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RECONSTRUCTION TECHNIQUES FOR CLASSICAL INVERSE STURM-LIOUVILLE PROBLEMS

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ABSTRACT. This paper gives constructive algorithms for the classical inverse Sturm-Liouville problem. It is shown that many of the formulations of this problem are equivalent to solving an overdetermined boundary value problem for a certain hyperbolic operator. Two methods of solving this latter problem are then provided, and numerical examples are presented.

1. INTRODUCTION

The classical inverse Sturm-Liouville problem consists of recovering the potential function $q(x)$ from

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) - hy(0) = 0, \tag{1.2}$$

$$y'(1) + Hy(1) = 0, \tag{1.3}$$

and a knowledge of spectral data. This data can take various forms, and this leads to a family of inverse problems. In each of these, one assumes that a complete spectrum, $\{\lambda_j\}_{j=1}^{\infty}$, for (1.1)–(1.3) is given. It is well known that this is insufficient for recovery of q, and thus some additional information must be provided. Some of the better known versions are:

The two-spectrum case. Here, in addition to the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ for (1.1)–(1.3), we assume that a second set of eigenvalues $\{\mu_j\}_{j=1}^{\infty}$ is given, where the value of H in (1.3) is replaced by $\tilde{H}$, $\tilde{H} \neq H$. Borg [4] showed that $q(x)$ is uniquely determined from the spectra $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\mu_j\}_{j=1}^{\infty}$. This result was extended by Levinson [13] to show that two spectra uniquely determine $q(x)$ and the boundary conditions, that is, the values h, H, and $\tilde{H}$.

Spectral function data. Here one seeks to reconstruct $q(x)$ from its spectral function. This is tantamount to providing the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ and the values of the ratios (norming constants) $\rho_j := \|\phi_j\|_{L^2}^2/\phi_j(0)^2$ for finite h, or else $\rho_j := \|\phi_j\|_{L^2}^2/\phi_j'(0)^2$ when $h = \infty$. Once again (Gel'fand and Levitan

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[5]), the spectral data $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\rho_j\}_{j=1}^{\infty}$ are sufficient to determine the set $q(x)$, h, H, and $\tilde{H}$.

Endpoint data. Recently (see Pöschel and Trubowitz [20] for example), it has been shown that in addition to a complete set of eigenvalues the spectral data could consist of certain information about the eigenfunctions at the endpoints of the interval. For example, if $h = H = \infty$, set $\kappa_j := \log(|\phi_j'(1)|/|\phi_j'(0)|)$. In [20] it is shown that the two sequences $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\kappa_j\}_{j=1}^{\infty}$ determine $q(x)$ uniquely. If $h = \infty$ and $H < \infty$, we define instead $\kappa_j := \log(|\phi_j(1)|/|\phi_j'(0)|)$, and so forth for other possible combinations of h and H.

The symmetric case. If it is known a priori that $q(x)$ is symmetric about the midpoint of the interval, $q(x) = q(1-x)$, and the boundary conditions obey the symmetry condition $h = H$, then knowledge of a single spectrum $\{\lambda_j\}_{j=1}^{\infty}$ is sufficient to recover $q(x)$. This result is an old one, and was noted by Borg in his original paper.

Partially known $q(x)$. If the values of q are given over at least a half of the interval, say for $\frac{1}{2} \le x \le 1$, then again a specification of a single spectrum suffices to recover the potential (Hochstadt and Lieberman [10]).

It is often the case that there are explicit formulas relating the data for one problem to that for another one. However, these often involve infinite product expansions, and unfortunately a finite amount of spectral data of one type does not usually correspond to a finite amount of spectral data of another type. For example, knowledge of ρ_j for $0 \le j \le N$ for some fixed N does not allow one to determine μ_j for $1 \le j \le N$.

Questions of existence and uniqueness for the inverse spectral problem have been studied rather thoroughly. In contrast, there seems to be a need for further investigation of numerical reconstruction techniques. The purpose of this article is to describe a class of such methods and to analyze their behavior. In comparison with previously developed computational methods, our approach has several advantages. First, there is a relatively low operation count, namely $O(N^3) + O(M^2)$, where M is the number of grid points on which $q(x)$ is to be represented. Second, there is great flexibility in passing from one version of the inverse spectral problem to another. In particular, all of the formulations described above can be handled without too much difficulty in a unified manner. In §5 we will comment further about the comparison of our method with other reconstruction techniques.

We now briefly discuss previous work on numerical techniques for the inverse spectral problem, referring the reader to the survey article [17] for analytic methods. In the original paper by Gel'fand and Levitan a formula is developed that at face value leads to a numerical scheme. This requires that one solve the integral equation

$$K(x,t) + \int_0^x K(x,s) f(s,t)\, ds = f(x,t), \qquad 0 \le t \le x, \tag{1.4}$$

for the function $K(x,t)$ at each fixed $x \in [0,1]$. Here the function $f(x,t)$ is obtained from spectral function data. The potential is then recovered from $q(x) = 2\frac{d}{dx}K(x,x)$. Assuming a discretization of the form $x_k = k/M$, one

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must solve M integral equations in order to recover K, and in addition to the high operation count associated with the method, there are several other difficulties that limit its effectiveness. Not the least of these is computation of the data function $f(x, t)$. Another type of solution method based on the Gel'fand-Levitan integral equation is studied by McLaughlin and Handelman [16]. Sacks [21] has given an iteration method that is based on some properties of the mapping from $K(x, x)$ to $K_t(x, 0)$. This method significantly reduces the operation count from the Gel'fand-Levitan technique and leads to an efficient reconstruction of $q(x)$. In 1973, Hochstadt [9] proposed an algorithm for solving both the symmetric and the two-spectrum case. The main idea is to reduce the problem to a finite system of nonlinear ordinary differential equations. Hald [6] showed that a revised version of this algorithm will always provide a solution of the inverse problem in the symmetric case, and illustrated his method by some numerical examples. More recently [1], Andersson has extended these techniques to the so-called "impedance case," the recovery of the function $p(x)$ in $-(pu')' = \lambda pu$. An iterative method for the case of symmetric potentials has also been suggested by Barcilon [2]. For this same case, Hald [7] developed a procedure based on a finite element approach. This reduces the inverse problem for the differential equation to a nonstandard discrete inverse eigenvalue problem; a convergence and a stability result are proven. In the book [20], Pöschel and Trubowitz, based on earlier work by Isaacson, McKean, and Trubowitz [11, 12], show that one can characterize the isospectral sets corresponding to Dirichlet data, that is, the set of potentials $q(x)$ that have a given (Dirichlet) spectrum. This information is then used to solve the endpoint-data version of the inverse Sturm-Liouville problem, and numerical examples are presented for this case. Finally, we should mention a version of the inverse Sturm-Liouville problem that lies outside of the scope of the present work. In [18], McLaughlin has shown that the potential can be uniquely determined from *nodal data*, that is, information on the zeros of the eigenfunctions of the operator given by (1.1)–(1.3). Hald and McLaughlin [8] have provided an efficient algorithm to reconstruct the potential $q(x)$ from this type of data.

The plan of the paper is as follows. In the next section we will give an overview of our method in the two-spectrum case with particular boundary conditions. We will show how to translate the eigenvalue sequences into Cauchy data for a certain hyperbolic equation. We then show that the resulting overposed boundary value problem can be solved by an iterative method for the function $q(x)$. This method can be of either a quasi-Newton or of a successive approximation type. In §3 we show what modifications should be made for other versions of the inverse spectral problem. The fourth section takes on some of the issues connected with the numerical implementation of this procedure, and we will show some numerical reconstructions of the function $q(x)$ from spectral data. Section 5 discusses the question of stability of the reconstruction with errors in the data. The final section will provide an analysis of the two iteration schemes, and a convergence result will be proven.

2. An overview of the method

We begin with a description of the reconstruction method, assuming the spectral data is known completely. For definiteness let us consider the two-spectrum

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case with $h = \infty$, $H = \infty$, and $\tilde{H} = 0$. Thus, the spectral data are the two sequences $\{\lambda_j\}$ and $\{\mu_j\}$, respectively the Dirichlet eigenvalues and the Dirichlet-Neumann eigenvalues. Let us recall the asymptotic forms for these two sequences [5]:

$$\lambda_j = j^2\pi^2 + \int_0^1 q(s)\,ds + a_j, \tag{2.1}$$

$$\mu_j = \left(j - \frac{1}{2}\right)^2 \pi^2 + \int_0^1 q(s)\,ds + b_j, \tag{2.2}$$

where the sequences $\{a_j\}$, $\{b_j\}$ belong to l^2.

