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Reconstruction of modified Gauss–Bonnet gravity for emergent universe

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We present a flat emergent universe model in modified Gauss–Bonnet gravity in four dimensions. The emergent universe model is free from a big-bang singularity and also describes the observed universe fairly well. It is assumed that the present universe emerged from a static Einstein universe phase that exists in the infinite past. To obtain the flat emergent universe model, we reconstruct mimetic modified $f(G)$-gravity (G representing Gauss–Bonnet terms). The functional form of $f(G)$-gravity is determined with or without matter, and can accommodate the early inflation and late accelerating phases satisfactorily. In contrast, in Einstein's general theory of relativity, a flat emergent universe was obtained within the modified matter sector.

Keywords: Flat emergent universe; modified gravity; Gauss–Bonnet gravity.

1. Introduction

In modern cosmology it is believed that the present universe emerged from an inflationary phase in the very early universe.[1–3] The big bang model of the universe, which is based on Einstein's general theory of relativity with a perfect fluid, fails to account for some of the conceptual issues in the observed universe. It is established that a semiclassical description of the theory or a higher order theory of gravity[4] admits an inflationary universe scenario which resolves some of the outstanding problems of big bang cosmology.Recent cosmological observations[5–7] predict another interesting feature of the observed universe, i.e. that it is passing through an accelerating phase. Theoretically, it is a challenge to describe the origin of the late accelerating phase as the physics of inflation,[8–12] and the introduction of a small cosmological constant for late acceleration are among the features that are not yet fully understood.[13,14] The known fields of the standard model particle physics are not suitable to describe present observations. For this, some new physics may emerge in future beyond the standard model at very high energy scales. Investigations have been carried out in several gravitational theories to realize early inflation and late acceleration either by a modification of the gravitational sector[15–17] or by a modification of the matter sector of Einstein's general relativity (GR). Thus there is enough motivation to explore an alternative cosmological model.

Astashenok *et al.*[18] investigated various scenarios of cosmological evolution in mimetic modified gravity models, with or without matter, to obtain different cosmological evolutions. Considering the simplest formulation based on the use of the Lagrange multiplier constraint, it can be shown that in some theories, it is possible to realize the accelerated expansion of the universe or even unified evolution, which includes inflation with dark energy, and dark matter described by the theory. Adopting general reconstruction schemes, it is found that mimetic gravity also generates bouncing cosmologies. The Gauss–Bonnet (GB) combination $G = R^2 - 4R_{\mu\delta}R^{\mu\delta} + R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta}$ plays an important role in cosmology.[19–21] Recently, in the literature,[18,22–24] cosmological models with a number of known evolutionary features were examined in a gravitational theory reconstructed with the GB functional form. The motivation of this paper is to reconstruct modified GB gravity which permits a singularity free cosmology, namely, a flat emergent universe (EU) model which encompasses the unified evolution of the universe.

The possibility of a cosmological model in which there is no big bang singularity, no beginning of time, and in which the universe effectively avoids a quantum regime for spacetime by staying large at all times was considered first by Ellis and Maartens.[25] In this model, the universe started out in the infinite past in an almost static Einstein universe state, and subsequently it entered into an expanding phase slowly, eventually evolving into a hot big bang era. Later Ellis *et al.*[26] constructed an EU scenario for a spatially closed universe with a minimally coupled scalar field ϕ, using a special form for the interaction potential $V(\phi)$. However, it was shown[27] immediately that such a potential can be recovered starting from a modified gravitational action with a polynomial Lagrangian $L = R + \alpha R^2$, making use of a suitable conformal transformation and identification of the field as $\phi = -\sqrt{3}\,\ln(1 + 2\alpha R)$. In the above however, one requires a negative coupling term α for a consistent EU scenario. Later in a flat spacetime, Mukherjee *et al.*[28] obtained new cosmological solutions with a static Einstein universe in the framework of semiclassical gravity.

It is known that the EU scenario promises to solve several conceptual and technical issues of the big bang model.

Recent cosmological observations predict that the universe is most likely spatially flat. Therefore, it is important to explore a flat EU model. In the framework of Einstein gravity, Mukherjee *et al.*[29] obtained an EU scenario in a flat universe making use of a cosmic fluid described by the nonlinear equation of state (EOS) $p = A\rho - B\sqrt{\rho}$ where A and B are arbitrary constants. The possibility of emergent cosmology using the effective potential formalism has been further explored recently showing that new models of emergent cosmology can be realized within limits set by the constraints imposed by the cosmic microwave background (CMB) radiation.[30] It has also been demonstrated that within the framework of modified gravity, the EU may be obtained in a spatially open or closed universe. Later it was found that a flat EU scenario can be obtained in a modified theory of gravity by including higher-order GB terms in the presence of a dilaton coupling.[31] The EU model in a flat universe has been explored in various theoretical frameworks, namely, the brane world.[32–34] Brans–Dicke theory,[35] as well as in the context of the nonlinear sigma model.[36,37] It was found that the EU scenario can be obtained with all the features of the observed universe. The EU model accommodates late-time de Sitter expansion and thus, naturally permits late-time acceleration of the universe. Such a scenario is promising from the perspective of offering unified early as well as late-time dynamics of the universe. Note, however, that the focal point of unification in such EU models lies in the choice of the EOS for the polytropic fluid, while several other models of unification rely more on scalar field dynamics in the presence of a suitable field potential.

