

RECOVERY OF A DENSITY FROM THE EIGENVALUES OF A NONHOMOGENEOUS MEMBRANE

C. Maeve McCarthy

Department of Mathematics & Statistics
Murray State University
Murray, Kentucky 42071
Email: maeve.mccarthy@murraystate.edu

ABSTRACT

The vibrating elastic membrane is a classical problem in Mathematical Physics which arises in a wide variety of physical applications. Since the geometry of the membrane is usually well defined for a particular problem, determination of the nature of any nonhomogeneity is critical. The eigenvalues of particular membranes are often quite accessible experimentally and so a method for the determination of the nonhomogeneity based on the available eigenvalues is of practical importance. Projection of the boundary value problem and its coefficients onto appropriate vector spaces leads to a matrix inverse problem. Although the matrix inverse problem is of nonstandard form, it can be solved by a fixed-point iterative method. Convergence of the method for a rectangular membrane is discussed and numerical evidence of the success of the method is presented.

Introduction

The modes of vibration or *eigenvalues* of any physical object depend on a variety of factors. Typically the geometry and boundary conditions governing a particular object can be determined by inspection of the object and its circumstances. For example, when a violin string is pulled taut over its bridge, it can be treated as a one-dimensional string of fixed length. A drum is typically described by a clamped membrane whose geometry is described by the shape of the drum. Thin plates and shells also have specific geometries and boundary conditions under different circumstances.

and Consider the situation where the geometry, boundary

conditions, and frequencies or *eigenvalues* are known and the parameters governing the material's constitution are sought. This class of problems are known as *inverse spectral problems*. Is it possible to recover the parameters that led to those eigenvalues? If an infinite number of eigenvalues is available, it may be possible. In general, if the number of eigenvalues is finite, the answer is no. However in certain circumstances it is possible to recover an approximation to the unknown nonhomogeneity, although this recovery may not be unique.

We begin by posing the inverse eigenvalue problem for the clamped membrane. A matrix inverse eigenvalue problem based on this is formulated. The solution of the matrix inverse problem via a fixed-point method is discussed. A number of numerical examples are presented.

The nonhomogeneous membrane inverse problem

Consider a clamped drum or nonhomogeneous membrane over a 2-dimensional region Ω . The vibrations of the membrane satisfy the boundary value problem

$$-\nabla(p\nabla u) + qu = \lambda ru, (x, y) \in \Omega, \quad (1)$$

$$u(x, y) = 0 \quad \text{on} \quad \partial\Omega. \quad (2)$$

The reader is referred to Love (1944) or Courant & Hilbert (1953) for further details. We seek to recover the unknown parameters p, q, r from the eigenvalues of (3-4).

There is extensive literature for one-dimensional inverse spectral problems, McLaughlin (1986). Much of the literature

on the two-dimensional problem concerns the recovery of a potential q when $p = r = 1$. The first general uniqueness result for the two-dimensional potential problem was not discovered until 1988 when Nachman, Sylvester and Uhlmann (1988) established the q is uniquely determined by the Dirichlet eigenvalues and the normal derivatives of the eigenfunctions on the boundary. Barcilon (1990) showed that when Ω is the unit disk and the boundary conditions are of Neumann type (i.e. the normal derivative is zero on $\partial\Omega$), the potential q can be recovered from the eigenvalues and eigenfunction data. El Badia (1989) established a uniqueness result for q independent of one direction on the unit square. Seidman (1988) established an approximation method for the recovery of rotationally symmetric q . Knobel & McLaughlin (1994) extended a one-dimensional technique due to Hald (1978) to the two-dimensional case with symmetric potential.

In this paper, we seek to generalize Hald's technique further to the recovery of a positive density on a known rectangle $R = (0, \pi/a) \times (0, \pi)$. That is, given the eigenvalues of the boundary value problem,

$$-\Delta u = \lambda \rho u \quad \text{on } R \quad (3)$$

$$u = 0 \quad \text{on } \partial R, \quad (4)$$

we seek to recover the density function $\rho(x, y) > 0$. The density ρ is assumed to be symmetric with respect to the midlines of the rectangle. Ultimately, the goal is to apply this approach to the general problem in future work.

Formulation of the Matrix Inverse Problem

Given the lowest m eigenvalues $\{\lambda_i(\rho)\}_{i=1}^m$ of the boundary value problem (3-4), we seek to recover an approximation $\hat{\rho}$ to the unknown symmetric density ρ . With finite data, it seems pragmatic to seek a finite-dimensional framework in which to proceed. Of particular interest is the method developed by Hald (1978) based on the conversion of the one-dimensional inverse potential problem to a finite dimensional matrix inverse problem.