From the fundamental paper of Gel'fand and Levitan [5], we know that for a given $q \in L^2(0, 1)$ there exists a function $K(x, t) = K(x, t; q)$ defined on the triangle $0 \le |t| \le x \le 1$ with the following properties. First, if $\phi(x, \lambda)$ satisfies

$$\phi'' + (\lambda - q(x))\phi = 0, \tag{2.3}$$

$$\phi(0, \lambda) = 0, \qquad \phi'(0, \lambda) = 1, \tag{2.4}$$

then

$$\phi(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x K(x, t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,dt. \tag{2.5}$$

The key point here is that K does not depend on λ. Furthermore, K is also the solution of the characteristic boundary value problem

$$K_{tt} - K_{xx} + q(x)K = 0, \qquad 0 \le |t| \le x \le 1, \tag{2.6}$$

$$K(x, \pm x) = \pm\frac{1}{2}\int_0^x q(s)\,ds, \qquad 0 \le x \le 1. \tag{2.7}$$

Note that the condition on the lower characteristic $\{t = -x\}$ could be replaced by $K(x, 0) = 0$. However it will simplify the appearance of some formulas below if we can regard $K(x, t)$ as being defined for $-x < t < x$.

Recovery of Cauchy data for K. The first step in our reconstruction procedure is to use the given spectral data to determine Cauchy data for K on $\{x = 1\}$, that is, the pair of functions $\{K(1, t), K_x(1, t)\}$ for $-1 \le t \le 1$. Since they are both odd functions of t, we only need to consider $0 \le t \le 1$.

Evaluate equation (2.5) with $\lambda = \lambda_j$ and $x = 1$ to get

$$\int_0^1 K(1, t)\sin\sqrt{\lambda_j}t\,dt = -\sin\sqrt{\lambda_j}. \tag{2.8}$$

Next differentiate both sides of (2.5) with respect to x and evaluate at $\lambda = \mu_j$ and $x = 1$ to obtain

$$\int_0^1 K_x(1, t)\sin\sqrt{\mu_j}t\,dt = -\sqrt{\mu_j}\cos\sqrt{\mu_j} - \frac{1}{2}\sin\sqrt{\mu_j}\int_0^1 q(s)\,ds. \tag{2.9}$$

From (2.1) or (2.2) we may clearly obtain $\int_0^1 q(s)\,ds$ from either sequence, hence unique recovery of the Cauchy data for K follows immediately from the following completeness property.

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Lemma 1 (Levinson [14]). *The sequences* $\{\sin\sqrt{\lambda_j}t\}$ *and* $\{\sin\sqrt{\mu_j}t\}$ *are complete in* $L^2(0, 1)$.

Recovery of q from Cauchy data. Equations (2.6) and (2.7) together with the Cauchy data for K on the line $\{x = 1\}$ may be regarded as an overposed boundary value problem for K, which we expect may be used to determine $q(x)$ uniquely. That this is indeed the case is indicated by the work of Suzuki [22], who has proved some uniqueness theorems by relating spectral data to the Cauchy data for a hyperbolic boundary value problem in a slightly different way.

It is also possible to regard $q(x)$ as the solution of a certain integral equation. From the usual d'Alembert solution formula for the inhomogeneous wave equation subject to the Cauchy data $\{K(1, t), K_x(1, t)\}$ on the line $x = 1$ we have

$$K(x, x) = \frac{1}{2}[K(1, 2x-1) + K(1, 1)] + \frac{1}{2}\int_1^{2x-1} K_x(1, s)\,ds - \frac{1}{2}\int_x^1 \int_y^{2x-y} q(y)K(y, s)\,ds\,dy. \tag{2.10}$$

Differentiating with respect to x and using (2.7) gives

$$q(x) = 2[K_t(1, 2x-1) + K_x(1, 2x-1)] - 2\int_x^1 q(y)K(y, 2x-y)\,dy, \tag{2.11}$$

which we emphasize is a nonlinear equation, since $K = K(x, t; q)$. It will be shown later (Theorem 1 of §6) that the function q is uniquely determined from the Cauchy data for K using equation (2.11).

Now set

$$G_1(t) = K_t(1, t; q), \qquad G_2(t) = K_x(1, t; q),$$
$$G(x) = 2[G_1(2x-1) + G_2(2x-1)],$$

so that (2.11) is the same as

$$q(x) = G(x) - 2\int_x^1 q(y)K(y, 2x-y)\,dy. \tag{2.12}$$

For the purpose of computing the solution q from the Cauchy data we consider two iterative procedures.

Successive approximation method. For a given $q(x)$, let $u = u(x, t; q)$ solve

$$u_{tt} - u_{xx} + q(x)u = 0, \qquad 0 \le |t| \le x \le 1, \tag{2.13}$$

$$u(1, t) = K(1, t), \quad u_x(1, t) = K_x(1, t), \qquad -1 \le t \le 1. \tag{2.14}$$

Now pick an initial guess q_0 and set

$$q_{n+1}(x) = 2\frac{d}{dx}u(x, x; q_n). \tag{2.15}$$

It is not difficult to check that (2.15) is equivalent to

$$q_{n+1}(x) = G(x) - 2\int_x^1 q_n(y)u(y, 2x-y; q_n)\,dy. \tag{2.16}$$

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Quasi-Newton method. For a given $q(x)$, define the mapping

$$F(q) = \{v_t(1, t; q), v_x(1, t; q)\},$$

where $v = v(x, t; q)$ solves the characteristic initial value problem

$$v_{tt} - v_{xx} + q(x)v = 0, \qquad 0 \le |t| \le x \le 1, \tag{2.17}$$

$$v(x, \pm x) = \pm\frac{1}{2}\int_0^x q(s)\,ds, \qquad 0 \le x \le 1. \tag{2.18}$$

The solution q satisfies $F(q) = \{K_t(1, t), K_x(1, t)\} = \{G_1, G_2\}$, and if Newton's method is applied to this equation we obtain the approximating sequence $\{q_n\}$ defined by

$$q_{n+1} = q_n - DF(q_n)^{-1}(F(q_n) - \{G_1, G_2\}), \tag{2.19}$$

where $DF(q)$ denotes the Fréchet derivative of F at q. This is unsuitable for computational purposes because $DF(q_n)^{-1}$ may be too complicated to compute. However, we will argue later that $DF(0)$ is an adequate approximation to $DF(q_n)$, and furthermore $DF(0)^{-1}$ can be expressed analytically. Thus the quasi-Newton method we will use is

$$q_{n+1} = q_n - DF(0)^{-1}(F(q_n) - \{G_1, G_2\}). \tag{2.20}$$

Computing $DF(0)\delta q$ amounts simply to solving (2.17) and (2.18) with the $q(x)v$ term in (2.17) dropped, and q replaced by δq in (2.18). The inverse operator $DF(0)^{-1}$ is then easily obtained. The resulting formula is

$$q_{n+1}(x) = q_n(x) + G(x) - v_t(1, 2x - 1; q_n) - v_x(1, 2x - 1; q_n). \tag{2.21}$$

It may be verified that (2.21) is the same as

$$q_{n+1}(x) = G(x) - 2\int_x^1 q_n(y)v(y, 2x - y; q_n)\,dy. \tag{2.22}$$

Discussion. The two methods each require the solution of a boundary value problem for equation (2.6) at each step of the iteration process. Note, however, that if we make the reasonable choice $q_0 \equiv 0$, then the two methods clearly both produce $q_1(x) = G(x)$, with no solution of (2.6) being necessary. As will be seen in the numerical examples of §4, if the exact $q(x)$ is not too large, then $q_1(x)$ is often a very good approximation already. This is nothing but the statement that linearization of the inverse problem at $q = 0$ may give a good result, but the point we wish to make is that the region of validity of this approximation is actually quite large.

To explain this, heuristically at least, consider the mapping $q \mapsto F(q)$ defined above. The value of $F(q)$ depends on q in two ways; namely, it may be regarded as a functional of q in the characteristic boundary condition (2.18), which is a linear relation, and q in equation (2.17) which is nonlinear. It is not hard to convince oneself that the latter dependence is small in comparison with the former, that is to say, the mapping F is a small nonlinear perturbation of the well-behaved linear mapping obtained by dropping the $q(x)v$ term from (2.17). Since neglecting this term amounts exactly to linearizing at $q = 0$, we expect that $F(q) \approx DF(0)q$ is a good approximation when q is of moderate size. By the same token, an iterative scheme based on this linear approximation

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can be expected to converge quite rapidly, and without restriction on the size of $q - q_0$.

There is a further sense in which $q_1 = G$ is always a good approximation to q. If $q \in L^\infty(0, 1)$, then as in Sacks [21, Lemma 4.1], one can show that $K \in W^{1,\infty}$. From equation (2.12) it then follows that $q - q_1 \in C([0, 1])$, which is to say that all discontinuities of the exact solution q are already present in the approximation q_1. More generally one can show that $q - q_1$ has one degree more smoothness than q itself has.

When further iteration is necessary, we have found there to be very little difference between the two update schemes. The successive approximation method is somewhat easier to analyze (see §6), while the quasi-Newton scheme is more easily adapted to other formulations of the inverse spectral problem (see §3).