It is known that the GB term without a dilaton field does not contribute to the dynamics[31] of the universe in four dimensions as it is an Euler number. The motivation of this paper is to obtain an EU scenario in a modified gravity with a modified GB gravity term without a dilaton field. We wish to accommodate a flat EU model of the universe, and reconstruct theoretically an acceptable modified gravity with GB terms in the gravitational action for accommodating the EU scenario. The EU model is interesting because of its various observed features. It permits a nonsingular solution of cosmology, an important and new property in understanding the early evolution of the universe. This is different from that obtained recently in the literature[18,22,24,38] where a modified functional form of GB gravity is determined to describe singular bounce solutions. The motivation of the paper is to determine the functional form of modified GB gravity that permits EU models which are singularity free. The modified form of the action is determined with or without matter.

The paper is organized as follows: In Sec. 2, we set up the relevant field equations starting from a gravitational action for modified GB gravity. In Sec. 3, the reconstruction of modified GB gravity in vacuum is discussed for a given scale factor of the universe. In Sec. 4, the modified GB theory is determined for a flat EU.

In Sec. 5, nonsingular bounce solution in mimetic $F(G)$ gravity is presented, and in Sec. 6, we present a brief discussion.

2. Gravitational Action and the Field Equations

The gravitational action for modified GB gravity theory is given by

$$I = -\frac{1}{2\kappa^2}\int (R + f(G))\sqrt{-g}\ d^4x + I_m, \tag{1}$$

where $\kappa^2 = \frac{1}{M_P^2}$ with the Planck mass represented by M_P, R represents the Ricci scalar and I_m represents the action for matter fields. The field equations are given by[22–24]

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}f(G) - (4R_{\mu\delta}R_{\nu}^{\delta} - 2RR_{\mu\nu} - 2R_{\mu}^{\delta\sigma\tau}R_{\nu\delta\sigma\tau} \\ + 4g^{\alpha\delta}g^{\beta\sigma}R_{\mu\alpha\nu\beta}R_{\delta\sigma})f'(G) - 2(\nabla_\mu\nabla_\nu f'(G))R \\ + 2g_{\mu\nu}(\Box f'(G))R - 4(\Box f'(G))R_{\mu\nu} + 4(\nabla_\mu\nabla_\nu f'(G))R_{\nu}^{\delta} \\ + 4(\nabla_\delta\nabla_\nu f'(G))R_{\mu}^{\delta} - 4g_{\mu\nu}(\nabla_\delta\nabla_\sigma f'(G))R^{\delta\sigma} \\ + 4(\nabla_\delta\nabla_\sigma f'(G))g^{\alpha\delta}g^{\beta\sigma}R_{\mu\alpha\nu\beta} = \kappa^2 T_{\mu\nu}, \end{aligned} \tag{2}$$

where the prime represents the derivative with respect to G, ∇_μ represents the covariant derivative, $\Box$ represents the covariant d'Alembertian, $\kappa^2 = 8\pi G$ and $T_{\mu\nu}$ represents the energy–momentum tensor of matter. We consider the Robertson–Walker metric for spatially homogeneous and isotropic spacetime which is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3}$$

in units in which $c = 1$ and $\hbar = 1$, where $a(t)$ represents the scale factor of the universe and $k = 0, +1, -1$ representing a flat, closed and open universe, respectively. We consider a flat spacetime $k = 0$ to obtain a functional form $f(G)$ which admits an EU as in Ref. 29. The Ricci scalar and the GB combinations in a flat spacetime are given by

$$R = 6\dot{H} + 12H^2, \quad G = 24H^2(\dot{H} + H^2), \tag{4}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and $\dot{()}$ is the time derivative. The gravitational field equations are given by[39]

$$6H^2 + f(G) - Gf'(G) + 24H^3\dot{G}f''(G) = 2\kappa^2\rho, \tag{5}$$

$$\begin{aligned} 4\dot{H} + 6H^2 + f(G) - Gf'(G) + 16H\dot{G}(\dot{H} + H^2)f''(G) \\ + 8H^2\ddot{G}f''(G) + 8H^2\dot{G}^2 f'''(G) = -2\kappa^2 p. \end{aligned} \tag{6}$$

In GR, the gravitational field equations in a flat Robertson–Walker metric can be written as

$$\rho_{\text{eff}} = \frac{3}{\kappa^2}H^2, \quad p_{\text{eff}} = -\frac{1}{\kappa^2}(2\dot{H} + 3H^2). \tag{7}$$

The effective energy density and pressure in the FRW background in $f(G)$-gravity can be expressed as

$$\rho_{\text{eff}} = \frac{1}{2\kappa^2}(6H^2 + f(G) - Gf'(G) + 24H^3\dot{G}f''(G)) + \rho, \tag{8}$$

$$\begin{aligned} p_{\text{eff}} = \frac{1}{2\kappa^2}(&\dot{H} + 6H^2 + f(G) - Gf'(G) + 8H^2\ddot{G}f''(G) \\ &+ 16H\dot{G}(\dot{H} + H^2)f''(G) + 8H^2\dot{G}^2 f'''(G)) + p, \end{aligned} \tag{9}$$

using Eq. (4). Now we define

$$\zeta(H, \dot{H}, \ldots) = p_{\text{eff}} - p \tag{10}$$

so that Eq. (15) can be written as

$$\zeta(H, \dot{H}, \ldots) = -\frac{1}{\kappa^2}[2\dot{H} + 3(A+1)H^2 - B\sqrt{3}H]. \tag{11}$$