Hald considered the one-dimensional potential problem

$$-u'' + qu = \lambda u, \quad 0 < x < \pi, \quad (5)$$

$$u(0) = u(\pi) = 0, \quad (6)$$

for potentials q even with respect to the midpoint $x = \pi/2$. Using the eigenfunctions of the $q = 0$ case, $\{\sin jx\}_{j=1}^n$, as a basis for an n -dimensional vector space V , the eigenfunction u was projected onto V . Using the even functions $\{\cos 2jx\}_{j=1}^m$, as a basis for an m -dimensional vector space W , the unknown potential q was projected onto W . Using these projections in a Rayleigh-Ritz formulation of the boundary value problem (3-4), Hald arrived at the matrix inverse problem:

Given scalars $\{\lambda_i\}_{i=1}^m$, find a vector $\alpha \in R^m$ such that the matrix $A(\alpha)$ given by

$$a_{ij}(\alpha) = i^2 \delta_{ij} + \sum_{k=1}^m \alpha_k \frac{4}{\pi} \int_0^\pi \cos 2kx \sin ix \sin jx dx \quad (7)$$

has $\{\lambda_i\}_{i=1}^m$ as its m smallest eigenvalues.

The solution α of the matrix inverse problem yields an approximation

$$\hat{q}(x) = 2 \sum_{k=1}^m \alpha_k \cos 2kx \quad (8)$$

to the unknown potential q .

This method was extended to the two-dimensional potential problem by Knobel and McLaughlin (1994). Following their work closely, we can extend the method to the density case. The eigenvalues and L^2 -orthonormal eigenfunctions of the case $\rho = 1$ are given by

$$\lambda_i^0 = a^2 n_i^2 + m_i^2 \quad (9)$$

$$\phi_i(x, y) = \frac{2\sqrt{a}}{\pi} \sin(an_i x) \sin(am_i y) \quad (10)$$

where n_i and m_i are positive integers, ordered such that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots + \infty$$

The set $\{\phi_i\}_{i=1}^n$ forms a basis for an n -dimensional vector space V . The eigenfunction u is projected onto V .

In order to take advantage of the structure of this particular problem, let us reformulate the boundary value problem (3-4). Let $\gamma = 1/\lambda$ and consider instead the boundary value problem

$$\rho(x, y)u = -\gamma \Delta u \quad \text{on } R \quad (11)$$

$$u = 0 \quad \text{on } \partial R, \quad (12)$$

The projection of u onto the n -dimensional vector space V leads to the finite-dimensional matrix eigenvalue problem

$$DU = \gamma \Lambda_n^0 U$$

where u is an n -vector, $\Lambda_n^0 = \text{diag}(\lambda_1^0, \dots, \lambda_n^0)$, and $D_{ij} = \int_R \rho \phi_i \phi_j$. The k -th eigenvalue of this system gives a lower bound approximation to $\gamma_k(\rho)$.

We seek approximations to symmetric density functions which are small perturbations of $\rho = 1$. Let us consider

$$\hat{\rho}(x, y) = 1 + r(x, y) = 1 + \sum_{i=1}^m \alpha_i \psi_i(x, y) \quad (13)$$

where $\{\psi_i\}_{i=1}^m$ are the symmetric L^∞ functions

$$\psi_i(x, y) = \frac{2\sqrt{a}}{\pi} \sin((2n_i - 1)ax) \sin((2m_i - 1)y)$$

where n_i, m_i are the integers used in (9). Essentially, we are using $\{\psi_i\}_{i=1}^m$ as a basis for an m -dimensional vector space W and projecting $\rho - 1$ onto W .

Use of this projection leads to the finite-dimensional matrix inverse problem:

Given scalars $\{\gamma_i\}_{i=1}^m$, find a vector $\alpha \in R^m$ such that the matrix $B(r)$ given by

$$b_{ij}(\alpha) = \frac{1}{\lambda_i^0} \left(\delta_{ij} + \sum_{k=1}^m \alpha_k \int_R \psi_k \phi_i \phi_j \right) \quad (14)$$

has $\{\gamma_i\}_{i=1}^m$ as its m largest eigenvalues.

The solution α of the matrix inverse problem yields the approximation (13) to the unknown density ρ .