3. Modification for other inverse problems

We now discuss how the method described in the last section can be modified for other formulations of the inverse spectral problem mentioned at the beginning of the paper. In some of the cases it is possible to derive Cauchy data for K on $\{x = 1\}$ directly from the spectral data, as was done for the particular problem of §2. We may then continue the reconstruction procedure with either of the iterative schemes. A second possibility is to transform the given problem to one we have already considered by some less obvious device. Finally, it may be that we can directly recover one of the two pieces of Cauchy data directly from spectral data, and the lack of the other one is compensated for by a priori restrictions on q.

Other boundary conditions at $x = 1$. Suppose the given spectral data are the sequences $\{\lambda_j\}$ and $\{\mu_j\}$, where λ_j is the jth eigenvalue for (1.1)–(1.3) with $h = \infty$ and a given value of H, while μ_j is the same when H is replaced by $\tilde{H}$. Then using (2.5) in the obvious way, we obtain the two equations

$$\int_0^1 (K_x(1, t) + HK(1, t)) \sin \sqrt{\lambda_j} t \, dt = \alpha_j, \tag{3.1}$$

$$\int_0^1 (K_x(1, t) + \tilde{H}K(1, t)) \sin \sqrt{\mu_j} t \, dt = \beta_j, \tag{3.2}$$

where α_j and β_j are known from the spectral data. Lemma 1 applies to the two sequences $\{\lambda_j\}$ and $\{\mu_j\}$, so for $H \neq \tilde{H}$ the Cauchy data are uniquely determined, and the reconstruction proceeds as above.

Endpoint data. We suppose that the spectral data consists of the Dirichlet eigenvalues $\{\lambda_j\}$ $(h = H = \infty)$ together with the numbers

$$\kappa_j = \log \left| \frac{\phi_j'(1)}{\phi_j'(0)} \right|, \qquad j = 1, 2, \ldots, \tag{3.3}$$

where ϕ_j is the jth eigenfunction. We still obtain $K(1, t)$ from equation (2.8),

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and using (2.5), we also find

$$\int_0^1 K_x(1,t)\sin\sqrt{\lambda_j}t\,dt = \sqrt{\lambda_j}((-1)^j e^{\kappa_j} - \cos\sqrt{\lambda_j}) - \frac{1}{2}\sin\sqrt{\lambda_j}\int_0^1 q(s)\,ds.$$

Thus from Lemma 1 the Cauchy data $\{K(1,t), K_x(1,t)\}$ are again uniquely determined by the spectral data, and so we may complete the recovery of $q(x)$ just as in §2.

The case of symmetric $q(x)$. Here we suppose that $q(x) = q(1-x)$ and only Dirichlet eigenvalues $\{\lambda_j\}$ are given. There are at least three ways to proceed. First, we clearly have $\kappa_j = 0$ for all j, so that the method for the endpoint data may be applied directly. Second, the sequence of Dirichlet eigenvalues is the same as alternating Dirichlet eigenvalues for the interval $[0, \frac{1}{2}]$ with Dirichlet-Neumann eigenvalues for $[0, \frac{1}{2}]$. Thus, this is equivalent to the special case of the two-spectrum problem studied in §2.

The third possibility is a modification of the quasi-Newton method. For a given $q(x)$ define the mapping $F(q) = v_t(1,t;q)$, where $v(x,t;q)$ satisfies (2.17) and (2.18). From the given spectral data we know $G_1(t) = K_t(1,t;q)$, and the quasi-Newton method analogous to (2.20) is

$$q_{n+1} = q_n - DF(0)^{-1}(F(q_n) - G_1). \tag{3.4}$$

Now $DF(0)$ is not invertible on the whole space, but it is when restricted to the subspace of even functions on $[0,1]$. The update scheme we obtain is

$$q_{n+1}(x) = q_n(x) + 2G_1(2x-1) - 2v_t(1, 2x-1; q_n). \tag{3.5}$$

Spectral function data. We suppose that the Dirichlet eigenvalues $\{\lambda_j\}$ are known along with the sequence of norming constants

$$\rho_j = \frac{\|\phi_j\|_{L^2}^2}{\phi_j'(0)^2}. \tag{3.6}$$

While an iterative method can be developed which uses this data directly (Sacks [21]), we have actually obtained better results by converting the norming constant data into endpoint data. This may be done as follows.

If we differentiate (2.3) with respect to λ, we obtain

$$-\dot\phi'' + q(x)\dot\phi = \phi + \lambda\dot\phi,$$

where $\dot\phi$ denotes $\frac{\partial\phi}{\partial\lambda}$. Multiplying this last equation by ϕ, equation (2.3) by $\dot\phi$, and subtracting, we obtain [20, p. 30]

$$\phi^2 = \phi''\dot\phi - \dot\phi''\phi.$$

Integrating this from $x = 0$ to $x = 1$ gives

$$\int_0^1 \phi^2(x,\lambda)\,dx = \phi'(1,\lambda)\dot\phi(1,\lambda).$$

Thus we have

$$\rho_j = \phi'(1,\lambda_j)\frac{\partial\phi}{\partial\lambda}(1,\lambda_j). \tag{3.7}$$

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If we differentiate (2.5) in λ, use (2.8) to simplify, and combine with (3.7), we get

$$\frac{\phi_j'(1)}{\phi_j'(0)} = \frac{2\rho_j\lambda_j}{\cos\sqrt{\lambda_j} + \int_0^1 tK(1,t)\cos\sqrt{\lambda_j}t\,dt}. \tag{3.8}$$

We use the spectrum $\{\lambda_j\}$ to compute the function $K(1,t)$, and then (3.8) gives the values of the endpoint data in terms of the norming constants.

An alternative method, using a more traditional approach, would be to use (3.7) and the expression (see [20, p. 39])

$$\phi(1,\lambda) = \prod_{i=1}^{\infty} \frac{\lambda_i - \lambda}{i^2\pi^2}.$$

From this last formula we get

$$\begin{aligned}\frac{\partial\phi}{\partial\lambda}(1,\lambda_j) &= \frac{-1}{j^2\pi^2} \prod_{i=1,\,i\neq j}^{\infty} \frac{\lambda_i - \lambda_j}{i^2\pi^2} \\ &= \frac{1}{j^2\pi^2 - \lambda_j} \prod_{i=1,\,i\neq j}^{\infty} \frac{\lambda_i - \lambda_j}{i^2\pi^2 - \lambda_j} \prod_{i=1}^{\infty} \left(1 - \frac{\lambda_j}{i^2\pi^2}\right).\end{aligned}$$

We recognize the second infinite product as $(\sin\sqrt{\lambda_j})/\sqrt{\lambda_j}$, and the final result

$$\frac{\phi_j'(1)}{\phi_j'(0)} = -\frac{\rho_j\sqrt{\lambda_j}(j^2\pi^2 - \lambda_j)}{\sin\sqrt{\lambda_j}} \prod_{i=1,\,i\neq j}^{\infty} \left(\frac{i^2\pi^2 - \lambda_j}{\lambda_i - \lambda_j}\right) \tag{3.9}$$

follows. For the purpose of numerical solution we will always arrange that $\lambda_i = i^2\pi^2$ for large enough i, so that the product in (3.9) is always finite. In order to control roundoff error, one must be careful in the application of (3.9). The first term should be written as

$$-\frac{\rho_j\sqrt{\lambda_j}(j^2\pi^2 - \lambda_j)}{\sin\sqrt{\lambda_j}} = (-1)^j\rho_j\sqrt{\lambda_j}(j\pi + \sqrt{\lambda_j})\frac{d_j}{\sin d_j},$$

where $d_j = \sqrt{\lambda_j} - j\pi$. The direct function evaluation, $(\sin d_j)/d_j$ should be used only if d_j is bounded away from zero, otherwise the quotient should be expanded as a Taylor series. With this safeguard, formula (3.9) gave good results even for large values of N. Some further improvement can be obtained by replacing the product by a sum of the logarithms of each term and then exponentiating the result.

Partially known $q(x)$. We suppose the Dirichlet eigenvalues $\{\lambda_j\}$ are known, and that $q(x)$ itself is known on, say, the right half-interval $[\frac{1}{2}, 1]$. From the one spectrum we are able to recover one piece of Cauchy data, namely $K(1,t)$. It is now not hard to check that if we set

$$D = \{(x,t): \tfrac{1}{2} < x < 1,\ 1 - x < t < x\},$$

then $K(x,t)$ is uniquely determined in D by the conditions (2.6) and (2.7) together with the known $K(1,t)$. But then the other piece of Cauchy data $K_x(1,t)$ is uniquely determined on $(0,1)$, and we may again proceed with the iteration step.

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A second approach, probably easier in practice, is to adapt the quasi-Newton scheme as follows. Define the mapping $F(q)$ as in the symmetric q case above. If we choose an initial guess q_0 to be equal to the known $q(x)$ on $[\frac{1}{2}, 1]$, then in the quasi-Newton formula (3.4) we need to compute the inverse of $DF(0)$ restricted to the subspace of functions vanishing on $[\frac{1}{2}, 1]$. This is again a straightforward calculation, and the resulting iteration scheme is

$$q_{n+1}(x) = q_n(x) + 4G_1(2x-1) - 4v_t(1, 2x-1; q_n), \qquad 0 \le x \le \tfrac{1}{2},$$

with $G_1(t) = K_t(1, t)$ as above.