Thus it is evident that the $\zeta(H, \dot{H}, \ldots)$ is a combination of H and its derivatives. The existence of a cosmological model can be ascertained by exploring the behavior of the function $\zeta(H, \dot{H}, \ldots)$. A number of cosmological solutions are presented in the literature,[40–75] to obtain cosmological models viable for present accelerating universe which has an initial singularity. Recently Ellis and Marteens[25] obtained a singularity free universe in a closed model of the universe in GR in the presence of a homogeneous scalar field with a specially constructed potential.[25] Subsequently it has been shown that such a potential exists in a modified GR with an R^2-term.[26] In this scenario, such a universe emerged from an Einstein static universe which exists in an infinitely past time. Thereafter it was found by Mukherjee *et al.*[29] that an EU can be obtained in a flat universe with a nonlinear EoS. The EoS is given by

$$p = A\rho - B\sqrt{\rho}. \tag{12}$$

In the EU model, H and $\dot{H}$ are nonvanishing, thus the effective pressure is different from that of the isotropic pressure as $\zeta(H, \dot{H}, \ldots) \neq 0$. The emergent model is interesting and it satisfies all the physical features of the present observed universe. The nonlinear equation is equivalent to three different cosmic fluids. The problem of the model was that once the EoS value or the EOS parameter is fixed, the universe does not accommodate the transition of the universe from a radiation-dominated to a matter-dominated phase. It has been shown[76] that a universe with interacting fluids however gives a realistic scenario. The flat EU can be obtained in GR modified with GB terms coupled with a dilaton field.[31] In four dimensions, the GB-term is a topological quantity and does not contribute to the dynamics of the evolution. The motivation of this paper is to include a functional form of GB gravity $f(G)$ in the gravitational action without a dilaton field to obtain an EU scenario both in vacuum and in the presence of matter described by a nonlinear EoS. The EU is interesting, which evolves from a static phase of the universe in the infinite past. It is free from a big bang singularity and exhibits the features of the past and present universe.[76–83] The functional form of modified gravity $f(G)$ will be reconstructed in this paper for obtaining a flat EU making use of the methods outlined in Refs. 18, 22, 24 and 38

3. Reconstruction of $f(G)$ Gravity

It is known that the GB terms in four-dimensions are Euler numbers which have no role in the dynamics of the universe. A GB term coupled with a dilaton field, however, plays an important role in understanding the dynamics of the universe.[31] We investigate $f(G)$ gravity models, which admit an EU model in the absence of a dilaton field. The reconstruction method is used here to obtain the modified gravity which permits a singularity free universe. The modified gravitational action[24] in the absence of matter is given by

$$I = \int \left[\frac{1}{2\kappa^2}(R + P(t)G + Q(t))\right]\sqrt{-g}d^4x, \tag{13}$$

where $P(t)$ and $Q(t)$ are two proper functions of a scalar field and t represents cosmic time. We investigate the $f(G)$ gravity models in which the flat EU model can be implemented. To determine the functional form of $f(G)$-gravity, we use the reconstruction method of modified gravity.[16,17,23] Using proper functions $P(t)$ and $Q(t)$ of a scalar field which we identify with the cosmic time, the action in Eq. (13) can be rewritten using the method adopted in Ref. 24. Now varying the action with respect to t, one obtains[23,24,52,53]

$$\frac{dP(t)}{dt}G + \frac{dQ(t)}{dt} = 0 \tag{14}$$

which can be used to find $t = t(G)$. Consequently by substituting t in Eq. (13), we can determine the functional form which is

$$f(G) = P(t)G + Q(t). \tag{15}$$

It is interesting to determine $P(t)$ and $Q(t)$ in terms of the Hubble parameter to study observed cosmological consequences. The modified action may work well to get early inflation as well as a late accelerating phase. Using Eqs. (5) and (15), one obtains

$$Q(t) = -6H^2(t) - 24H^3(t)\frac{dP(t)}{dt}. \tag{16}$$

Finally using Eqs. (6), (15) and (16), one obtains a second-order differential equation neglecting the matter which is given by

$$2H^2(t)\frac{d^2P(t)}{dt^2} + 2H(t)(2\dot{H}(t) - H^2(t))\frac{dP(t)}{dt} + \dot{H} = 0. \tag{17}$$

This second-order differential equation can be solved if the scale factor is given. In Sec. 4 we consider the scale factor of an EU obtained in GR in the presence of a nonlinear EOS to determine the modified $f(G)$-functional form which accommodates EU.

4. Gauss–Bonnet Modified Gravity and The EU

The flat EU model in Einstein's gravity is obtained with a nonlinear EoS. The field equations (7) are given by

$$\rho = 3H^2, \quad p = -(2\dot{H} + 3H^2), \tag{18}$$

where H represents Hubble parameter, $\kappa^2 = 1$ and $\zeta(H, \dot{H}, \ldots) = 0$. The conservation equation is given by

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0. \tag{19}$$

The conservation equation can be integrated using the nonlinear EoS given in Eq. (18) which determines the energy density in terms of the scale factor as

$$\rho(a) = \frac{1}{(A+1)^2}\left(B + \frac{K}{a^{\frac{3(A+1)}{2}}}\right)^2. \tag{20}$$

It is interesting to note that by expanding the expression for the energy density, it is possible to identify the three terms with three different types of fluid which are determined by the EoS parameters A and B. The total energy density is made up of three terms with three different types of EoS,

$$\rho(a) = \sum_{i=1}^{3} \rho_i \quad \text{and} \quad p(a) = \sum_{i=1}^{3} p_i. \tag{21}$$

