j,\ \mathcal{H}^0(X))$, we have $v_2 w_2 = v_1 w_1$.

There is a morphism $\phi\colon Y_j \longrightarrow Y_{i_1}$ such that the triple (w_1, id, ϕ) is a morphism of exact triangles:

$$\begin{array}{ccccccc} P_j^{\oplus l} & \xrightarrow{v_1 \circ w_1} & X & \xrightarrow{y} & Y_j & \longrightarrow & P_j^{\oplus l}[1] \\ {\scriptstyle w_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle w_1[1]} \\ P_{i_1}^{\oplus k_1} & \xrightarrow{v_1} & X & \xrightarrow{y_1} & Y_{i_1} & \longrightarrow & P_{i_1}^{\oplus k_1}[1], \end{array}$$

i.e. $\phi y = y_1$.

Since Y_j and Y_{i_1} have cohomology in the segment $[-a, -1]$, by the induction hypothesis, the following square is commutative:

$$\begin{array}{ccc} Y_j & \xrightarrow{\phi} & Y_{i_1} \\ {\scriptstyle f_{Y_j}}\downarrow & & \downarrow{\scriptstyle f_{Y_{i_1}}} \\ F(Y_j) & \xrightarrow{F(\phi)} & F(Y_{i_1}). \end{array}$$

Denote by $f_X^j, f_X^{i_1}, f_X^{i_2}$ the unique morphisms constructed as above to make the diagram (10) commutative for v equal, respectively $v = v_1 w_1$, $v = v_1$, $v = v_2$. We have:

$$F(y_1) f_X^j = F(\phi y) f_X^j = F(\phi) F(y) f_X^j = F(\phi) F_{Y_j} y = f_{Y_{i_1}} \phi y = f_{Y_{i_1}} y_1.$$

It follows that $f_X^j = f_X^{i_1}$. Similarly, since $v_1 w_1 = v_2 w_2$ we have $f_X^j = f_X^{i_2}$. Therefore, the morphism f_X does not depend on the choice of i and of the morphism $u\colon P_i^{\oplus k} \longrightarrow \mathcal{H}^0(X)$.

By means of the translation functor, we obtain the only possible extension of f_a to $\mathcal{D}_{a+1}$ in the obvious way. Let us prove that it is indeed a natural transformation from j_{a+1} to $F|_{\mathcal{D}_{a+1}}$, i.e. that for any morphism $\phi\colon X \longrightarrow Y$, X, Y being in $\mathcal{D}_{a+1}$, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ F(X) & \xrightarrow{F(\phi)} & F(Y). \end{array} \tag{11}$$

We shall reduce the problem to the case when both X and Y are in $\mathcal{D}_a$.

There are two working possibilities that we shall utilize for this.

Case 1. Suppose that the upper bound, say 0 (without loss of generality), of cohomology for X is greater than that for Y. Take a surjective morphism $u\colon P_i^{\oplus k} \longrightarrow \mathcal{H}^0(X)$ satisfying (a), (b), (c) and construct the morphism $v\colon P_i^{\oplus k} \longrightarrow X$ related to u as above. We have an exact triangle:

$$P_i^{\oplus k} \xrightarrow{v_1} X \xrightarrow{\alpha} Z \longrightarrow P_i^{\oplus k}[1].$$

If we take i sufficiently negative, then $\mathrm{Hom}(P_i^{\oplus k}, Y) = 0$. Applying the functor

$\mathrm{Hom}(-,\ Y)$ to this triangle we found that there exists a morphism $\psi\colon Z\longrightarrow Y$ such that $\phi=\psi\alpha$. We know that f_X, defined above, satisfies the equation $F(\alpha)f_X=f_Z\alpha$.

If we assume that $F(\psi)f_Z=f_Y\psi$, then

$$F(\phi)f_X=F(\psi)F(\alpha)f_X=F(\psi)f_Z\alpha=f_Y\psi\alpha=f_Y\phi.$$

This means that, for this case in verifying commutativity of (11), we can substitute X by an object Z such that the upper bound of its cohomology is, by one, less than that for X. Moreover, one can easily see that if X belongs to $\mathcal{D}_k$, with $k>1$, then Z does to $\mathcal{D}_{k-1}$, and if it is in $\mathcal{D}_1$ then so is Z.

Case 2. Suppose now that the upper bound, say 0 (again without loss of generality), of cohomology for Y is greater than or equal to that for X. Take a surjective morphism $u: P_i^{\oplus k}\longrightarrow\mathcal{H}^0(Y)$ with i satisfying (a), (b), (c) (with Y instead of X) and construct a morphism $v: P_i^{\oplus k}\longrightarrow Y$ related to u. Consider an exact triangle

$$P_i^{\oplus k}\overset{v}{\longrightarrow}Y\overset{\beta}{\longrightarrow}W\longrightarrow P_i^{\oplus k}.$$

Denote the composition $\beta\circ\phi$ by ψ.

If we assume that $F(\psi)f_X=f_W\psi$, then, since $F(\beta)f_Y=f_W\beta$, we have

$$F(\beta)(f_Y\phi-F(\phi)f_X)=f_W\beta\phi-f(\beta\phi)f_X=f_W\psi-F(\psi)f_X=0. \tag{12}$$

We again take i sufficiently negative so that $\mathrm{Hom}(X,\ P_i^{\oplus k})=0$. As $F(P_i^{\oplus k})$ is isomorphic to $P_i^{\oplus k}$, then $\mathrm{Hom}(X,\ F(P_i^{\oplus k}))=0$. Applying the functor $\mathrm{Hom}(X,\ F(-))$ to the above triangle, we found that the composition with $F(\beta)$ gives an inclusion of $\mathrm{Hom}(X,\ F(Y))$ into $\mathrm{Hom}(X,\ F(W))$. It follows from (15) that $f_Y\phi=F(\phi)f_X$.

Thus, for this case, in verifying commutativity of (11), we can substitute Y by an object W such that the upper bound of its cohomology is less by one than that for Y. If Y belongs to $\mathcal{D}_k$, $k>1$, then W does to $\mathcal{D}_{k-1}$, if Y belongs to $\mathcal{D}_1$, then so does W.

Suppose now that X and Y are in $\mathcal{D}_{a+1}$, $a>1$. Depending on which case, (1) or (2), we are in, we can substitute either X or Y by an object lying in $\mathcal{D}_a$. Then, if necessary, repeating the procedure, we can lower the upper bound of the cohomology of the object to such a point that the other case is applicable. Then we shorten the cohomology segment of the second object and come to the situation when both objects are in $\mathcal{D}_a$, i.e. to the induction hypothesis.

At every step of the construction we always made the only possible choice for the morphism f_X. This means that the natural transformation with required properties is unique. This finishes the proof of the proposition. □

Remark. As was mentioned by the reviewer the same argument as in the proof of the proposition proves that for any pair (F_1, F_2) of exact functors in the category

$\mathcal{D}^b(\mathcal{A})$ such that $F_1|_{\mathcal{C}}$ and $F_2|_{\mathcal{C}}$ are isomorphic to the identity functor, the restriction map

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(F_1, F_2) \to \mathrm{Hom}_{\mathcal{C}}(F_1|_{\mathcal{C}}, F_2|_{\mathcal{C}})$$

is bijective.

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