Other boundary conditions at $x = 0$. In the case that the boundary condition at $x = 0$ is (1.2) for some finite h, we can make the following modification to the above procedure. Define $\phi(x, \lambda)$ to be the solution of (2.3) with

$$\phi'(0, \lambda) = h, \qquad \phi(0, \lambda) = 1.$$

Then there exists $K(x, t) = K(x, t; q)$ such that

$$\phi(x, \lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x, t)\cos\sqrt{\lambda}t\, dt,$$

where K solves the characteristic boundary value problem

$$K_{tt} - K_{xx} + q(x)K = 0, \qquad 0 \le |t| \le x \le 1,$$
$$K(x, \pm x) = h + \frac{1}{2}\int_0^x q(s)\, ds, \qquad 0 \le x \le 1.$$

One can now proceed as in the $h = \infty$ case. Given suitable spectral data, we may uniquely recover the Cauchy data $\{K(1, t), K_x(1, t)\}$, and the successive approximation or quasi-Newton method may be adapted to recover q from the resulting overposed boundary value problem.

4. Numerical implementation and examples

In this section we shall present the results of some of the numerical computations we performed to show the performance of the algorithm described in §2. In all of these experiments we obtained spectral data, that is, we solved the direct problem, by using the FORTRAN software package *SLEIGN* kindly provided by its authors [3]. This program uses a modified Prüfer transformation to transform the second-order differential operator into a more numerically tractable system. A shooting method (using the adaptive scheme RKF45 as the integrator) is then used to approximate the eigenvalues and eigenfunctions of the equation.

Our goal is to reconstruct the function $q(x)$ as best as we can from a finite number of pieces of spectral data, and we shall always assume that this corresponds to the first N eigenvalues (or eigenvalues and norming constants, etc.). Given that uniqueness requires a complete set of spectral data, it is however necessary to make some assumptions about the missing spectral data.

Consider the case, for example, of Dirichlet eigenvalues λ_j. From the asymptotic form (2.1) we see that in the absence of any further information, a reasonable approximation is that

$$\lambda_j = (j\pi)^2 + \int_0^1 q(s)\, ds \quad \text{for } j > N. \tag{4.1}$$

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As long as

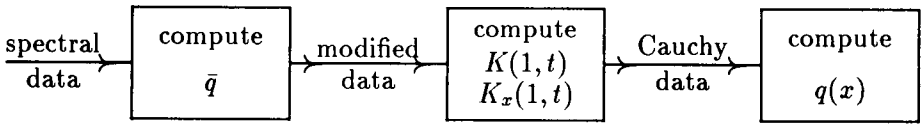
$$\lambda_1 < \lambda_2 < \cdots < \lambda_N < ((N+1)\pi)^2 + \int_0^1 q(s)\,ds, \tag{4.2}$$

it then follows from the known existence theory for the inverse Dirichlet problem (e.g., [17, 20]) that the entire infinite sequence defined by (4.1)–(4.2) is the Dirichlet spectrum of some q. The smoother q is, the faster the sequence $\{a_j\}$ in (2.1) will decay, and the more accurate this approximation will be. Of course, we do not know the mean value $\overline{q} := \int_0^1 q(s)\,ds$ either, and strictly speaking, nothing at all can be said about it from the knowledge of a finite number of eigenvalues. That is, for any $q(x)$ there exists $\hat{q}(x)$ with any prescribed mean and having any finite number of pieces of spectral data in common with q. On the other hand, if N is large enough, and q is reasonably well behaved, we expect that a good estimate for the value of $\overline{q}$ can be obtained from the identity

$$\overline{q} = \lim_{j\to\infty} (\lambda_j - j^2\pi^2). \tag{4.3}$$

For now, we will regard $\overline{q}$ as a quantity to be estimated as part of the solution, distinct from q itself, and we assume always that (4.1) holds. In the next section we will show examples of the effect that choice of $\overline{q}$ has on the reconstruction, and we will comment further on the error in q due to the approximation (4.1). All other spectral data will be approximated similarly for $j > N$, that is, taken to be the value one obtains in replacing q by the constant $\overline{q}$.

Our reconstruction procedure may be divided into the three steps indicated in the flowchart.



Computation of $\overline{q}$. If $\overline{q}$ is not known from some independent source, we must approximate it on the basis of the given spectral data. The most straightforward estimate would be

$$\overline{q} = \lambda_N - N^2\pi^2, \tag{4.4}$$

and we have found this to work well in most situations. If the data happens to be of the two-spectrum type, we get one estimate from each spectrum and the average of the two could be used. Once the value of $\overline{q}$ has been obtained, we follow the usual procedure of working with the modified data obtained by subtracting $\overline{q}$ from the given eigenvalues. We will thus reconstruct $q(x) - \overline{q}$, and then add the mean back on at the end.

Computation of the Cauchy data $\{K(1,t), K_x(1,t)\}$. According to the discussion of §§2 and 3, the Cauchy data for K are uniquely determined by systems of equations of the form

$$\int_0^1 f(t) \sin\sqrt{\lambda_j}\, t\,dt = \alpha_j, \qquad j = 1, 2, \ldots. \tag{4.5}$$

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To approximate f using N such equations, we let $\{u_k\}_{k=1}^N$ be a set of basis functions on $[0, 1]$, and look for f in the form

$$f(t) = \sum_{k=1}^{N} f_k u_k(t). \tag{4.6}$$

Then the coefficients $\vec{f} := (f_1, \ldots, f_N)$ can be computed from the system

$$A\vec{f} = \vec{\alpha}, \tag{4.7}$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$ and A is the $N \times N$ matrix with entries

$$a_{jk} = \int_0^1 u_k(t) \sin \sqrt{\lambda_j}\, t\, dt. \tag{4.8}$$

For certain choices of basis functions $\{u_k\}$, it is not hard to verify that the matrix A is always nonsingular. For example, when $u_k = \sin \sqrt{\lambda_k}\, t$, this follows easily from the linear independence of the u_k's. For some purposes, for example when the λ_j's are Dirichlet eigenvalues, a better basis is $u_k(t) = \sin k\pi t$. In this case, A is guaranteed to be nonsingular provided that $\lambda_N < (N+1)^2\pi^2$. Note that this is not really a new hypothesis but is already implied by (4.1) and the requirement (in the case that $\overline{q} = 0$) that $\lambda_1, \ldots, \lambda_N$, $(N+1)^2\pi^2, (N+2)^2\pi^2, \ldots$ be a legitimate sequence of Dirichlet eigenvalues. Note that having made the reduction to the case $\overline{q} = 0$, we will have $K(1, 0) = K(1, 1) = 0$, so this choice of basis works particularly well for recovery of $K(1, t)$ from Dirichlet eigenvalues. For either of these choices of basis, the matrix A tends to be strongly diagonally dominant, with condition number very close to 1.

When the λ_j's in (4.5) are eigenvalues corresponding to $h = \infty$ and some finite H, then a good choice of basis is to take u_k to be the kth eigenfunction of (1.1)–(1.3) with $q = 0$, namely $u_k(t) = \sin \nu_k t$, where ν_k is the solution of $z = -H \tan z$ in the range $n\frac{\pi}{2} \le z < (n+1)\frac{\pi}{2}$. However, this will require computation of the ν_k's. Comparable accuracy and conditioning will generally be obtained by taking $u_k(t) = \sin(k - \frac{1}{2})\pi t$ (i.e., the eigenfunctions when $H = 0$).

In general, there are two considerations to keep in mind in selecting the basis $\{u_k\}$. First, we want a well-conditioned matrix A, and second it should be appropriate for the particular choice of $f(t)$ in (4.5). For example, as long as $h = \infty$, it follows that $K(1, 0) = K_x(1, 0) = 0$, so it is natural to have $u_k(0) = 0$ for recovery of any combination of the Cauchy data. At the other end we will have $K(1, 1) = 0$, but $K_x(1, 1) \neq 0$ (unless $q(1) = q(0)$), so the basis $\sin k\pi t$ is best for recovery of $K(1, t)$, but not for $K_x(1, t) + HK(1, t)$.

When $h < \infty$, there are cosines instead of sines in (4.5), and we make the appropriate modifications. We may choose the basis functions either to satisfy the prescribed boundary condition or else $u_k'(0) = 0$, depending on the situation. For example, if $f(t) = K(1, t)$, then $f'(0) = 0$, so the second choice seems more natural. Note that we can always reduce to the case $K(1, 1) = 0$ without changing the other boundary condition by subtraction of a linear or quadratic function.