It is noted that the parameter A plays an important role in understanding different types of cosmic fluids present in the universe.[29] Later, considering interactions among the fluids, it is shown that a physically viable EU can be obtained for different strengths of interactions among them accommodating the observed universe.[76] The scale factor of the universe is obtained using Eqs. (18) and (19), which is

$$a(t) = \left[\frac{3K(A+1)}{2}\left(\sigma + \frac{2}{\sqrt{3}B}e^{\frac{\sqrt{3}}{2}Bt}\right)\right]^{\frac{2}{3(A+1)}}, \tag{22}$$

where σ and K are integration constants. It is evident that in the infinitely past, i.e. $t \to -\infty$, the scale factor becomes $a = a_o$, a static Einstein universe, which evolves later into a dynamical universe scenario[29–37] encompassing the present observed universe. The interesting aspect of the EU is that there is no singularity, and the early inflationary universe can be realized in addition to the present accelerating phase. The universe however emerged out of an Einstein static phase. It is also shown that the static Einstein phase can be realized in the context of wormholes in GR with exotic matter. The throat of the wormhole expanded into a large universe after a fairly long time determined by a cosmological term Λ.[77] The functional form of the modified GB gravity will be determined to implement the flat EU model.

Now we construct an equation with Hubble parameter which admits EU solutions corresponding to the scale factor in Eq. (22). The equation is

$$\dot{H} = \alpha H - \beta H^2, \tag{23}$$

where $\alpha = \frac{\sqrt{3}B}{2}$ and $\beta = \frac{3(A+1)}{2}$. For simplicity, we rewrite the scale factor given by Eq. (22) subsequently as

$$a(t) = a_o[\eta + e^{\alpha t}]^{\frac{1}{\beta}}, \tag{24}$$

by denoting $a_o = (\frac{3K(A+1)}{2})^{\frac{2}{3(A+1)}} \frac{2}{\sqrt{3}B}$, $\eta = \frac{\sqrt{3}B\sigma}{2}$. This form of the scale factor will be used in Sec. 4. In this case, in the infinitely past, i.e. $t \to -\infty$, the scale factor becomes $a \to a_o\eta^{\frac{1}{\beta}}$. Thus, a static Einstein universe emerged, which later evolves into a dynamical universe[29,76] encompassing the present observed universe. The Hubble parameter is given by $H = \frac{\alpha e^{\alpha t}}{\beta(\eta + e^{\alpha t})}$. In this case, we determine the functional form of modified GB gravity which permits an EU without matter. Thus we look for the existence of Eq. (23) in Eq. (17) without matter. The general solution can be obtained which accommodates an EU solution in modified gravity in vacuum which is given by

$$P(t) = -\frac{\eta\beta^2(\eta + 2e^{\alpha t})}{\alpha^2(1+\beta)e^{2\alpha t}} + C_1 a_o^{1+2\beta} \int \frac{(\eta + e^{\alpha t})^{2+\frac{1}{\beta}}}{e^{2\alpha t}} dt + C_2 \tag{25}$$

and consequently, from Eq. (16), one obtains

$$Q(t) = -\frac{6\alpha^2 e^{\alpha t}}{\beta^2(\eta + e^{\alpha t})^2}[8\eta\beta + e^{\alpha t}] + \frac{24 C_1 a_o^{1+2\beta}\alpha^3 e^{\alpha t}}{\beta^3(\eta + e^{\alpha t})^{1-\frac{1}{\beta}}}, \tag{26}$$

where C_1 and C_2 are integration constants. The GB combination can be expressed as a function of cosmic time t which is now given by

$$G(t) = \frac{24\alpha^4}{\beta^4} \frac{e^{3\alpha t}(\eta\beta + e^{\alpha t})}{(\eta + e^{\alpha t})^4}. \tag{27}$$

The GB term G is a function of t which is highly nonlinear. An inverse function of t in terms of G namely, $t = t(G)$ can be obtained which is given by

$$t = \pm\left[\frac{\ln\eta}{\alpha} - \frac{1}{\alpha}\ln\left(\left(\frac{24\alpha^4}{\beta^4 G}\right)^{\frac{1}{4}} \mp 1\right)\right]. \tag{28}$$

Using the above expression, we get the functional form of modified GB theory which is

$$f(G) = -\sqrt{\frac{3G}{2}} - \frac{2\beta^2}{\alpha^2(1+\beta)}\left(\frac{\alpha}{\beta}\left(\frac{24}{G}\right)^{\frac{1}{4}} - 1\right)G - \frac{4\beta\eta^2\sqrt{6G}}{\left(\frac{\alpha}{\beta}\left(\frac{24}{G}\right)^{\frac{1}{4}} - 1\right)} - \frac{\beta^2 G}{\alpha^2(1+\beta)\left(\frac{\alpha}{\beta}\left(\frac{24}{G}\right)^{\frac{1}{4}} - 1\right)} + f_0 \tag{29}$$

for $C_1 = 0$, $C_2 = 0$ and f_0 a constant. The interesting feature of the EU solution is that there is no initial singularity and it admits the observed features of the universe which was shown in Refs. 29 and 76. The functional form of the modified $f(G)$ gravity obtained here is in vacuum.