Solution of the Matrix Inverse Problem

Generally speaking matrix inverse problems have multiple solutions. In fact, when $m = n$ there may be as many as n^n solutions to the problem. There are many algorithms for the solution of the special cases of the inverse eigenvalue problem, see Chu's survey article (1998). Ji (1998) developed an algorithm for the construction of all of the solutions by treating the problem as a multi-parameter eigenvalue problem. For examples of the many Newton-based iterative methods that have been developed, see Biegler-König (1981) and Friedland, Nocedal and Overton (1987). In the case where $m < n$, least-square formulations are typically used to generalize other algorithms.

The approach used here is to reformulate the problem as a system of non-linear equations and to use a fixed-point approach. By writing the non-linear system in fixed-point form $\beta = F(\beta)$, existence of small solutions and convergence as $n \rightarrow \infty$ of the matrix inverse problem to solutions of the inverse boundary value problems originally posed can be established. Proof of the convergence of this algorithm will appear in a more general framework (McCarthy, 1999). The relevant assumptions are stated here for completeness.

Let B_j be the sub-matrix of B formed by deleting the j th row and j th column from B , and let V_j be the $(n - 1)$ -column vector formed from the j th column of B by deleting b_{jj} . Then the matrix

$$\tilde{B} = \begin{bmatrix} b_{jj} & V_j^T \\ V_j & B_j \end{bmatrix}$$

is similar to B and has the same eigenvalues. For the remainder of this paper we shall assume that the eigenvalues of the boundary value problem (3-4) with $\rho = 1$ are simple. Thus the minimal eigenvalue separation defined by

$$v = \min_{1 \leq i \neq j \leq m+1} \left| \frac{1}{\lambda_i^0} - \frac{1}{\lambda_j^0} \right|$$

is strictly positive, i.e. $v > 0$. If

$$\|\alpha\|_2 < \frac{v\pi\lambda_1^0}{4\sqrt{a}} \quad \text{and} \quad 0 < \left| \gamma_j - \frac{1}{\lambda_j^0} \right| < \frac{v}{2}$$

then it can be shown that $(B_j - \gamma_j)$ is nonsingular. It follows that γ_j is an eigenvalue of \tilde{B} when

$$(b_{jj} - \gamma_j) - V_j^T (B_j - \gamma_j)^{-1} V_j = 0, \quad j = 1, \dots, m. \quad (15)$$

The solution of the matrix eigenvalue problem (14) satisfies

$$\alpha = \Psi^{-1} (\Lambda_m^0 (\Gamma + G(\alpha)) - I) \quad (16)$$

where

$$\begin{aligned} \Psi_{ij} &= \int_R \psi_j \phi_i^2 \\ \Lambda_m^0 &= \text{diag}(\lambda_1^0, \dots, \lambda_m^0) \\ \Gamma &= (\gamma_1, \dots, \gamma_m)^T \\ G &= (G_1, \dots, G_m)^T \\ G_j(\alpha) &= V_j^T(\alpha) (B_j(\alpha) - \gamma_j)^{-1} V_j(\alpha) \end{aligned}$$

In order to establish existence of a solution (14), the problem must be restricted to a sufficiently small ball in R^m that the function

$$F(\alpha) = \Psi^{-1} (\Lambda_m^0 (\Gamma + G(\alpha)) - I) \quad (17)$$

maps onto itself and is a contraction. If

$$\|\Gamma - (\Lambda^{0m})^{-1}\|_2 \leq \frac{\pi^2}{128 \|\Psi^{-1}\|_2^2 a \sqrt{m}} \left(\frac{\lambda_1^0}{\lambda_m^0} \right) \nu$$

then $F : B_\delta \rightarrow B_\delta$ where B_δ is a ball of radius δ . If

$$\kappa = \lambda_m^0 \left(\frac{1}{2\lambda_1^0 \pi} + \frac{1}{32 \|\Psi^{-1}\|_2} \right) < 1$$

then F is also a contraction. It should be noted that for the choice of ψ and hence the matrix Ψ in this paper, these conditions amount to the number of eigenvalues being measured and hence the number of terms being recovered for the approximation (13) being small. In fact we must restrict $m < 8$ in order to guarantee the existence and uniqueness of small solutions to the matrix inverse problem.

Each matrix inverse problem has a solution α^n associated with it that depends on the dimension n of the underlying system. Convergence of these solutions can be proved under the assumption that n is large enough.