Finally, one can also try nontrigonometric basis functions, such as splines or orthogonal polynomials. These will give rise to a nonsingular matrix A as long

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as no function in the span of the u_k's is orthogonal to one of the $\sin\sqrt{\lambda_j}t$'s. In the case of splines, the possibilities include both linear and cubic spline bases. For smooth functions these gave reconstructions that were almost as good as those using trigonometric bases, and linear splines were only marginally poorer than cubic ones. For rough functions $q(x)$ these bases performed quite badly. We also tried Chebyshev polynomials, assuming that within the class of polynomial bases these would offer the most likely chance of giving reasonable results, and we modified them to match the correct boundary conditions. The results were rather poor—considerably poorer than even linear spline functions in all the cases we tried. In order to obtain the best performance from the method, it is essential that the choice of basis elements reflect the known boundary conditions satisfied by the function $f(t)$. In order to satisfy this constraint in the case of Chebyshev polynomials, we were forced, for example, to take only the even ones, and this necessitated the use of rather high-degree polynomial approximation.

In summary, we found no reason to deviate from the more obvious choice of trigonometric bases, since these consistently gave the best results, and these bases were used to produce all the numerical results shown in the paper.

Computation of q. The final step is to compute the solution q by either of the two iteration schemes described earlier. A convergence result for the successive approximation method is given in §6. Very rapid convergence was usually obtained by either method, as the examples below show. Carrying out this part of the reconstruction is completely straightforward, once one has a numerical method for approximation of the wave field functions $u(x, t; q)$ or $v(x, t; q)$ defined in §2. In the computations used to obtain the examples below, a second-order accurate finite difference discretization of (2.6) was used. See §3 of Sacks [21], where a more detailed discussion is given for a similar computation.

Examples. To illustrate the results of this paper we have chosen three particular potential functions, $q^{(1)}(x)$, $q^{(2)}(x)$, and $q^{(3)}(x)$ as shown in Figure 1.

The first of these represents a smooth function, the second a function with discontinuities, and the third is a continuous function without a continuous derivative and symmetric about $x = \frac{1}{2}$.

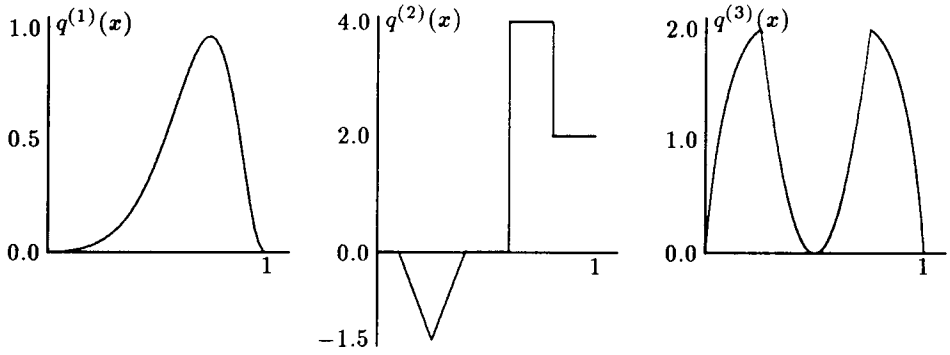

FIGURE 1
Functions used to test the algorithm

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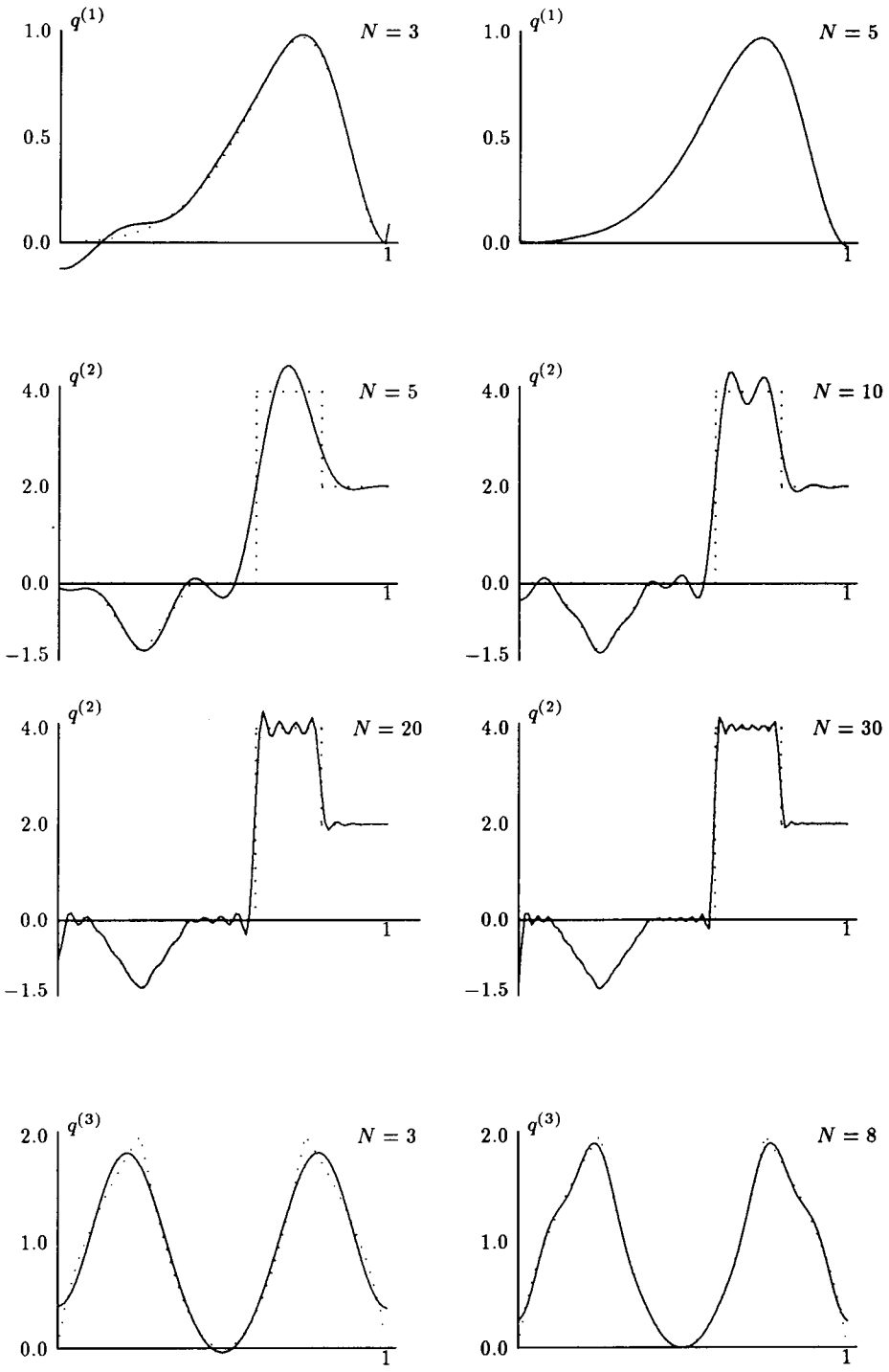

FIGURE 2
Reconstruction of $q(x)$ for various values of N

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Figure 2 shows the results of applying the algorithm to each of the functions $q^{(1)}(x)$, $q^{(2)}(x)$, and $q^{(3)}(x)$. In the case of the functions $q^{(1)}$ and $q^{(2)}$, the data was from two spectra, the first using Dirichlet conditions at both ends, the second using a Dirichlet condition at $x = 0$ and a Neumann condition at $x = 1$. For the symmetric function $q^{(3)}$ we used a single spectrum coming from Dirichlet conditions at both ends. The symmetry of the potential was then used to convert the inverse problem into one using endpoint data, as discussed in the previous section. In the case of $q^{(3)}(x)$ we provided the algorithm with the exact value of the mean $\overline{q}$.

The successive approximation method was used to compute q from the Cauchy data $K(1, t)$, $K_x(1, t)$, and this scheme was stopped at the third iteration after an initial guess of $q_0 = 0$. As the analysis of the next section will show, we require relatively fewer spectral data (smaller values of N) to reconstruct a smooth $q(x)$.

Rate of convergence of the iteration scheme. Table 1 shows the convergence rate of the iterates $q_n(x)$ to $q(x)$ for the case $q = q^{(1)}$ for the three types of data used. The results show the rapid convergence of the method, this time using the quasi-Newton method. A marginally better result would have been obtained by the successive approximation method. The slightly higher accuracy of the two-spectrum reconstruction is probably due to the forward solver producing more accurate values for the eigenvalues than for the other kinds of spectral data. This same rapid convergence holds for the less smooth functions $q^{(2)}$ and $q^{(3)}$.

TABLE 1

Convergence of the iterates $q_n(x)$ for the test function $q = q^{(1)}(x)$ using the quasi-Newton method.

Relative L^∞ and L^2 norm convergence of $q_n(x)$, $N = 10$			
n	Two spectrum	Spectral function	Endpoint data
0	1.0000 1.0000	1.0000 1.0000	1.0000 1.0000
1	0.0176 0.0182	0.0177 0.0184	0.0182 0.0185
2	0.0036 0.0009	0.0050 0.0011	0.0050 0.0015
3	0.0036 0.0008	0.0050 0.0010	0.0051 0.0015

To give some idea of the running time of our algorithm, consider the case of the potential $q^{(2)}(x)$. To obtain the reconstruction of this function shown in Figure 2 with $N = 20$ on a Sun 3/360 took 1.75 seconds. We used three iterations of the method and took $M = 100$, that is, a stepsize of 0.01. The computational cost of the algorithm is quite insensitive to the function $q(x)$ or to the value of N, at least over the ranges under consideration. The choice of stepsize in the successive approximation or quasi-Newton scheme is the dominant factor. In comparison, the cost of computing the spectral data accurate to 10^{-5} using *SLEIGN* was 134 seconds. Computing eigenvalues involves many

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calls to evaluate $q(x)$, and so the actual running time of the program is highly dependent on how expensive it is to compute q.