5. EU in Mimetic $f(G)$ Gravity

We determine the functional form of mimetic $f(G)$-gravity following the parametrization of the metric in the gravitational action given by Eq. (1).[18,23] To obtain mimetic $f(G)$ gravity, one assumes the parametrization of metric,[43–45,52–60] which is

$$g_{\mu\nu} = -\hat{g}^{\rho\tau}\partial_\rho\phi\,\partial_\tau\phi\,\hat{g}_{\mu\nu}, \tag{30}$$

where ϕ is the mimetic scalar field instead of $g_{\mu\nu}$. The following equation is obtained by varying the metric

$$\delta g_{\mu\nu} = \hat{g}^{\rho\tau}\delta g_{\tau\omega}\hat{g}^{\omega\sigma}\partial_\rho\phi\,\partial_\sigma\phi\,\hat{g}_{\mu\nu} - \hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\,\delta\hat{g}_{\mu\nu} - 2\hat{g}^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma(\delta\phi)\,\hat{g}_{\mu\nu}. \tag{31}$$

The following field equation is obtained now by varying the action with respect to the redefined metric $\hat{g}_{\rho\sigma}$ instead of the standard Jordan frame metric $g_{\mu\nu}$[18,84]

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 8\Big[R_{\mu\delta\nu\sigma} + R_{\delta\nu}g_{\sigma\mu} - R_{\delta\sigma}g_{\nu\mu} \\ - R_{\mu\nu}g_{\sigma\delta} + R_{\mu\sigma}g_{\nu\delta} + \frac{1}{2}(g_{\mu\nu}g_{\sigma\delta} - g_{\mu\sigma}g_{\nu\delta})R\Big]\nabla^\delta\nabla^\sigma f_G \\ + (f_G G - f(G))g_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi \\ \left(-R + 8\left(-R_{\delta\sigma} + \frac{1}{2}g_{\delta\sigma}R\right)\nabla^\delta\nabla^\sigma f_G + 4(f_G G - f(G))\right) \\ = T_{\mu\nu} + \partial_\mu\phi\,\partial_\nu\phi\,T, \end{aligned} \tag{32}$$

where $2\kappa^2 = 1$, $f_G = \frac{df}{dG}$, $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu}V_\lambda$, the energy–momentum tensor $T_{\mu\nu} = \mathrm{diag.}(\rho, -p, -p, -p)$, with ρ the energy density and p the pressure. Now varying the action with respect to ϕ one obtains

$$\nabla^\mu \left[\partial_\mu \phi \left(-R - 8E_{\delta\sigma}\nabla^\delta\nabla^\sigma f_G + 4(f_G G - f(G)) - T\right)\right] = 0, \tag{33}$$

where $E_{\delta\sigma} = R_{\delta\sigma} - \frac{1}{2} g_{\delta\sigma} R$ is the Einstein tensor. From Eq. (30) one gets

$$g^{\mu\nu}\partial_\mu \phi \, \partial_\nu \phi = -1,$$

and for a homogeneous scalar field ϕ which depends only on the cosmic time, this leads to the constraint $\phi = t$. Using this constraint in a flat universe, the time–time component of the field equation in Eq. (32) can be expressed as

$$2\dot{H} + 3H^2 + 16H(\dot{H} + H^2)\frac{df_G}{dt} + 8H^2\frac{d^2 f_G}{dt^2} - (f_G G - f) = -p. \tag{34}$$

It may be mentioned here that the same equation is obtained for the space–space components. By integrating Eq. (33), one gets

$$-R - 8\left(R_{\delta\sigma} - \frac{R}{2} g_{\delta\sigma}\right)\nabla^\delta\nabla^\sigma f_G + 4(f_G G - f(G)) = -\rho + 3p - \frac{C}{a^3}. \tag{35}$$

Using the expression of the Ricci scalar with H in the RW metric, it can be rewritten as

$$\dot{H} + 2H^2 + \frac{2}{3}(f_G G - f(G)) + 4H(2\dot{H} + 3H^2)\frac{df_G}{dt} + 4H^2\frac{d^2 f_G}{dt^2} = -\frac{\rho}{6} + \frac{p}{2} - \frac{C}{a^3}, \tag{36}$$

where C is an integration constant. Using Eqs. (34) and (36), one obtains

$$\dot{H} + 4H^2\frac{d^2 f_G}{dt^2} + 4H(2\dot{H} - H^2)\frac{df_G}{dt} = -\frac{1}{2}(\rho + p) - \frac{C}{a^3}. \tag{37}$$

For simplicity the constant C is redefined here. We now introduce a function $g(t) = \frac{df_G}{dt}$, and the above equation becomes

$$4H^2\frac{dg(t)}{dt} + 4H(2\dot{H} - H^2)g(t) = -\dot{H} - \frac{(\rho + p)}{2} - \frac{C}{a^3}. \tag{38}$$

To obtain a flat EU as mentioned in Eq. (23), the above equation will generate the modified functional form of GB-gravity. The general solution now can be obtained from Eq. (38), corresponding to a functional form by integrating the differential equation.

In this section, we reconstruct the modified GB functional form which admits a flat EU emergent universe[29] obtained in Einstein's gravity (GR) with nonlinear EoS. In the absence of dark radiation ($C = 0$), Eq. (38) can be used to determine the functional form of $f(G)$ to accommodate the EU solution. In this case we consider the right-hand side of Eq. (38) as

$$\dot{H} + \frac{1}{2}(\rho + p) = \frac{1}{2}H^2. \tag{39}$$

which is obtained in GR. The EoS, $p = \frac{-2\beta-2}{3}\rho - 2\alpha\sqrt{\frac{\rho}{3}}$ for $\beta > -1$ is relevant here. The modified $f(G)$ functional forms in the presence of matter will be determined in Sec. 6. Equation (38) reduces to a differential equation

$$\frac{dg(t)}{dt} + \frac{2\dot{H} - H^2}{H} g(t) = -\frac{1}{8}. \tag{40}$$

The solution of this equation is

$$g(t) = -\frac{1}{8H_o^2}\left(\frac{H}{H_o}\right)^2 e^{\int H dt} \int H^2 e^{-\int H dt} dt + C_1 \left(\frac{H}{H_o}\right)^2 e^{\int H dt}. \tag{41}$$