Numerical Results

Each of the following examples is on the rectangle $R = (0, \pi/a) \times (0, \pi)$ with $a = \sqrt{0.7}$ which for the boundary value problem (13) with $\rho = 1$ has a minimal eigenvalue separation $\nu > 0.1$ for the first 56 eigenvalues. The computations were all carried out with $n = 64$ basis functions for the vector space V giving 64×64 matrices. The number of basis functions m for the vector space W is varied in order to show the affects of allowing the theoretical assumptions guaranteeing our contractive map to break down. Let

$$d_h(x, y) = h^2 - \left(x - \frac{\pi}{2a}\right)^2 - 4\left(y - \frac{\pi}{2}\right)^2$$

and define

$$r_h(x, y) = \begin{cases} e^{-1/d_h(x, y)} & \text{if } d(x, y) > 0 \\ 0 & \text{otherwise} \end{cases}$$

The data was generated using Matlab's PDE Toolbox which uses a finite element method with a piecewise linear triangulation, and the recovery algorithm was implemented in Matlab.

Example 1

We seek to recover an approximation to the density

$$\rho_1(x, y) = 1 + r_{3\pi/8}(x, y)$$

from the first m eigenvalues of the boundary value problem (13). Figure 1 shows the function ρ_1 . The eigenvalues of (3-4) with $\rho = \rho_1$ are given in Table 1. Figures 2, 3 and 4 show the recovery of ρ_1 from 4, 8 and 16 eigenvalues respectively. Notice that the use of fewer eigenvalues gives the better result since $m < 8$ guarantees existence and uniqueness of small solutions to the matrix inverse problem. Increasing m amplifies the error in the approximation. Figure 5 shows the best possible projection by this method by showing $\rho_1 = 1 + \hat{r}$ where \hat{r} is the truncated Fourier sine series of r , the perturbation of the density from 1.