5. Continuous dependence on the data

The central idea in this paper is the reconstruction of the potential $q(x)$ from a finite number, N, of spectral data. Thus, by necessity, we only obtain an approximation to $q(x)$, and as a practical matter we must consider the further possibility of these spectral measurements containing errors. This section considers the dependence of the reconstruction on the number N, and on errors in the spectral data. Furthermore, our algorithm requires the mean of $q(x)$ and knowledge of the boundary conditions, although in theory these quantities can be obtained as part of the solution from a complete set of spectral data. As we indicated in the last section, we can usually estimate the value of $\bar{q}$ even from limited data, but we assumed that an independent measurement of the boundary coefficients had been made. Thus we should understand the consequence of an incorrect estimate in $\bar{q}$ and investigate the ability of the algorithm to reconstruct $q(x)$ in the presence of errors in h, H, and $\tilde{H}$.

Error due to missing spectral data. For definiteness, let us consider the two-spectrum problem of §2, and assume we have already reduced to the case $\bar{q} = 0$. From knowledge of the spectral data $\{\lambda_j\}_{j=1}^N$ and $\{\mu_j\}_{j=1}^N$ we may approximate the Cauchy data for K by solving two $N \times N$ systems as described in §4. The exact error we commit depends of course on the particular basis functions $\{u_k\}$ which are chosen.

If, for example, we choose $u_k(t) = \sin k\pi t$ in the system for $K(1, t)$, then it is not hard to see that the approximation obtained in this way is precisely the same as the solution of the full system (2.8) if $\lambda_j = (j\pi)^2$ for $j > N$. Similarly, using $u_k(t) = \sin(k - \frac{1}{2})\pi t$ in the system for $K_x(1, t)$ yields the exact solution of (2.9) if $\mu_j = (j - \frac{1}{2})^2\pi^2$ for $j > N$. According to the analysis of Hochstadt [9], if the corresponding approximate solution is q_{est}, then the difference $q - q_{est}$ is a certain combination of the jth eigenfunctions for q_{est} and q and their derivatives for $j > N$. Since either of these eigenfunctions looks very much like $\sin j\pi x$, this amounts to saying that the error is completely in the high (higher than λ_N) frequency components of q.

If we use instead the basis $u_k(t) = \sin\sqrt{\lambda_j}t$ for recovery of $K(1, t)$ (respectively $u_k(t) = \sin\sqrt{\mu_j}t$ for recovery of $K_x(1, t)$), then the approximate Cauchy data is simply the projection of the exact Cauchy data onto the finite-dimensional subspaces spanned by these basis functions. Thus, the L^2 error in the Cauchy data will tend to zero as $N \to \infty$ at a rate which is dependent on the smoothness of K, which is itself dependent on the smoothness of the exact coefficient $q(x)$. According to the analysis of §6, the corresponding L^2 error in q will also tend to zero at the same rate. In terms of fixed frequency components, it seems clear that neither of the iteration schemes can produce an updated guess with significantly more content than is already present in the data. Thus band-limited spectral data will also yield an (essentially) band-limited approximate solution for these choices of basis.

We may expect, therefore, that the error in the approximate solution due to the finiteness of the spectral data is something like the error in approximating a function by a partial sum of its Fourier series. In fact, we have observed this

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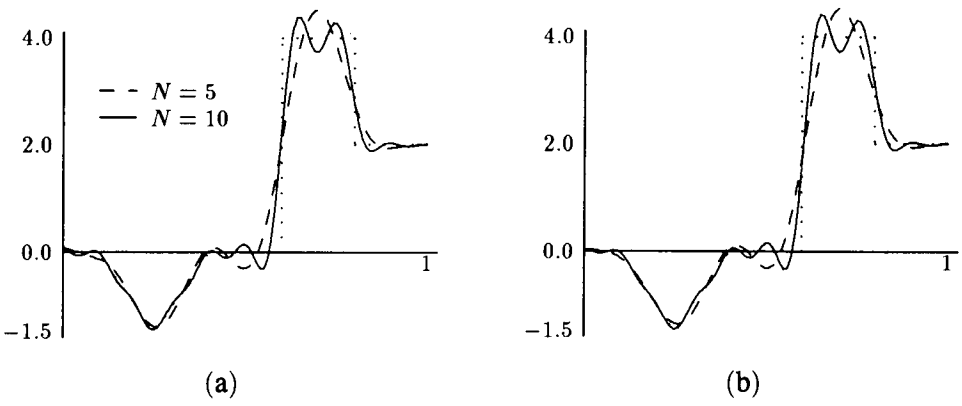

FIGURE 3
Comparison of algorithm and Fourier series reconstruction by algorithm using Fourier cosine series

to be a surprisingly accurate description. See Hald [7] for more direct analysis of the recovery of Fourier coefficients of q.

The left graph in Figure 3 shows the reconstruction of $q^{(2)}$ using our algorithm (except that we have provided the exact value of $\overline{q}$) with two spectra and $N = 5$ and $N = 10$. The graph on the right shows the corresponding 10- and 20-term partial sums of the Fourier cosine series for $q^{(2)}$ (recall a certain value of N corresponds to $2N$ pieces of spectral data).

Error in spectral data. Let us next consider the effect of error in the given spectral data, again for the two-spectra problem of §2 with $\overline{q} = 0$. We shall write the eigenvalues for the Dirichlet spectrum in the form $\sqrt{\lambda_j} = j\pi + d_j$ and the measured eigenvalues in the form $\sqrt{\lambda_j} = j\pi + d_j + \delta_j$, to indicate that one should consider the measurement errors to be of magnitude d_j rather than $\sqrt{\lambda_j}$. So the difference in the actual value of $\alpha_j := \sin\sqrt{\lambda_j}$ and its value $\overline{\alpha}_j$ based on the measured spectrum is

$$\alpha_j - \overline{\alpha}_j = (-1)^j[\sin(d_j + \delta_j) - \sin d_j],$$

so that, assuming $d_j \neq 0$,

$$|1 - \overline{\alpha}_j/\alpha_j| \leq |1 - \sin(d_j + \delta_j)/\sin d_j| \approx |\delta_j|/d_j$$

if $\delta_j < d_j \ll \pi$. This says that the relative error in α_j, and hence in the reconstructed data $K(1, t)$, due to error in the free term of (4.7) is approximately the value of the ratio of δ_j to d_j. A similar result can be derived for the effect of errors in the Dirichlet-Neumann eigenvalues $\{\mu_j\}$. Actually, these errors also cause an error in the coefficient matrix A, but these may be expected to lead to further relative error in the computed Cauchy data of a similar magnitude. Note that since $d_j \to 0$ as $j \to \infty$, and $\lambda_j \to \infty$, the relative error in λ_j itself needs to be extremely small to have useful information in it. See McLaughlin [19] for further results about continuous dependence of the solution on the spectral data.

Recovery of boundary conditions. As was mentioned earlier, if complete spectral data is available, then it is in theory possible to determine the boundary conditions as part of the solution of the problem. We do not believe, however, that this is numerically feasible in most cases. Let us use the two-spectrum case with

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finite h, H, $\tilde{H}$ as an example; similar conclusions would be obtained for the other versions of the problem.

For $q(x) \in L^2(0, 1)$, the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\tilde{\lambda}_j\}_{j=1}^{\infty}$ must satisfy the asymptotic formulae [15]

$$\lambda_j = j^2\pi^2 + A_0 + a_j, \qquad \tilde{\lambda}_j = j^2\pi^2 + \tilde{A}_0 + \tilde{a}_j, \tag{5.1}$$

where $\{a_j\}$ and $\{\tilde{a}_j\}$ are in l^2 and the constants A_0 and $\tilde{A}_0$ satisfy

$$A_0 = 2(h + H) + \int_0^1 q(s)\,ds, \qquad \tilde{A}_0 = 2(h + \tilde{H}) + \int_0^1 q(s)\,ds.$$

Furthermore, the smoother q is, the faster the decay of the sequences a_j, $\tilde{a}_j$ to 0 will take place, e.g., order of $1/j^2$ if q is C^2. Can we really compute anything from these asymptotic formulae? We may clearly expect to be able to estimate the values of A_0 and $\tilde{A}_0$ from a reasonable finite amount of spectral data, with less data being necessary when q is smoother (see discussion below). We may thus obtain approximations, say to $\bar{q}$ and $\tilde{H}$ if h and H are known. However, to recover all four of these constants, we would need to be able to accurately estimate higher-order coefficients in the asymptotic expansions of the eigenvalues, and this seems unlikely unless a large number of eigenvalues are quite accurately known.