Here g_t is determined for a given scale factor of the universe. We consider an EU[29] with the scale factor given by

$$a = a_o(\eta + e^{\alpha t})^{\frac{1}{\beta}}. \tag{42}$$

The Hubble parameter for the EU corresponding to the scale factor given by Eq. (24) is

$$H = \frac{H_0 e^{\alpha t}}{\eta + e^{\alpha t}}, \quad H_0 = \frac{\alpha}{\beta}.$$

Thus the general solution in terms of t is given by

$$g(t) = \frac{g_0(\eta + e^{\alpha t})^{\frac{H_0}{\alpha}+2}}{e^{2\alpha t}} + \frac{(\eta + e^{\alpha t})^2}{8H_0 e^{2\alpha t}} - \frac{\eta(\eta + e^{\alpha t})}{8(\alpha + H_o)e^{2\alpha t}}, \tag{43}$$

where g_0 is an integration constant. The functional form of modified GB gravity is then determined in terms of t as

$$f_G(t) = \int g(t) dt. \tag{44}$$

The functional form of $g(t)$ given in Eq. (43) will be used here to determine the function $f_G(t)$ for a singularity free universe for a given α.

The GB term can be expressed as a function of t for the scale factor given by Eq. (24) as

$$G = \frac{24\alpha^4}{\beta^4}\left(\frac{\beta\eta + e^{\alpha t}}{(\eta + e^{\alpha t})^4}\right) e^{3\alpha t}. \tag{45}$$

For simplicity we take $\alpha = H_0$ which leads to

$$e^{H_0 t} = \frac{\eta}{\left(\frac{G_0}{G}\right)^{1/3} - 1}, \tag{46}$$

where $G_0 = 24H_0^4$. We can now invert it to express t as a function of G, which can be written as

$$t = \frac{1}{H_0} \ln \left[\frac{\eta}{\left(\frac{G_0}{G}\right)^{1/3} - 1} \right]. \tag{47}$$

The functional form of modified GB gravity is determined as

$$f(G) = \int f_G(G) dG.$$

Integrating Eq. (44) once again, we determine the functional form of modified gravity, which is

$$\begin{aligned} f(G) = a_1 G^{1/3} - a_2 G^{2/3} + a_3 G + a_4 \ln \left[1 - \left(\frac{G_0}{G}\right)^{1/3} \right] \\ - a_5 G \ln \left[\left(\frac{G_0}{G}\right)^{1/3} - 1 \right] + a_6 \ln G \\ - a_7 \ln \left[\left(\frac{G_0}{G}\right)^{1/3} - 1 \right] + a_8, \end{aligned} \tag{48}$$

where $a_1 = (\frac{1}{32H_0^2} - \frac{3\eta g_0}{2H_0})G_0^{\frac{2}{3}}$, $a_2 = (\frac{1}{16H_0^2} + \frac{3\eta g_0}{H_0})G_0^{\frac{1}{3}}$, $a_3 = \frac{3\eta g_0}{2H_0} + \frac{5}{32H_0^2} + \frac{3\eta g_0 \ln \eta}{H_0} + \frac{\ln \eta}{8H_0^2} + g_1$, $a_4 = (\frac{3\eta g_0}{H_0} + \frac{1}{8H_0^2})G_0$, $a_5 = \frac{3\eta g_0}{H_0} + \frac{1}{8H_0^2}$, $a_6 = \frac{\eta g_0}{H_0} + \frac{1}{24H_0^2}$, $a_7 = \frac{3\eta g_0}{H_0} G_0$, $a_8 = g_2 - \frac{G_0}{3}(\frac{3\eta g_0}{H_0} + \frac{1}{8H_0^2}) \ln G_0$, where g_2 is an integration constant. Thus the modified GB gravity in the presence of matter described by an EoS is determined.

6. Discussion

In this paper, modified gravity with GB terms is reconstructed which accommodates a flat EU model in the presence or absence of matter. In the literature, a number of cosmological models with cosmological evolution having a bouncing solution are reported with a reconstructed mimetic gravity.[18,22,23] In this work we reconstructed modified GB gravity to accommodate the EU model, which is a singularity free universe. The present universe emerged from a static Einstein universe in the infinite past. As the GB terms in four dimensions is a topological term which arises in string theory, it does not contribute to the dynamics of the universe. In that case, a dilaton field coupled to the GB combination is generally considered to investigate the evolutionary features of the observed universe. In a four-dimensional universe, we determined the nonlinear functional form of the GB terms in the gravitational action without a dilaton field which accommodates the EU scenario. We note that in the absence of a dilaton field, a nonlinear combination of GB terms in four dimensions accommodates an EU scenario. We reconstructed the modified gravity action with $f(G)$ which is relevant in the early and also in the late universe. It is found that a number of modified gravity models can be obtained with different values of $\beta = \frac{3(A+1)}{2}$, a parameter which may be related to the EoS parameters of the EU model in Einstein gravity with the nonlinear EoS $p = A\rho - B\sqrt{\rho}$. It is found that for a given $\alpha = \frac{\sqrt{3}B}{2}$, the Hubble parameter decreases with an increase in β at the present epoch.