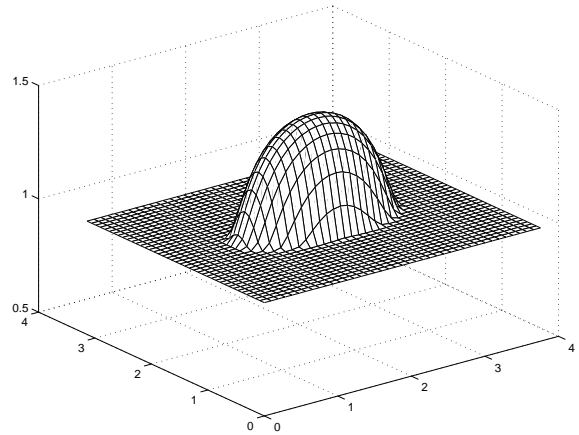


Figure 1. The function ρ_1

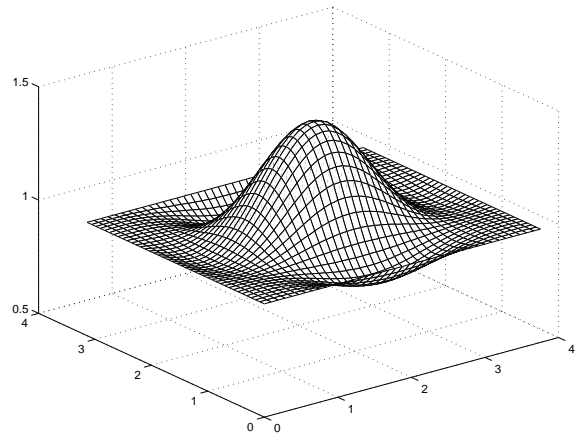


Figure 2. The recovery of ρ_1 using 4 eigenvalues

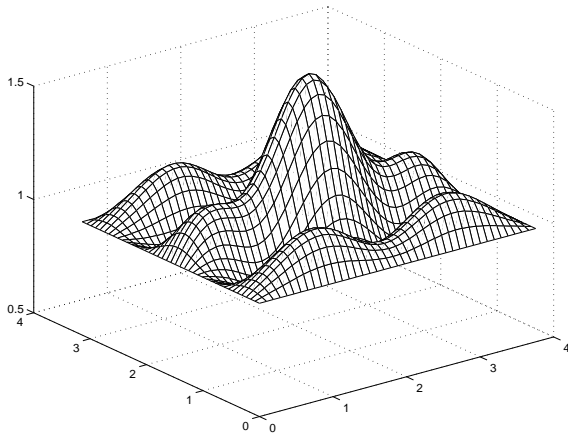


Figure 3. The recovery of ρ_1 using 8 eigenvalues

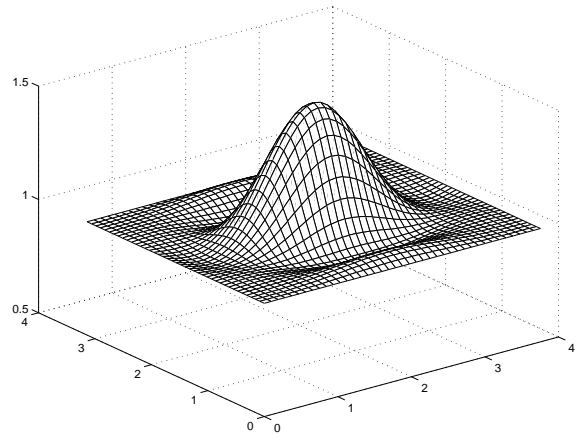


Figure 5. $1 + \hat{\rho}$, where $\hat{\rho}$ is the Fourier sine series of $\rho_1 - 1$ with 8 terms

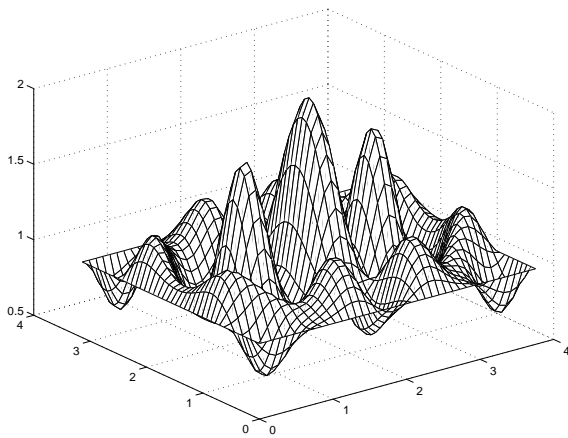


Figure 4. The recovery of ρ_1 using 16 eigenvalues

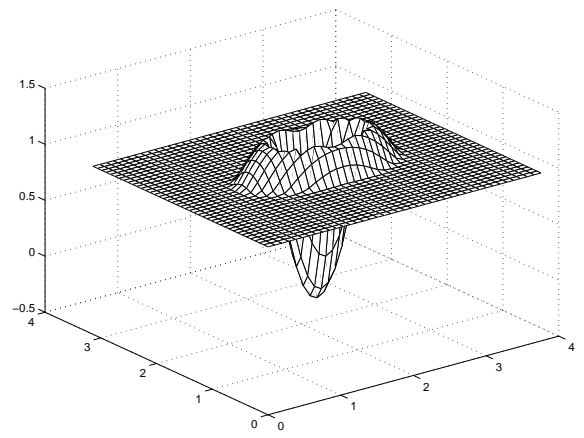


Figure 6. The function ρ_2

Example 2

We seek to recover an approximation to the density

$$\rho_2(x, y) = 1 + r_{3\pi/8}(x, y) - 8.25r_{\pi/4}$$

from the first m eigenvalues of the boundary value problem (13). Figure 6 shows the function ρ_2 . The eigenvalues of (3-4) with $\rho = \rho_2$ are given in Table 1. Figures 7, 8 and 9 show the recovery of ρ_2 from 4, 8 and 16 eigenvalues respectively. In this case, the higher number of prescribed eigenvalues gives the better result. Both recoveries locate the perturbation, but neither one recovers the amplitude of the perturbation well. Figure 10 shows the best possible projection by this method by showing $\rho_2 = 1 + \hat{r}$ where \hat{r} is the truncated Fourier sine series of r , the perturbation of the density from 1.

Conclusions

The recovery of an approximation to a symmetric density for a nonhomogeneous membrane is possible given a limited number of eigenvalues. It may be possible to apply the method to non-symmetric densities through an appropriate choice of the vector space W . The success of the method relies on the assumption of a small perturbation from the density $\rho = 1$. The issue of the nonexistence of multiple eigenvalues for the $\rho = 1$ case remains unaddressed. Knobel and McLaughlin relied heavily on the symmetry of q to address this issue, and it is anticipated that their result can be applied to this problem.