Error in $\bar{q}$ and the boundary conditions. As we described in the previous section, our algorithm requires an estimate of the mean value $\bar{q}$, and for reasons described above we really must make the assumption that the boundary conditions are given independently of the spectral data. Now we must ask the questions: how well can we expect to compute $\bar{q}$, what are the penalties for a bad estimate, and what is the effect of measurement error in the boundary coefficients?

In Figure 4 we show the values of the quantity $\lambda_j - j^2\pi^2 - \int_0^1 q(s)\,ds$ for j in the range $4 \le j \le 30$. Note the dramatic difference in the rate of convergence of $\lambda_j - j^2\pi^2$ in each of the three cases. In the function $q^{(1)}(x)$ it is possible to make an evaluation of $\bar{q}$ correct to three or four decimal places with N about 10. For $N \approx 30$, we may even be able to compute a reasonable value for the next term in the asymptotic expansion. For the case of function $q^{(2)}$ we can really only expect to compute the mean to within a single decimal place. Of course, this behavior is simply a consequence of the fact, mentioned above, that the decay rate of the sequence $\{a_j\}$ in (5.1) is governed by the smoothness of q. There are numerous ways one can attempt to compute the value $\bar{q}$ from the data illustrated in Figure 4. Some of the more sophisticated methods can give poorer results than the simplest, as the values for the function $q^{(2)}(x)$ show. Our practice of computing the mean by using only the last term, that is, $\lambda_N - N^2\pi^2$, gave, on average, as good a result as any other method we tried. A comparison of Figures 2 and 3(a) shows the type of error that can be expected from choosing the value of the mean as described in the last section; almost all the error is made at the endpoints (in this case only the left endpoint, but this varies with the function being reconstructed and the type of spectral data).

A further example of the sensitivity of the algorithm to errors in both the computation of the mean and the boundary condition is shown in Figure 5.

In the graph on the left in Figure 5 the dashed curve represents the reconstruction of $q^{(2)}(x)$ from two spectra using $N = 10$ and the correct value

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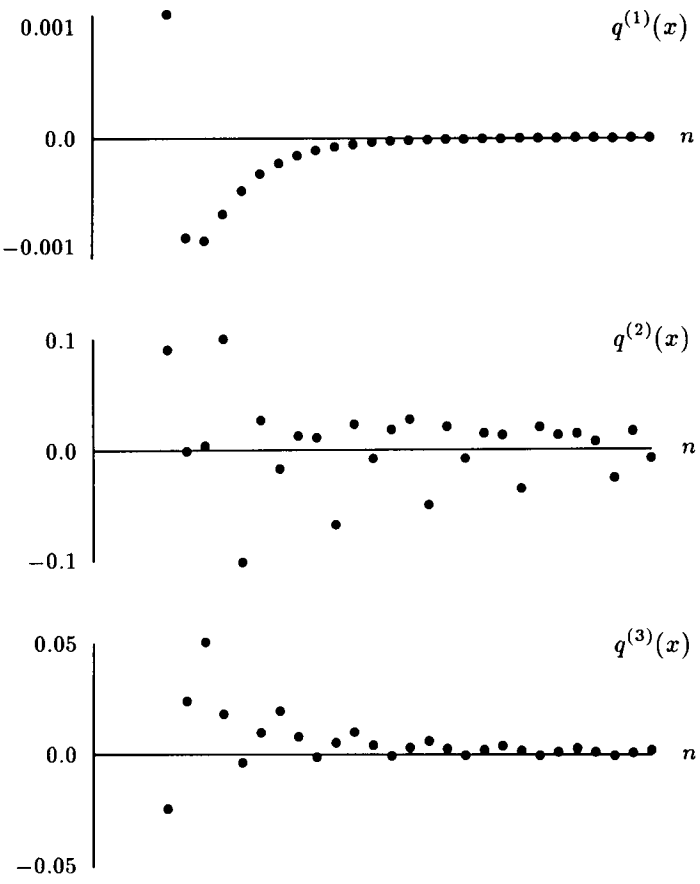
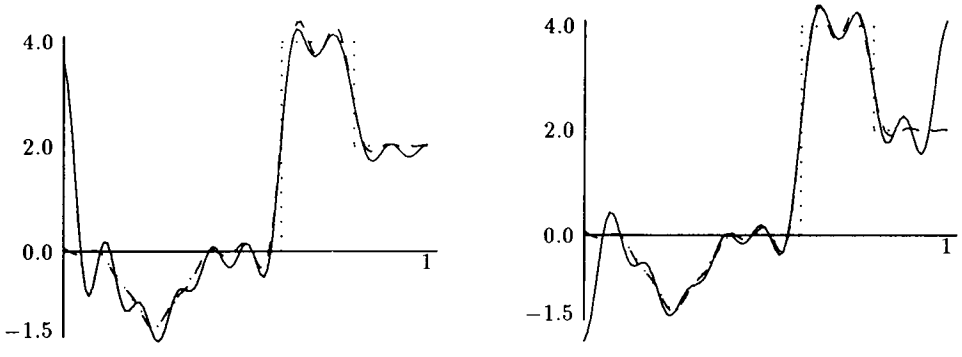


FIGURE 4
Values of $\lambda_n - n^2\pi^2 - \int_0^1 q$, $4 \le n \le 30$



With 10% error in the mean With 0.05 error in H

FIGURE 5
Reconstruction of $q^{(2)}(x)$ *with error in the mean and boundary condition*

$\overline{q} = 0.985$. The solid line shows the reconstruction using the value $\overline{q} = 1.085$, a relative error of about 10%. This represents the greatest deviation of $\lambda_j - j^2\pi^2$

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from $\overline{q}$ over the range $4 \leq j \leq 30$. This worst case gives encouragement that a highly accurate value of $\overline{q}$ is not essential to the reconstruction of the potential function. However, the algorithm does produce a solution with the prescribed value of $\overline{q}$, so that the L^2 error in q cannot be less than the error in $\overline{q}$.

If an error is made in determining the boundary condition, what is the effect? If we fix the left-hand boundary condition to be Dirichlet $(h = \infty)$, then for finite H the eigenvalues $\{\mu_j\}$ satisfy

$$\mu_j = (j - \tfrac{1}{2})^2\pi^2 + A_0 + a_j,$$

where $\{a_j\} \in l^2$ and $A_0 = 2H + \overline{q}$. From this we may expect that errors in $\overline{q}$ and H are somewhat interchangeable in the sense that we will be computing $2H + \overline{q}$ from A_0 and will interpret an error in H as an incorrect $\overline{q}$ with the same magnitude of error. For example, if the correct value of H is 0.05, but we believe it to be $H = 0$, then the reconstructed q, using the correct $\overline{q}$, is shown in the right-hand graph of Figure 5, again using $N = 10$. The magnitude of the errors in the two reconstructions, which correspond to identical errors in A_0, are comparable, although distributed somewhat differently.

Comparison with other reconstruction methods. To conclude this section, we would like to mention a typical source of computational difficulty which is *not* present in the algorithms discussed here. Consider, for example, the case when the data are Dirichlet eigenvalues and norming constants, $\{\lambda_j\}, \{\rho_j\}$. Solution methods based on the Gel'fand-Levitan integral equation (1.4), and also on the approach of Sacks [21], require the computation from the spectral data of the auxiliary function $g(t)$ defined by

$$g(t) = \sum_{1}^{\infty} (2\pi j \sin j\pi t - (\sin\sqrt{\lambda_j}t)/\sqrt{\lambda_j}\rho_j).$$

For example, in [21], the function $q_1(x) = 2g(2x)$ is exactly the first approximation to the solution q in the iterative procedure described there. In any case, the success of the algorithm is limited by any errors present in g. Now g is actually a series for a reasonably well-behaved function (exact smoothness depending on the smoothness of q), but is plainly the difference of two rather singular functions, and so convergence of the series can be quite slow. This is especially true if the differences $\lambda_j - j^2\pi^2$ or the corresponding quantity for the norming constants are not tending to zero very rapidly. Poor convergence is also to be expected if the endpoint values are inconsistent with the terms of the series, as happens, for example, if $q(0) \neq 0$.

The point then is that if the method requires a direct computation of $g(t)$ from the above definition, we may get a poor result because g is represented in the "wrong" basis, and there is no apparent way of expressing g in terms of a more suitable basis. A great advantage of the method which we have used in this paper is the flexibility to use more appropriate basis functions. As a dramatic example of this, we have obtained far better reconstruction of $q(x) = \cos 2\pi x$ with $N = 2$ using the method of this paper that can be obtained by the method of [21] with $N = 20$. An attempt to solve the same problem using a more direct discretization of the Gel'fand-Levitan integral equation we expect would give an even worse result because the function g is used in defining the function $f(x, t)$, and this appears both as the free term and as the kernel of the integral equation.