Thus GB terms play an important and significant role in understanding the observed features of the universe which predicts the existence of dark matter and dark energy. Thus a rich structure of GR modified with nonlinear combinations of GB terms in the Einstein–Hilbert action can be unearthed to understand the evolution of the universe. The cosmological reconstruction method is adopted to obtain the modified GB term in $f(G)$ gravity. Two alternative representations for the action are used, with and without matter for this purpose. It turns out that the cosmological solutions in the representation with matter follow from the wider class of theories. The cosmological solution without a big bang singularity, e.g. the EU model, is reconstructed here. This can also be obtained in Einstein gravity with matter described by a nonlinear EoS in the original formulation. The result obtained here is different from the cosmologies with a big bang singulairity and big rip singularity. We found both of them are applicable here with different functional forms. It is shown that cosmological solutions with a bounce can be reconstructed in the representations of modified gravity. We have reconstructed modified $f(G)$ gravity in the presence and absence of a definite EoS and found that a polynomial mixed with logarithmic functional forms in powers of GB-terms are obtained, which are interesting in cosmological model building for $G \neq G_0$. The functional form obtained for the EU is different from that obtained for a universe with bounce only.[18,22,23]

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References

1. E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, MA, 1990).
2. S. Dodelson, *Modern Cosmology* (Academic Press, 2003).
3. S. Weinberg *Cosmology* (Oxford University Press, 2008).
4. A. A. Starobinsky, *Phys. Lett. B* **91** (1980) 99.
5. A. G. Riess *et al.*, *Astron. J.* **116** (1998) 1009.
6. S. Perlmutter *et al.*, *Nature* **51** (1998) 391.

7. S. Perlmutter*et al.*, *Astrophys. J.* **517** (1999) 565.
8. A. H. Guth, *Phys. Rev. D* **23** (1981) 347.
9. K. Sato, *Mon. Not. R. Astron. Soc.* **195** (1981) 467.
10. A. D. Linde, *Phys. Lett. B* **108** (1982) 389.
11. A. Albrecht and P. J. Steinhardt, *Phys. Rev. Lett.* **48** (1982) 1220.
12. A. D. Linde, *Phys. Lett. B* **129** (1983) 177.
13. R. H. Dicke, P. J. E. Peebles, P. J. Roll and D. T. Wilkinson, *Astrophys. J. Lett.* **142** (1965) 414.
14. S. M. Carroll, *Living Rev. Relativ.* **4** (2001) 1.
15. S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, *Phys. Rev. D* **70** (2004) 043528.
16. S. Nojiri and S. D. Odintsov, *J. Phys. Conf. Ser.* **66** (2007) 012005.
17. S. Nojiri and S. D. Odintsov, *J. Phys. A* **40** (2007) 6725.
18. A. V. Astashenok, S. D. Odintsov and V. K. Oikonomou, *Class. Quantum Grav.* **32** (2015) 185007.
19. B. C. Paul and S. Mukherjee, *Phys. Rev. D* **42** (1990) 2595.
20. B. C. Paul, S. Mukherjee and A. Beesham, *Pramana J. Phys.* **44** (1995) 133.
21. B. C. Paul and M. Sami, *Phys. Rev. D* **70** (2004) 027301.
22. K. Bamba, A. N. Makarenko, A. N. Myagky and S. D. Odintsov, *Phys. Lett. B* **732** (2014) 349.
23. K. Bamba, S. Nojiri and S. D. Odintsov, *J. Cosmol. Astropart. Phys.* **0810** (2008) 045.
24. K. Bamba, S. D. Odintsov, L. Sebastiani and S. Zerbini, *Euro. Phys. J. C* **67** (2010) 295.
25. G. F. R. Ellis and R. Maartens, *Class. Quantum Grav.* **21** (2004) 223.
26. G. F. R. Ellis, J. Murugan and C. G. Tsagas, *Class. Quantum Grav.* **21** (2004) 233.
27. D. J. Mulryne, R. Tavakol, J. E. Lidsey and G. F. R. Ellis, *Phys. Rev. D* **71** (2005) 123512.
28. S. Mukherjee, B. C. Paul, S. D. Maharaj and A. Beesham, arXiv:gr-qc/0505103.
29. S. Mukherjee, B. C. Paul, N. Dadhich, S. D. Maharaj and A. Beesham, *Class. Quantum Grav.* **23** (2006) 6927.
30. S. Bag, V. Sahn, Y. Shtanov and S. Unnikrishnan, *J. Cosmol. Astropart. Phys.* **07** (2014) 034.
31. B. C. Paul and S. Ghose *Gen. Relativ. Gravit.* **42** (2010) 795.
32. A. Banerjee, T. Bandyopadhyay and S. Chakraborty *Gen. Relativ. Gravit.* **40** (2008) 1603.
33. A. Banerjee, T. Bandyopadhyay and S. Chakraborty, *Gravit. Cosmol.* **13** (2007) 290.
34. U. Debnath, *Class. Quantum Grav.* **25** (2008) 205019.
35. S. del Campo, R. Herrera and P. Labrana, *J. Cosmol. Astropart. Phys.* **30** (2007) 0711.
36. A. Beesham, S. V. Chervon and S. D. Maharaj *Class. Quantum Grav.* **26** (2009) 075017.
37. S. V. Chervon, S. D. Maharaj, A. Beesham and A. S. Kubasov, *Gravit. Cosmol.* **20** (2014) 176.
38. V. K. Oikonomou, *Phys. Rev. D* **92** (2015) 124027.
39. S. Nojiri, S. D. Odintsov, A. Toporensky and P. Tretyakov, *Gen. Relativ. Gravit.* **42** (2010) 1997.
40. S. Nojiri, S. D. Odintsov and O. G. Gorbunova, *J. Phys. A* **39** (2006) 6627.
41. G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, *Phys. Rev. D* **73** (2006) 084007.