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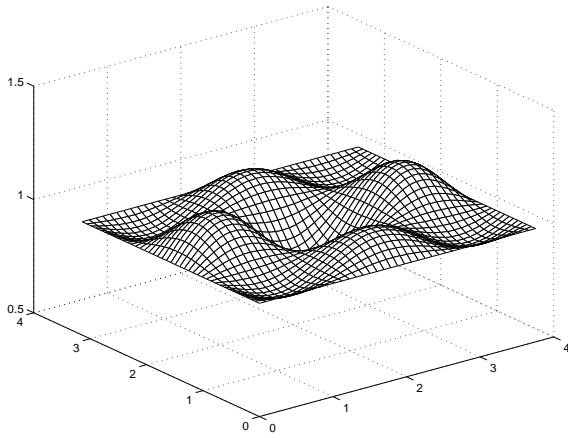


Figure 7. The recovery of ρ_2 using 4 eigenvalues

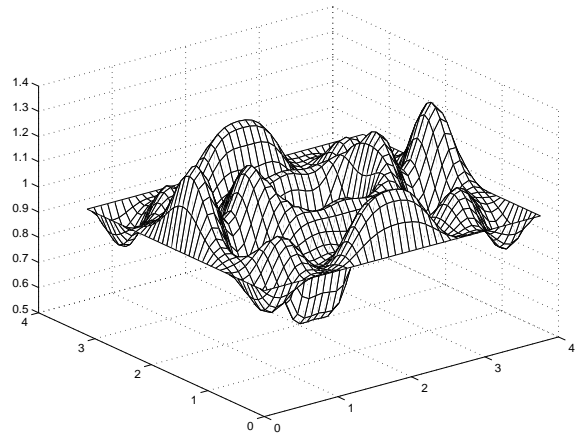


Figure 9. The recovery of ρ_2 using 16 eigenvalues

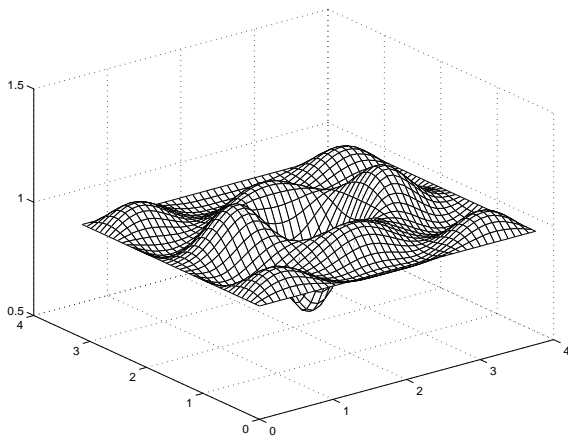


Figure 8. The recovery of ρ_2 using 8 eigenvalues

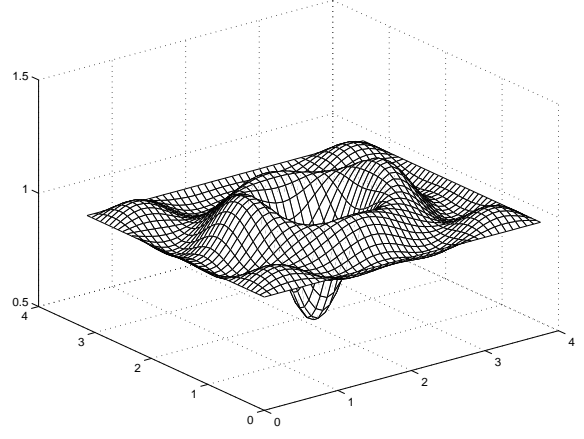


Figure 10. $1 + \hat{\rho}$, where $\hat{\rho}$ is the Fourier sine series of $\rho_2 - 1$ with 8 terms

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Table 1. Eigenvalue Data

i	(n_i, m_i)	λ_i^0	$\lambda_i(\rho_1)$	$\lambda_i(\rho_2)$
1	(1,1)	1.7	1.4860	1.7250
2	(2,1)	3.8	3.5696	3.6724
3	(1,2)	4.7	4.5737	4.6255
4	(2,2)	6.8	6.7400	6.7509
5	(3,1)	7.3	6.7425	7.419
6	(1,3)	9.7	9.0075	9.9915
7	(3,2)	10.3	10.1307	10.2309
8	(2,3)	11.8	10.7096	11.5009
9	(4,1)	12.2	11.8523	12.0626
10	(4,2)	15.2	14.5549	15.0285
11	(3,3)	15.3	14.9412	15.8789
12	(1,4)	16.7	15.5623	16.1735
13	(5,1)	18.5	17.1382	18.4452
14	(2,4)	18.8	18.3369	18.4939
15	(4,3)	20.2	19.1837	20.1705
16	(5,2)	21.5	20.9599	21.3119

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