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6. Uniqueness and convergence results

In this section we return to the problem of recovering q from Cauchy data for K, and prove results about uniqueness and convergence of the successive approximation method. As the analysis is somewhat similar to corresponding material in Sacks [21], we will be brief.

Let us assume that the left boundary condition is Dirichlet $(h = \infty)$, and the Cauchy data $\{K(1, t), K_x(1, t)\}$ has been constructed from the spectral data in one of its forms, in the manner discussed earlier. Recall that the function q is a fixed point of the mapping

$$q \mapsto 2\frac{d}{dx}u(x, x; q) := T(q), \tag{6.1}$$

where $u(x, t; q)$ is defined by (2.13) and (2.14).

Theorem 1. *The mapping T has at most one fixed point in $L^\infty(0, 1)$.*

Proof. For a fixed $M > 0$, define $C_M = \{q \in L^2(0, 1): |q(x)| \le M \text{ a.e.}\}$. Let P_M denote the operator of projection onto C_M, that is,

$$P_M h(x) = \begin{cases} h(x) & \text{if } |h(x)| \le M, \\ \pm M & \text{if } \pm h(x) \ge M. \end{cases}$$

Suppose q and $\hat{q}$ are fixed points of T, and choose M so that $|q|, |\hat{q}| < M$. Thus, q and $\hat{q}$ are also fixed points of $P_M T$, and we are done if we show that $q \mapsto P_M T$ is a contraction on C_M in the weighted norm

$$\|q\|_\lambda^2 := \int_0^1 q^2(x) e^{2\lambda(x-1)}\, dx,$$

for some sufficiently large λ.

We clearly have

$$\|P_M T(q) - P_M T(\hat{q})\|_\lambda \le \|T(q) - T(\hat{q})\|_\lambda, \tag{6.2}$$

and using (2.16), we get

$$T(q)(x) - T(\hat{q})(x) = 2\int_x^1 (\hat{q}(y) - q(y)) u(y, 2x - y; q)\, dy + 2\int_x^1 \hat{q}(y)[u(y, 2x - y; \hat{q}) - u(y, 2x - y; q)]\, dy.$$

By introducing the Riemann function for $Lu = u_{xx} - u_{tt} + q(x)u$, the second term on the right may be rewritten in the form

$$\int_x^1 Q(x, y)(\hat{q}(y) - q(y))\, dy, \tag{6.3}$$

with a bounded kernel Q depending on q and $\hat{q}$. We see, therefore, that

$$|T(q) - T(\hat{q})| \le C\int_x^1 |q(y) - \hat{q}(y)|\, dy, \tag{6.4}$$

and so by a standard calculation it follows that

$$\|T(q) - T(\hat{q})\|_\lambda \le \frac{C}{\sqrt{\lambda}}\|q - \hat{q}\|_\lambda. \tag{6.5}$$

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Thus, by choosing λ sufficiently large, we obtain the necessary contracting property of $P_M T$. □

As a corollary, we have a new proof of uniqueness for all of the forms of the inverse spectral problem mentioned in §§2 and 3.

The proof of Theorem 1 shows that $P_M T$ has at most one fixed point q, and clearly if $|q(x)| < M$, then q is also a fixed point of T. Therefore, if the inverse spectral problem has a solution q with $|q(x)| < M$, then it may be obtained as the unique fixed point of $P_M T$. Furthermore, the sequence $\{q_n\}$ defined by $q_{n+1} = P_M T q_n$ is guaranteed to converge to q in $L^2(0, 1)$ for any choice of $q_0 \in L^\infty(0, 1)$. The successive approximation procedure described in §2 is the iteration of T rather than $P_M T$, but these are of course the same, provided the sequence of iterates remains bounded and M is chosen sufficiently large. The role of the projection operator is to ensure that all of the action takes place on a bounded set in $L^\infty(0, 1)$. If the solution being sought is itself bounded, then we expect that it does not hurt to truncate q_{n+1} wherever it gets too large. Of course, knowing when it is safe to truncate requires some a priori knowledge of q, but a crude upper bound for $|q(x)|$ can always be obtained from the spectral data (e.g., [6, 19]). We remark also, that with only a finite number of pieces of spectral data available, we have construed the problem in such a way that the solution we seek must always be bounded. In practice, we have never found the extra step of truncation to be necessary.

Theorem 2. *Let $q \in L^\infty(0, 1)$ be the solution of the inverse spectral problem and choose $M > \|q\|_{L^\infty}$. Pick $q_0 \in L^\infty(0, 1)$ and for $n > 0$ define $q_{n+1} = P_M T q_n$. Then $q_n \to q$ uniformly on $[0, 1]$.*

Proof. By the above remarks and the proof of Theorem 1 it follows that $q_n \to q$ in $L^2(0, 1)$. To obtain the uniform convergence, we observe that since the sequence $\{q_n\}$ is uniformly bounded, the corresponding solutions $u(\cdot, \cdot; q_n)$ are uniformly bounded in $W^{1,\infty}$ (see [21, Lemma 4.1]). From (2.12) and (2.16) it follows that the sequence $\{T(q_n) - q\}$ is an equicontinuous family, and so the $\{q_n - q\}$ are equicontinuous. By the Arzelà-Ascoli theorem, the uniform convergence follows. □

We have been unable to prove a corresponding convergence result for the quasi-Newton method, which was also described in §2. The principal obstacle is that (6.4) is no longer valid when $u(x, t; q)$ is replaced by $v(x, t; q)$—a similar equation is obtained except that the integration is from 0 to 1. Thus (2.22) is in the nature of an iterative scheme for a Fredholm integral equation, rather than a Volterra equation. As mentioned earlier, we have never found a significant difference between (2.16) and (2.22), and the explanation, no doubt, is that (2.22) is a small perturbation of a Volterra update method.

Bibliography

1. L. E. Andersson, *Algorithms for solving inverse eigenvalue problems for Sturm-Liouville equations*, Inverse Methods in Action (P. C. Sabatier, ed.), Springer-Verlag, 1990, pp. 138–145.
2. V. Barcilon, *Iterative solution of the inverse Sturm-Liouville problem*, J. Math. Phys. **15** (1974), 429–436.
3. P. B. Bailey, B. S. Garbow, H. G. Kaper, and A. Zettl, *Eigenvalue and eigenfunction computations for Sturm-Liouville problems*, Trans. Math. Software (to appear).

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4. G. Borg, *Eine Umkehrung der Sturm-Liouville Eigenwertaufgabe*, Acta Math. **76** (1946), 1–96.
5. I. M. Gel'fand and B. M. Levitan, *On the determination of a differential equation from its spectral function*, Amer. Math. Soc. Transl. **1** (1951), 253–304.
6. O. H. Hald, *The inverse Sturm-Liouville problem with symmetric potentials*, Acta Math. **141** (1978), 263–291.
7. ____, *The inverse Sturm-Liouville problem and the Rayleigh-Ritz method*, Math. Comp. **32** (1978), 687–705.
8. O. H. Hald and J. R. McLaughlin, *Solutions of inverse nodal problems*, Inverse Problems **5** (1989), 307–347.
9. H. Hochstadt, *The inverse Sturm-Liouville problem*, Comm. Pure Appl. Math. **26** (1973), 715–729.
10. H. Hochstadt and B. Lieberman, *An inverse Sturm-Liouville problem with mixed given data*, SIAM J. Appl. Math. **34** (1976), 676–680.
11. E. L. Isaacson and E. Trubowitz, *The inverse Sturm-Liouville problem*. I, Comm. Pure Appl. Math. **36** (1983), 767–783.
12. E. L. Isaacson, H. P. McKean, and E. Trubowitz, *The inverse Sturm-Liouville problem*. II, Comm. Pure Appl. Math. **37** (1984), 1–11.
13. N. Levinson, *The inverse Sturm-Liouville problem*, Math. Tidsskr. B **25** (1949), 25–30.
14. ____, *Gap and density theorems*, Amer. Math. Soc. Colloq. Publ., vol. 26, Amer. Math. Soc., Providence, RI, 1940; reprinted 1968.
15. B. M. Levitan, *Inverse Sturm-Liouville problems*, VNU Science Press, Utrecht, 1987.
16. J. R. McLaughlin and G. H. Handelman, *Sturm-Liouville inverse eigenvalue problems*, Mechanics Today, the Reissner Volume, Pergamon Press, New York, 1980, pp. 281–295.
17. J. R. McLaughlin, *Analytic methods for recovering coefficients in differential equations from spectral data*, SIAM Rev. **28** (1986), 53–72.
18. ____, *Inverse spectral theory using nodal points as data—a uniqueness result*, J. Differential Equations **73** (1988), 354–362.
19. ____, *Stability theorems for two inverse spectral problems*, Inverse Problems **4** (1988), 529–540.
20. J. Pöschel and E. Trubowitz, *Inverse spectral theory*, Academic Press, London, 1987.
21. P. E. Sacks, *An iterative method for the inverse Dirichlet problem*, Inverse Problems **4** (1988), 1055–1069.
22. T. Suzuki, *Gel'fand-Levitan's theory, deformation formulas and inverse problems*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **32** (1985), 223–271.

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