42. S. Nojiri, S. D. Odintsov and M. Sami, *Phys. Rev. D* **74** (2006) 046004.
43. S. Nojiri, S. D. Odintsov and V. K. Oikonomou, *Phys. Rep.* **692** (2017) 1.
44. L. Sebastiani, S. Vagnozzi and R. Myrzakulov, *Adv. High Energy Phys.* **2017** (2017) 3156915.
45. Vagnozzi S *Class. Quantum Grav.* **34** (2017) 185006.
46. G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, *Phys. Rev. D* **75** (2007) 086002.
47. B. Li, J. D. Barrow and D. F. Mota, *Phys. Rev. D* **76** (2007) 044027.
48. A. De Felice and S. Tsujikawa, *Phys. Lett. B* **675** (2009) 1.
49. M. Alimohammadi and A. Ghalee, *Phys. Rev. D* **79** (2009) 063006.
50. C. G. Boehmer and F. S. N. Lobo, *Phys. Rev. D* **79** (2009) 067504.
51. K. Uddin, J. E. Lidsey and R. Tavakol, *Gen. Relativ. Gravit.* **41** (2009) 2725.
52. A. H. Chamseddine and V. Mukhanov, *J. High Energy Phys.* **11** (2013) 135.
53. A. H. Chamseddine, V. Mukhanov and A. Vikman, *J. Cosmol. Astropart. Phys.* **06** (2014) 017.
54. A. Golovnev, *Phys. Lett. B***728** (2014) 39.
55. N. Deruelle and J. Rua, *J. Cosmol. Astropart. Phys.* **04** (2014) 002.
56. D. Momeni, A. Altaibayeva and R. Myrzakulov , *Int. J. Geom. Methods Mod. Phys.* **11** (2014) 1450091.
57. J. Matsumoto, S. D. Odintsov and S. V. Sushkov, *Phys. Rev. D* **91** (2015) 064062.
58. G. Leon and S. N. Saridakis, *J. Cosmol. Astropart. Phys.* **04** (2015) 031.
59. R. Myrzakulov and L. Sebastiani, *Gen. Relativ. Gravit.* **47** (2015) 89.
60. D. Momeni, R. Myrzakulov and E. Güdekli, *Int. J. Geom. Methods Mod. Phys.* **12** (2015) 1550101.
61. S. Y. Zhou, E. J. Copeland and P. M. Saffin, *J. Cosmol. Astropart. Phys.* **0907** (2009) 009.
62. A. De Felice and T. Suyama, *J. Cosmol. Astropart. Phys.* **0906** (2009) 034.
63. J. Sadeghi, M. R. Setare and A. Banijamali, *Phys. Lett. B* **679** (2009) 302.
64. A. De Felice and A. Tsujikawa, *Phys. Rev. D* **80** (2009) 063516.
65. K. Bamba, C. Q. Geng, S. Nojiri and S. D. Odintsov, arXiv:0909.4397.
66. G. Cognola, M. Gastaldi and S. Zerbini, *Int. J. Theor. Phys.* **47** (2008) 898.
67. S. Nojiri, S. D. Odintsov and P. V. Tretyakov, *Prog. Theor. Phys. Suppl.* **172** (2008) 81.
68. N. Goheer, J. Larena and P. K. S. Dunsby, *Phys. Rev. D* **80** (2009) 061301.
69. G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, *Euro. Phys. J. C* **64** (2009) 483.
70. K. Bamba, Z.-K. Guo and N. Ohta, *Prog. Theor. Phys.* **118** (2007) 879.
71. S. Nojiri and S. D. Odintsov, *Phys. Rep.* **505** (2011) 59.
72. S. Nojiri and S. D. Odintsov, *Int. J. Geom. Methods Mod. Phys.* **11** (2014) 1460006.
73. V. Faraoni and S. Capozziello, *Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics*, Fundamental Theories of Physics, Vol. 170 (Springer, New York, 2010).
74. S. Capozzielo and M. De Laurentis, *Phys. Rep.* **509** (2011) 167.
75. G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini *Phys. Rev. D* **73** (2006) 084007.
76. B. C. Paul and A. S. Majumdar, *Class. Quantum Grav.* **32** (2015) 115001.
77. B. C. Paul and A. S. Majumdar, *Class. Quantum Grav.* **35** (2018) 065001.
78. B. C. Paul, P. Thakur and S. Ghose, *Mon. Not. R. Astron. Soc.* **407** (2010) 415.
79. B. C. Paul, S. Ghose and P. Thakur, *Mon. Not. R. Astron. Soc.* **413** (2011) 686.

80. B. C. Paul and P. Thakur, *Astrophys. Space Sci.* **362** (2017) 73.
81. S. Ghose, P. Thakur and B. C. Paul, *Mon. Not. R. Astron. Soc.* **421** (2012) 20.
82. B. C. Paul, P. Thakur and A. Beesham *Astrophys. Space Sci.* **361** (2016) 336.
83. B. C. Paul and P. Thakur, *J. Cosmol. Astropart. Phys.* **11** (2013) 052.
84. S. Nojiri and S. D. Odintsov, *Phys. Lett. B* **631** (2005) 1.